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Research article

On parametric types of Apostol Bernoulli-Fibonacci, Apostol Euler-Fibonacci, and Apostol Genocchi-Fibonacci polynomials via Golden calculus

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Abstract: This paper aims to give generating functions for the new family of polynomials, which are called parametric types of the Apostol Bernoulli-Fibonacci, the Apostol Euler-Fibonacci, and the Apostol Genocchi-Fibonacci polynomials by using Golden calculus. Numerous properties of these polynomials with their generating functions are investigated. These generating functions give us a generalization of some well-known generating functions for special polynomials such as Apostol Bernoulli-Fibonacci, Apostol Euler-Fibonacci, and Apostol Genocchi-Fibonacci polynomials. Using the Golden differential operator technique, the functional equation method for generating function, we present some properties of these newly established polynomials.

Keywords: Appell polynomials; Golden calculus; generating functions; determinants
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1. Introduction

Special polynomials, generating functions, and the trigonometric functions are used not only in mathematics but also in many branches of science such as statistics, mathematical physics, and engineering.

Let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ indicate the set of positive integers, the set of integers, the set of real numbers, and the set of complex numbers, respectively. Let $\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \in \mathbb{C}$ (or $\mathbb{R}$).

The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following exponential generating function (see [1–3]):

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt}, \tag{1.1}$$

$$(\lambda \in \mathbb{C};\ \ |t| < 2\pi,\ \text{when } \lambda = 1;\ \text{and } |t| < \left|\log \lambda\right|,\ \text{when } \lambda \neq 1). \tag{1.2}$$

Note that $\mathcal{B}_n^{(\alpha)}(x;1) = \mathcal{B}_n^{(\alpha)}(x)$ denote the Bernoulli polynomials of order α and $\mathcal{B}_n^{(\alpha)}(0;\lambda) = \mathcal{B}_n^{(\alpha)}(\lambda)$ denote the Apostol-Bernoulli numbers of order α, respectively. Setting $\alpha = 1$ into (1.1), we get $\mathcal{B}_n^{(1)}(\lambda) = \mathcal{B}_n(\lambda)$ which are the so-called Apostol-Bernoulli numbers.

The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following exponential generating function (see [4, 5]):

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt}, \tag{1.3}$$

$$\left(|t| < \pi \text{ when } \lambda = 1;\ \ |t| < |\log(-\lambda)| \text{ when } \lambda \neq 1; 1^{\alpha} := 1\right). \tag{1.4}$$

By virtue of (1.3),we have $\mathcal{E}_n^{(\alpha)}(x;1) = \mathcal{E}_n^{(\alpha)}(x)$ denote the Euler polynomials of order α and $\mathcal{E}_n^{(\alpha)}(0;\lambda) = \mathcal{E}_n^{(\alpha)}(\lambda)$ denote the Apostol-Euler numbers of order α, respectively. Setting $\alpha = 1$ into (1.3), we get $\mathcal{E}_n^{(1)}(\lambda) = \mathcal{E}_n(\lambda)$ which are the so-called Apostol-Euler numbers.

The Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order α are defined by means of the following exponential generating function (see [6]):

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt}, \tag{1.5}$$

$$\left(|t| < \pi \text{ when } \lambda = 1;\ \ |t| < |\log(-\lambda)| \text{ when } \lambda \neq 1; 1^{\alpha} := 1\right). \tag{1.6}$$

By virtue of (1.5),we have $\mathcal{G}_n^{(\alpha)}(x;1) = \mathcal{G}_n^{(\alpha)}(x)$ denote the Genocchi polynomials of order α and $\mathcal{G}_n^{(\alpha)}(0;\lambda) = \mathcal{G}_n^{(\alpha)}(\lambda)$ denote the Apostol-Genocchi numbers of order α, respectively. Setting $\alpha = 1$ into (1.5), we get $\mathcal{G}_n^{(1)}(\lambda) = \mathcal{G}_n(\lambda)$ which are the so-called Apostol-Genocchi numbers.

In recent years, many generalizations of these polynomials have been studied by mathematicians. See for example [7–20]. With the aid of these polynomials two parametric kinds of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi of order α defined by Srivastava et al. [15, 19] whose generating functions are given by

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(c,\alpha)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt}\cos(yt), \tag{1.7}$$

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(s,\alpha)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt}\sin(yt), \tag{1.8}$$

and

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(c,\alpha)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt}\cos(yt), \tag{1.9}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(s,\alpha)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt}\sin(yt), \tag{1.10}$$

and

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(c,\alpha)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt}\cos(yt), \tag{1.11}$$

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(s,\alpha)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt}\sin(yt). \tag{1.12}$$

Remark 1.1. Note that the symbols c and s occurring in the superscripts on the left-hand sides of these last Eqs (1.7)–(1.12) indicate the presence of the trigonometric cosine and the trigonometric sine functions, respectively, in the generating functions on the corresponding right-hand sides.

The motivation of this paper is to obtain F-analogues of the Eqs (1.7)–(1.12) with the help of the Golden calculus. Namely, we define the parametric Apostol Bernoulli-Fibonacci, the Apostol Euler-Fibonacci, and the Apostol Genocchi-Fibonacci polynomials by means of the Golden Calculus. Utilizing the Golden-Euler formula and these generating functions with their functional equations, numerous properties of these polynomials are given. The special cases of these polynomials and numbers are studied in detail. The rest of this paper is structured as follows. In Section 2, we present some key definitions and properties that are crucial to Golden calculus. Then, with the help of the Golden calculus, we mention some polynomials that have been previously defined in the literature. In Section 3, considering the properties of Golden calculus, we introduce six families of two-parameter polynomials with the help of Golden trigonometric functions and exponential functions. Then, in the three subsections of this section, we examine the various properties of these polynomials defined with the help of generating functions and their functional equations.

2. Golden calculus

In this part of the our paper, we mention some definitions and properties related to Golden calculus (or F-calculus).

The Fibonacci sequence is defined by means of the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

where $F_0 = 0$, $F_1 = 1$. Fibonacci numbers can be expressed explicitly as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. $\alpha \approx 1,6180339\ldots$ is called Golden ratio. The golden ratio is frequently used in many branches of science as well as mathematics. Interestingly, this mysterious number also appears in architecture and art. Miscellaneous properties of Golden calculus have been defined and studied in detail by Pashaev and Nalci [21]. Therefore, [21] is the key reference for Golden calculus. In addition readers can also refer to Pashaev [22], Krot [23], and Ozvatan [24].

The product of Fibonacci numbers, called F-factorial was defined as follows:

$$F_1 F_2 F_3 \ldots F_n = F_n!, \tag{2.1}$$

where $F_0! = 1$. The binomial theorem for the F-analogues (or-Golden binomial theorem) are given by

$$(x+y)_F^n = \sum_{k=0}^{n} (-1)^{\binom{k}{2}} \binom{n}{k}_F x^{n-k} y^k, \tag{2.2}$$

in terms of the Golden binomial coefficients, called as Fibonomials

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}! F_k!}$$

with n and k being nonnegative integers, $n \geq k$. Golden binomial coefficients (or-Fibonomial coefficients) satisfy the following identities as follows:

$$\binom{n}{k}_F = \beta^k \binom{n-1}{k}_F + \alpha^{n-k} \binom{n-1}{k-1}_F,$$

and

$$\binom{n}{k}_F = \alpha^k \binom{n-1}{k}_F + \beta^{n-k} \binom{n-1}{k-1}_F.$$

The Golden derivative defined as follows:

$$\frac{\partial_F}{\partial_F x} (f(x)) = \frac{f(\alpha x) - f\left(-\frac{x}{\alpha}\right)}{\left(\alpha - \left(-\frac{1}{\alpha}\right)\right) x} = \frac{f(\alpha x) - f(\beta x)}{(\alpha - \beta) x}. \tag{2.3}$$

The Golden Leibnitz rule and the Golden derivative of the quotient of $f(x)$ and $g(x)$ can be given as

$$\frac{\partial_F}{\partial_F x} (f(x)g(x)) = \frac{\partial_F}{\partial_F x} f(x)g(\alpha x) + f(-\frac{x}{\alpha}) \frac{\partial_F}{\partial_F x} g(x),$$

$$\frac{\partial_F}{\partial_F x} \left(\frac{f(x)}{g(x)}\right) = \frac{\frac{\partial_F}{\partial_F x} f(x)g(\alpha x) - \frac{\partial_F}{\partial_F x} g(x) f(\alpha x)}{g(\alpha x) g(\frac{-x}{\alpha})},$$

respectively. The first and second type of Golden exponential functions are defined as

$$e_F^x = \sum_{n=0}^{\infty} \frac{(x)_F^n}{F_n!}, \tag{2.4}$$

and

$$E_F^x = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{(x)_F^n}{F_n!}. \tag{2.5}$$

Briefly, we use the following notations throughout the paper

$$e_F^x = \sum_{n=0}^{\infty} \frac{x^n}{F_n!}, \tag{2.6}$$

and

$$E_F^x = \sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \frac{x^n}{F_n!}. \tag{2.7}$$

Using the Eqs (2.2), (2.4), and (2.5), the following equation can be given

$$e_F^x E_F^y = e_F^{(x+y)_F}. \tag{2.8}$$

The Fibonacci cosine and sine (Golden trigonometric functions) are defined by the power series as

$$\cos_F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{F_{2n}!}, \tag{2.9}$$

and

$$\sin_F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{F_{2n+1}!}. \tag{2.10}$$

For arbitrary number k, Golden derivatives of e_F^{kx}, E_F^{kx}, $\cos_F(kx)$, and $\sin_F(kx)$ functions are

$$\frac{\partial_F}{\partial_F x}\left(e_F^{kx}\right) = k e_F^{kx}, \tag{2.11}$$

$$\frac{\partial_F}{\partial_F x}\left(E_F^{kx}\right) = k E_F^{-kx}, \tag{2.12}$$

$$\frac{\partial_F}{\partial_F x}\left(\cos_F(kx)\right) = -k\sin_F(kx), \tag{2.13}$$

and

$$\frac{\partial_F}{\partial_F x}\left(\sin_F(x)\right) = k\cos_F(kx). \tag{2.14}$$

Using (2.4), Pashaev and Ozvatan [25] defined the Bernoulli-Fibonacci polynomials and related numbers. After that Kus et al. [26] introduced the Euler-Fibonacci numbers and polynomials. Moreover they gave some identities and matrix representations for Bernoulli-Fibonacci polynomials and Euler-Fibonacci polynomials. Very recently, Tuglu and Ercan [27] (also, [28]) defined the generalized Bernoulli-Fibonacci polynomials and generalized Euler-Fibonacci polynomials, namely, they studied the Apostol Bernoulli-Fibonacci and Apostol Euler-Fibonacci of order α as follows:

$$\left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_{n,F}^{\alpha}(x;\lambda)\frac{t^n}{F_n!},$$

and

$$\left(\frac{2}{\lambda e_F^t + 1}\right)^{\alpha} e_F^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_{n,F}^{\alpha}(x;\lambda)\frac{t^n}{F_n!}.$$

3. Generalized parametric types of Apostol Bernoulli-Fibonacci, Apostol Euler-Fibonacci, and Apostol Genocchi-Fibonacci polynomials via Golden calculus

Krot [23] defined the fibonomial convolution of two sequences as follows. Let a_n and b_n are two sequences with the following generating functions

$$A_F(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{F_n!} \text{ and } B_F(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{F_n!},$$

then their fibonomial convolution is defined as

$$c_n = a_n * b_n = \sum_{l=0}^{n} \binom{n}{k}_F a_l b_{n-l}.$$

So, the generating function takes the form

$$C_F(t) = A_F(t)B_F(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{F_n!}.$$

Let $p, q \in \mathbb{R}$. The Taylor series of the functions $e_F^{pt}\cos_F(qt)$ and $e_F^{pt}\sin_F(qt)$ can be express as follows:

$$e_F^{pt}\cos_F(qt) = \sum_{n=0}^{\infty} C_{n,F}(p,q)\frac{t^n}{F_n!}, \tag{3.1}$$

and

$$e_F^{pt}\sin_F(qt) = \sum_{n=0}^{\infty} \mathcal{S}_{n,F}(p,q)\frac{t^n}{F_n!}, \tag{3.2}$$

where

$$C_{n,F}(p,q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k}_F p^{n-2k}q^{2k}, \tag{3.3}$$

$$\mathcal{S}_{n,F}(p,q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1}_F p^{n-2k-1}q^{2k+1}. \tag{3.4}$$

By virtue of above definitions of $C_{n,F}(p,q)$ and $\mathcal{S}_{n,F}(p,q)$ and the numbers $\mathcal{B}_{n,F}^{(\alpha)}(\lambda)$, $\mathcal{E}_{n,F}^{(\alpha)}(\lambda)$ and $\mathcal{G}_{n,F}^{(\alpha)}(\lambda)$, we can define two parametric types of the Apostol Bernoulli-Fibonacci polynomials, the Apostol Euler-Fibonacci polynomials, and the Apostol Genocchi-Fibonacci polynomials of order α, as follows:

$$\begin{aligned}
\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda) &= \mathcal{B}_{n,F}^{(\alpha)}(\lambda) * C_{n,F}(p,q), \\
\mathcal{B}_{n,F}^{(s,\alpha)}(p,q;\lambda) &= \mathcal{B}_{n,F}^{(\alpha)}(\lambda) * \mathcal{S}_{n,F}(p,q), \\
\mathcal{E}_{n,F}^{(c,\alpha)}(p,q;\lambda) &= \mathcal{E}_{n,F}^{(\alpha)}(\lambda) * C_{n,F}(p,q), \\
\mathcal{E}_{n,F}^{(s,\alpha)}(p,q;\lambda) &= \mathcal{E}_{n,F}^{(\alpha)}(\lambda) * \mathcal{S}_{n,F}(p,q), \\
\mathcal{G}_{n,F}^{(c,\alpha)}(p,q;\lambda) &= \mathcal{G}_{n,F}^{(\alpha)}(\lambda) * C_{n,F}(p,q), \\
\mathcal{G}_{n,F}^{(s,\alpha)}(p,q;\lambda) &= \mathcal{G}_{n,F}^{(\alpha)}(\lambda) * \mathcal{S}_{n,F}(p,q),
\end{aligned}$$

whose exponential generating functions are given, respectively, by

$$\left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{pt}\cos_F(qt) = \sum_{n=0}^{\infty} \mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)\frac{t^n}{F_n!}, \tag{3.5}$$

$$\left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{pt}\sin_F(qt) = \sum_{n=0}^{\infty} \mathcal{B}_{n,F}^{(s,\alpha)}(p,q;\lambda)\frac{t^n}{F_n!}, \tag{3.6}$$

$$\left(\frac{2}{\lambda e_F^t + 1}\right)^{\alpha} e_F^{pt}\cos_F(qt) = \sum_{n=0}^{\infty} \mathcal{E}_{n,F}^{(c,\alpha)}(p,q;\lambda)\frac{t^n}{F_n!}, \tag{3.7}$$

$$\left(\frac{2}{\lambda e_F^t + 1}\right)^{\alpha} e_F^{pt}\sin_F(qt) = \sum_{n=0}^{\infty} \mathcal{E}_{n,F}^{(s,\alpha)}(p,q;\lambda)\frac{t^n}{F_n!}, \tag{3.8}$$

$$\left(\frac{2t}{\lambda e_F^t + 1}\right)^{\alpha} e_F^{pt}\cos_F(qt) = \sum_{n=0}^{\infty} \mathcal{G}_{n,F}^{(c,\alpha)}(p,q;\lambda)\frac{t^n}{F_n!}, \tag{3.9}$$

$$\left(\frac{2t}{\lambda e_F^t + 1}\right)^{\alpha} e_F^{pt}\sin_F(qt) = \sum_{n=0}^{\infty} \mathcal{G}_{n,F}^{(s,\alpha)}(p,q;\lambda)\frac{t^n}{F_n!}. \tag{3.10}$$

Remark 3.1. By virtue of (3.5) and (3.6), when $\lambda \neq 1$, $\mathcal{B}_{0,F}^{(c,\alpha)}(p,q;\lambda) = 0$ and when $\lambda = 1$, $\mathcal{B}_{0,F}^{(c,\alpha)}(p,q;1) = 1$. Moreover for $\forall\lambda \in \mathbb{C}$, $\mathcal{B}_{0,F}^{(s,\alpha)}(p,q;\lambda) = 0$.

Remark 3.2. By virtue of (3.7), when $\lambda = -1$, $\mathcal{E}_{0,F}^{(c,\alpha)}(p,q;-1)$ is undefined and when $\lambda \neq -1$, $\mathcal{E}_{0,F}^{(c,\alpha)}(p,q;\lambda) = \left(\frac{2}{\lambda+1}\right)^{\alpha}$. Also, from (3.8), when $\lambda \neq -1$, $\mathcal{E}_{0,F}^{(s,\alpha)}(p,q;\lambda) = 0$. For $\lambda = -1$, $\mathcal{E}_{0,F}^{(s,\alpha)}(p,q;-1)$ is determined according to the values of α.

Remark 3.3. By virtue of (3.9) and (3.10), when $\lambda \neq -1$, $\mathcal{G}_{0,F}^{(c,\alpha)}(p,q;\lambda) = 0$ and $\lambda = -1$, $\mathcal{G}_{0,F}^{(c,\alpha)}(p,q;-1) = (-2)^{\alpha}$. Moreover for $\forall\lambda \in \mathbb{C}$, $\mathcal{G}_{0,F}^{(s,\alpha)}(p,q;\lambda) = 0$.

Remark 3.4. If we take $\alpha = 1$ and $q = 0$ in (3.5), (3.7), and (3.9), we get Apostol Bernoulli-Fibonacci polynomials, Apostol Euler-Fibonacci polynomials, and Apostol Genocchi-Fibonacci polynomials

$$\frac{te_F^{pt}}{\lambda e_F^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_{n,F}(p;\lambda)\frac{t^n}{F_n!}, \tag{3.11}$$

$$\frac{2e_F^{pt}}{\lambda e_F^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_{n,F}(p;\lambda)\frac{t^n}{F_n!}, \tag{3.12}$$

$$\frac{2te_F^{pt}}{\lambda e_F^t + 1} = \sum_{n=0}^{\infty} \mathcal{G}_{n,F}(p;\lambda)\frac{t^n}{F_n!}, \tag{3.13}$$

respectively. If we take $p = 0$ in (3.11)–(3.13), we obtain Apostol Bernoulli-Fibonacci numbers $\mathcal{B}_{n,F}(\lambda)$, Apostol Euler-Fibonacci numbers $\mathcal{E}_{n,F}(\lambda)$, and Apostol Genocchi-Fibonacci numbers $\mathcal{G}_{n,F}(\lambda)$.

3.1. Basic properties of $\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)$ *and* $\mathcal{B}_{n,F}^{(s,\alpha)}(p,q;\lambda)$

Theorem 3.1. *The following identities hold true:*

$$\mathcal{B}_{n,F}^{(c,\alpha)}(p+r,q;\lambda) = \sum_{k=0}^{n} (-1)^{\binom{n-k}{2}} \binom{n}{k}_F \mathcal{B}_{k,F}^{(c,\alpha)}(p,q;\lambda)\, r^{n-k}, \tag{3.14}$$

and

$$\mathcal{B}_{n,F}^{(s,\alpha)}(p+r,q;\lambda) = \sum_{k=0}^{n} (-1)^{\binom{n-k}{2}} \binom{n}{k}_F \mathcal{B}_{k,F}^{(s,\alpha)}(p,q;\lambda)\, r^{n-k}. \tag{3.15}$$

Proof. By applying (3.5), we first derive the following functional equation:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{B}_{n,F}^{(c,\alpha)}(p+r,q;\lambda)\frac{t^n}{F_n!} &= \left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{(p+r)_F t}\cos_F(qt) \\ &= \left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{pt}\cos_F(qt)\, E_F^{rt},\end{aligned}$$

which readily yields

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{B}_{n,F}^{(c,\alpha)}(p+r,q;\lambda)\frac{t^n}{F_n!} &= \left(\sum_{n=0}^{\infty}\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)\frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty}(-1)^{\binom{n}{2}}\frac{(rt)^n}{F_n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{\binom{n-k}{2}}\binom{n}{k}_F\mathcal{B}_{k,F}^{(c,\alpha)}(p,q;\lambda)\,r^{n-k}\right)\frac{t^n}{F_n!}.\end{aligned}$$

Comparing the coefficients of t^n on both sides of this last equation, we have

$$\mathcal{B}_{n,F}^{(c,\alpha)}(p+r,q;\lambda)=\sum_{k=0}^{n}(-1)^{\binom{n-k}{2}}\binom{n}{k}_F\mathcal{B}_{k,F}^{(c,\alpha)}(p,q;\lambda)\,r^{n-k},$$

which proves the result (3.14). The assertion (3.15) can be proved similarly. □

Remark 3.5. We claim that

$$\mathcal{B}_{n,F}^{(c,\alpha)}(p+1,q;\lambda)-\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)=\sum_{k=0}^{n-1}(-1)^{\binom{n-k}{2}}\binom{n}{k}_F\mathcal{B}_{k,F}^{(c,\alpha)}(p,q;\lambda),$$

and

$$\mathcal{B}_{n,F}^{(s,\alpha)}(p+1,q;\lambda)-\mathcal{B}_{n,F}^{(s,\alpha)}(p,q;\lambda)=\sum_{k=0}^{n-1}(-1)^{\binom{n-k}{2}}\binom{n}{k}_F\mathcal{B}_{k,F}^{(s,\alpha)}(p,q;\lambda).$$

Theorem 3.2. *For every $n\in\mathbb{N}$, following identities hold true:*

$$\frac{\partial_F}{\partial_F p}\left\{\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)\right\}=F_n\mathcal{B}_{n-1,F}^{(c,\alpha)}(p,q;\lambda),\tag{3.16}$$

$$\frac{\partial_F}{\partial_F p}\left\{\mathcal{B}_{n,F}^{(s,\alpha)}(p,q;\lambda)\right\}=F_n\mathcal{B}_{n-1,F}^{(s,\alpha)}(p,q;\lambda),\tag{3.17}$$

$$\frac{\partial_F}{\partial_F q}\left\{\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)\right\}=-F_n\mathcal{B}_{n-1,F}^{(s,\alpha)}(p,q;\lambda),\tag{3.18}$$

and

$$\frac{\partial_F}{\partial_F q}\left\{\mathcal{B}_{n,F}^{(s,\alpha)}(p,q;\lambda)\right\}=F_n\mathcal{B}_{n-1,F}^{(c,\alpha)}(p,q;\lambda).\tag{3.19}$$

Proof. Using (3.5) and applying the Golden derivative operator $\frac{\partial_F}{\partial_F p}$, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial_F}{\partial_F p}\left\{\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)\right\}\frac{t^n}{F_n!} &= \frac{\partial_F}{\partial_F p}\left\{\left(\frac{t}{\lambda e_F^t-1}\right)^{\alpha}e_F^{pt}\right\}\cos_F(qt)\\ &= t\left(\frac{t}{\lambda e_F^t-1}\right)^{\alpha}e_F^{pt}\cos_F(qt)\\ &= \sum_{n=0}^{\infty}\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)\frac{t^{n+1}}{F_n!}\end{aligned}$$

$$= \sum_{n=1}^{\infty} \mathcal{B}_{n-1,F}^{(c,\alpha)}(p,q;\lambda)\frac{t^n}{F_{n-1}!}.$$

By comparing the coefficients of t^n on both sides of this last equation, we arrive at the desired result (3.16). To prove (3.18), using (3.5) and applying the Golden derivative operator $\frac{\partial_F}{\partial_F q}$, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\partial_F}{\partial_F q}\left\{\mathcal{B}_{n,F}^{(c,\alpha)}(p,q;\lambda)\right\}\frac{t^n}{F_n!} &= \frac{\partial_F}{\partial_F q}\left\{\left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{pt}\cos_F(qt)\right\} \\
&= \frac{\partial_F}{\partial_F q}\{\cos_F(qt)\}\left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{pt} \\
&= -t\sin_F(qt)\left(\frac{t}{\lambda e_F^t - 1}\right)^{\alpha} e_F^{pt} \\
&= \sum_{n=1}^{\infty} -\mathcal{B}_{n-1,F}^{(s,\alpha)}(p,q;\lambda)\frac{t^n}{F_{n-1}!}.
\end{aligned}$$

Comparing the coefficients of t^n on both sides of this last equation, we arrive at the desired result (3.18). Equations (3.17) and (3.19) can be similarly derived. □

Theorem 3.3. *The following identities hold true:*

$$\mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda) = \sum_{k=0}^{n}\binom{n}{k}_F \mathcal{B}_{k,F}(\lambda)\,C_{n-k}(p,q), \tag{3.20}$$

and

$$\mathcal{B}_{n,F}^{(s,1)}(p,q;\lambda) = \sum_{k=0}^{n}\binom{n}{k}_F \mathcal{B}_{k,F}(\lambda)\,\mathcal{S}_{n-k}(p,q). \tag{3.21}$$

Proof. Setting $\alpha = 1$ in (3.5) and using (3.1), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda)\frac{t^n}{F_n!} &= \frac{t}{\lambda e_F^t - 1} e_F^{pt}\cos_F(qt) \\
&= \left(\sum_{n=0}^{\infty} \mathcal{B}_{n,F}(p;\lambda)\frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty} C_{n,F}(p,q)\frac{t^n}{F_n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F \mathcal{B}_{k,F}(\lambda)\,C_{n-k}(p,q)\right)\frac{t^n}{F_n!}.
\end{aligned}$$

Comparing the coefficients of t^n on both sides of this last equation, we arrive at the desired result (3.20). Equation (3.21) can be similarly derived. □

Theorem 3.4. *The following identities hold true:*

$$\mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k q^{2k}\binom{n}{2k}_F \mathcal{B}_{n-2k,F}(p;\lambda), \tag{3.22}$$

and

$$\mathcal{B}_{n,F}^{(s,1)}(p,q;\lambda)=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k q^{2k+1}\binom{n}{2k+1}_F \mathcal{B}_{n-2k-1,F}(p;\lambda). \tag{3.23}$$

Proof. Setting $\alpha = 1$ in (3.5) and using (2.9), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda)\frac{t^n}{F_n!} &= \frac{t}{\lambda e_F^t-1}e_F^{pt}\cos_F(qt)\\
&= \left(\sum_{n=0}^{\infty}\mathcal{B}_{n,F}(p;\lambda)\frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n q^{2n}\frac{t^{2n}}{F_{2n}!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k q^{2k}\binom{n}{2k}_F\mathcal{B}_{n-2k,F}(p;\lambda)\right)\frac{t^n}{F_n!}.
\end{aligned}$$

Comparing the coefficients of t^n on both sides of this last equation, we arrive at the desired result (3.22). Equation (3.23) can be similarly derived. □

Theorem 3.5. *The following identities hold true:*

$$\mathcal{C}_{n,F}(p,q)=\lambda\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F\mathcal{B}_{n-k,F}^{(c,1)}(p,q;\lambda)+\frac{(\lambda-1)}{F_{n+1}}\mathcal{B}_{n+1,F}^{(c,1)}(p,q;\lambda), \tag{3.24}$$

and

$$\mathcal{S}_{n,F}(p,q)=\lambda\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F\mathcal{B}_{n-k,F}^{(s,1)}(p,q;\lambda)+\frac{(\lambda-1)}{F_{n+1}}\mathcal{B}_{n+1,F}^{(s,1)}(p,q;\lambda). \tag{3.25}$$

Proof. Using the following equation for the proof of (3.24), we have

$$\begin{aligned}
e_F^{pt}\cos_F(qt) &= \frac{\lambda e_F^t-1}{t}\frac{t}{\lambda e_F^t-1}e_F^{pt}\cos_F(qt)\\
\sum_{n=0}^{\infty}\mathcal{C}_{n,F}(p,q)\frac{t^n}{F_n!} &= \left(\lambda\sum_{n=0}^{\infty}\frac{t^{n-1}}{F_n!}-\frac{1}{t}\right)\left(\sum_{n=0}^{\infty}\mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda)\frac{t^n}{F_n!}\right)\\
&= \left(\lambda\sum_{n=1}^{\infty}\frac{t^{n-1}}{F_n!}+\frac{\lambda-1}{t}\right)\left(\sum_{n=0}^{\infty}\mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda)\frac{t^n}{F_n!}\right).
\end{aligned}$$

Considering $\mathcal{B}_{0,F}^{(c,1)}(p,q;\lambda)=0$, and doing some calculations, we arrive at the desired result (3.24). Equation (3.25) can be similarly derived. □

Theorem 3.6. *The following identities hold true:*

$$\mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda)=\sum_{k=0}^{n}p^k\binom{n}{k}_F\mathcal{B}_{n-k,F}^{(c,1)}(q;\lambda), \tag{3.26}$$

and

$$\mathcal{B}_{n,F}^{(s,1)}(p,q;\lambda)=\sum_{k=0}^{n}p^k\binom{n}{k}_F\mathcal{B}_{n-k,F}^{(s,1)}(q;\lambda). \tag{3.27}$$

Proof. By applying (3.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda)\frac{t^n}{F_n!} &= \frac{t}{\lambda e_F^t-1}e_F^{pt}\cos_F(qt)\\ &= \left(\sum_{n=0}^{\infty}\mathcal{B}_{n,F}^{(c,1)}(q;\lambda)\frac{t^n}{F_n!}\right)\left(\sum_{n=0}^{\infty}p^n\frac{t^n}{F_n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}p^k\binom{n}{k}_F\mathcal{B}_{n-k,F}^{(c,1)}(q;\lambda)\right)\frac{t^n}{F_n!}.\end{aligned}$$

Comparing the coefficients of t^n on both sides of this last equation, we arrive at the desired result (3.26). Equation (3.27) can be similarly derived. □

Theorem 3.7. *Determinantal forms of the cosine and sine Apostol Bernoulli-Fibonacci polynomials are given by*

$$\mathcal{B}_{n+1,F}^{(c,1)}(p,q;\lambda)=\frac{1}{(\lambda-1)^{n+2}}\begin{vmatrix} F_{n+1}C_{n,F}(p,q) & \lambda F_{n+1} & \lambda F_{n+1}\binom{n}{1}_F & \cdots & \lambda\binom{n}{n}_F\\ F_nC_{n-1,F}(p,q) & \lambda-1 & \lambda F_n\binom{n-1}{0}_F & \cdots & \lambda\binom{n-1}{n-1}_F\\ F_{n-1}C_{n-2,F}(p,q) & 0 & \lambda-1 & \cdots & \lambda\binom{n-2}{n-2}_F\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ F_0C_{-1,F}(p,q) & 0 & 0 & \cdots & \lambda-1\end{vmatrix},$$

and

$$\mathcal{B}_{n+1,F}^{(s,1)}(p,q;\lambda)=\frac{1}{(\lambda-1)^{n+2}}\begin{vmatrix} F_{n+1}S_{n,F}(p,q) & \lambda F_{n+1} & \lambda F_{n+1}\binom{n}{1}_F & \cdots & \lambda\binom{n}{n}_F\\ F_nS_{n-1,F}(p,q) & \lambda-1 & \lambda F_n\binom{n-1}{0}_F & \cdots & \lambda\binom{n-1}{n-1}_F\\ F_{n-1}S_{n-2,F}(p,q) & 0 & \lambda-1 & \cdots & \lambda\binom{n-2}{n-2}_F\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ F_0S_{-1,F}(p,q) & 0 & 0 & \cdots & \lambda-1\end{vmatrix}.$$

Proof. Equation (3.24) cause the system of unknown $(n+2)$-equations with $\mathcal{B}_{n,F}^{(c,1)}(p,q;\lambda)$, $(n=0,1,2,\ldots)$. Then we apply the Cramer's rule to solve this equation. We obtain the desired result. In a similar way, we can obtain the determinantal form for sine Apostol Bernoulli-Fibonacci polynomials. □

In subsections 3.2 and 3.3, we give the some basic properties of the polynomials $\mathcal{E}_{n,F}^{(c,\alpha)}(p,q;\lambda)$, $\mathcal{E}_{n,F}^{(s,\alpha)}(p,q;\lambda)$, $\mathcal{G}_{n,F}^{(c,\alpha)}(p,q;\lambda)$, and $\mathcal{G}_{n,F}^{(s,\alpha)}(p,q;\lambda)$. Their proofs run parallel to those of the results presented in this subsection; so, the proofs are omitted.

3.2. Basic properties of $\mathcal{E}_{n,F}^{(c,\alpha)}(p,q;\lambda)$ and $\mathcal{E}_{n,F}^{(s,\alpha)}(p,q;\lambda)$

Theorem 3.8. *The following identities hold:*

$$\mathcal{E}_{n,F}^{(c,\alpha)}(p+r,q;\lambda)=\sum_{k=0}^{n}(-1)^{\binom{n-k}{2}}\binom{n}{k}_F\mathcal{E}_{k,F}^{(c,\alpha)}(p,q;\lambda)\,r^{n-k}$$

and

$$\mathcal{E}^{(s,\alpha)}_{n,F}(p+r,q;\lambda)=\sum_{k=0}^{n}(-1)^{\binom{n-k}{2}}\binom{n}{k}_F\mathcal{E}^{(s,\alpha)}_{k,F}(p,q;\lambda)\,r^{n-k}.$$

Remark 3.6. We claim that

$$\mathcal{E}^{(c,\alpha)}_{n,F}(p+1,q;\lambda)-\mathcal{E}^{(c,\alpha)}_{n,F}(p,q;\lambda)=\sum_{k=0}^{n-1}(-1)^{\binom{n-k}{2}}\binom{n}{k}_F\mathcal{E}^{(c,\alpha)}_{k,F}(p,q;\lambda),$$

$$\mathcal{E}^{(s,\alpha)}_{n,F}(p+1,q;\lambda)-\mathcal{E}^{(s,\alpha)}_{n,F}(p,q;\lambda)=\sum_{k=0}^{n-1}(-1)^{\binom{n-k}{2}}\binom{n}{k}_F\mathcal{E}^{(s,\alpha)}_{k,F}(p,q;\lambda).$$

Theorem 3.9. *For every $n\in\mathbb{N}$, following identities hold true:*

$$\frac{\partial_F}{\partial_F p}\left\{\mathcal{E}^{(c,\alpha)}_{n,F}(p,q;\lambda)\right\}=F_n\mathcal{E}^{(c,\alpha)}_{n-1,F}(p,q;\lambda),$$

$$\frac{\partial_F}{\partial_F p}\left\{\mathcal{E}^{(s,\alpha)}_{n,F}(p,q;\lambda)\right\}=F_n\mathcal{E}^{(s,\alpha)}_{n-1,F}(p,q;\lambda),$$

$$\frac{\partial_F}{\partial_F q}\left\{\mathcal{E}^{(c,\alpha)}_{n,F}(p,q;\lambda)\right\}=-F_n\mathcal{E}^{(s,\alpha)}_{n-1,F}(p,q;\lambda),$$

and

$$\frac{\partial_F}{\partial_F q}\left\{\mathcal{E}^{(s,\alpha)}_{n,F}(p,q;\lambda)\right\}=F_n\mathcal{E}^{(c,\alpha)}_{n-1,F}(p,q;\lambda).$$

Theorem 3.10. *The following identities hold true:*

$$\mathcal{E}^{(c,1)}_{n,F}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}_F\mathcal{E}_{k,F}(\lambda)\,C_{n-k}(p,q),$$

and

$$\mathcal{E}^{(s,1)}_{n,F}(p,q;\lambda)=\sum_{k=0}^{n}\binom{n}{k}_F\mathcal{E}_{k,F}(\lambda)\,\mathcal{S}_{n-k}(p,q).$$

Theorem 3.11. *The following identities hold true:*

$$\mathcal{E}^{(c,1)}_{n,F}(p,q;\lambda)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k q^{2k}\binom{n}{2k}_F\mathcal{E}_{n-2k,F}(p;\lambda),$$

and

$$\mathcal{E}^{(s,1)}_{n,F}(p,q;\lambda)=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k q^{2k+1}\binom{n}{2k+1}_F\mathcal{E}_{n-2k-1,F}(p;\lambda).$$

Theorem 3.12. *The following identities hold true:*

$$C_{n,F}(p,q)=\frac{1}{2}\mathcal{E}^{(c,1)}_{n,F}(p,q;\lambda)+\frac{\lambda}{2}\sum_{k=0}^{n}\binom{n}{k}_F\mathcal{E}^{(c,1)}_{n-k,F}(p,q;\lambda),$$

and

$$\mathcal{S}_{n,F}(p,q)=\frac{1}{2}\mathcal{E}^{(s,1)}_{n,F}(p,q;\lambda)+\frac{\lambda}{2}\sum_{k=0}^{n}\binom{n}{k}_F\mathcal{E}^{(s,1)}_{n-k,F}(p,q;\lambda).$$

Theorem 3.13. *The following identities hold true:*

$$\mathcal{E}_{n,F}^{(c,1)}(p,q;\lambda)=\sum_{k=0}^{n}p^{k}\binom{n}{k}_{F}\mathcal{E}_{n-k,F}^{(c,1)}(q;\lambda),$$

and

$$\mathcal{E}_{n,F}^{(s,1)}(p,q;\lambda)=\sum_{k=0}^{n}p^{k}\binom{n}{k}_{F}\mathcal{E}_{n-k,F}^{(s,1)}(q;\lambda).$$

Theorem 3.14. *Determinantal forms of the cosine and sine Apostol Euler-Fibonacci polynomials are given by*

$$\mathcal{E}_{n,F}^{(c,1)}(p,q;\lambda)=\left(\frac{2}{\lambda+1}\right)^{n+1}\begin{vmatrix} C_{n,F}(p,q) & \frac{\lambda}{2}\binom{n}{1}_F & \frac{\lambda}{2}\binom{n}{2}_F & \cdots & \frac{\lambda}{2}\binom{n}{n}_F \\ C_{n-1,F}(p,q) & \frac{\lambda+1}{2} & \frac{\lambda}{2}\binom{n-1}{1}_F & \cdots & \frac{\lambda}{2}\binom{n-1}{n-1}_F \\ C_{n-2,F}(p,q) & 0 & \frac{\lambda+1}{2} & \cdots & \frac{\lambda}{2}\binom{n-2}{n-2}_F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{0,F}(p,q) & 0 & 0 & \cdots & \frac{\lambda+1}{2} \end{vmatrix},$$

and

$$\mathcal{E}_{n,F}^{(s,1)}(p,q;\lambda)=\left(\frac{2}{\lambda+1}\right)^{n+1}\begin{vmatrix} S_{n,F}(p,q) & \frac{\lambda}{2}\binom{n}{1}_F & \frac{\lambda}{2}\binom{n}{2}_F & \cdots & \frac{\lambda}{2}\binom{n}{n}_F \\ S_{n-1,F}(p,q) & \frac{\lambda+1}{2} & \frac{\lambda}{2}\binom{n-1}{1}_F & \cdots & \frac{\lambda}{2}\binom{n-1}{n-1}_F \\ S_{n-2,F}(p,q) & 0 & \frac{\lambda+1}{2} & \cdots & \frac{\lambda}{2}\binom{n-2}{n-2}_F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{0,F}(p,q) & 0 & 0 & \cdots & \frac{\lambda+1}{2} \end{vmatrix}.$$

3.3. *Basic properties of $\mathcal{G}_{n,F}^{(c,\alpha)}(p,q;\lambda)$ and $\mathcal{G}_{n,F}^{(s,\alpha)}(p,q;\lambda)$*

Theorem 3.15. *The following identities hold true:*

$$\mathcal{G}_{n,F}^{(c,\alpha)}(p+r,q;\lambda)=\sum_{k=0}^{n}(-1)^{\binom{n-k}{2}}\binom{n}{k}_{F}\mathcal{G}_{k,F}^{(c,\alpha)}(p,q;\lambda)\,r^{n-k},$$

and

$$\mathcal{G}_{n,F}^{(s,\alpha)}(p+r,q;\lambda)=\sum_{k=0}^{n}(-1)^{\binom{n-k}{2}}\binom{n}{k}_{F}\mathcal{G}_{k,F}^{(s,\alpha)}(p,q;\lambda)\,r^{n-k}.$$

Remark 3.7. We claim that

$$\mathcal{G}_{n,F}^{(c,\alpha)}(p+1,q;\lambda)-\mathcal{G}_{n,F}^{(c,\alpha)}(p,q;\lambda)=\sum_{k=0}^{n-1}(-1)^{\binom{n-k}{2}}\binom{n}{k}_{F}\mathcal{G}_{k,F}^{(c,\alpha)}(p,q;\lambda),$$

$$\mathcal{G}_{n,F}^{(s,\alpha)}(p+1,q;\lambda)-\mathcal{G}_{n,F}^{(s,\alpha)}(p,q;\lambda)=\sum_{k=0}^{n-1}(-1)^{\binom{n-k}{2}}\binom{n}{k}_{F}\mathcal{G}_{k,F}^{(s,\alpha)}(p,q;\lambda).$$

Theorem 3.16. *For every $n\in\mathbb{N}$, following identities hold true:*

$$\frac{\partial_F}{\partial_F p}\left\{\mathcal{G}_{n,F}^{(c,\alpha)}(p,q;\lambda)\right\}=F_n\mathcal{G}_{n-1,F}^{(c,\alpha)}(p,q;\lambda),$$

$$\frac{\partial_F}{\partial_F p}\left\{\mathcal{G}_{n,F}^{(s,\alpha)}(p,q;\lambda)\right\} = F_n\mathcal{G}_{n-1,F}^{(s,\alpha)}(p,q;\lambda),$$

$$\frac{\partial_F}{\partial_F q}\left\{\mathcal{G}_{n,F}^{(c,\alpha)}(p,q;\lambda)\right\} = -F_n\mathcal{G}_{n-1,F}^{(s,\alpha)}(p,q;\lambda),$$

and

$$\frac{\partial_F}{\partial_F q}\left\{\mathcal{G}_{n,F}^{(s,\alpha)}(p,q;\lambda)\right\} = F_n\mathcal{G}_{n-1,F}^{(c,\alpha)}(p,q;\lambda).$$

Theorem 3.17. *The following identities hold true:*

$$\mathcal{G}_{n,F}^{(c,1)}(p,q;\lambda) = \sum_{k=0}^{n}\binom{n}{k}_F \mathcal{G}_k(\lambda)\,C_{n-k}(p,q),$$

and

$$\mathcal{G}_{n,F}^{(s,1)}(p,q;\lambda) = \sum_{k=0}^{n}\binom{n}{k}_F \mathcal{G}_{k,F}(\lambda)\,\mathcal{S}_{n-k}(p,q).$$

Theorem 3.18. *The following identities hold true:*

$$\mathcal{G}_{n,F}^{(c,1)}(p,q;\lambda) = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k q^{2k}\binom{n}{2k}_F \mathcal{G}_{n-2k,F}(p;\lambda),$$

and

$$\mathcal{G}_{n,F}^{(s,1)}(p,q;\lambda) = \sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor}(-1)^k q^{2k+1}\binom{n}{2k+1}_F \mathcal{G}_{n-2k-1,F}(p;\lambda).$$

Theorem 3.19. *The following identities hold true:*

$$C_{n,F}(p,q) = \frac{\lambda}{2}\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F \mathcal{G}_{n-k,F}^{(c,1)}(p,q;\lambda) + \frac{\lambda+1}{2F_{n+1}}\mathcal{G}_{n+1,F}^{(c,1)}(p,q;\lambda),$$

and

$$\mathcal{S}_{n,F}(p,q) = \frac{\lambda}{2}\sum_{k=0}^{n}\frac{1}{F_{k+1}}\binom{n}{k}_F \mathcal{G}_{n-k,F}^{(s,1)}(p,q;\lambda) + \frac{\lambda+1}{2F_{n+1}}\mathcal{G}_{n+1,F}^{(s,1)}(p,q;\lambda).$$

Theorem 3.20. *The following identities hold true:*

$$\mathcal{G}_{n,F}^{(c,1)}(p,q;\lambda) = \sum_{k=0}^{n}p^k\binom{n}{k}_F \mathcal{G}_{n-k,F}^{(c,1)}(q;\lambda),$$

and

$$\mathcal{G}_{n,F}^{(s,1)}(p,q;\lambda) = \sum_{k=0}^{n}p^k\binom{n}{k}_F \mathcal{G}_{n-k,F}^{(s,1)}(q;\lambda).$$

Theorem 3.21. *Determinantal forms of the cosine and sine Apostol Genocchi-Fibonacci polynomials are given by*

$$\mathcal{G}^{(c,1)}_{n+1,F}(p,q;\lambda)=\left(\frac{2}{\lambda+1}\right)^{n+2}\begin{vmatrix} F_{n+1}C_{n,F}(p,q) & \frac{\lambda}{2}F_{n+1}\binom{n}{0}_F & \frac{\lambda}{2}F_{n+1}\binom{n}{1}_F & \cdots & \frac{\lambda}{2}\binom{n}{n}_F \\ F_nC_{n-1,F}(p,q) & \frac{\lambda+1}{2} & \frac{\lambda}{2}F_n\binom{n-1}{0}_F & \cdots & \frac{\lambda}{2}\binom{n-1}{n-1}_F \\ F_{n-1}C_{n-2,F}(p,q) & 0 & \frac{\lambda+1}{2} & \cdots & \frac{\lambda}{2}\binom{n-2}{n-2}_F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\lambda+1}{2}\end{vmatrix},$$

and

$$\mathcal{G}^{(s,1)}_{n+1,F}(p,q;\lambda)=\left(\frac{2}{\lambda+1}\right)^{n+2}\begin{vmatrix} F_{n+1}S_{n,F}(p,q) & \frac{\lambda}{2}F_{n+1}\binom{n}{0}_F & \frac{\lambda}{2}F_{n+1}\binom{n}{1}_F & \cdots & \frac{\lambda}{2}\binom{n}{n}_F \\ F_nS_{n-1,F}(p,q) & \frac{\lambda+1}{2} & \frac{\lambda}{2}F_n\binom{n-1}{0}_F & \cdots & \frac{\lambda}{2}\binom{n-1}{n-1}_F \\ F_{n-1}S_{n-2,F}(p,q) & 0 & \frac{\lambda+1}{2} & \cdots & \frac{\lambda}{2}\binom{n-2}{n-2}_F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\lambda+1}{2}\end{vmatrix}.$$

4. Conclusions

Our aim in this article is to define the F-analogues of the parametric types of the Apostol Bernoulli, the Apostol Euler, and the Apostol Genocchi polynomials studied by Srivastava et al. [15,19]. Namely, we have defined parametric types of the Apostol Bernoulli-Fibonacci, the Apostol Euler-Fibonacci, and the Apostol Genocchi-Fibonacci polynomials using the Golden calculus and investigated their properties. In our future work, we plan to define the parametric types of some special polynomials with the help of Golden calculus and to obtain many combinatorial identities with the help of their generating functions.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. Q. M. Luo, On the Apostol-Bernoulli polynomials, *Centr. Eur. J. Math.*, **2** (2004), 509–515. https://doi.org/10.2478/BF02475959
2. Q. M. Luo, H. M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, *J. Math. Anal. Appl.*, **308** (2005), 290–302. https://doi.org/10.1016/j.jmaa.2005.01.020
3. Q. M. Luo, H. M. Srivastava, Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, *Comput. Math. Appl.*, **51** (2006), 631–642. https://doi.org/10.1016/j.camwa.2005.04.018

4. D. Q. Lu, H. M. Srivastava, Some series identities involving the generalized Apostol type and related polynomials, *Comput. Math. Appl.*, **62** (2011), 3591–3602. https://doi.org/10.1016/j.camwa.2011.09.010

5. Q. M. Luo, Apostol-Euler polynomials of higher order and Gaussian hyper-geometric functions, *Taiwanese J. Math.*, **10** (2006), 917–925. https://doi.org/10.11650/twjm/1500403883

6. Q. M. Luo, Extensions of the Genocchi polynomials and their Fourier expansions and integral representations, *Osaka J. Math.*, **48** (2011), 291–309.

7. M. A. Ozarslan, Unified Apostol-Bernoulli, Euler and Genocchi polynomials, *Comput. Math. Appl.*, **62** (2011), 2452–2462. https://doi.org/10.1016/j.camwa.2011.07.031

8. H. Ozden, Y. Simsek, H. M. Srivastava, A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials, *Comput. Math. Appl.*, **60** (2010), 2779–2787. https://doi.org/10.1016/j.camwa.2010.09.031

9. N. Kilar, Y. Simsek, Computational formulas and identities for new classes of Hermite-based Milne-Thomson type polynomials: analysis of generating functions with Euler's formula, *Math. Meth. Appl. Sci.*, **44** (2021), 6731–6762. https://doi.org/10.1002/mma.7220

10. Y. Simsek, Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q-integrals, *Turk. J. Math.*, **42** (2018), 557–577. https://doi.org/10.3906/mat-1703-114

11. Y. Simsek, Computation methods for combinatorial sums and Euler-type numbers related to new families of numbers, *Math. Methods Appl. Sci.*, **40** (2017), 2347–2361. https://doi.org/10.1002/mma.4143

12. H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inf. Sci.*, **5** (2011), 390–444.

13. S. Jin, M. C. Dagli, F. Qi, Degenerate Fubini-type polynomials and numbers, degenerate Apostol-Bernoulli polynomials and numbers, and degenerate Apostol-Euler polynomials and numbers, *Axioms*, **11** (2022), 477. https://doi.org/10.3390/axioms11090477

14. Y. He, T. Kim, General convolution identities for Apostol-Bernoulli, Euler and Genocchi polynomials, *J. Nonlinear Sci. Appl.*, **9** (2016), 4780–4797.

15. H. M. Srivastava, M. Masjed-Jamei, M. R. Beyki, Some new generalizations and applications of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, *Rocky Mountain J. Math.*, **49**(2019), 681–697.

16. M. Masjed-Jamei, M. R. Beyki, W. Koepf, A new type of Euler polynomials and numbers, *Mediterr. J. Math.*, **15** (2018), 1–17. https://doi.org/10.1007/s00009-018-1181-1

17. M. Masjed-Jamei, M. R. Beyki, E. Omey, On a parametric kind of Genocchi polynomials, *J. Inequal. Spec. Funct.*, **9** (2018), 68–81.

18. M. Masjed-Jamei, W. Koepf, Symbolic computation of some power-trigonometric series, *J. Symb. Comput.*, **80** (2017), 273–284. https://doi.org/10.1016/j.jsc.2016.03.004

19. H. M. Srivastava, M. Masjed-Jamei, M. R. Beyki, A parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, *Appl. Math. Inf. Sci.*, **12** (2018), 907–916.

20. H. M. Srivastava, C. Kizilates, A parametric kind of the Fubini-type polynomials, *RACSAM*, **113** (2019), 3253–3267. https://doi.org/10.1007/s13398-019-00687-4

21. O. K. Pashaev, S. Nalci, Golden quantum oscillator and Binet-Fibonacci calculus, *J. Phys. A: Math. Theor.*, **45** (2012), 1–23. https://doi.org/10.1088/1751-8113/45/1/015303

22. O. K. Pashaev, Quantum calculus of Fibonacci divisors and infinite hierarchy of bosonic-fermionic golden quantum oscillators, *Internat. J. Geom. Methods Modern Phys.*, **18** (2021), 1–32. https://doi.org/10.1142/S0219887821500754

23. E. Krot, An introduction to finite fibonomial calculus, *Centr. Eur. J. Math.*, **2** (2004), 754–766. https://doi.org/10.2478/BF02475975

24. M. Ozvatan, *Generalized golden-Fibonacci calculus and applications*, Ph.D. thesis, Izmir Institute of Technology, 2018.

25. O. K. Pashaev, M. Ozvatan, Bernoulli-Fibonacci polynomials, *arXiv*, 2020. https://doi.org/10.48550/arXiv.2010.15080

26. S. Kus, N. Tuglu, T. Kim, Bernoulli F-polynomials and Fibo-Bernoulli matrices, *Adv. Differ. Equ.*, **2019** (2019), 145. https://doi.org/10.1186/s13662-019-2084-6

27. N. Tuglu, E. Ercan, *Some properties of Apostol Bernoulli Fibonacci and Apostol Euler Fibonacci polynomials*, Icmee-2021, 32-34.

28. E. Gulal, N. Tuglu, Apostol Bernoulli-Fibonacci polynomials, Apostol Euler-Fibonacci polynomials, and their generating functions, unpublished paper.

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