Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 8274–8293.
DOI: 10.3934/math.2023417
Received: 29 November 2022
Revised: 12 January 2023
Accepted: 18 January 2023
Published: 01 February 2023

Research article

Some results in function weighted *b*-metric spaces

Budi Nurwahyu*, Naimah Aris and Firman

Department of Mathematics, Hasanuddin University, Tamalanrea KM 10, Makassar, Indonesia

* **Correspondence**: Email: budinurwahyu@unhas.ac.id.

Abstract: In this paper, we introduce F-b-metric space (function weighted b-metric space) as a generalization of the F-metric space (the function weighted metric space). We also propose and prove some topological properties of the F-b-metric space, the theorems of fixed point and the common fixed point for the generalized expansive mappings, and an application on dynamic programing.

Keywords: fixed point; common fixed point; function weighted metric; function weighted b-metric; expansive mapping
Mathematics Subject Classification: 47H09, 47H10, 54H25

1. Introduction

In this paper, we intend to introduce a function weighted b-metric which can be named as F-b metric. In addition, we propose and prove theorems related to the topological properties in F-b metric space and the fixed point theorems for an expansive mapping and the common fixed point for two expansive mappings. The expansive mapping used is a generalized Banach expansive mapping. The concept of function weighted b-metric is inspired from the concept of b-metric introduced by Bakhtin in 1989 and the concept of function weighted metric introduced by Jleli and Samet in 2018.

b-metric space was first introduced by Bakhtin in 1989 [1], then used by Czerwick in 1993 [2] in the fixed point theorem for generalized contraction mappings. George in 2015 [3–5] introduced the rectangular b-metric space which is a generalization of the b-metric space, by replacing the triangle inequality condition with rectangular inequality. However, in 2010 Khamsi and Hussein introduced a generalization of the b-metric space known as the s-relaxed metric space. In this definition, the triangle inequality condition in the b-metric space is replaced by an s-metric polygon inequality [6]. In addition, to the above authors, generalizations to the metric space have also been introduced by other authors [7–9].

In general, the authors in proposing and proving the fixed point in the generalized metric space and the use of the generalized contraction mappings. While the use of generalization of metric space at fixed points for expansive mappings is not much, some of them have been done by several authors [10–13]. Currently, the generalization of the metric space that is being developed is the *F*-metric space. This space was introduced by Jleli and Samet which is also called the function weighted metric space [14]. In addition, Jleli and Samet also prove some topological properties and fixed point theorems of the contraction mapping [15]. There have been several other authors who have used the *F*-metric space for the fixed point theorem of the contraction mapping in that space [16,17]. The concept of the contraction mapping was introduced by Banach in 1922, while the expansive mapping was introduced by Wang in 1984 [18]. However, the use of the contractive mapping by authors is generally more popular for determining the existence of a fixed point or allied fixed points than the use of the expansive mapping for the existence of a fixed point in a generalized metric space.

To prove a function weighted metric is a function weighted *b*-metric, it must show that the third condition in function weighted *b*-metric applies to function weighted metric. The generalization of the third condition is important because there are several function weighted *b*-metrics are not metric weighted function.

Moreover, it should be noted that the results in this paper which use the function weighted *b*-metric are generalizations of the results using the function weighted metric [15]. This paper also provides some examples of weighted metric and function weighted *b*-metric and some examples of fixed point theorems of a generalized Wang [18] expansive mapping and common fixed points for two mappings, and an application on dynamic programing.

2. Preliminaries

In this paper we need some definitions to be used for the results section and some required examples.

Definition 2.1. [14] Let $f\colon (0,+\infty) \to \mathbb{R}$ be a function, non-decreasing, f is called *logarithmic-like* if it satisfies $\lim_{t\to 0} f(t) = -\infty$.

We denote $F = \{f\colon (0,+\infty) \to \mathbb{R} \mid f \text{ non}-\text{decreasing, and logarithmic}-\text{like}\}$. From this definition, it follows that, if $f \in \mathfrak{F}$, then for any $r > 0$ and $K \geq 0$, there is $\delta > 0$, so that for any $0 < s < \delta$, then $f(s) < f(r) - K$.

Some examples of functions that satisfy the logarithmic-like property are: $f(x) = ln\, x$, $f(x) = x - \frac{1}{x}$, $f(x) = x - e^{\frac{1}{x}}$, $x \in (0,+\infty)$.

Definition 2.2. [14] Let X be a non-empty set, $f \in F$, $0 \leq K < +\infty$. A mapping $\rho\colon X \times X \to [0,+\infty)$ is called a function weighted metric (F-*metric*), if for any $x, y \in X$, ρ satisfies the following conditions:

A1. $\rho(x,y) = 0$, if and only if $x = y$,

A2. $\rho(x,y) = \rho(y,x)$,

A3. $\rho(x,y) > 0$, then $f\big(\rho(x,y)\big) \leq f(\sum_{j=1}^{N-1} \rho\big(a_j, a_{j+1}\big) + K$,

for every $\{a_1 = x,\ a_2,\ a_3, \dots, a_N = y\} \subset X$ and $N \in \mathbb{N}, N \geq 2$.
The pair (X,ρ) is called a function weighted metric space (F *metric space*).

In the following, we define a function weighted b-metric which is a generalization of the function weighted metric, as follows:

Definition 2.3. Let X be a non-empty set, $f \in F$, $0 \le K < +\infty$, $b \ge 1$. A mapping $\rho: X \times X \to [0, +\infty)$ is called a *function weighted b-metric* (*F-b* metric), if for any $x, y \in X$, ρ satisfies the following conditions:

B1. $\rho(x,y) = 0$, if and only if $x = y$,

B2. $\rho(x,y) = \rho(y,x)$,

B3. $\rho(x,y) > 0$ then

$$f\big(\rho(x,y)\big) \le f\big(\textstyle\sum_{j=1}^{N-1} b^j \rho\big(a_j, a_{j+1}\big)\big) + K,$$

for every

$$\{a_1 = x,\ a_2,\ a_3, \dots, a_N = y\} \subset X \text{ and } N \in \mathbb{N}, N \ge 2.$$

The pair (X, ρ) is called a function weighted b-metric space (*F-b* metric space).

If $b = 1$, then the above definition is the definition of the function weighted metric. However, the function weighted b-metric becomes b-metric, if $f(x) = \ln x$, $K = 0$, and $N = 2$.

The function weighted b-metric space is not necessarily a function of the weighted metric space. This can be shown in the following examples:

Example 2.1. Let $X = R$ be a set of all real numbers. Define a function $\rho(x,y) = |x-y|^p$ with $p \ge 2$ and $x, y \in X$, then ρ is a b-metric with $b = 2^{p-1}$. However, ρ is not a function weighted metric, this can be shown as follows.

Suppose ρ is a function weighted metric, it means that there is a function $f \in F$ that satisfies the of axiom A3 of the definition of function weighted metric.

We choose $t_j \in X$ and take any $m \in \mathbb{N}$. Define $t_j = \frac{2j}{m}$ for $j = 0,1,2, \dots, m-1$. From the of axiom A3, we have

$$f(\rho(0,2)) \le f\Big(\sum_{j=0}^{m-1} \big|t_j - t_{j+1}\big|^p\Big) + K = f\Big(\sum_{j=0}^{m-1} \left|\frac{2j}{m} - \frac{2(j+1)}{m}\right|^p\Big) + K$$

$$= f\big(\tfrac{1}{m^p} \textstyle\sum_{j=0}^{m-1} |2j - 2(j+1)|^p\big) + K = f\big(\tfrac{2^p}{m^{p-1}}\big) + K. \tag{2.1}$$

So, from (2.1) we get

$$f(4) = f\big(\rho(0,2)\big) \le f\left(\frac{2^p}{m^{p-1}}\right) + K. \tag{2.2}$$

Since f has a *logarithm-like property*, then we have $\frac{2^p}{m^{p-1}} \to 0$, as $m \to +\infty$ and consequently from (2.2) we have $f\big(\frac{2^p}{m^{p-1}}\big) + K \to -\infty$, as $m \to +\infty$. Which is a contradiction. So, ρ is not a function weighted metric.

However, ρ is a function weighted b-metric. It is shown as follows:

To prove condition B3, we use a property

$$|x-y|^p \leq 2^{p-1}(|x-a|^p + |a-y|^p).$$

Let $N \in \mathbb{N}, N \geq 2$, and a set $\{a_1 = x,\ a_2,\ a_3, \ldots, a_N = y\} \subset X$.
Then we have

$$\rho(x,y) = |x-y|^p = |a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + a_4 - \cdots a_{N-1} - a_N|^p$$
$$\leq \begin{pmatrix} 2^{p-1}|a_1-a_2|^p + (2^{p-1})^2|a_2-a_3|^p + (2^{p-1})^3|a_3-a_4|^p \\ + \cdots + (2^{p-1})^{N-1}|a_{N-1}-a_N|^{p-1} \end{pmatrix}$$
$$= \sum_{j=1}^{N-1}(2^{p-1})^j\left|a_j - a_{j+1}\right|^p = \sum_{j=1}^{N-1}(2^{p-1})^j\,\rho\big(a_j, a_{j+1}\big).$$

So, we obtain

$$\rho(x,y) \leq \sum_{j=1}^{N-1}(2^{p-1})^j\,\rho\big(a_j, a_{j+1}\big).$$

Thus, ρ is a function weighted b-metric with $f(x) = \ln x$, $b = 2^{p-1}$ and $K = 0$.

Example 2.2. Let $X = [0,1]$, define a function $\rho(x,y) = |x-y|^2 e^{|x-y|}$ for all $x, y \in X$.
To prove condition B3, we use a property

$$|x-y|^2 \leq 2(|x-a|^2 + |a-y|^2).$$

Let $N \in \mathbb{N}, N \geq 2$, and a set $\{a_1 = x, a_2, a_3, \ldots, a_N = y\} \subset X$.
Then we have

$$\rho(x,y) = |x-y|^2 e^{|x-y|}$$
$$= |a_1 - a_2 + a_2 - a_3 + \cdots + a_{N-1} - a_N|^2 e^{|a_1-a_2+a_2-a_3+\cdots+a_{N-1}-a_N|}$$
$$\leq (2|a_1-a_2|^2 + (2)^2|a_2-a_3|^2 + \cdots + (2)^{N-1}|a_{N-1}-a_N|^2)e^{|a_1-a_2+a_2-a_3+\cdots+a_{N-1}-a_N|}$$
$$\leq \big(2|a_1-a_2|^2 e^{|a_1-a_2|+|a_2-a_3+a_{N-1}-a_N|} + (2)^2|a_2-a_3|^2 e^{|a_2-a_3|+|a_1-a_2+a_3-a_N|} + \cdots$$
$$+ (2)^{N-1}|a_{N-1}-a_N|^2\big)e^{|a_{N-1}-a_N|+|a_1-a_{N-1}|}$$

$$= \big(2|a_1-a_2|^2 e^{|a_1-a_2|+|a_2-a_N|} + (2)^2|a_2-a_3|^2 e^{|a_2-a_3|+|a_1-a_2+\cdots+a_{N-1}-a_N|} + \cdots$$
$$+ (2)^{N-1}|a_{N-1}-a_N|^2 e^{|a_{N-1}-a_N|+|a_1-a_2+a_{N-1}-a_N|}\big)$$

Since $a_1 = x, a_2, a_3, \ldots, a_N = y \in [0,1]$, then we have

$$\rho(x,y) = |x-y|^2 e^{|x-y|}$$
$$\leq 2|a_1-a_2|^2 e^{|a_1-a_2|}e^2 + (2)^2|a_2-a_3|^2 e^{|a_2-a_3|}e^2 + \cdots + (2)^{N-1}|a_{N-1}-a_N|^2 e^2$$
$$= e^2\big(2|a_1-a_2|^2 e^{|a_1-a_2|} + (2)^2|a_2-a_3|^2 e^{|a_2-a_3|} + \cdots + (2)^{N-1}|a_{N-1}-a_N|^2\big)$$

$$= e^2 \left(\sum_{j=1}^{N-1}(2)^j \left|a_j - a_{j+1}\right|^2 e^{|a_j - a_{j+1}|}\right) = e^2 \left(\sum_{j=1}^{N-1}(2)^j \rho\left(a_j, a_{j+1}\right)\right). \quad (2.3)$$

So, from (2.3) we obtain

$$\rho(x,y) \le e^2 \sum_{j=1}^{N-1}(2^{p-1})^j \rho\left(a_j, a_{j+1}\right).$$

So, we have

$$ln\, \rho(x,y) \le 2 + \ln \sum_{j=1}^{N-1}(2^{p-1})^j \rho\left(a_j, a_{j+1}\right).$$

Thus, ρ is a function weighted b-metric with the function $f(x) = ln\, x$, $b = 2^{p-1}$ and $K = 2$.

Example 2.3. Let $X = [1,2]$, define the function

$$\rho(x,y) = \begin{cases} 0, & x = y \\ e^{|x-y|}, & x \ne y \end{cases}$$

for all $x, y \in X$.

For conditions B1 and B2, it is clearly satisfied. To prove condition B3, choose a function $f(t) = t - \frac{1}{t}$ for any $t \in (0, +\infty)$, then $f \in F$.

Let (a_n) be any finite sequence in $X = [1,2]$, where $(a_1, a_N) = (x, y)$ for all $x, y \in X$, and $\rho(x,y) > 0$, for $n = 1,2,3, \dots, N$. So, we have

$$e + 1 + f\left(\sum_{j=1}^{N-1} b^j \rho\left(a_j, a_{j+1}\right)\right) - f\left(\rho(x,y)\right)$$

$$= e + 1 + \sum_{j=1}^{N-1} b^j \rho\left(a_j, a_{j+1}\right) - \frac{1}{\sum_{j=1}^{N-1} b^j \rho\left(a_j, a_{j+1}\right)} - \rho(x,y) + \frac{1}{\rho(x,y)}$$

$$= e + 1 + \sum_{j=1}^{N-1} b^j e^{|a_j - a_{j+1}|} - \frac{1}{\sum_{j=1}^{N-1} b^j e^{|a_j - a_{j+1}|}} - e^{|x-y|} + \frac{1}{e^{|x-y|}}. \quad (2.4)$$

Since $b \ge 1$ and $x, y \in [1,2]$, then from (2.4) we obtain

$$e + 1 + f\left(\sum_{j=1}^{N-1} b^j \rho\left(a_j, a_{j+1}\right)\right) - f\left(\rho(x,y)\right)$$

$$\ge e + 1 - \frac{1}{\sum_{j=1}^{N-1} b^j e^{|a_j - a_{j+1}|}} - e^{|x-y|} \ge e + 1 - 1 - e = 0. \quad (2.5)$$

So, from (2.5) we get

$$f(\rho(x,y)) \leq f\left(\sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1})\right) + e + 1.$$

Thus, ρ is a function weighted b-metric with $b \geq 1$, $f(t) = t - \frac{1}{t}$, $t > 0$, and $K = e + 1$.

Definition 2.4. [13,19] Let X be non-empty set and $T_1, T_2: X \to X$ are functions that satisfy: If $y = T_1x = T_2x$ for an $x \in X$, then x is called a *coincidence point* of T_1, T_2, and y is called a *point of coincidence* of T_1 and T_2.

Definition 2.5. [13, 19] Let X be non-empty set and $T_1, T_2: X \to X$ be a function that satisfy: for every $x \in X$, if $T_1x = T_2x$, then $T_2T_1x = T_1T_2x$, $\{T_1, T_2\}$ is called a *weakly compatible.*

Definition 2.6. Let (X, ρ) be a function weighted b-metric space (*F-b metric*) and $\{a_n\}$ be a sequence in X.

a. $\{a_n\}$ is said to converge (*F-b convergent*) to $a \in X$, *if* $\rho(a_n, a) \to 0$, as $n \to +\infty$.
b. $\{a_n\}$ is said a Cauchy sequence (*F-b Cauchy*) in X, if $\rho(a_n, a_m) \to 0$, as $n, m \to +\infty$.

(X, ρ) is said a *complete*, if for every *Cauchy* sequence in *F-b* metric space is *F-b* convergent in X.

Definition 2.7. Let (X, ρ) be a function weighted b-metric (*F-b metric*) and $p \in X$, then $N_r(p) = \{x \in X \mid \rho(x, p) < r\}$ is called an *F-b open neighborhood* of p.

$G \subset X$ is called *F-b open* in X, if for any $y \in G$, there is $N_r(y)$, such that $N_r(y) \subset G$. Let $K \subset X$, if K is *F-b* open in X, then K^c is called *F-b closed* in X.

Preposition 2.1. *Let τ be a collection of all F-b open sets in X, then τ is topology F-b topology on X.*

3. Results

In this section, we present some prepositions about the properties of the function weighted b-metric space (*F-b*-metric space) and some theorems about fixed points and common fixed points for generalized expansive mapping.

Preposition 3.1. *Let $(f, K) \in F \times [0, +\infty)$, (X, ρ) be a function weighted b-metric space and $\{a_n\}$ be a sequence in X.*

If $\rho(a_n, a) \to 0$, as $n \to \infty$, then for any G open in X containing a, there is a positive integer N, such that for any $n \geq N$, then $a_n \in G$.

Proof. Since $a \in G$ and G open in *F-b metric space X*, then there is a open neighborhood $N_r(a)$ such that $N_r(a) \subset G$. Since $\rho(a_n, a) \to 0$, as $n \to \infty$, then there is a positive integer *N*, such that for any $n \geq N$, we have $\rho(a_n, a) < \frac{1}{2nb}$.

Let $N_{\frac{1}{2nb}}(a_n)$ be an open neighborhood of a_n in *X*. We will show that $N_{\frac{1}{2nb}}(a_n) \subset N_r(a)$. Taking any $x \in N_{\frac{1}{2nb}}(a_n)$, $x \neq a$, we have $\rho(x, a) > 0$, then by using of axiom B3, we obtain

$$f(\rho(x,a)) \leq f\left(b(\rho(x,a_n)+\rho(a_n,a))\right)+K$$

$$\leq f\left(b\left(\frac{1}{2nb}+\frac{1}{2nb}\right)\right)+K=f\left(\frac{1}{n}\right)+K. \quad (3.1)$$

Since $f \in F$ and $r > 0$, then there is $\sigma > 0$ such that for any $0 < t < \sigma$ the following holds

$$f(t) < f(r) - K.$$

For the next, we choose a positive integer N, such that for any $n \geq N$, $\frac{1}{n} < \sigma$, then we get

$$f\left(\frac{1}{n}\right) < f(r) - K. \quad (3.2)$$

So, we get

$$f(\rho(x,a)) < f(r).$$

Since f is non-decreasing, we obtain $\rho(x,a) < r$. This means that $x \in N_r(a)$. So, we get that

$$N_{\frac{1}{2nb}}(a_n) \subset N_r(a) \subset G$$

for any $n \geq N$. So it is proved that for any $n \geq N$, then $a_n \in$ G.

Preposition 3.2. *Let* $(f,K) \in F \times [0,+\infty)$ *and* (X,ρ) *be a function weighted b-metric space (F-b metric space)*. *Supose* (a_n) *is a sequence in X which satisfies:*

$$\rho(a_n,\ a_{n+1}) \leq \frac{c}{b}\rho(a_{n-1},\ a_n), \quad (3.3)$$

where $0 < c < 1$.

Then (a_n) *is Cauchy sequence in F-b metric space X.*

Proof. By using iteration, it can be obtained

$$\rho(a_n,a_{n+1}) \leq \frac{c^n}{b^n}\rho(a_0,a_1).$$

So, we have

$$b^n\rho(a_n,a_{n+1}) \leq c^n\rho(a_0,a_1). \quad (3.4)$$

Let $m > n$, then from (3.4) we get

$$\sum_{i=n+1}^{m} b^i\rho(a_i,a_{i+1}) \leq \sum_{i=n+1}^{m} c^i\rho(a_0,a_1) \leq \frac{c^n}{1-c}\rho(a_0,a_1). \quad (3.5)$$

Since $0 < c < 1$, then from (3.5), if $n \to +\infty$, then $\frac{c^n}{1-c}\rho(a_0,a_1) \to 0$.

This means, that for any $\gamma > 0$, there is $N \in \mathbb{N}$ such that for any $n \geq N$ we have

$$0 < \frac{c^n}{1-c}\rho(a_0, a_1) < \gamma. \tag{3.6}$$

Since $(f, K) \in F \times [0, +\infty)$, so f is non-decreasing and logarithmic-like function. So, for every $\varepsilon > 0$ there exists $\gamma > 0$ such that for any $s \in (0, \gamma)$, we have $f(s) < f(\varepsilon) - K$.

Therefore, from (3.5) and (3.6), and for $m > n \geq N$, we have

$$f\left(\textstyle\sum_{i=n+1}^{m} b^i \rho(a_i, a_{i+1})\right) \leq f\left(\frac{c^n}{1-c}\rho(a_0, a_1)\right) < f(\varepsilon) - K. \tag{3.7}$$

By using of axiom B3, and (3.7) we obtain

$$\rho(a_m, a_n) > 0 \text{ then } f\big(\rho(a_m, a_n)\big) \leq f\Big(\sum_{i=n+1}^{m} b^i \rho(a_i, a_{i+1}) + K < f(\varepsilon).$$

Since f is a non-decreasing, then $\rho(a_m, a_n) < \varepsilon$ for any $m > n \geq N$. It shows that (a_n) is a Cauchy sequence in *F-b* metric space X.

Preposition 3.3. *Suppose* $(f, K) \in F \times [0, +\infty)$ *and let* (X, ρ) *be a complete function weighted b-metric space.*

If (a_n) *is a convergent sequence in* F*-b metric space X, then the limit of* (a_n) *is unique.*

Proof. Suppose $\lim_{n\to+\infty} \rho(a_n, a^*) = 0$, $\lim_{n\to+\infty} \rho(a_n, s^*) = 0$, and $a^* \neq s^*$.

Since $\rho(a^*, s^*) > 0$, then from of axiom B3, we have

$$f\big(\rho(a^*, s^*)\big) \leq f\Big(b\big(\rho(a^*, a_n) + \rho(a_n, s^*)\big)\Big) + K. \tag{3.8}$$

Since $\lim_{n\to+\infty} \rho(a_n, a^*) = 0$ and $\lim_{n\to+\infty} \rho(a_n, s^*) = 0$, then we have

$$b\big(\rho(a^*, a_n) + \rho(a_n, s^*)\big) \to 0, \text{ as } n \to +\infty.$$

Then from (3.8) and by using the logarithmic-like property of f, we get

$$\lim_{n\to\infty} f\big(c\rho(a^*, a_n) + b\rho(a_n, s^*)\big) + K = -\infty,$$

which is a contradiction.

Preposition 3.4. *Suppose* $(f, K) \in F \times [0, +\infty)$ *and let* (X, ρ) *be a function weighted b-metric space.*

If $(a_n) \subset X$ *is a convergence sequence in F-b metric space X, then* (a_n) *is a Cauchy sequence in F-b metric space* X.

Proof. Let $\varepsilon > 0$, $(a_n) \subset X$ be a sequence converges to $a \in X$. It means

$$\lim_{n\to+\infty} \rho(a_n, a) = 0 \text{ and } \lim_{m\to+\infty} \rho(a_m, a) = 0. \tag{3.9}$$

Since $(f, K) \in F \times [0, +\infty)$, this means that f is a non-decreasing function and has a logarithmic-like

property. It implies there is $\gamma > 0$ such that for any $s \in (0, \gamma)$ we have $f(s) < f(\varepsilon) - K$.
From (3.9), we can choose a non-negative integer N, such that for any $n, m \geq N$,

$$b\big(\rho(a_n, a) + \rho(a_m, a)\big) < \gamma,$$

and holds

$$f\left(b\big(\rho(a_n, a) + \rho(a_m, a)\big)\right) < f(\varepsilon) - K. \tag{3.10}$$

Let $n, m \geq N$, and $\rho(a_n, a_m) > 0$, then by using of axiom B3 and (3.10), we have

$$f\big(\rho(a_n, a_m)\big) \leq f\left(b\big(\rho(a_n, a) + \rho(a, a_m)\big)\right) + K < f(\varepsilon).$$

Since f non-decreasing, then we get $\rho(a_n, a_m) < \varepsilon$. Thus, (a_n) is a Cauchy sequence in *F-b* metric space X.

Preposition 3.5. *Suppose $(f, K) \in F \times [0, +\infty)$ and let (X, ρ) be a function weighted b-metric space. Let τ be a topology on X.*

$A \subset X$ is closed ($A^c \in \tau$) if and only if for any sequence $(a_n) \subset A$ converges to a, then $a \in A$.

Proof. Let $A \subset X$ be a closed set and $(a_n) \subset A$ be a sequence that converges to a.

Suppose $a \notin A$, since A^c is open in X, it means there is a neighborhood of a, namely,

$$N_r(a) = \{x \in X \mid \rho(a.x) < r\}$$

such that $N_r(a) \cap A = \emptyset$.

Since $(a_n) \subset A$ and converges to a, there is a non-negative integer N such that for any $n \geq N$, then $a_n \in N_r(a)$. A contradiction with $N_r(a) \cap A = \emptyset$.

Converse, let $p \in A^c$, we will prove that there exists a $N_r(p)$, such that $N_r(p) \cap A = \emptyset$.

Suppose for any open neighborhood $N_r(p)$, $N_r(p) \cap A \neq \emptyset$. Let be taken $n \geq 1$ and chosen $b_n \in N_{\frac{1}{n}}(p) \cap A$. So, we can get a sequence $(b_n) \subseteq A$ that converges to p, where $p \in A^c$. It is a contradiction, because for any sequence $(a_n) \subset A$ converges to a, then $a \in A$.

Theorem 3.1. *Suppose $(f, K) \in F \times [0, +\infty)$ and let (X, ρ) be a complete function weighted b-metric space. Let $\varphi: X \to X$ be a continuous mapping that satisfies:*

$$\begin{aligned} &\rho\big(\varphi(x), \varphi(y)\big) \\ &\geq p\rho\big(y, \varphi(y)\big) + q\rho(x, y) - r\rho\big(x, \varphi(x)\big) \\ &\quad + s\rho(\varphi(x), y) - t\left(\rho\big(y, \varphi(y)\big)\rho(\varphi(x), y)\right)^{\frac{1}{2}} \end{aligned} \tag{3.11}$$

for all $x, y \in X$, where $q, t > 0, 0 < 1 + r < p + q$, and $p, s > 1$. Then φ has a unique fixed point in X.

Proof. Let $a_0 \in X$, and define a sequence $(a_n) \subset$X as follows:

$$a_n = \varphi(a_{n-1}).$$

From (3.11) we have

$$\begin{aligned}\rho(a_n,a_{n+1}) &= \rho(\varphi(a_{n-1}),\varphi(a_n))\\ &\geq \begin{pmatrix} p\rho(a_n,a_{n+1})+q\rho(a_{n-1},a_n)-r\rho(a_{n-1},a_n)\\ +s\rho(a_n,a_n)-t\left(\rho(a_n,\varphi(a_n))\rho(\varphi(a_{n-1}),a_n)\right)^{\frac{1}{2}}\end{pmatrix}\\ &\geq \begin{pmatrix} p\rho(a_n,a_{n+1})+q\rho(a_{n-1},a_n)-r\rho(a_{n-1},a_n)\\ +s\rho(a_n,a_n)-t\left(\rho(a_n,a_{n+1})\rho(a_n,a_n)\right)^{\frac{1}{2}}\end{pmatrix}\\ &= \left(p\rho(a_n,a_{n+1})+q\rho(a_{n-1},a_n)-r\rho(a_{n-1},a_n)\right)\\ &= \left(p\rho(a_n,a_{n+1})+q\rho(a_{n-1},a_n))-r\rho(a_{n-1},a_n)\right). \end{aligned} \tag{3.12}$$

From (3.12) we can get

$$\rho(a_n,a_{n+1}) \leq \frac{(r-q)}{(p-1)}\rho(a_{n-1},a_n) = \beta\rho(a_{n-1},a_n).$$

Where $\beta = \frac{(r-q)}{p-1}$, and since $0 < \frac{(r-q)}{p-1} < 1$, we get

$$\lim_{n\to+\infty}\rho(a_n,a_{n+1}) = 0. \tag{3.13}$$

By using Preposition 3.2, we get (a_n) is a Cauchy sequence in *F-b* metric space X.

Since (X,ρ) is complete then there is $a^* \in X$ (a^* is unique, Preposition 3.3) such that

$$\lim_{n\to+\infty}\rho(a_n,a^*) = 0. \tag{3.14}$$

Next, we show that a^* is a fixed point of φ.

Suppose $\varphi(a^*) \neq a^*$, it means $\rho(\varphi(a^*),a^*) > 0$, by using of axiom B3, then we have

$$f\left(\rho(\varphi(a^*),a^*)\right) \leq f\left(b\left(\rho\left(\varphi(a^*),\varphi(a_n)\right)+\rho(\varphi(a_n),a^*)\right)\right)+K.$$

Since φ is a continuous mapping and (a_n) converges to a^*, then we have

$$b\left(\rho\left(\varphi(a^*),\varphi(a_n)\right)+\rho(a_{n+1},a^*)\right) \to 0, \text{ as } n\to+\infty.$$

By using the logarithmic-like property of f, we get

$$\lim_{n\to+\infty} f\left(b\left(\rho\left(\varphi(a^*),\varphi(a_n)\right)+\rho(a_{n+1},a^*)\right)\right)+K = -\infty.$$

Which is a contradiction. Thus a^* is a fixed point of φ. Now, we show the uniqueness of the fixed point of φ.

Suppose a^* and s^* are the fixed point of φ and $a^* \neq s^*$. From (3.11) we have

$\rho(a_{n+1}, s^*)) = \rho(\varphi(a_n), \varphi(s^*)))$

$\geq \left(p\rho(s^*, \varphi(s^*)) + q\rho(a_n, s^*) - r\rho(a_n, \varphi(a_n)) + s\rho(\varphi(a_n), s^*) - t\left(\rho(s^*, \varphi(s^*))\rho(\varphi(a_n), s^*)\right)^{\frac{1}{2}}\right)$

$\geq \left(p\rho(s^*, s^*) + q\rho(a_n, s^*) - r\rho(a_n, a_{n+1}) + s\rho(a_{n+1}, s^*) - t\rho(s^*, s^*)\rho(a_{n+1}, s^*)\right)$

$= -r\rho(a_n, a_{n+1}) + s\rho(a_{n+1}, s^*) - t\rho(s^*, s^*)\rho(a_{n+1}, s^*)$

$= -r\rho(a_n, a_{n+1}) + s\rho(a_{n+1}, s^*)$

So, we get

$$\rho(\varphi(a_n), \varphi(s^*))) = \rho(a_{n+1}, s^*)) \leq \frac{r\rho(a_n, a_{n+1})}{s-1}. \tag{3.15}$$

Since $\rho(a^*, s^*) > 0$, by using of axiom B3, and using (3.15) we get

$$f(\rho(a^*, s^*)) \leq f\left(b\left(\rho(a^*, \varphi(a_n)) + \rho(\varphi(a_n), s^*)\right)\right) + K$$

$$\leq f\left(b\left(\rho(a^*, a_{n+1}) + \frac{r\rho(a_{n-1}, a_n)}{s-1}\right)\right) + K. \tag{3.16}$$

Since $s > 1$, from (3.13) and (3.14) we have

$$b\left(\rho(a^*, a_{n+1}) + \frac{r\rho(a_{n-1}, a_n)}{s-1}\right) \to 0, \text{ as } n \to +\infty.$$

By using logarithmic-like property of f we get

$$f\left(b\left(\rho(a^*, a_{n+1}) + \frac{r\rho(a_n, a_{n+1})}{s-1}\right)\right) \to -\infty.$$

Which is a contradiction with (3.16).

Example 3.1. Let $X = [0, \frac{1}{2}]$, and define $\rho(x, y) = (x - y)^2$, *for all* $x, y \in X$.

From Example 2.1, ρ is a function weighted *b*-metric, with $b = 2$, $f(\theta) = ln\,\theta$, $\theta \in (0, +\infty)$, and $K = 0$.

Define a function $\varphi(x) =$, for every $x \in X = [0, \frac{1}{2}]$, and choose $p = 3$, $q = \frac{1}{16}$, $r = 1$, $s = 3$, $t = 8$.

So we have $q, t > 0$, $0 < 1 + r < p + q$, and $p, s > 1$. So, from (3.11) we show that

$$3\rho(y, \varphi(y)) + \frac{1}{16}\rho(x, y) - \rho(x, \varphi(x)) + 3\rho(\varphi(x), y) - 8\left(\rho(x, \varphi(y))\rho(\varphi(x), y)\right)^{\frac{1}{2}} \leq \rho(\varphi(x), \varphi(y)).$$

From $x, y \in [0, \frac{1}{2}]$ we get

$$3\rho\big(y,\varphi(y)\big)+\frac{1}{16}\rho(x,y)-\rho\big(x,\varphi(x)\big)+3\rho(\varphi(x),y)-8\left(\rho\big(x,\varphi(y)\big)\rho(\varphi(x),y)\right)^{\frac{1}{2}}$$

$$=3\left(y-\frac{y^2}{2}\right)^2+\frac{1}{16}(x-y)^2-\left(x-\frac{x^2}{2}\right)^2+3\left(\frac{x^2}{2}-y\right)^2-8\left|\frac{y^2}{2}-y\right|\left|\frac{x^2}{2}-y\right|$$

$$\leq 3\left(y-\frac{y^2}{2}\right)^2+3\left(\frac{x^2}{2}-y\right)^2+\frac{1}{16}(x-y)^2-6\left(y-\frac{y^2}{2}\right)\left(\frac{x^2}{2}-y\right)-\left(x-\frac{x^2}{2}\right)^2 -2\left|\frac{y^2}{2}-y\right|\left|\frac{x^2}{2}-y\right|$$

$$\leq 3\left(\left(\frac{y^2}{2}-y\right)-\left(\frac{x^2}{2}-y\right)\right)^2+\frac{1}{16}(x-y)^2-\left(x-\frac{x^2}{2}\right)^2-2\left|\frac{y^2}{2}-y\right|\left|\frac{x^2}{2}-y\right|$$

$$=\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{1}{16}(x-y)^2+2\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\left(x-\frac{x^2}{2}\right)^2-2\left|\frac{y^2}{2}-y\right|\left|\frac{x^2}{2}-y\right|.$$

Since $x,y\in\left[0,\frac{1}{2}\right]$, then we have

$$\left|\frac{x^2}{2}-y\right|\geq\left|\frac{x^2}{2}-y^2\right|,\ \left|y-\frac{y^2}{2}\right|\geq\frac{y^2}{2}.$$

So we get

$$\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{1}{16}(x-y)^2+2\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\left(x-\frac{x^2}{2}\right)^2-2\left|\frac{y^2}{2}-y\right|\left|\frac{x^2}{2}-y\right|$$

$$\leq\left(\frac{x^2}{2}-y^2\right)^2+\frac{1}{16}(x-y)^2+2\left(\frac{x^2}{2}-y^2\right)^2-\left(x-\frac{x^2}{2}\right)^2-2\left|y-\frac{y^2}{2}\right|\left|\frac{x^2}{2}-y^2\right|$$

$$=\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\frac{1}{8}xy+\frac{x^4}{4}-\frac{15}{16}x^2+x^3+\frac{1}{16}y^2-\frac{y^4}{2}$$

$$\leq\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\frac{1}{8}xy+\frac{x^2}{16}-\frac{15}{16}x^2+\frac{x^2}{2}+\frac{1}{16}y^2-\frac{y^4}{2}$$

$$\leq\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\frac{6x^2}{16}+\frac{1}{16}y^2-\frac{y^4}{2}$$

$$\leq\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2$$

$$\leq\rho\big(\varphi(x),\varphi(y)\big).$$

Since $y\in\left[0,\frac{1}{2}\right]$, we have $\frac{1}{16}y^2-\frac{y^4}{2}\leq 0$, so we get

$$\left(3\rho(y,\varphi(y))+\frac{1}{16}\rho(x,y)-\rho(x,\varphi(x))+3\rho(\varphi(x),y)-8\rho(x,\varphi(y))\rho(\varphi(x),y)\right)$$

$$\leq\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\frac{6x^2}{16}+\frac{1}{16}y^2-\frac{y^4}{2}$$

$$\leq\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2\leq\rho(\varphi(x),\varphi(y)).$$

From $a_n=\varphi(a_{n-1})=\frac{(a_{n-1})^2}{2}$, we get $a_n=\frac{(a_0)^{2n}}{2^{2^n-1}}$. Since $a_0\in[0,\frac{1}{2}]$, then we get $a_n\to 0$, as $n\to +\infty$.

So, it's clear $x=0$ is a fixed point of φ.

Theorem 3.2. *Suppose* $(f,K)\in F\times[0,+\infty)$ *and let* (X,ρ) *be a complete function weighted b-metric space. Let* $u,v:X\to X$ *be self-mappings such that* $u(X)\subseteq v(X)$ *and* $v(X)$ *closed, which satisfy as follows:*

$$\rho(v(x),v(y))\geq\alpha\rho(v(y),u(y))+\gamma\rho(u(x),v(x))+\delta\rho(u(x),v(y))$$
$$+\omega\rho(u(x),u(y))+\theta\rho(u(x),v(x))\rho(u(x),v(y)),\tag{3.17}$$

for all $x,y\in X$, where $\alpha,\gamma,\omega,\theta>0$, $\delta+\omega>1$, and $\alpha+\gamma+\omega>1$.

If $\{u,v\}$ *is weak compatible, then* u,v *have a unique common fixed point in* X.

Proof. Let $x_0\in X$, since $u(X)\subseteq v(X)$, we define a sequence $\{a_n\}$ and $\{b_n\}$ in X, such that

$$b_n=u(a_{n-1})=v(a_n),\text{ for }n=1,2,\cdots.$$

From (3.17) we have

$$\begin{aligned}\rho(b_{n-1},b_n)&=\rho(v(a_{n-1}),v(a_n))\\&\geq\alpha\rho(v(a_n),u(a_n))+\gamma\rho(u(a_{n-1}),v(a_{n-1}))+\delta\rho(u(a_{n-1}),v(a_n))\\&+\omega\rho(u(a_{n-1}),u(a_n))+\theta\gamma\rho(u(a_{n-1}),v(a_{n-1}))\rho(u(a_{n-1}),v(a_n))\\&\geq\alpha\rho(b_n,b_{n+1})+\gamma\rho(b_n,b_{n-1})+\delta\rho(b_n,b_n)\\&+\omega\rho(b_n,b_{n+1})+\theta(\rho(b_n,b_{n-1})\rho(b_n,b_n))\\&=\alpha\rho(b_n,b_{n+1})+\gamma\rho(b_n,b_{n-1})+\omega\rho(b_n,b_{n+1}).\end{aligned}$$

So we get

$$\rho(b_n,b_{n+1})\leq\frac{(1-\gamma)}{\alpha+\omega}\rho(b_n,b_{n-1}).\tag{3.18}$$

Since $\alpha+\gamma+\omega>1$, we have $0<\frac{(1-\gamma)}{\alpha+\omega}<1$, so from (3.18), we get

$$\lim_{n\to+\infty} \rho(b_n, b_{n+1}) = 0. \tag{3.19}$$

So, we get that (b_n) is a Cauchy sequence in *F-b* metric space X.

Since (X, ρ) is complete then there is $b^* \in X$, such that

$$\lim_{n\to+\infty} \rho(b_n, b^*) = 0. \tag{3.20}$$

Since $v(X)$ is closed, then $b^* \in v(X)$. It implies, there exists $a^* \in X$ such that $b^* = v(a^*)$.

We will show that $u(a^*) = v(a^*)$.

From (3.17) we have

$$\begin{aligned}
&\rho\big(v(a^*), v(a_n)\big)\\
&\geq \alpha\rho\big(v(a_n), u(a_n)\big) + \gamma\rho\big(u(a^*), v(a^*)\big) + \delta\rho\big(u(a^*), v(a_n)\big)\\
&\quad +\omega\rho\big(u(a^*), u(a_n)\big) + \theta\left(\rho\big(u(a^*), v(a^*)\big)\rho\big(u(a^*), v(a_n)\big)\right)\\
&\geq \alpha\rho\big(v(a_n), u(a_n)\big) + \omega\rho\big(u(a^*), u(a_n)\big).
\end{aligned}$$

So, we get

$$\rho\big(u(a^*), u(a_n)\big) \leq \frac{1}{\omega}\Big(\rho\big(v(a^*), v(a_n)\big) - \alpha\rho\big(v(a_n), u(a_n)\big)\Big). \tag{3.21}$$

Suppose $u(a^*) \neq v(a^*)$, then $\rho(u(a^*), v(a^*)) > 0$.

By using of B3,we can get

$$\begin{aligned}
&f\left(\rho\big(u(a^*), v(a^*)\big)\right)\\
&\leq f\left(b\rho\big(u(a^*), u(a_n)\big) + b^2\rho\big(u(a_n), v(a_n)\big) + b^2\rho\big(v(a_n), v(a^*)\big)\right).
\end{aligned} \tag{3.22}$$

By using (3.21) and (3.22) then we have

$$\begin{aligned}
&f\left(\rho\big(u(a^*), v(a^*)\big)\right)\\
&\leq f\left(\frac{b}{\omega}\Big(\rho\big(v(a^*), v(a_n)\big) - \alpha\rho\big(v(a_n), u(a_n)\big)\Big) + b^2\rho\big(u(a_n), v(a_n)\big) + b^2\rho\big(v(a_n), v(a^*)\big)\right) + K\\
&= f\left(\frac{b}{\omega}\big(\rho(b^*, b_n) - \alpha\rho(b_n, b_{n+1})\big) + b^2\rho(b_{n+1}, b_n) + b^2\rho(b_n, b^*)\right) + K.
\end{aligned}$$

By using (3.19) and (3.20), the we get

$$\frac{b}{\omega}\big(\rho(b^*, b_n) - \alpha\rho(b_n, b_{n+1})\big) + b^2\rho(b_{n+1}, b_n) + b^2\rho(b_n, b^*) \to 0, \text{ as } n \to +\infty.$$

So, by applying of the logarithmic-like property of f, then we obtain

$$f\left(\frac{b}{\omega}\big(\rho(b^*,b_n)-\alpha\rho(b_n,b_{n+1})\big)+b^2\rho(b_{n+1},b_n)+b^2\rho(b_n,b^*)\right)+K\to-\infty.$$

Which is a contradiction. Thus we have $u(a^*)=v(a^*)=b^*$.

Since $\{u,v\}$ is weakly compatible, then we have $vu(a^*)=uv(a^*)$, so we have $v(b^*)=u(b^*)$.

$$\begin{aligned}\rho(v(b^*),b^*)&=\rho\big(v(b^*),v(a^*)\big)\\&\geq\alpha\rho\big(v(a^*),u(a^*)\big)+\gamma\rho\big(u(b^*),v(b^*)\big)+\delta\rho\big(u(b^*),v(a^*)\big)\\&+\omega\rho\big(u(b^*),u(a^*)\big)+\theta\left(\rho\big(u(b^*),v(b^*)\big)\rho\big(u(b^*),v(a^*)\big)\right)\\&=\alpha\rho(b^*,b^*)+\gamma\rho\big(v(b^*),v(b^*)\big)+\delta\rho(v(b^*),b^*)+\omega\rho(v(b^*),b^*)\\&+\theta\left(\rho\big(u(b^*),v(b^*)\big)\rho(v(b^*),b^*)\right)\\&\geq\delta\rho(v(b^*),b^*)+\omega\rho(v(b^*),b^*).\end{aligned}$$

So, we get

$$\rho(v(b^*),b^*)\geq(\delta+\omega)\rho(v(b^*),b^*).$$

Since $\delta+\omega>1$, it implies $\rho(v(b^*),b^*)=0$. Thus $v(b^*)=b^*$.

Since $v(b^*)=u(b^*)$, then we have b^* is a common fixed point of u and v. Next, we show that u and v have a unique common fixed point.

Suppose c^* is another common fixed point of u and v. From (3.17), then we have

$$\begin{aligned}\rho(b^*,c^*)&=\rho\big(v(b^*),v(c^*)\big)\\&\geq\alpha\rho\big(v(c^*),u(c^*)\big)+\gamma\rho\big(u(b^*),v(b^*)\big)+\delta\rho\big(u(b^*),v(c^*)\big)\\&+\omega\rho\big(u(b^*),u(c^*)\big)+\theta\left(\rho\big(v(c^*),u(c^*)\big)\rho\big(u(b^*),v(c^*)\big)\right)\\&\geq\alpha\rho(c^*,c^*)+\gamma\rho(b^*,b^*)+\delta\rho(b^*,c^*)+\omega\rho(b^*,c^*)+\theta\big(\rho(b^*,b^*)\rho(b^*,c^*)\big)\\&\geq\delta\rho(b^*,c^*)+\omega\rho(b^*,c^*)=(\delta+\omega)\rho(b^*,c^*).\end{aligned}$$

Since $\delta+\omega>1$, it implies $\rho(b^*,c^*)=0$. Thus $c^*=b^*$.

Example 3.2. Let $X=[0,\frac{1}{2}]$, and define $\rho(x,y)=(x-y)^2$, for all $x,y\in X$.

From Example 2.1, ρ is a function weighted b-metric, with $b=2$, $f(s)=\ln s$, $s\in(0,+\infty)$, and $\mathrm{K}=0$.

Define the functions $u(x)=\frac{x^2}{4}$ and $v(x)=\frac{x^2}{2}$ for every $x\in X=[0,\frac{1}{2}]$, so we have $u(X)\subset v(X)$ and $v(X)=[0,\frac{1}{4}]$ is closed. Let $\alpha=\frac{1}{4},\delta=\frac{1}{2},\gamma=\frac{1}{4},\omega=\frac{3}{4}$, and $\theta=\frac{1}{8}$. These parameters satisfy $\delta+\omega>1$, and $\alpha+\gamma+\omega>1$.

From (3.17) and for all $x,y\in\left[0,\frac{1}{2}\right]$, we have

$$\alpha\rho\left(\frac{y^2}{2},\frac{y^2}{4}\right)+\gamma\rho\left(\frac{x^2}{4},\frac{x^2}{2}\right)+\delta\rho\left(\frac{x^2}{4},\frac{y^2}{2}\right)+\omega\rho\left(\frac{x^2}{4},\frac{y^2}{4}\right)+\theta\left(\rho\left(\frac{x^2}{4},\frac{x^2}{2}\right)\rho\left(\frac{x^2}{4},\frac{y^2}{2}\right)\right)$$

$$=\frac{1}{4}\left(\frac{y^2}{2}-\frac{y^2}{4}\right)^2+\frac{1}{4}\left(\frac{x^2}{4}-\frac{x^2}{2}\right)^2+\frac{1}{2}\left(\frac{x^2}{4}-\frac{y^2}{2}\right)^2+\frac{3}{4}\left(\frac{x^2}{4}-\frac{y^2}{4}\right)^2+\frac{1}{8}\left(\frac{x^2}{4}-\frac{y^2}{2}\right)^2$$

$$=\frac{1}{16}\left(\frac{y^2}{2}\right)^2+\frac{1}{16}\left(\frac{x^2}{2}\right)^2+\frac{1}{2}\left(\frac{x^2}{4}-\frac{y^2}{2}\right)^2+\frac{3}{4}\left(\left(\frac{x^2}{4}\right)^2-2\frac{x^2}{4}\frac{y^2}{4}+\left(\frac{y^2}{4}\right)^2\right)+\frac{1}{8}\left(\left(\frac{x^2}{4}\right)^2-\frac{x^2}{2}\frac{y^2}{2}+\left(\frac{y^2}{2}\right)^2\right)$$

$$=\frac{1}{16}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{1}{8}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{3}{16}\left(\left(\frac{x^2}{2}\right)^2+\left(\frac{y^2}{2}\right)^2\right)+\frac{1}{8}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\frac{3}{8}\frac{x^2}{2}\frac{y^2}{2}-\frac{3}{8}\left(\frac{x^2}{4}\right)^2$$

$$=\frac{1}{16}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{1}{8}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{3}{16}\left(\left(\frac{x^2}{2}\right)^2-2\frac{x^2}{2}\frac{y^2}{2}+\left(\frac{y^2}{2}\right)^2\right)+\frac{1}{8}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\frac{3}{8}\left(\frac{x^2}{4}\right)^2$$

$$=\frac{1}{16}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{1}{8}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{3}{16}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2+\frac{1}{8}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\frac{3}{8}\left(\frac{x^2}{4}\right)^2$$

$$=\frac{1}{2}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2-\frac{3}{8}\left(\frac{x^2}{4}\right)^2\leq\frac{1}{2}\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2\leq\left(\frac{x^2}{2}-\frac{y^2}{2}\right)^2=\rho\big(v(x),v(y)\big)$$

From $b_n=u(a_{n-1})=v(a_n)$, where $u(x)=\frac{x^2}{4}$ and $v(x)=\frac{x^2}{2}$, then we get

$$b_n=u(a_{n-1})=\frac{(a_{n-1})^2}{4}=v(a_n)=\frac{{a_n}^2}{2}.$$

So we get $b_n=\frac{(b_0)^{2n}}{4^{2^n-1}}$. Since $b_0\in[0,\frac{1}{2}]$, then we obtain $b_n\to 0$, as $n\to+\infty$. It's clear that $x=0$ is a common fixed point of u and v.

4. An application in dynamic programing

In this section, an application of the main results related to Theorem 3.2 on dynamic programming is presented, which is to find the common solution of two functional equations.

A dynamic programming system involves having two spaces as its main components, namely the decision space (DS) and the state space (SS). The decision space is a collection of possible solutions of the problem that can occur, and the state space is a collection of states, initial states, state actions and state transitions.

Let U and V the DS and SS, respectively. We assume a problem of dynamic programming formulated in the form of functional equations as follows:

$$K(t)=\min_{v\in V}\left\{Q(t,v)+f\left(t,v,K\big(\gamma(t,v)\big)\right)\right\},\ \text{for } t\in U. \tag{4.1}$$

$$L(t)=\min_{v\in V}\left\{Q(t,v)+f\left(t,v,L\big(\gamma(t,v)\big)\right)\right\},\ \text{for } t\in U. \tag{4.2}$$

Where A and B are Banach spaces such that $U \subseteq A, V \subseteq B$ and

$$\gamma : U \times V \to U,$$

$$Q : U \times V \to R,$$

$$f : U \times V \times R \to R.$$

Assume that the decision spaces and state spaces are *U* and *V*, respectively. We show that the functional equations (4.1) and (4.2) has a unique common solution.

Let *Z* (*U*) be a set of all bounded real-valued mappings on *U*. For all $g \in Z(U)$, define

$$\|g\| = \max_{u \in U} |g(u)|.$$

Then $(Z(U), \|.\|)$ is a Banach space.

Defined a function $\rho : Z(U) \times Z(U) \to R^+$ as follows:

$$\rho(g, h) = \max_{u \in U} |g(u) - h(u)|^2$$

Based on Example 2.1, ρ is a complete function weighted *b*-metric in *Z* (*U*).

Let the following conditions hold:

C1: Q and f are bounded.

C2: For $u \in U$ and $g, h \in Z(U)$ define functions

$$S : Z(U) \to Z(U) \text{ by}$$

$$Sg(u) = \min_{v \in V} \{Q(u, v) + f(u, v, g(\gamma(u, v)))\},$$

$$Sh(u) = \min_{v \in V} \{Q(u, v) + f(u, v, h(\gamma(u, v)))\},$$

and there exists a function $T : Z(U) \to Z(U)$ such that $Tg \subseteq Sg$ and if $Tg(u) = Sg(u)$ then $STg(u) = TSg(u)$.

It is clear that *S* is well-defined, since Q, f are bounded. $\{S, T\}$ is weakly compatible.

C3: For $(u, v) \in U \times V$, $g, h \in Z(U)$ and $x \in U$, we write

$$|f(u, v, g(x)) - f(u, v, h(x))|^2 \geq M(g, h),$$

where

$$M(g, h) = \rho(Sg, Sh)$$
$$\geq \alpha\rho(Sh, Th) + \gamma\rho(Tg, Sh) + \delta\rho(Tg, Sh)$$
$$+\omega\rho(Tg, Th) + \theta\rho(Tg, Sh)\rho(Tg, Sh),$$

for all $g, h \in Z(U)$, here $\alpha, \gamma, \omega, \theta > 0$, $\delta + \omega > 1$, and $\alpha + \gamma + \omega > 1$.

Theorem 4.1. *If the conditions* (C1)–(C3) *hold, then Eqs* (4.1) *and* (4.2) *have a unique common bounded solution.*

Proof. Suppose $\varepsilon > 0$ and $g, h \in Z(U)$. Take $u \in U$ and $v_1, v_2 \in V$ such that

$$F(u, v_1) + f(u, v, g(\gamma(u, v_1))) < Sg + \epsilon. \tag{4.3}$$

$$F(u, v_2) + f(u, v, h(\gamma(u, v_2))) < Sh + \epsilon. \tag{4.4}$$

And

$$F(u, v_2) + f(u, v_2, g(\gamma(u, v_2))) \geq Sg. \tag{4.5}$$

$$F(u, v_1) + f(u, v, h(\gamma(u, v_1))) \geq Sh. \tag{4.6}$$

Then, using (4.3) and (4.6), we get

$$f(u, v_1, g(\gamma(u, v_1))) - f(u, v, h(\gamma(u, v_1))) \leq Sg(u) - Sh(u) + \epsilon \leq |Sg(u) - Sh(u)| + \epsilon. \tag{4.7}$$

Similarly, by (4.4) and (4.5), we get

$$f(u, v_2, h(\gamma(u, v_2))) - f(u, v_2, g(\gamma(u, v_2))) \leq Sh(u) - Sg(u) + \epsilon \leq |Sg(u) - Sh(u)| + \epsilon. \tag{4.8}$$

From (4.7) and (4.8) we have

$$\left|ff_1(u, v, g(\gamma(u, v_2))) - f(u, v, h(\gamma(u, v_2)))\right| \leq |Sg(u) - Sh(u)| + \epsilon.$$

So we have

$$\left|f(u, v, g(\gamma(u, v))) - f(u, v, h(\gamma(u, v)))\right|^2$$

$$\leq |Sg(u) - Sh(u)|^2 + \epsilon.$$

Thus for all $\epsilon > 0$, we get

$$\left|f(u, v, g(\gamma(u, v))) - f(u, v, h(\gamma(u, v)))\right|^2 \leq d(Sg, Sh).$$

Using C3, we have

$$M(g, h) \leq \left|f(u, v, g(\gamma(u, v))) - f(u, v, h(\gamma(u, v)))\right|^2 \leq d(Sg, Sh).$$

So, we have

$$d(Sg, Sh) \geq \alpha\rho(Sh, Th) + \gamma\rho(Tg, Sh) + \delta\rho(Tg, Sh) + \omega\rho(Tg, Th) + \theta\rho(Tg, Sh)\rho(Tg, Sh).$$

Therefore, based on Theorem 3.2, then the functional equation on (4.1) and (4.2) have a unique common solution.

5. Conclusions

In this paper, we have revealed that the function weighted b-metric is a generalization of the function weighted metric. Some topological properties of the space of the function weighted b-metric related to open sets, closed sets, convergent sequences, the uniqueness of the limit of the sequences and the metrizable properties of the function weighted b-metric are also given. Furthermore, the results regarding the fixed point of an expansive mapping and its corresponding fixed point for two mappings are also revealed. Some examples to clarify the theorem are given. An application of the common fixed point theorem to dynamic programming is also provided.

Acknowledgments

The author is very grateful to the anonymous referee and our colleagues who have provided very useful comments for completing this paper. This work is financially supported by the PDU Research Project 2022, Hasanuddin University, Indonesia, No. 1474/UN.22.2/PT01/2022.

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. I. A. Bakhtin, The contraction mapping principle in quasi- metric spaces, *Funct. Anal.*, **30** (1989), 26–37.
2. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inf. Univ. Ostrav.*, **1** (1993), 5–11.
3. R. George, S. Radenović, K. P. Reshma, S. Shukla, Rectangular b-metric space and contraction principles, *J. Nonlinear Sci. Appl.*, **8** (2015), 1005–1013. https://doi.org/10.22436/jnsa.008.06.11
4. R. George, A. Belhenniche, S. Benahmed, Z. D. Mitrović, N. Mlaiki, L. Guran, On an open question in controlled rectangular b-metric spaces, *Mathematics*, **8** (2020), 2239. https://doi.org/10.3390/math8122239
5. A. Belhenniche, L. Guran, S. Benahmed, F. L. Pereira, Solving nonlinear and dynamic programming equations on extended b-metric spaces with the fixed-point technique, *Fixed Point Theory Algorithms Sci. Eng.*, **2022** (2022), 24. https://doi.org/10.1186/s13663-022-00736-5
6. M. A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, *Fixed Point Theory Appl.*, **2010** (2010), 315398. https://doi.org/10.1155/2010/315398
7. A. Branciari, A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37. https://doi.org/10.5486/PMD.2000.2133
8. R. Fagin, R. Kumar, D. Sivakumar, Comparing top k lists, *SIAM J. Discrete Math.*, **17** (2003), 134–160. https://doi.org/10.1137/S0895480102412856
9. Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.*, **7** (2006), 289–297.
10. X. Huang, C. Zhu, X. Wen, Fixed point theorems for expanding mappings in cone metric spaces, *Math. Rep.*, **14** (2012), 141–148.

11. C. Chen, C. Zhu, Fixed point theorems for times reasonable expansive mapping, *Fixed Point Theory Appl.*, **2008** (2008), 302617. https://doi.org/10.1155/2008/302617
12. M. Akkouchi, A common fixed point theorem for expansive mappings under strict implicit conditions on b-metric spaces, *Acta Univ. Palacki. Olomuc., Fac. Rerum Nat. Math.*, **50** (2011), 5–15.
13. J. Rashmi, R. D. Daheriya, M. Ughade, Fixed point, coincidence point and common fixed point theorems under various expansive conditions in b-metric spaces, *Int. J. Sci. Innovative Math. Res.*, **3** (2015), 26–34.
14. M. Jleli, B. Samet, On a new generalization of metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 128. https://doi.org/10.1007/s11784-018-0606-6
15. M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, *Fixed Point Theory Algorithm Sci. Eng.*, **2015** (2015), 61. https://doi.org/10.1186/s13663-015-0312-7
16. A. Hussain, H. Al-Sulami, H. Hussain, H. Farooq, Newly fixed disc results using advanced contractions on F-metric space, *J. Appl. Anal. Comput.*, **10** (2020), 2313–2322. https://doi.org/10.11948/20190197
17. H. Işik, N. Hussain, A. R. Khan, Endpoint results for weakly contractive mappings in F-metric spaces with an application, *Int. J. Nonlinear Anal. Appl.*, **11** (2020), 351–361. https://doi.org/10.22075/ijnaa.2020.20368.2148
18. S. Z. Wang, B. Y. Li, Z. M. Gao, K. Iseki, Some fixed point theorems on expansion mappings, *Math. Jpn.*, **29** (1984), 631–636.
19. M. Abbas, B. E. Rhoades, Common fixed point theorems for occasionally weakly compatible mappings satisfying a generalized contractive condition, *Math. Commun.*, **13** (2008), 295–301.