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Research article

Geometric properties of holomorphic functions involving generalized distribution with bell number

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Abstract: One of the statistical tools used in geometric function theory is the generalized distribution which has recently gained popularity due to its use in solving practical issues. In this work, we obtained a new subclass of holomorphic functions, which defined by the convolution of generalized distribution and incomplete beta function associated with subordination in terms of the bell number. Further, we estimate the coefficient inequality and upper bound for a subclass of holomorphic functions. Our findings show a clear relationship between statistical theory and geometric function theory.

Keywords: holomorphic functions; generalized distribution; bell number; incomplete beta functions
Mathematics Subject Classification: 30C80, 30C45

1. Introduction

Let $\mathcal{N}(U)$ be the class of function that is holomorphic in the open unit disk $U := \{z \in C : |z| < 1\}$, and Let A be the subset of $\mathcal{N}(U)$ that consists of functions

$$f(z) = z + \sum_{i=2}^{\infty} a_i\, z^i, z \in U. \tag{1.1}$$

Let $f_j(z) = \sum_{i=0}^{\infty} a_{i,j}\, z^i$ $(j = 1, 2)$ which are holomorphic in $\mathcal{N}(U)$, then well known hadamard product of f_1 and f_2 is given by

$$(f_1 * f_2)(z) = \sum_{i=0}^{\infty} a_{i,1} a_{i,2}\, z^i, \quad z \in U.$$

If holomorphic functions of $f, g \in \mathcal{N}(U)$, then f is subordinate to g, this is implies $f \prec g$, if there presents a schwarz function $u \in \mathcal{N}(U)$ with $|u(z)| \leq 1, z \in U$ and $u(0) = 0$ like $f(z) = g\,(u(z))$ for all $z \in U$. In specifically if g is univalent in U, then we have the following condition holds ture;

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad and \quad f(U) \subset g(U).$$

Let P denote the class of holomorphic functions Φ in U with $Re(\Phi(z)) > 0$, and $\Phi(0) = 1$ of the form is given by

$$\Phi(z) = 1 + \sum_{i=1}^{\infty} d_i z^i, \quad z \in U. \tag{1.2}$$

Generalized distribution series were recently proposed by Porwal [20], who also obtained some necessary and sufficient criteria for a few classes of univalent functions. Because this distribution is a generalization of all discrete probability distributions, the study of generalized distribution series is of particular interest. In 2021, Abiodun Tinuoye Oladipo [19] studied analytic univalent functions defined by a generalized discrete probability distribution. For a thorough analysis, see [20]. We now go back to the generalized distribution definition.

The probability mass function of a generalized discrete probability distribution is below as follows

$$p(i) = \frac{a_i}{S}, \quad i = 0, 1, 2, ...$$

where $p(i)$ is the probability mass function, since $\sum_i p(i) = 1$ and $p(i) \geq 0$ and

$$S = \sum_{i=0}^{\infty} a_i. \tag{1.3}$$

Let

$$\psi(x) = \sum_{i=0}^{\infty} a_i x^i. \tag{1.4}$$

It is clear from (1.3) that the series generated by (1.4) is convergent for $|x| < 1$ and convergent for $x = 1$. Binomial distribution, Yule-Simmon distribution, Poisson distribution, Bernoulli distribution, and Logarithmic distribution are different well-known discrete probability distributions, which can be obtained in of specific concern is the polynomial whose coefficients are probabilities of the generalized distribution study in [20] and as follows

$$M_\psi(z) = z + \sum_{i=2}^{\infty} \frac{a_{i-1}}{s}, \; z^i \; z \in U, \tag{1.5}$$

where$S = \sum\limits_{i=0}^{\infty} a_i$.

Definition 1. Let $\phi(a, c; z)$ be the incomplete beta function provided by

$$\phi(a, c; z) = z + \sum_{i=2}^{\infty} \frac{(a)_{i-1}}{(c)_{i-1}} z^i = z + \sum_{i=2}^{\infty} \varphi_{i-1} z^i, \quad c \neq 0, -1, -2....$$

where $\varphi_{i-1} = \frac{(a)_{i-1}}{(c)_{i-1}}$, and $(y)_i-$ Pochkammer symbol defined interms of the gamma function by

$$(y)_i = \frac{\Gamma(y+i)}{\Gamma(y)}$$

$$(y)_i = \begin{cases} 1 & \text{if } i = 0 \\ y(y+1)(y+2).....(y+i-1) & \text{if } i \in \mathbb{N}. \end{cases} \tag{1.6}$$

For $f \in A$, by using Carlson and Shaffer operator [7] $\mathcal{L}_c^a f(z) : A \longrightarrow A$ defined by

$$\begin{aligned}\mathcal{L}_c^a f(z) = \quad & \phi(a, c; z) * f(z) \\ & = z + \sum_{i=2}^{\infty} \frac{(a)_{i-1}}{(c)_{i-1}} a_i z^i, \; z \in U\end{aligned}$$

$$\mathcal{L}_c^a f(z) = z + \sum_{i=2}^{\infty} \varphi_{i-1} a_i z^i, \; z \in U \tag{1.7}$$

where $\varphi_{i-1} = \frac{(a)_{i-1}}{(c)_{i-1}}$, and notice that

$$z(\mathcal{L}_c^a f(z))' = a\mathcal{L}_c^{a+1} - (a-1)\mathcal{L}_c^a f(z), for\; z \in U.$$

Remark 1. We will focus the operator $\mathcal{L}(a, c)$ of some specific cases is given below:
(i) $\mathcal{L}_a^a = f(z)$;
(ii) $\mathcal{L}_1^2 = zf'(z)$;
(iii) $\mathcal{L}_1^3 = zf'(z) + \frac{1}{2}z^2 f''(z)$;
(iv) $\mathcal{L}_1^m + 1 = D^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z), m > -1, m \in z$ is the well known Ruscheweyh derivative of $f(z)$ [23].

Definition 2. For a permanent non-negative integer i, the bell numbers B_i count the possible disjoint partitions of a set with i elements into non-empty subsets.

The numbers B_i are labeled the bell numbers later Eric temple bell (1883–1960) ([1, 2]) who named them the exponential numbers. The bell numbers $B_i, (i \geqq 0)$ satisfy a recurrence relation involving binomial coefficients $B_{i+1} = \sum_{k=0}^{i} \binom{i}{k} B_k$. Clearly $B_0 = B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, and B_6 = 203$. The function $e^{e^z - 1}$ that as follows:

$$\mathcal{Q}(z) = e^{e^z - 1} = \sum_{i=0}^{\infty} B_i \left(\frac{z^i}{i!} \right) = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + ... \tag{1.8}$$

Recently Kumar et al. and coauthors studied coefficient bounds for starlike functions to the bell number [8, 14]. We introduced a linear operator $\mathcal{K}_{c,\psi}^a f(z)$ by convolution product of generalized distribution is given (1.5) and the incomplete beta function in (1.7), defined as follows

$$\mathcal{K}_{c,\psi}^a f(z) = z + \sum_{i=2}^{\infty} \varphi_{i-1} \frac{a_{i-1}}{S} z^i, \; z \in U \tag{1.9}$$

where $\varphi_{i-1} = \frac{(a)_{i-1}}{(c)_{i-1}}$,
and notice that

$$z(\mathcal{K}_{c,\psi}^a f(z))' = a\mathcal{K}_{c,\psi}^{a+1} - (a-1)\mathcal{K}_{c,\psi}^a f(z), for\; z \in U.$$

Remark 2. We will focus the operator $\mathcal{K}_{c,\psi}^a f(z)$ of some specific cases is given below:
(i) $\mathcal{K}_{a,1}^a = f(z)$;
(ii) $\mathcal{K}_{1,1}^2 = zf'(z)$.

Motivated by the articles of [6,10–13,21,22,24], using the concept of subordination and the linear operator [18] $\mathcal{K}^a_{c,}$, we defined a subclass of A by $\mathcal{K}^a_{c,\psi}f(z)$. For this subclass, we obtained coefficient inequality and sharp bounds of $\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right|$.

Murugusundaramoorthy et al. [18] defined the subclasses as given by:

Definition 3. For $0 \leq \delta \leq 1$, Let $\mathcal{MK}^a_{c,\psi}(\delta, \mathcal{Q})$ with $c \neq 0, -1, -2......$ and $\mathcal{Q}(z) = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + ..., z \in U$ indicate the subclass of function. f is belongs to the class A that satisfies the subordination condition

$$\frac{z(\mathcal{K}^a_{c,\psi}f(z))'}{(1-\delta)\mathcal{K}^a_{c,\psi}f(z) + \delta z} \prec \mathcal{Q}(z). \tag{1.10}$$

Remark 3. (i) For $\delta = 0$, let $\mathcal{MK}^a_{c,\psi}(0, \mathcal{Q}) = \mathcal{SK}^a_{c,\psi}(\mathcal{Q})$ indicate the subclass of A, the elements of which are follows by (1.1) and convince the subordination condition

$$\frac{z(\mathcal{K}^a_{c,\psi}f(z))'}{\mathcal{K}^a_{c,\psi}f(z)} \prec \mathcal{Q}(z).$$

(ii) For $\delta = 1$, let $\mathcal{MK}^a_{c,\psi}(1, \mathcal{Q}) = \mathcal{RK}^a_{c,\psi}(\mathcal{Q})$ indicate the subclass of A, the elements of which are follows by (1.1) and convince the subordination condition

$$z(\mathcal{K}^a_{c,\psi}f(z))' \prec \mathcal{Q}(z).$$

Using the techiques of Zlotkiewicz and libera [15, 16] and Koepf [12] and Caglar et al. [4] concerted with the help of Mapple^{TM} software and obtain coefficient inequality and sharp bounds of $\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right|$ for the function f comes under the class $\mathcal{MK}^a_{c,\psi}(\delta, \mathcal{Q})$.

2. Preliminarities

To prove our main results, we recall the below lemmas as follows. The lemma 2.1 is well known caratheodory lemma (see also [9] corollary 2.3) and is used for this study:

Lemma 1. [7] If $c \in P$ is given by (1.2), then $|d_k| \leq 2$, for all $k \geq 1$ and the results is best possible for $\Phi(z) = \frac{1+\rho z}{1-\rho z}$ and $|\rho| = 1$.

Lemma 2. [5] Let $\Phi \in P$ as in (1.2). Then,

$$|d_2 - vd_1^2| \leq 2max\{1; |2v-1|\}, \quad where\ v \in C \tag{2.1}$$

the results is sharp for the functions as follows by $\Phi_1(z) = \frac{1+\rho z}{1-\rho z}$ and $\Phi_2(z) = \frac{1+\rho^2 z^2}{1-\rho^2 z^2}$ with $|\rho| = 1$.

Lemma 3. [17] (Lemma 1 and Remark, pp. 162 and 163) If Φ given by (1.2) is a element of the class P, then

$$|d_2 - vd_1^2| \leq (y)_i = \begin{cases} 2, & \text{if } 0 \leq v \leq 1 \\ 4v-2, & \text{if } v \geq 1. \end{cases} \tag{2.2}$$

Where $v < 0$ or $v > 1$, the equality holds if and only if Φ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if Φ is $\frac{1+z^2}{1-z^2}$ or one of its rotations.if $v = 0$, the equality holds if and only if

$$\Phi_3(z) = \left(\frac{1}{2} + \frac{\eta}{2}\right) + \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\eta}{2}\right)\left(\frac{1-z}{1+z}\right), 0 \leq \eta \leq 1$$

or one of its rotations. If $v = 1$, the equality holds if and only if Φ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Although, when $0 < v < 1$, the above upper bound is sharp, it can be enhanced as follows:

$$|d_2 - vd_1^2| + v|d_1^2| \leq 2,\ if\ 0 < v \leq \frac{1}{2} \tag{2.3}$$

and

$$|d_2 - vd_1^2| + (1 - v)|d_1^2| \leq 2,\ if \frac{1}{2} \leq v \leq 1. \tag{2.4}$$

Lemma 4. [3] Let $\Phi \in P$ as follows in (1.2). Then

$$d_2 = \frac{1}{2}\left[d_1^2 + (4 - d_1^2)x\right], \tag{2.5}$$

and

$$d_3 = \frac{1}{4}\left[d_1^2 + 2(4 - d_1^2)d_1x - (4 - d_1^2)d_1 + x^2 + 2(4 - d_1^2)(1 - |x|^2z)\right] \tag{2.6}$$

for some complex numbers x, z convincing $|z| \leq 1$ and $|x| \leq 1$.

3. Coefficient inequality and upper bound

In this section, we first obtain the coefficient inequality for $f \in \mathcal{MK}_{c,\psi}^{a}(\delta, \Omega)$ and this tends to solve the upper bound $\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right|$ for the subclass $\mathcal{MK}_{c,\psi}^{a}(\delta, \Omega)$.

Theorem 1. If $f \in \mathcal{MK}_{c,\psi}^{a}(\delta, \Omega)$ and is in (1.1), then

$$\left|\frac{a_1}{S}\right| \leq \left|\frac{c}{a}\right| \frac{1}{1 + \delta},$$

$$\left|\frac{a_2}{S}\right| \leq \left|\frac{(c)_2}{(a)_2}\right| \frac{1}{(2 + \delta)} max\left\{1; \left|\frac{\delta + 3}{(1 + \delta)}\right|\right\},$$

$$\left|\frac{a_3}{S}\right| \leq \left|\frac{(c)_3}{(a)_3}\right| \frac{1}{2(3 + \delta)},$$

where S denote the sum of the convergent series.

Proof. If $f \in \mathcal{MK}_{c,\psi}^{a}(\delta, \Omega)$ from (1.10), then a function exists $\omega \in \mathcal{H}(U)$ with $\omega(0) = 0,\ z \in U$ and $|\omega(z)| < 1$ such that

$$\frac{z(\mathcal{K}_{c,\psi}^{a} f(z))'}{(1 - \lambda)\mathcal{K}_{c,\psi}^{a} f(z) + \lambda z} = \Omega\left(\omega(z)\right). \tag{3.1}$$

Define the function Φ by

$$\Phi(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + d_1z + d_2z^2 + ... \quad , z \in U.$$

Which is

$$\omega(z)=\frac{\Phi(z)-1}{\Phi(z)+1},\ z\in U \tag{3.2}$$

and, since $\omega\in\mathcal{N}(U)$ with $|\omega(z)|<1$ and $\omega(0)=0,\ z\in D$ and it follows that $\Phi\in P$ substitute for ω from (3.2) on the R.H.S of (3.1) and we get

$$\mathcal{Q}\left(\omega(z)\right)=\mathcal{Q}\left(\frac{\Phi(z)-1}{\Phi(z)+1}\right)=1+\frac{d_1}{2}z+\frac{d_2}{2}z^2+\left(\frac{d_3}{2}-\frac{d_1^3}{48}\right)z^3+....,z\in U \tag{3.3}$$

and, with help of (1.7), the L.H.S of (3.1) will be

$$\begin{aligned}\frac{z(\mathcal{K}^a_{c,\psi}f(z))'}{(1-\delta)\mathcal{K}^a_{c,\psi}f(z)+\delta z}&=1+(1+\delta)\varphi_1\frac{a_1}{S}+\left[(2+\delta)\varphi_2\frac{a_2}{S}+(\delta^2-1)\varphi_1^2\frac{a_1^2}{S^2}\right]z^2\\&+\left[(3+\delta)\varphi_3\frac{a_3}{S}+(2\delta^2+\delta-3)\varphi_1\varphi_2\frac{a_1}{S}\frac{a_2}{S}+(\delta^3-\delta^2-\delta+1)\varphi_1^3\frac{a_1^3}{S^3}\right]z^3+...\end{aligned} \tag{3.4}$$

Where $\varphi_n, n\in N$, is in (1.7).

Therefore, substituting (3.3) and (3.4) in (3.1) and compare the coefficients of z, z^2 and z^3, we get

$$\frac{a_1}{S}=\frac{c}{a}.\frac{d_1}{2(1+\delta)}, \tag{3.5}$$

$$\frac{a_2}{S}=\frac{(c)_2}{(a)_2}\frac{1}{2(2+\delta)}\left(d_2-\frac{(\delta-1)}{2(1+\delta)}d_1^2\right), \tag{3.6}$$

$$\frac{a_2}{S}=\frac{(c)_3}{(a)_3}\left(d_3-\frac{2\delta^2+\delta-3}{2(1+\delta)(2+\delta)}d_1d_2-\left(\frac{5\delta^3+12\delta^2+9\delta-1}{4(2+\delta)(1+\delta)^3}\right)d_1^3\right). \tag{3.7}$$

Thus, from Lemma 1, we have

$$\left|\frac{a_1}{S}\right|\le\left|\frac{c}{a}\right|\frac{1}{(1+\delta)},$$

$$\left|\frac{a_2}{S}\right|\le\left|\frac{(c)_2}{(a)_2}\right|\frac{1}{2(2+\delta)}\left|d_2-\frac{(\delta-1)}{2(1+\delta)}d_1^2\right|,$$

and according to lemma 2, it follows that

$$\left|\frac{a_2}{S}\right|\le\left|\frac{(c)_2}{(a)_2(2+\delta)}\right|max\left\{1;\left|\frac{\delta+3}{1+\delta}\right|\right\},$$

and

$$\frac{a_3}{S}=\frac{(c)_3}{(a)_3}\frac{1}{2(3+\delta)}\left(d_3-\frac{2\delta^2+\delta-3}{2(1+\delta)(2+\delta)}d_1d_2+\frac{5\delta^2-15\delta+4}{24(1+\delta)(2+\delta)}d_1^3\right). \tag{3.8}$$

Substitute the values of d_2 and d_3 is follows by the relations (2.5) and (2.6) in (3.8), respectively, and, indicating $d=d_1$, we get

$$\frac{a_3}{S}=\frac{(c)_3}{(a)_3}\frac{1}{2(3+\delta)}\left[\frac{-(\delta^2+3\delta-34)}{24(1+\delta)(2+\delta)}d^3+\frac{5\delta+7}{4(1+\delta)(2+\delta)}(4-p^2)px-\frac{1}{4}(4-d^2)dx^2+\frac{1}{2}(4-d^2)(1-|x|^2)z\right],$$

for few complex numbers z and x, with $|z| \leq 1$ and $|x| < 1$. Using the triangle's inequality and substitute $|x| = y$, we get

$$\frac{a_3}{S} \leq \frac{(c)_3}{(a)_3}\frac{1}{4(3+\delta)}\left[\frac{|\delta^2+3\delta-34|}{24(1+\delta)(2+\delta)}d^3 + \frac{5\delta+7}{4(1+\delta)(2+\delta)}(4-d^2)dy - \frac{1}{4}(4-d^2)dy^2 + \frac{1}{2}(4-d^2)(1-y^2)\right]$$
$$= \mathcal{F}(d,y), (0 \leq d \leq 2, \quad 0 \leq y \leq 1).$$

Now, we will identify the maximum of $\mathcal{F}(d,y)$ on the closed version rectangle $[0,2]\times[0,1]$

$$\mathcal{G}(d,y) = \frac{|\delta^2+3\delta-34|}{24(1+\delta)(2+\delta)}d^3 + \frac{5\delta+7}{4(1+\delta)(2+\delta)}(4-d^2)dy - \frac{1}{4}(4-d^2)dy^2 + \frac{1}{2}(4-d^2)(1-y^2),$$

and using the MAPLE^{TM} software, we get

$$max\,\{\mathcal{G}(d,y) : (d,y) \in [0,2]\times[0,1]\} = max\left\{2, \frac{\delta^2+3\delta-34}{3(1+\delta)(2+\delta)}\right\},$$

and

$$\mathcal{G}(0,0) = 2, \mathcal{G}(2,y) = \frac{\delta^2+3\delta-34}{3(1+\delta)(2+\delta)}.$$

A simple computation shows that $2 > \frac{\delta^2+3\delta-34}{3(1+\delta)(2+\delta)}$whenever $\delta \geq 0$;therefore,

$$max\,\{\mathcal{G}(d,t) : (d,t) \in [0,2]\times[0,1]\} = \mathcal{G}(0,0) = 2,$$

which implies that

$$max\,\{\mathcal{G}(d,y) : (d,y) \in [0,2]\times[0,1]\} = \frac{(c)_3}{(a)_3}\frac{1}{2(3+\delta)} = \mathcal{F}(0,0).$$

and the proof is complete.

Theorem 2. If $f \in \mathcal{MK}^a_{c,\psi}(\delta,\mathcal{Q})$ and is of the form (1.1), then $\mu \in C$, we have

$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \leq \left|\frac{(c)_2}{(a)_2}\right|\frac{1}{2+\delta}max\left\{1, \frac{|(2\delta)a(c+1)+\mu(2+\delta)c(a+1)|}{(1+\delta)^2|a(c+1)|}\right\}.$$

Proof. $f \in \mathcal{MK}^a_{c,\psi}(\delta,\mathcal{Q})$ is of the form (1.1), from (3.5) and (3.6), we get

$$\frac{a_2}{S} - \mu\frac{a_1^2}{S^2} = \frac{1}{2(2+\delta)}\frac{(c)_2}{(a)_2}(d_2 - vd_1^2)$$

where

$$v = \frac{(\delta^2-1)a(c+1)+\mu(2+\delta)c(a+1)}{2(1+\delta)^2a(c+1)}.$$

Taking the modules of both sides , with the use of the inequality (2.1) of Lemma 2, then we obtain the necessary estimate.

The above theorem For $a = c$, reduces to the given special case:

Corollary 1. If $f \in \mathcal{MK}_\psi(\delta,\mathcal{Q})$ and is of the form (1.1), then $\mu \in C$,

$$\left|\frac{a_2}{S} - \mu\frac{a_1^2}{S^2}\right| \leq \frac{1}{2+\delta}max\left\{1, \frac{|(2\delta)+\mu(2+\delta)|}{(1+\delta)^2}\right\}.$$

Remark 4. If $f \in \mathcal{MK}_{\psi}(\delta, \mathcal{Q})$ and is of the form (1.1), then, for the special case $\mu = 1$,

$$\left|\frac{a_2}{S} - \frac{a_1^2}{S^2}\right| \leq \frac{1}{2+\delta} max\left\{1, \frac{|(2+3\delta)|}{(1+\delta)^2}\right\}.$$

4. Conclusions

This paper deals with the geometric properties of holomorphic functions involving generalized distribution with bell numbers. Also, we found that coefficient inequality and sharp bound to be in the subclass of holomorphic functions. Further, Hankel determinant may be investigated to this distribution in the future. We anticipate that this distribution series will be important in several fields related to mathematics, science, and technology.

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Conflict of interest

The authors declare that they have no conflicts of interests.

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