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Research article

Generalizing the concept of decreasing impatience

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Abstract: The **framework** of this paper is behavioral finance and, more specifically, intertemporal choice when individuals exhibit decreasing impatience in their decision-making processes. After characterizing the two main types of decreasing impatience (moderately and strongly decreasing impatience), the **main objective** of this paper is to generalize these concepts when the criterion of time increase is given by an arbitrary function which describes such increments. In general, the **methodology** is mathematical calculus but particularly the concept of derivative according to the function which rules the increase of time. The **main contribution** of this paper is the characterization of this extension of the concept of decreasing impatience by using the aforementioned novel derivative and the well-known Prelec's index.

Keywords: intertemporal choice; discount function; decreasing impatience; inconsistency
Mathematics Subject Classification: 68M10

1. Introduction

Intertemporal choice refers to the process by which an individual or a group of individuals (e.g., an organization) have to select one among a set of dated outcomes. The simplest case is when the subject has to choose one of two dated rewards, and the more complex situation is when the individual must decide about his/her preferred sequence of rewards [4, 12]. Typically, a problem in intertemporal choice considers an immediate or short-term advantage and a long-term disadvantage in a specific choice option, or vice versa. Because the consequences of a decision are not only rewards (in the traditional sense), this choice does not necessarily have to involve money or some material or economic consequence, but it may refer to non-monetary decisions such as health-related outcomes, substance abuse and others [14].

From an economic point of view, intertemporal choice may be analyzed by using either preferences or a discount function able to value the different offered options. The bridge between both perspectives can be found in Fishburn and Rubinstein [9]: If order, monotonicity, continuity, impatience, and separability* hold, and the set of rewards X is an interval, then there are continuous real-valued functions u on X and F on the time interval T such that

$$(y, r) \geq (z, s) \text{ if, and only if, } u(y)F(r) \geq u(z)F(s). \tag{1.1}$$

Additionally, $u(0) = 0$ and u is increasing, whilst F is decreasing and positive. Focusing on function F, Samuelson [18] proposed the exponential discount ($F(t) = \exp\{-kt\}$, $k > 0$) as the criterion to value dated outcomes in his well-known Discounted Utility (DU) model. This discount function is characterized by a constant discount rate:

$$\delta(t) = k, \tag{1.2}$$

which leads to consistency in the decisions governed by this discount function. However, this model (stationary and consistent) is not able to explain several paradoxes or anomalies shown in intertemporal choice (see, e.g., Read [11, 15, 16]: delay effect, interval effect, magnitude effect, direction effect, sign effect, sequence effect, date-delay effect and frame effect) and, consequently, this justifies the need to take into account other decision models. To do this, we will focus on inconsistent preferences and, more specifically, on those decisions guided by a criterion of strongly or moderately decreasing impatience.

Undoubtedly, between these two types of decreasing impatience and variable impatience there are lot of behavioral situations which is convenient to be analyzed. In effect, the actual distinction between moderately and strongly decreasing impatience reflects that, in the context of intertemporal choice, the individuals exhibit different patterns of decreasing impatience. This makes us to think about the possibility of the existence of much more types of decreasing impatience and so the need for broaden this research line. Therefore, the main objective of this paper is to generalize the concepts of moderately and strongly decreasing impatience by introducing the impatience variable according to a certain function g. From a methodological point of view, we will use the techniques of the differential calculus, in particular, the derivative according to a given function (see [1]) which reflects different ways of increasing the variable "time".

This paper has been structured as follows (see Figure 1). After this Introduction, Section 2 presents the preliminary concepts which are necessary for the later treatment of inconsistency, in particular, decreasing impatience and its main modalities: moderately and strongly decreasing impatience. After characterizing these noteworthy types of decreasing impatience, Section 3 introduces other criteria for increasing the variable "time" through a general function, denoted by g, which rules the increments of time. Finally, Section 4 summarizes and concludes.

*Separability means that the preference of a dated outcome, $F(x, t)$, can be represented as a separable discount utility, $F(x, t) = u(x)F(t)$, where time and money are evaluated separately.

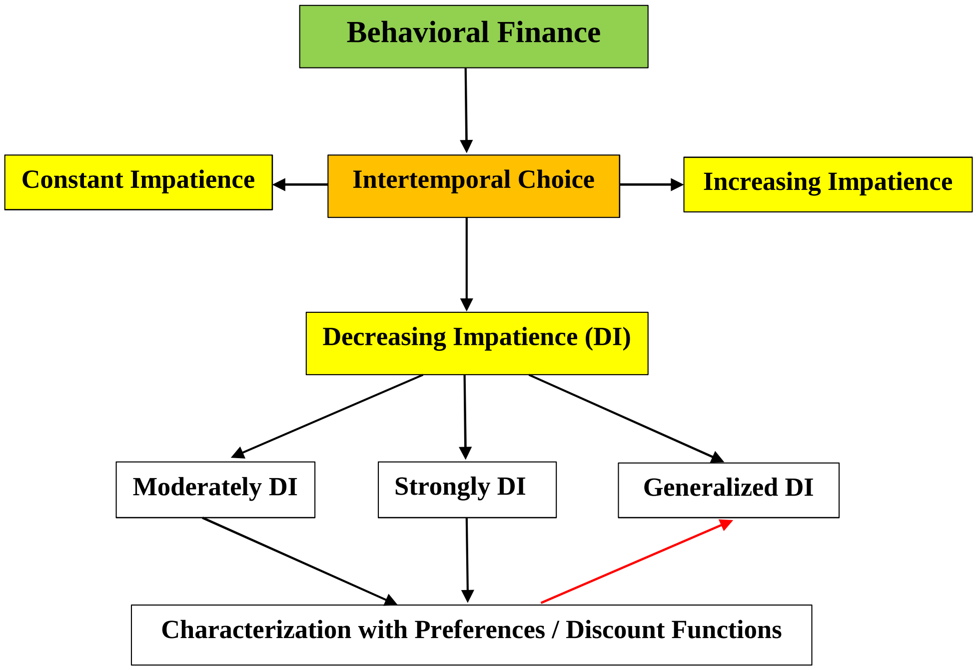

Figure 1. Structure of the manuscript. Source: Own elaboration.

2. Impatience and inconsistency

As indicated, in behavioral finance, *intertemporal choice* refers to a process by which an individual chooses one of two possible dated rewards (x, r) and (y, s), where $x < y$ and $r < s$. Contrarily[†], the choice is obvious if the subject is rational. A concept implicit in the process of intertemporal choice is *impatience*. Many scholars define impatience in the following way. If an individual chooses (x, r), we will say that she exhibits impatience because she is not willing to wait for a bigger amount. Otherwise, if she chooses (y, s), we will say that she shows patience. In relative terms, we will say that an individual A is *more impatient* than another subject B if A chooses (x, r) but B is not willing to receive x and consequently requires a greater amount at time r instead of receiving (y, s).

This situation allows us to define the *patience* associated to an intertemporal choice. In effect, given an amount z and two instants r and s, let $x := x(z, r, s)$[‡] be the amount such that the subject prefers (v, r) to (y, s), for every $v > x(z, r, s)$, and prefers (y, s) to (v, r), for every $v < x(z, r, s)$. All these situations will be written as:

1. $(x(z, r, s), s) \sim (z, s)$,
2. $(v, r) \succ (y, s)$, for every $v > x(z, r, s)$, and
3. $(v, r) \prec (y, s)$, for every $v < x(z, r, s)$.

Under these circumstances, we can define the *patience* associated to an interval $[r, s]$ as the ratio

$$f(r, s) := \frac{x(z, r, s)}{z},$$

[†]That is to say, if $x \geq y$ or $s \geq t$.
[‡]The value x will be a function of z, r and s.

whilst the impatience associated to $[r, s]$ remains as

$$d(r, s) := 1 - f(r, s).$$

Observe that both the patience and the impatience vary from 0 to 1. In a beginning, we will assume that the preference is linear, that is to say:

$$x(kz, r, s) = kx(z, r, s),$$

for every $k \in \mathbb{R}$, and so $f(r, s)$ and $d(r, s)$ are well-defined.

A concept related to impatience is *inconsistency* which can be referred to either amount or time [2, 3, 5, 6, 8]:

1. The intertemporal choice is inconsistent with respect to the amount if $(x, s) \sim (y, t)$ implies $(kx, s) \succ (ky, t)$, for every $k > 1$, which is called the *magnitude effect*.
2. The intertemporal choice is inconsistent with respect to time if $(x, s) \sim (y, t)$ implies $(x, s + k) \nsim (y, t + k)$, for some $k > 0$. A particular case is when $(x, s) \sim (y, t)$ implies $(x, s + k) \prec (y, t + k)$, for every $k > 0$, which is called the *delay effect*.

In this paper, we will focus on the second of these inconsistencies, viz the delay effect [20, 21]. It is well known that a preference satisfying completeness and continuity gives rise to a continuous discount function $F(x, t)$. Moreover, if F is linear:

$$F(x, t) = xF(1, t) := xF(t),$$

and differentiable, the following identity holds:

$$F(t) = \exp\left\{-\int_0^t \delta(z)\mathrm{d}z\right\},$$

where

$$\delta(z) = -\frac{\mathrm{d}\ln F(z)}{\mathrm{d}z}$$

is the so-called *instantaneous discount rate*. It is immediate to show that the delay effect is equivalent to require that $\delta'(z) < 0$, whereby it will be also called *decreasing impatience*. Contrarily, if $\delta'(z) > 0$, $F(t)$ exhibits *increasing impatience*, and, finally, if $\delta'(z) = 0$, $F(t)$ is the well-known *exponential* discount function.

Searching a parallelism with the magnitude effect, a stronger condition will be

$$(x, s) \sim (y, t) \text{ implies } (x, \lambda s) \prec (y, \lambda t),$$

for every $\lambda > 1$. In effect, assume that $(x, s) \sim (y, t)$. For every $k > 0$,

$$(x, s + k) = \left(x, s\left(1 + \frac{k}{s}\right)\right) \prec \left(y, t\left(1 + \frac{k}{s}\right)\right) = \left(y, t + k\frac{t}{s}\right) \prec (y, t + k).$$

Therefore, we can distinguish two modalities of decreasing impatience [17]:

1. $(x, s) \sim (y, t)$ implies $(x, s + k) \prec (y, t + k)$, for every $k > 0$, but not $(x, \lambda s) \prec (y, \lambda t)$, for every $\lambda > 1$. In this case, we will say that the decreasing impatience is *moderate* or that the preferences exhibit a *moderately decreasing impatience*.
2. $(x, s) \sim (y, t)$ implies $(x, \lambda s) \prec (y, \lambda t)$, for every $\lambda > 1$. In this case, we will say that the decreasing impatience is *strong* or that the preferences exhibit a *strongly decreasing impatience*.

3. Generalizing the concept of decreasing impatience

In this section, we are going to consider dated rewards (y, r) and (z, s), where $0 < y < z$ and $r < s$. Let 0 denote the current time and h the calendar time at time 0 and $\mathbb{R}$ the set of non-negative real numbers. The following definition is based on [7, 19, 22].

Definition 1. *A deformation of calendar time at instant t is a function*

$$g : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$

$$(t, h) \mapsto g(t, h),$$

such that

1. $g(t, 0) = 0$,
2. $g(0, h) = h$, *and*
3. *g is continuous and strictly increasing with respect to h.*

Observe the different nature of t (dates) and h (interval time). In effect, h represents the incremental time starting from t.

Definition 2. *A preference relation $\succeq$ is said to satisfy decreasing impatience relatively to the time deformation $g(t, h)$ if*

$$(y, r) \sim (z, s) \tag{1}$$

implies

$$(y, r + g(r, h)) \prec (z, s + g(s, h)), \tag{2}$$

for every $h > 0$.

Let F be the discount function derived from the relation of preference $\succeq$ (see Introduction). This allows to enunciate the following theorem, where F is assumed to be differentiable.

Theorem 1. *A preference relation $\succeq$ exhibits decreasing impatience relatively to the differentiable time deformation $g(t, h)$ if, and only if,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \left. \frac{\partial g(t, h)}{\partial h} \right|_{h=0} \leq P(t),$$

where $P(t) := -\frac{\mathrm{d}}{\mathrm{d}t} \ln \delta(t)$ denotes the so-called Prelec's convexity index [13].

Proof. **Necessity**. In effect, starting from the equivalence (1), one has:

$$yF(r) = zF(s). \tag{3}$$

Moreover, by the preference (2), the following inequality holds:

$$yF(r + g(r, h)) < zF(s + g(s, h)). \tag{4}$$

By dividing the left-hand and the right-hand sides of (3) and (4), respectively, one has:

$$\frac{F(r + g(r, h))}{F(r)} < \frac{F(s + g(s, h))}{F(s)}. \tag{5}$$

Observe that the inequality (5) can be successively written as:

$$\frac{F(r+g(r,h))}{F(r)}-1<\frac{F(s+g(s,h))}{F(s)}-1,$$

$$\frac{F(r+g(r,h))-F(r)}{F(r)}<\frac{F(s+g(s,h))-F(s)}{F(s)}$$

and, finally,

$$\frac{g(r,h)}{h}\frac{F(r+g(r,h))-F(r)}{g(r,h)F(r)}<\frac{g(s,h)}{h}\frac{F(s+g(s,h))-F(s)}{g(s,h)F(s)}. \tag{6}$$

Letting $h\to 0$, by continuity, one has $g(r,h)\to 0$ and $g(s,h)\to 0$. Therefore, as $g(r,0)=g(s,0)=0$, then [1]:

$$\left.\frac{\partial g(r,h)}{\partial h}\right|_{h=0}\left.\frac{\mathrm{d}}{\mathrm{d}t}\ln F(t)\right|_{t=r}\leq\left.\frac{\partial g(s,h)}{\partial h}\right|_{h=0}\left.\frac{\mathrm{d}}{\mathrm{d}t}\ln F(t)\right|_{t=s}$$

or, equivalently,

$$\left.\frac{\partial g(r,h)}{\partial h}\right|_{h=0}\delta(r)\geq\left.\frac{\partial g(s,h)}{\partial h}\right|_{h=0}\delta(s).$$

Observe that equality is only possible at a numerable number of points. This means that the function of t

$$\left.\frac{\partial g(t,h)}{\partial h}\right|_{h=0}\delta(t)$$

is strictly decreasing. Therefore, its derivative is non-positive:

$$\left.\frac{\partial^2 g(t,h)}{\partial t\partial h}\right|_{h=0}\delta(t)+\left.\frac{\partial g(t,h)}{\partial h}\right|_{h=0}\delta'(t)\leq 0.$$

As $\left.\frac{\partial g(t,h)}{\partial h}\right|_{h=0}>0$ and $\delta(t)>0$, we can write:

$$\frac{\left.\frac{\partial^2 g(t,h)}{\partial t\partial h}\right|_{h=0}}{\left.\frac{\partial g(t,h)}{\partial h}\right|_{h=0}}\leq-\frac{\delta'(t)}{\delta(t)}$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left.\frac{\partial g(t,h)}{\partial h}\right|_{h=0}\leq-\frac{\mathrm{d}}{\mathrm{d}t}\ln\delta(t):=P(t). \tag{7}$$

Sufficiency. By integrating both sides of inequality (7) between r and s, one has:

$$\ln\left.\frac{\partial g(s,h)}{\partial h}\right|_{h=0}-\ln\left.\frac{\partial g(r,h)}{\partial h}\right|_{h=0}\leq-\ln\delta(s)+\ln\delta(r),$$

from where:

$$\frac{\left.\frac{\partial g(s,h)}{\partial h}\right|_{h=0}}{\left.\frac{\partial g(r,h)}{\partial h}\right|_{h=0}}\leq\frac{\delta(r)}{\delta(s)}$$

and, consequently,

$$\delta(r)\left.\frac{\partial g(r,h)}{\partial h}\right|_{h=0}\geq\delta(s)\left.\frac{\partial g(s,h)}{\partial h}\right|_{h=0} \tag{8}$$

or

$$\delta(r + g(r, h)) \geq \delta(s + g(s, h)). \tag{9}$$

Therefore, δ is decreasing with respect to increments given by $g(t, h)$ and so $\succeq$ exhibits decreasing impatience relatively to the time deformation $g(t, h)$. □

Example 1. *Assume that* $g(t, h) = h(t + 1)$ *and* $F(t) = \frac{1}{1+kt}$*, where* $k > 0$*. In this case,*

- $\frac{d}{dt} \ln \frac{\partial g(t,h)}{\partial h}\Big|_{h=0} = \frac{1}{t+1}$.
- $-\frac{d}{dt} \ln \delta(t) = \frac{k}{1+kt}$.

Observe that inequality (7) holds if, and only if, $k > 1$.

Example 2. *Assume that* $g(t, h) = \frac{h}{t+1}$ *(percentage increments of time) and* $F(t) = \frac{1}{1+kt}$*, where* $k > 0$*. In this case,*

- $\frac{d}{dt} \ln \frac{\partial g(t,h)}{\partial h}\Big|_{h=0} = -\frac{1}{t+1}$.
- $-\frac{d}{dt} \ln \delta(t) = \frac{1}{1+kt}$.

Observe that inequality (7) holds for every value of t.

Example 3. *Assume that* $g(t, h) = \ln(\exp\{t\} + h)$ *and* $F(t) = \exp\{-kt\}$*, where* $k > 0$*. In this case,*

- $\frac{d}{dt} \ln \frac{\partial g(t,h)}{\partial h}\Big|_{h=0} = -1$.
- $-\frac{d}{dt} \ln \delta(t) = 0$.

Observe that inequality (7) holds for every value of t.

In some economic contexts, Karpoff [10], when searching the relationship between the price and the volume of transactions of an asset in a stock market, suggests quadratic and logarithmic increments:

Example 4. *Quadratic increments aim to determine the variation of the volume when quadratic changes in the price of an asset have been considered. In this case,* $g(t, h) = (t + h)^2 - t^2$*. If* $F(t) = \frac{1}{1+kt}$*, where* $k > 0$*, then:*

- $\frac{d}{dt} \ln \frac{\partial g(t,h)}{\partial h}\Big|_{h=0} = \frac{1}{t}$.
- $-\frac{d}{dt} \ln \delta(t) = \frac{1}{1+kt}$.

Observe that now inequality (7) never holds.

Example 5. *Logarithmic increments aim to find the variation of the volume when considering quadratic changes in the price of an asset. In this case,* $g(t, h) = \ln(t + h) - \ln t$*. If* $F(t) = \frac{1}{1+kt}$*, where* $k > 0$*, then:*

- $\frac{d}{dt} \ln \frac{\partial g(t,h)}{\partial h}\Big|_{h=0} = -\frac{1}{t}$.
- $-\frac{d}{dt} \ln \delta(t) = \frac{1}{1+kt}$.

Observe that inequality (7) holds for every value of t.

3.1. Noteworthy particular cases

1. Let $g(t,h) = h$. In this case, $\frac{\partial g(r,h)}{\partial h}\Big|_{h=0} = \frac{\partial g(s,h)}{\partial h}\Big|_{h=0} = 1$ and inequality (8) remains $\delta(s) < \delta(r)$. Therefore, if $g(t,h) = h$, then Definition 2 is the usual decreasing impatience.
2. Let $g(t,h) = h(t+1) - t$. In this case, $\frac{\partial g(r,h)}{\partial h}\Big|_{h=0} = r+1$ and $\frac{\partial g(s,h)}{\partial h}\Big|_{h=0} = s+1$ and inequality (8) remains $\delta(r)(r+1) < \delta(s)(s+1)$. Therefore, if $g(t,h) = h(t+1) - t$, then Definition 2 is strongly decreasing impatience [17].

4. Conclusions

Discount function have proved to be an excellent tool to analyze the human preferences involving intertemporal choices. In effect, a noteworthy property of preferences is the so-called impatience exhibited by individual in the process of decision-making. More specifically, in this paper we have considered increasing and decreasing (moderate and strong) impatience as a characteristic inherent to the choices involving monetary or non-monetary rewards.

In effect, the concept of decreasing impatience is linked to absolute increments of time. Recently, Rohde [17] introduced the concepts of moderately and strongly decreasing impatience where absolute increments are compared to proportional increments of time. This distinction is very important in economics and, particularly, in econometrics and finance where several models of variation can be applied to different variables.

The main contribution of this paper is Theorem 1 which generalizes the former concepts to other general patterns of time variation and characterizes decreasing impatience for a generic incremental time. Thus, some examples illustrating the findings are provided at the end of this manuscript.

A further research could be describing some noteworthy cases of incremental times by using suitable measures involving rewards and times, that is to say, by using the reverse path used when characterizing moderately and strongly decreasing impatience (see red arrow in Figure 1).

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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