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Research article

Some best proximity point results on best orbitally complete quasi metric spaces

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Abstract: In this paper, we first introduce the concepts of d- and d^{-1}-proximal Ćirić contraction mappings. Also, we present new definitions and notations by taking into account the lack of symmetry property of quasi-metric spaces. Moreover, we give some examples to support our definitions and notations. Then, we prove some right and left best proximity point results for these mappings on best orbitally complete quasi-metric spaces. Hence, we obtain some generalizations of famous results in the literature.

Keywords: best proximity point; orbitally continuous mapping; quasi-metric space
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1. Introduction and prelimineries

A real valued function d defined on $\mathfrak{I} \times \mathfrak{I}$ that satisfies the axioms of a metric with the exception that the distance between distinct points is nonzero is known as a pseudo metric on a non-empty set $\mathfrak{I}$. Numerous well-known results on metric space, such as the Baire category, Cantor intersection, and Banach fixed-point theorems, are applicable in this situation and are readily extended to pseudo-metric contexts. But, these extensions are not as simple as in the pseudo metric spaces, when the symmetry condition is eliminated. Despite this, a number of authors have focused on ignoring the symmetry constraint since unsymmetric distance functions have a wide range of applications in both mathematics and many other fields [1–3]. Wilson [4] introduced the idea of quasi-metric in this context for the first time. Then, Kelly [5] succeeded in generalizing a number of well-known conclusions, including the Baire category theorem and the Urysohn lemma, by taking into consideration biotopological spaces, which are intimately related to quasi-metric spaces. In the same paper, the Cauchy sequence for a quasi pseudo metric space has been also defined. Reilly et al. [6] pointed out that any convergent sequence might not be Cauchy according to Kelly's definition. They suggested a variety of definitions of the Cauchy sequence in a quasi-metric space to address this drawback. Categorizing concepts of Cauchyness and completeness, Altun et al. [7] recently established a few fixed-point theorems on quasi-metric spaces. Many intriguing and notable results can be found in the literature [8–13]. Now, we recall some notations and definitons in quasi-metric space. Let $d : \Im \times \Im \to [0, \infty)$ be a function where $\Im$ is a nonempty set. Consider the following conditions

(i) $d(\check{s}, \check{s}) = 0$,
(ii) $d(\check{s}, z) \leq d(\check{s}, \breve{u}) + d(\breve{u}, z)$,
(iii) $d(\check{s}, \breve{u}) = d(\breve{u}, \check{s}) = 0 \Longleftrightarrow \check{s} = \breve{u}$,
(iv) $d(\check{s}, \breve{u}) = 0 \Longleftrightarrow \check{s} = \breve{u}$,

for all $\check{s}, \breve{u}, z \in \Im$.

- If it meets the requirements (i) and (ii), d is referred to as quasi-pseudometric on $\Im$.
- If it meets the requirements (ii) and (iii), then d is referred to as quasi-metric on $\Im$.
- If it meets the requirements (ii) and (iv), then d is referred to as T_1-quasi-metric on $\Im$.

It is evident that every quasi-metric is a quasi-pseudo metric, and that every T_1-quasi-metric is a quasi-metric. Indeed, let $\Im = [0, \infty)$ and $d : \Im \times \Im \to \mathbb{R}$ be a function defined by $d(\check{s}, \breve{u}) = \max\{\breve{u} - \check{s}, 0\}$. Then, $(\Im, d)$ is a quasi-metric space, but it is not a T_1-quasi metric space. If we take $\check{s} = 2$ and $\breve{u} = 1$, then we have $d(2, 1) = 0$, but $2 \neq 1$.

Let $(\Im, d)$ be a quasi-metric space and $d^{-1} : \Im \times \Im \to [0, \infty)$ and $d^s : \Im \times \Im \to [0, \infty)$ be mappings defined by

$$d^{-1}(\check{s}, \breve{u}) = d(\breve{u}, \check{s})$$

and

$$d^s(\check{s}, \breve{u}) = \max\{d(\check{s}, \breve{u}), d^{-1}(\check{s}, \breve{u})\}$$

for all $\check{s}, \breve{u} \in \Im$. Then, d^{-1} is a quasi-metric (called a conjugate of d) and d^s is an ordinary metric on $\Im$. The subset Γ of $\Im$ is said to be d-open if for all $\check{s} \in \Gamma$ there exists $r > 0$ such that

$$B_d(\check{s}, r) = \{\breve{u} \in \Im : d(\check{s}, \breve{u}) < r\} \subseteq \Gamma;$$

the subset Γ of $\Im$ is said to be d^{-1}-open if for all $\check{s} \in \Gamma$ there exists $r > 0$ such that

$$B_{d^{-1}}(\check{s}, r) = \{\breve{u} \in \Im : d(\breve{u}, \check{s}) < r\} \subseteq \Gamma.$$

If τ_d ($\tau_{d^{-1}}$) denotes the family of all d-open subsets of $\Im$ (the family of all d^{-1}-open subsets of $\Im$), then τ_d ($\tau_{d^{-1}}$) is a T_0-topology on $\Im$. If d is a T_1-quasi-metric, then τ_d is a T_1-topology on $\Im$.

Let $\{\check{s}_t\}$ be sequence in $\Im$. It is clear that the sequence $\{\check{s}_t\}$ converges to $\check{s} \in \Im$ with respect to τ_d if and only if $d(\check{s}, \check{s}_t) \to 0$ as $t \to \infty$.

Now, we provide several definitions of Cauchyness and completeness in quasi metric spaces.

Definition 1 ([7]). *Let $\{\check{s}_t\}$ be a sequence in a quasi-metric space $(\Im, d)$. A sequence $\{\check{s}_t\}$ is called*

(i) a right K-Cauchy sequence if for every $\varepsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that $d(\check{s}_r, \check{s}_t) < \varepsilon$ for all $r \geq t \geq t_0$,

(ii) *a left K-Cauchy sequence if for every* $\varepsilon > 0$ *there exists* $t_0 \in \mathbb{N}$ *such that* $d(\check{s}_t, \check{s}_r) < \varepsilon$ *for all* $r \geq t \geq t_0$.

Definition 2 ([14])**.** *A quasi-metric space* $(\Im, d)$ *is called*

(i) *a right (left) d-complete if every right (left) K-Cauchy sequence is convergent to a point in* $\Im$ *with respect to* d,

(ii) *a right (left)* d^{-1}*-complete if every right (left) K-Cauchy sequence is convergent to a point in* $\Im$ *with respect to* d^{-1}.

On the other hand, using the nonself mappings $H : \Gamma \to \Lambda$ where Γ and Λ are nonempty subsets of a metric space, different from the literature, the metric fixed-point theory has been developed. There is no fixed-point of the mapping H in the situation of $\Gamma \cap \Lambda = \emptyset$. In this situation, it makes sense to check to see if there is a point $\check{s}$ in Γ such that $d(\check{s}, H\check{s}) = d(\Gamma, \Lambda)$, known as the best proximity point of H. Thus, Basha and Veeramani [15] demonstrated various best proximity point results for multivalued mappings and derived an optimal solution to the minimization problem $\min_{\check{s}\in\Gamma} d(\check{s}, H\check{s})$. Many authors have recently explored this subject because the best proximity point theory incorporates the fixed-point theory in a particular situation $\Gamma = \Lambda = \Im$ [16–24]. Now, we present some notations and definitions about the best proximity point theory. Let Γ and Λ be subsets of a metric space $(\Im, d)$. Then, consider the following sets:

$$\Gamma_0 = \{\check{s} \in \Gamma : d(\check{s}, \breve{u}) = d(\Gamma, \Lambda) \text{ for some } \breve{u} \in \Lambda\},$$

and

$$\Lambda_0 = \{\breve{u} \in \Lambda : d(\check{s}, \breve{u}) = d(\Gamma, \Lambda) \text{ for some } \check{s} \in \Gamma\}.$$

Definition 3. *[25] Let* $(\Im, d)$ *be a metric space,* $H : \Gamma \to \Lambda$ *be a mapping where* $\emptyset \neq \Gamma, \Lambda \subseteq \Im$ *and* $\check{s} \in \Gamma$. *Then, the set of iterative sequences*

$$O_H(\check{s}) = \{\{\check{s}_t\} \subseteq \Gamma : \check{s}_0 = \check{s} \text{ and } d(\check{s}_{t+1}, H\check{s}_t) = d(\Gamma, \Lambda) \text{ for all } t \in \mathbb{N}\}$$

is called the orbit of $\check{s}$.

Note that, when $\Gamma = \Lambda = \Im$ in Definition 3, it becomes

$$O_H(\check{s}) = \{\{H^t\check{s}\} : t \in \mathbb{N}\}.$$

Definition 4. *[25] Let* $(\Im, d)$ *be a metric space and* $H : \Gamma \to \Lambda$ *be a mapping where* $\emptyset \neq \Gamma, \Lambda \subseteq \Im$. *If for each* $\check{s} \in \Gamma$ *and* $\{\check{s}_t\} \in O_H(\check{s})$ *the implication*

$$\check{s}_{t_i} \to \check{s}^* \Rightarrow H\check{s}_{t_i} \to H\check{s}^* \text{ as } i \to \infty$$

holds for any subsequence $\{\check{s}_{t_i}\}$ *of* $\{\check{s}_t\}$, *then* H *is called best orbitally continuous at a point* $\check{s}^* \in \Gamma$.

Definition 5. *[25] Let* $(\Im, d)$ *be a metric space and* $\emptyset \neq \Gamma, \Lambda \subseteq \Im$. *Assume that* $H : \Gamma \to \Lambda$ *and* $g : \Gamma \to \mathbb{R}$ *are two mappings. We say that* g *is best orbitally lower semicontinuous at* $\check{s}^*$ *in* Γ *if for each* $\check{s} \in \Gamma$ *and* $\{\check{s}_t\} \in O_H(\check{s})$ *the implication*

$$\check{s}_{t_i} \to \check{s}^* \text{ as } i \to \infty \Rightarrow g(\check{s}^*) \leq \liminf_{i\to\infty} g(\check{s}_{t_i})$$

holds for any subsequence $\{\check{s}_{t_i}\}$ *of* $\{\check{s}_t\}$.

Definition 6. *[25] Let $(\Im, d)$ be a metric space, $\emptyset \neq \Gamma, \Lambda \subseteq \Im$ and $H : \Gamma \to \Lambda$ be a mapping. If for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H(\check{s})$, every Cauchy subsequence $\{\check{s}_{t_i}\}$ of $\{\check{s}_t\}$ converges to a point in Γ_0, then Γ is said to be H-best orbitally complete.*

In this paper, we first introduce the concepts of d- and d^{-1}-proximal Ćirić contraction mappings. Then, we obtain some right and left best proximity point results for these mappings in best orbitally complete quasi metric spaces. Also, we present new definitions and notations by taking into account the lack of symmetry property of quasi-metric spaces. Moreover, we give some examples to support our definitions and notations.

2. Main results

First, we recall some notations and definitions about best proximity points in a quasi-metric space. Let $(\Im, d)$ be a quasi-metric space and $\Gamma, \Lambda \subseteq \Im$. Then, consider the following sets:

$$\Gamma_0^{\ell} = \{\check{s} \in \Gamma : d(\check{s}, \breve{u}) = d(\Gamma, \Lambda) \text{ for some } \breve{u} \in \Lambda\},$$
$$\Gamma_0^{r} = \{\check{s} \in \Gamma : d(\breve{u}, \check{s}) = d(\Lambda, \Gamma) \text{ for some } \breve{u} \in \Lambda\},$$

and

$$\Lambda_0^{\ell} = \{\breve{u} \in \Lambda : d(\check{s}, \breve{u}) = d(\Gamma, \Lambda) \text{ for some } \check{s} \in \Gamma\},$$
$$\Lambda_0^{r} = \{\breve{u} \in \Lambda : d(\breve{u}, \check{s}) = d(\Lambda, \Gamma) \text{ for some } \check{s} \in \Gamma\}.$$

Definition 7. *[25] Let $(\Im, d)$ be a quasi-metric space, $\emptyset \neq \Gamma, \Lambda \subseteq \Im$ and $H : \Gamma \to \Lambda$ be a mapping. A point $\check{s}$ is called a right (left) best proximity point of H if $d(H\check{s}, \check{s}) = d(\Lambda, \Gamma)$ ($d(\check{s}, H\check{s}) = d(\Gamma, \Lambda)$).*

Now, we present some definitions.

Definition 8. *Let $(\Im, d)$ be a quasi-metric space, $H : \Gamma \to \Lambda$ be a mapping where $\emptyset \neq \Gamma, \Lambda \subseteq \Im$ and $\check{s} \in \Gamma$. Then, the set of iterative sequences*

$$O_H^{\rho}(\check{s}) = \{\{\check{s}_t\} \subseteq \Gamma : \check{s}_0 = \check{s} \text{ and } \rho(\check{s}_{t+1}, H\check{s}_t) = \rho(\Gamma, \Lambda) \text{ for all } t \in \mathbb{N}\}$$

is called the $\rho(d$ or $d^{-1})$-orbit of $\check{s}$.

Definition 9. *Let $(\Im, d)$ be a quasi-metric space and $H : \Gamma \to \Lambda$ be a mapping where $\emptyset \neq \Gamma, \Lambda \subseteq \Im$. We say that Γ is*

i) H-best ρ-orbitally right d-complete if for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{\rho}(\check{s})$, every right K-Cauchy subsequence $\{\check{s}_{t_i}\}$ of $\{\check{s}_t\}$ converges to a point in Γ_0^r with respect to d.

iii) H-best ρ-orbitally left d-complete if for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{\rho}(\check{s})$, every left K-Cauchy subsequence $\{\check{s}_{t_i}\}$ of $\{\check{s}_t\}$ converges to a point in Γ_0^{ℓ} with respect to d.

Definition 10. *Let $(\Im, d)$ be a quasi-metric space and $H : \Gamma \to \Lambda$ be a mapping where $\emptyset \neq \Gamma, \Lambda \subseteq \Im$. Then, a function $g : \Gamma \to \mathbb{R}$ is called*

i) d-best ρ-orbitally lower semicontinuous at a point $\check{s}^* \in \Gamma$ if for each $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{\rho}(\check{s})$, the implication

$$\check{s}_{t_i} \xrightarrow{d} \check{s}^* \text{ as } i \to \infty \Rightarrow g(\check{s}^*) \leq \liminf_{i\to\infty} g(\check{s}_{t_i})$$

holds for any subsequence $\{\check{s}_{t_i}\}$ of $\{\check{s}_t\}$.

ii) d^{-1}*-best* ρ*-orbitally lower semicontinuous at a point* $\check{s}^* \in \Gamma$ *if for each* $\check{s} \in \Gamma$ *and* $\{\check{s}_t\}$ *in* $O_H^{\rho}(\check{s})$*, the implication*

$$\check{s}_{t_i} \xrightarrow{d^{-1}} \check{s}^* \text{ as } i \to \infty \Rightarrow g(\check{s}^*) \leq \lim_{i\to\infty} \inf g(\check{s}_{t_i})$$

holds for any subsequence $\{\check{s}_{t_i}\}$ *of* $\{\check{s}_t\}$*.*

Now, we present the definition of a d-proximal Ćirić-type contraction.

Definition 11. *Let* $(\Im, d)$ *be a quasi-metric space and* $\emptyset \neq \Gamma, \Lambda \subseteq \Im$*. A mapping* $H : \Gamma \to \Lambda$ *is said to be a d-proximal Ćirić-type contraction if there exists* $k \in [0, 1)$ *such that*

$$\left.\begin{array}{l} d(u_1, H\check{s}_1) = d(\Gamma, \Lambda) \\ d(u_2, H\check{s}_2) = d(\Gamma, \Lambda) \end{array}\right\}$$

implies

$$d(u_1, u_2) \leq kM_d(\check{s}_1, \check{s}_2, u_1, u_2) \tag{2.1}$$

for all $u_1, u_2, \check{s}_1, \check{s}_2 \in \Gamma$ *where*

$$M_d(\check{s}_1, \check{s}_2, u_1, u_2) = \max\left\{\begin{array}{c} d(\check{s}_1, \check{s}_2), d(\check{s}_1, u_1), d(\check{s}_2, u_2), \\ \frac{1}{2}(d(\check{s}_1, u_2) + d(\check{s}_2, u_1)) \end{array}\right\}.$$

Now, we present the following theorem which is our first main result.

Theorem 1. *Let* $(\Im, d)$ *be a quasi-metric space,* $\emptyset \neq \Gamma, \Lambda \subseteq \Im$*,* $\Gamma_0^r \neq \emptyset$ *and* $H : \Gamma \to \Lambda$ *be a* d^{-1}*-proximal Ćirić-type contraction satisfying* $H(\Gamma_0^r) \subseteq \Lambda_0^r$*. If* Γ *is H-best* d^{-1}*-orbitally right* d^{-1}*-complete and a function* $g : \Gamma \to \mathbb{R}$ *given as* $g(\check{s}) = d(H\check{s}, \check{s})$ *is* d^{-1}*-best* d^{-1}*-orbitally lower semicontinuous on* Γ*, then H has a right best proximity point in* Γ*.*

Proof. Let $\check{s}_0 \in \Gamma_0^r$ be any given point. Since $H\check{s}_0 \in H(\Gamma_0^r) \subseteq \Lambda_0^r$, there exists $\check{s}_1 \in \Gamma_0^r$ such that

$$d(H\check{s}_0, \check{s}_1) = d(\Lambda, \Gamma).$$

Similarly, there exists $\check{s}_2 \in \Gamma_0^r$ such that

$$d(H\check{s}_1, \check{s}_2) = d(\Lambda, \Gamma).$$

Continuing this process, we can construct a sequence $\{\check{s}_t\}$ such that

$$d(H\check{s}_t, \check{s}_{t+1}) = d(\Lambda, \Gamma) \tag{2.2}$$

for all $t \in \mathbb{N}$, that is, $\{\check{s}_t\} \in O_H^{d^{-1}}(\check{s}_0)$. If there exists $t_0 \in \mathbb{N}$ such that $\check{s}_{t_0} = \check{s}_{t_0+1}$, then the proof is complete. Then, we suppose that $\check{s}_t \neq \check{s}_{t+1}$ for all $t \geq 1$. Due to the fact that H is a d^{-1}-proximal Ćirić-type contraction, we have

$$d(\check{s}_{t+1}, \check{s}_t) \leq kM_{d^{-1}}(\check{s}_{t-1}, \check{s}_t, \check{s}_t, \check{s}_{t+1})$$

where

$$\begin{aligned}
M_{d^{-1}}(\check{s}_{t-1}, \check{s}_t, \check{s}_t, \check{s}_{t+1}) &= \max\left\{ \begin{array}{c} d(\check{s}_t, \check{s}_{t-1}), d(\check{s}_t, \check{s}_{t-1}), d(\check{s}_{t+1}, \check{s}_t), \\ \frac{1}{2}(d(\check{s}_{t+1}, \check{s}_{t-1}) + d(\check{s}_t, \check{s}_t)) \end{array} \right\} \\
&= \max\left\{ d(\check{s}_t, \check{s}_{t-1}), d(\check{s}_{t+1}, \check{s}_t), \frac{1}{2} d(\check{s}_{t+1}, \check{s}_{t-1}) \right\} \\
&\leq \max\left\{ \begin{array}{c} d(\check{s}_t, \check{s}_{t-1}), d(\check{s}_{t+1}, \check{s}_t), \\ \frac{1}{2}(d(\check{s}_{t+1}, \check{s}_t) + d(\check{s}_t, \check{s}_{t-1}) \end{array} \right\} \\
&= \max\{d(\check{s}_t, \check{s}_{t-1}), d(\check{s}_{t+1}, \check{s}_t)\}
\end{aligned}$$

for all $t \geq 1$. If $M_{d^{-1}}(\check{s}_{t_0-1}, \check{s}_{t_0}, \check{s}_{t_0}, \check{s}_{t_0+1}) = d(\check{s}_{t_0+1}, \check{s}_{t_0})$ for some $t_0 \in \mathbb{N}$, then we have

$$\begin{aligned}
d(\check{s}_{t_0+1}, \check{s}_{t_0}) &\leq kd(\check{s}_{t_0+1}, \check{s}_{t_0}) \\
&< d(\check{s}_{t_0+1}, \check{s}_{t_0})
\end{aligned}$$

which is a contradiction. Then, we get $M_{d^{-1}}(\check{s}_{t-1}, \check{s}_t, \check{s}_t, \check{s}_{t+1}) = d(\check{s}_t, \check{s}_{t-1})$ for all $t \geq 1$.

$$d(\check{s}_{t+1}, \check{s}_t) \leq kd(\check{s}_t, \check{s}_{t-1})$$

for all $t \geq 1$. Therefore, we have

$$\begin{aligned}
d(\check{s}_{t+1}, \check{s}_t) &\leq kd(\check{s}_t, \check{s}_{t-1}) \\
&\leq k^2 d(\check{s}_{t-1}, \check{s}_{t-2}) \\
&\vdots \\
&\leq k^t d(\check{s}_1, \check{s}_0)
\end{aligned}$$

for all $t \in \mathbb{N}$. Then, we have

$$\lim_{t\to\infty} d(\check{s}_{t+1}, \check{s}_t) = 0. \tag{2.3}$$

Then, we get, for all $r > t$,

$$\begin{aligned}
d(\check{s}_r, \check{s}_t) &= d(\check{s}_r, \check{s}_{r-1}) + d(\check{s}_{r-1}, \check{s}_{r-2}) + \cdots + d(\check{s}_{t+1}, \check{s}_t) \\
&\leq k^{r-1} d(\check{s}_1, \check{s}_0) + k^{r-2} d(\check{s}_1, \check{s}_0) + \cdots + k^t d(\check{s}_1, \check{s}_0) \\
&= k^t d(\check{s}_1, \check{s}_0)\left(1 + k + \cdots + k^{r-t-1}\right) \\
&= k^t d(\check{s}_1, \check{s}_0)\frac{1 - k^{r-t}}{1-k} \\
&\leq \frac{k^t d(\check{s}_1, \check{s}_0)}{1-k},
\end{aligned}$$

and so $\{\check{s}_t\}$ is a right K-Cauchy sequence in Γ. Then, there exists $\check{s}^* \in \Gamma_0^r$ such that $d(\check{s}_t, \check{s}^*) \longrightarrow 0$ as $t \to \infty$, since Γ is H-best d^{-1}-orbitally right d^{-1}-complete. On the other hand, from (2.2) and (2.3) we have

$$\begin{aligned}
d(\Lambda, \Gamma) &\leq d(H\check{s}_t, \check{s}_t) \\
&\leq d(H\check{s}_t, \check{s}_{t+1}) + d(\check{s}_{t+1}, \check{s}_t)
\end{aligned}$$

$$= \quad d(\Lambda,\Gamma) + d(\check{s}_{t+1}, \check{s}_t)$$

for all $t \in \mathbb{N}$. For the limit as $t \to \infty$, we have

$$\lim_{t\to\infty} d(H\check{s}_t, \check{s}_t) = d(\Lambda,\Gamma). \tag{2.4}$$

Then, since g is d^{-1}-best d^{-1}-orbitally lower semicontinuous on Γ, we have

$$\begin{aligned} d(\Lambda,\Gamma) &\le d(H\check{s}^*, \check{s}^*) \\ &= g(\check{s}^*) \\ &\le \liminf_{t\to\infty} g(\check{s}_t) \\ &= \liminf_{t\to\infty} d(H\check{s}_t, \check{s}_t) \\ &= d(\Lambda,\Gamma), \end{aligned}$$

and so we get $d(H\check{s}^*, \check{s}^*) = d(\Lambda,\Gamma)$. Hence, $\check{s}^*$ is a right best proximity point of H. □

The following example is given to show the effectiveness of Theorem 1.

Example 1. *Let $\mathfrak{I} = [0,\infty) \times [0,\infty)$ and $d : \mathfrak{I} \times \mathfrak{I} \to \mathbb{R}$ be a function defined by*

$$d(\check{s},\check{u}) = \begin{cases} 0 & , \quad \check{s} = \check{u} \\ 2\check{s}_1 + \check{u}_1 + |\check{s}_2 - \check{u}_2| & , \quad \check{s} \neq \check{u} \end{cases}$$

for all $\check{s} = (\check{s}_1,\check{s}_2), \check{u} = (\check{u}_1,\check{u}_2) \in \mathfrak{I}$. Hence, $(\mathfrak{I}, d)$ is a quasi-metric space. Consider the sets $\Gamma = \{0\} \times [0,\infty)$, $\Lambda = \{1\} \times [0,\infty)$ and a mapping $H : \Gamma \to \Lambda$ given as $H(0,\check{s}) = \left(1, \frac{\check{s}}{3}\right)$. Then, we have $d(\Lambda,\Gamma) = 2$, and so we get

$$\begin{aligned} O_H^{d^{-1}}((0,\check{s})) &= \left\{ \{(0,\check{s}_t)\} \subseteq \Gamma : \begin{array}{c} (0,\check{s}) = (0,\check{s}_0) \textit{ and} \\ d(H(0,\check{s}_t),(0,\check{s}_{t+1})) = d(\Lambda,\Gamma) \\ \textit{for all } t \in \mathbb{N} \end{array} \right\} \\ &= \left\{ \{(0,\check{s}_t)\} \subseteq \Gamma : \begin{array}{c} \check{s} = \check{s}_0 \textit{ and} \\ d\left(\left(1,\frac{\check{s}_t}{3}\right),(0,\check{s}_{t+1})\right) = 2 \\ \textit{for all } t \in \mathbb{N} \end{array} \right\} \\ &= \left\{ \{(0,\check{s}_t)\} \subseteq \Gamma : \check{s} = \check{s}_0 \textit{ and } \check{s}_{t+1} = \frac{\check{s}_t}{3} \textit{ for all } t \in \mathbb{N} \right\} \\ &= \left\{ \left\{ \left(0, \frac{\check{s}}{3^t}\right) \right\} \subseteq \Gamma : t \in \mathbb{N} \right\}. \end{aligned}$$

Also, we obtain $\Gamma_0^r = \Gamma$, $\Lambda_0^r = \Lambda$ and $H(\Gamma_0^r) \subseteq \Lambda_0^r$. Let for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{d^{-1}}((0,\check{s}))$, $\{\check{s}_{t_i}\}$ be a right K-Cauchy subsequence of $\{\check{s}_t\}$. Since every sequence in $O_H^{d^{-1}}((0,\check{s}))$ is convergent with respect to d^{-1}, the subsequence $\{\check{s}_{t_i}\}$ is convergent to $(0,0)$ with respect to d^{-1}. Hence, we have that $\check{s}_{t_i} \overset{d^{-1}}{\to} (0,0) \in \Gamma$. Then, Γ is H-best d^{-1}-orbitally right d^{-1}complete. It is clear that a mapping $g : \Gamma \to \mathbb{R}$ given as

$g(\check{s}) = d(H\check{s}, \check{s})$ is d^{-1}-best d^{-1}-orbitally lower semicontinuous on Γ. Now, we will show that H is a d^{-1}-proximal Ćirić-type contraction for $k = \frac{1}{3}$. Let $\check{s}_1, \check{s}_2, u_1, u_2 \in \Gamma$ satisfying

$$\begin{array}{c} d(H\check{s}_1, u_1) = d(\Lambda, \Gamma) \\ d(H\check{s}_2, u_2) = d(\Lambda, \Gamma) \end{array}.$$

Hence, we have that $\check{s}_1 = (0, a), \check{s}_2 = (0, b), u_1 = \left(0, \frac{a}{3}\right)$ and $u_2 = \left(0, \frac{b}{3}\right)$ where $a, b \in [0, \infty)$. Then,

$$\begin{aligned} d(u_1, u_2) &= \left|\frac{a}{3} - \frac{b}{3}\right| \\ &= \frac{1}{3} d(\check{s}_1, \check{s}_2) \\ &\leq \frac{1}{3} M_d(\check{s}_1, \check{s}_2, u_1, u_2). \end{aligned}$$

Therefore, all conditions of Theorem 1 hold. Hence, H is a right best proximity point $\check{s}^*$ in Γ which is $\check{s}^* = (0, 0)$.

Now, we give the following definition.

Definition 12. *Let $(\Im, d)$ be a quasi-metric space and $H : \Gamma \to \Lambda$ be a mapping where $\emptyset \neq \Gamma, \Lambda \subseteq \Im$. We say that H is d-d-best ρ (d or d^{-1})-orbitally continuous at a point $\check{s}^* \in \Gamma$ if for every $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^\rho(\check{s})$, the implication*

$$\check{s}_{t_i} \xrightarrow{d} \check{s}^* \Rightarrow H\check{s}_{t_i} \xrightarrow{d} H\check{s}^*, \text{ as } i \to \infty$$

holds for any subsequence $\{\check{s}_{t_i}\}$ of $\{\check{s}_t\}$.

The following example is important to better understand Definition 12.

Example 2. *Let $\Im = [0, \infty) \times [0, \infty)$ and $d : \Im \times \Im \to \mathbb{R}$ be a function defined as*

$$d(\check{s}, \check{u}) = \max\{\check{u}_1 - \check{s}_1, 0\} + |\check{s}_2 - \check{u}_2|$$

for all $\check{s} = (\check{s}_1, \check{s}_2), \check{u} = (\check{u}_1, \check{u}_2) \in \Im$. Then, $(\Im, d)$ is a quasi-metric space. Consider the sets $\Gamma = \left\{\left(1 + \frac{1}{t}, 0\right) : t \in \mathbb{N}\right\} \cup \{(1, 0)\}$, $\Lambda = \left\{\left(\frac{1}{t}, 1\right) : t \in \mathbb{N}\right\}$ and a mapping $H : \Gamma \to \Lambda$ given as

$$H\check{s} = \begin{cases} \left(\frac{1}{2}, 1\right) & \check{s} = (1, 0) \\ \\ \left(\frac{1}{t+1}, 1\right) & \check{s} = \left(1 + \frac{1}{t}, 0\right) \end{cases}.$$

Then, we have that $d(\Gamma, \Lambda) = 1$, and so we get that $O_H^d(\check{s})$ is the set of all sequences in Γ. Also, it can be seen that the sequence $\{\check{s}_t\}$ in $O_H^d(\check{s})$ converges to $(1, 0)$ with respect to d^{-1}. Also, we have

$$\begin{aligned} \lim_{t\to\infty} d(H\check{s}, H\check{s}_t) &= \lim_{t\to\infty} d\left(\left(\frac{1}{2}, 1\right), \left(\frac{1}{t+1}, 1\right)\right) \\ &= \lim_{t\to\infty} \max\left\{\frac{1}{t+1} - \frac{1}{2}, 0\right\} \end{aligned}$$

$$= \quad 0.$$

Therefore, H is d^{-1}-d-best d-orbitally continuous on Γ. *But, we have*

$$\begin{aligned}
\lim_{t\to\infty} d(H\check{s}_t, H\check{s}) &= \lim_{t\to\infty} d\left(\left(\frac{1}{t+1}, 1\right), \left(\frac{1}{2}, 1\right)\right) \\
&= \lim_{t\to\infty} \max\left\{\frac{1}{2} - \frac{1}{t+1}, 0\right\} \\
&= \frac{1}{2}.
\end{aligned}$$

Hence, H is not d^{-1}-d^{-1}-best d-orbitally continuous on Γ.

Proposition 1. *Let $(\Im, d)$ be a quasi-metric space and $\emptyset \neq \Gamma, \Lambda \subseteq \Im$. Assume that $H : \Gamma \to \Lambda$ is a mapping and $g : \Gamma \to \mathbb{R}$ is a function given as $g(\check{s}) = d(\check{s}, H\check{s})$. Then, the following statements are true.*

i) If H is d-d^{-1}-best ρ-orbitally continuous at a point $\check{s}^* \in \Gamma$, then g is d-best ρ-orbitally lower semicontinuous at a point $\check{s}^* \in \Gamma$.
ii) If H is d^{-1}-d-best ρ-orbitally continuous at a point $\check{s}^* \in \Gamma$, then g is d^{-1}-best ρ-orbitally lower semicontinuous at a point $\check{s}^* \in \Gamma$.

Proof. Let $\check{s} \in \Gamma$, $\{\check{s}_t\}$ be a sequence in $O_H^\rho(\check{s})$ and $\{\check{s}_{t_i}\}$ be a subsequence of $\{\check{s}_t\}$ such that $d(\check{s}^*, \check{s}_{t_i}) \to 0$ as $i \to \infty$. Since H is d-d^{-1}-best ρ-orbitally continuous on Γ, we have that $d(H\check{s}_{t_i}, H\check{s}^*) \to 0$ as $i \to \infty$. Hence, we get

$$\begin{aligned}
g(\check{s}^*) &= d(\check{s}^*, H\check{s}^*) \\
&\leq d(\check{s}^*, \check{s}_{t_i}) + d(\check{s}_{t_i}, H\check{s}_{t_i}) + d(H\check{s}_{t_i}, H\check{s}^*).
\end{aligned}$$

Taking the limit inferior as $i \to \infty$, we obtain

$$g(\check{s}^*) \leq \lim_{i\to\infty} \inf d(\check{s}_{t_i}, H\check{s}_{t_i}) = \lim_{i\to\infty} \inf g(\check{s}_{t_i}).$$

The proof of (i) is complete. We can prove ii) in a way that is similar to the above one. □

The converse of Proposition 1 may not be true. We give an example to demonstrate this fact.

Example 3. *Let $\Im = [0, \infty) \times [0, \infty)$ and $d : \Im \times \Im \to \mathbb{R}$ be a function defined by*

$$d(\check{s}, \check{u}) = \begin{cases} 0 & , \quad \check{s} = \check{u} \\ \\ \check{s}_1 + |\check{s}_2 - \check{u}_2| & , \quad \check{s} \neq \check{u} \end{cases}$$

for all $\check{s} = (\check{s}_1, \check{s}_2), \check{u} = (\check{u}_1, \check{u}_2) \in \Im$. Then, $(\Im, d)$ is a quasi-metric space. Consider the sets $\Gamma = \{0\} \times [0, \infty)$, $\Lambda = \{1\} \times [0, \infty)$ and a mapping $H : \Gamma \to \Lambda$ given as $H(0, \check{s}) = \left(1, \frac{\check{s}}{2}\right)$. Then, we have that $d(\Gamma, \Lambda) = 0$, and so we get

$$O_H^d((0, \check{s})) = \left\{ \{(0, \check{s}_t)\} \text{ in } \Gamma : \begin{array}{r} (0, \check{s}) = (0, \check{s}_0) \text{ and} \\ d((0, \check{s}_{t+1}), H(0, \check{s}_t)) = d(\Gamma, \Lambda) \\ \text{for all } t \in \mathbb{N} \end{array} \right\}$$

$$= \left\{ \{(0,\check{s}_t)\} \textit{ in } \Gamma : \begin{array}{c} \check{s} = \check{s}_0 \textit{ and} \\ d((0,\check{s}_{t+1}),\left(1,\frac{\check{s}_t}{2}\right)) = 0 \\ \textit{for all } t \in \mathbb{N} \end{array} \right\}$$

$$= \left\{ \{(0,\check{s}_t)\} \textit{ in } \Gamma : \check{s} = \check{s}_0 \textit{ and } \check{s}_{t+1} = \frac{\check{s}_t}{2} \textit{ for all } t \geq 1 \right\}$$

$$= \left\{ \left\{ \left(0, \frac{\check{s}}{2^t}\right) \right\} \textit{ in } \Gamma : t \geq 1 \right\}.$$

Now, we will show that a mapping $g : \Gamma \to \mathbb{R}$ *given as* $g(\check{s}) = d(\check{s}, H\check{s})$ *is d-best d-orbitally lower semicontinuous. Let* $\{\check{s}_{t_i}\}$ *be a convergent subsequence of* $\{\check{s}_t\}$ *in* $O_H^d((0,\check{s}))$ *with respect to d. From the definition of d, it can be seen that every sequence in* $O_H^d((0,\check{s}))$ *converges to* $(0,0)$ *with respect to d. Thus, we have that* $\check{s}_{t_i} \xrightarrow{d} (0,0)$*. Hence, we have*

$$\begin{aligned} g((0,0)) &= d((0,0), H(0,0)) \\ &= 0 \\ &= \liminf_{i\to\infty} g(\check{s}_{t_i}), \end{aligned}$$

and so g is d-best d-orbitally lower semicontinuous. Also, we can show that g is d^{-1}*-best d-orbitally lower semicontinuous. However, H is not d-*d^{-1}*-best d-orbitally continuous. Indeed, for* $\check{s} = (0,1) \in \Gamma$*, we have*

$$O_H^d((0,\check{s})) = \left\{ \left(0, \frac{1}{2^t}\right)\} \ : t \in \mathbb{N} \right) \right\}.$$

Hence, if we take $\{\check{s}_t\} = \left\{(0, \frac{1}{2^t})\right\}$ *in* $O_H^d((0,\check{s}))$*, then although we have* $\check{s}_t \xrightarrow{d} 0$*, we get*

$$\begin{aligned} d(H\check{s}_t, H\check{s}) &= d\left(\left(1, \frac{\check{s}}{2^{t+1}}\right), (1,0)\right) \\ &= 1 + \frac{\check{s}}{2^{t+1}} \end{aligned}$$

which implies that

$$\lim_{t\to\infty} d(H\check{s}_t, H\check{s}) = 1.$$

Hence, the sequence $\{H\check{s}_t\}$ *is not convergent to* $H\check{s}$ *with respect to* d^{-1}*; so, H is not d-*d^{-1}*-best d-orbitally continuous on* Γ*. It can be seen that H is not d-d-best d-orbitally continuous,* d^{-1}*-d-best d-orbitally continuous or* d^{-1}*-*d^{-1}*-best d-orbitally continuous.*

Now, we present the following result by using Proposition 1.

Corollary 1. *Let* $(\mathfrak{I}, d)$ *be a quasi-metric space,* $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{I}$*,* $\Gamma_0^r \neq \emptyset$ *and* $H : \Gamma \to \Lambda$ *be a* d^{-1}*-proximal Ćirić-type contraction satisfying* $H(\Gamma_0^r) \subseteq \Lambda_0^r$*. Then, H has a right best proximity point in* Γ *provided that* Γ *is H-best* d^{-1}*-orbitally right* d^{-1}*-complete and H is* d^{-1}*-d-best* d^{-1}*-orbitally continuous on* Γ*.*

If we take $\Gamma = \Lambda = \mathfrak{I}$ in Theorem 1 and Corollary 1, then we present the following fixed-point results which are generalizations of [26] in T_1-quasi-metric spaces.

Corollary 2. *Let* $(\Im, d)$ *be a* T_1*-quasi-metric space and* $H : \Im \to \Im$ *be a* d^{-1}*-proximal Ćirić-type contraction. If* $\Im$ *is* H*-best* d^{-1}*-orbitally right* d^{-1}*-complete and a function* $g : \Im \to \mathbb{R}$ *given as* $g(\check{s}) = d(H\check{s}, \check{s})$ *is* d^{-1}*-best* d^{-1}*-orbitally lower semicontinuous on* $\Im$*, then* H *has a fixed-point in* $\Im$*.*

Corollary 3. *Let* $(\Im, d)$ *be a* T_1*-quasi-metric space and* $H : \Im \to \Im$ *be a* d^{-1}*-proximal Ćirić-type contraction. If* $\Im$ *is* H*-best* d^{-1}*-orbitally right* d^{-1}*-complete and* H *is* d^{-1}*-*d*-best* d^{-1}*-orbitally continuous on* $\Im$*, then* H *has a fixed-point in* $\Im$*.*

Now, we present some left best proximity point results.

Theorem 2. *Let* $(\Im, d)$ *be a quasi-metric space,* $\emptyset \neq \Gamma, \Lambda \subseteq \Im$*,* $\Gamma_0^\ell \neq \emptyset$ *and* $H : \Gamma \to \Lambda$ *be a* d*-proximal Ćirić-type contraction mapping satisfying* $H(\Gamma_0^\ell) \subseteq \Lambda_0^\ell$*. If* Γ *is* H*-best* d*-orbitally left* d^{-1}*-complete and a function* $g : \Gamma \to \mathbb{R}$ *given as* $g(\check{s}) = d(\check{s}, H\check{s})$ *is* d^{-1}*-best* d*-orbitally lower semicontinuous on* Γ*, then* H *has a left best proximity point in* Γ*.*

Proof. Let $\check{s}_0 \in \Gamma_0^\ell$ be an arbitrary point. Since $H\check{s}_0 \in H(\Gamma_0^\ell) \subseteq \Lambda_0^\ell$, there exists $\check{s}_1 \in \Gamma_0^\ell$ such that

$$d(\check{s}_1, H\check{s}_0) = d(\Gamma, \Lambda).$$

Similarly, there exists $\check{s}_2 \in \Gamma_0^l$ such that

$$d(\check{s}_2, H\check{s}_1) = d(\Gamma, \Lambda).$$

Continuing this process, we can construct a sequence $\{\check{s}_t\}$ such that

$$d(\check{s}_{t+1}, H\check{s}_t) = d(\Gamma, \Lambda) \tag{2.5}$$

for all $t \in \mathbb{N}$, that is, $\{\check{s}_t\} \in O_H^d(\check{s}_0)$. If there exists $t_0 \in \mathbb{N}$ such that $\check{s}_{t_0} = \check{s}_{t_0+1}$, then the proof is complete. So, we assume that $\check{s}_t \neq \check{s}_{t+1}$ for all $t \in \mathbb{N}$. Similar to the proof of Theorem 1, we can obtain that

$$\lim_{t\to\infty} d(\check{s}_t, \check{s}_{t+1}) = 0 \tag{2.6}$$

and the sequence $\{\check{s}_t\}$ is a left K-Cauchy sequence. There exists $\check{s}^* \in \Gamma_0^\ell$ such that $d(\check{s}_t, \check{s}^*) \longrightarrow 0$ as $t \to \infty$, due to the fact that Γ is H-best d-orbitally left d^{-1}-complete. On the other hand, from (2.5),

$$\begin{aligned} d(\Gamma, \Lambda) &\leq d(\check{s}_t, H\check{s}_t) \\ &\leq d(\check{s}_t, \check{s}_{t+1}) + d(\check{s}_{t+1}, H\check{s}_t) \\ &= d(\check{s}_t, \check{s}_{t+1}) + d(\Gamma, \Lambda) \end{aligned}$$

for all $t \in \mathbb{N}$. For the limit as $t \to \infty$, we have, from (2.6),

$$\lim_{t\to\infty} d(\check{s}_t, H\check{s}_t) = d(\Gamma, \Lambda).$$

Then, since g is d^{-1}-best d-orbitally lower semicontinuous on Γ, we have

$$\begin{aligned} d(\Gamma, \Lambda) &\leq d(\check{s}^*, H\check{s}^*) \\ &= g(\check{s}^*) \\ &\leq \liminf_{t\to\infty} g(\check{s}_t) \\ &= \liminf_{t\to\infty} d(\check{s}_t, H\check{s}_t) \\ &= d(\Gamma, \Lambda), \end{aligned}$$

and so we get that $d(\check{s}^*, H\check{s}^*) = d(\Gamma, \Lambda)$. Hence, $\check{s}^*$ is a left best proximity point of H. □

Now, we present the following result by using Proposition 1.

Corollary 4. *Let* $(\mathfrak{I}, d)$ *be a quasi-metric space,* $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{I}$*,* $\Gamma_0^\ell \neq \emptyset$ *and* $H : \Gamma \to \Lambda$ *be a d-proximal Ćirić-type contraction satisfying* $H(\Gamma_0^\ell) \subseteq \Lambda_0^\ell$*. Then,* H *has a right best proximity point in* Γ *provided that* Γ *is H-best d-orbitally left* d^{-1}*-complete and* H *is* d^{-1}*-d-best d-orbitally continuous on* Γ*.*

3. Conclusions

Fixed point theory is an exciting area of research for metric spaces and generalized metric spaces. Fixed-point theorems are mainly useful when dealing with problems that arise in the theory of the existence of differential equations, integral equations, partial differential equations, dynamic programming, fractal modeling, chaos theory and various other disciplines of mathematics, statistics, engineering, economics and approximation theory. Also, best proximity point results are a generalization of the corresponding fixed-point results. Therefore, we aimed to extend some results existing in the literature with the aid of best proximity point theory for best orbitally complete quasi-metric spaces. We first presented new definitions and notations by taking into account the lack of symmetry property of quasi-metric spaces. Moreover, we gave some examples to support our definitions and notations. Hence, many researchers can extend some best proximity point and fixed-point results obtained with the help of nonlinear functions in different metric spaces to quasi-metric spaces by using these definitions in future works. We also introduced the concepts of d- and d^{-1}-proximal Ćirić contraction mappings. Then, we proved some right and left best proximity point results for these mappings on best orbitally complete quasi-metric spaces. Therefore, we obtained some generalizations of the famous fixed-point and best proximity point results in the literature.

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Conflict of interest

All authors declare no conflicts of interest regarding the publication of this paper.

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