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Research article

A novel pessimistic multigranulation roughness by soft relations over dual universe

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Abstract: A multigranulation rough set over two universes delivers a unique perspective on the combination of multigranulation information. This paper presents the pessimistic multignualtion rough set over dual universes based on soft binary relations. Firstly, a new pessimistic multigranualtion rough set over dual universes based on two soft binary relations has been developed, and their properties are derived. Then we extend this idea and present pessimistic multigranulation roughness over dual universes based on the finite number of soft binary relations. Finally, we present an example to illustrate our proposed multigranualtion rough set model.

Keywords: rough set; soft set; multigranulation rough set; soft relation and approximation by soft binary relation
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1. Introduction

Every day in our environment, we meet with various problems involving uncertainty. For example, the beauty concept is uncertain because it is impossible to categorize beauty into two categories beautiful and ugly. Therefore, the concept of beauty is not precise but somewhat uncertain. The classic set requires precision for all mathematics. Numerous theories exist to address uncertainty, like probability theory, vague set theory, interval mathematics, intuitionistic fuzzy sets, and fuzzy set theory to deal with uncertainty, but each of these theories has its difficulties as mentioned in [15]. Soft set theory was invited by Molodtsov [15], which is a helpful approach to dealing with uncertainty.

Another approach to address uncertainty is the rough set theory proposed by Pawlak in 1982 [17]. Numerous advantageous applications of rough set theory exist. In particular, in the areas of decision-making, machine learning, pattern recognition, information acquisition, expert systems, knowledge creation from databases, inductive reasoning, cognitive sciences, and artificial intelligence. The rough set techniques are essential [17, 18, 42, 50]. Rough set theory's major benefit in data analysis is that it does not need any preliminary or supplemental data information like statistical probability. The Pawlak rough set is constructed using the equivalence classes as its building blocks. However, for many practical applications, the equivalence relationship is also restrictive in many situations. To tackle this problem, the Pawlak rough set model can be generalized in several ways by substituting other relations for the equivalence relation, tolerance relation, [24, 48], similarity relation [25], dominance relation [8], neighborhood system [35]. Addressing the rough set using general binary relations mentioned in [39, 41, 46, 47], the soft, rough set can be seen in [2, 7, 21], covering based discussion in [4, 11, 38, 43]. Rough approximation based on soft binary relation in [10], Xu et al. [39] expanded this idea and approximated a set with respect to the aftersets and foresets of the binary relation over the universe U. Recently, Shabir et al. [23] took this idea a step further and approximated a subset of one universe, V, in another universe, U. Utilizing the aftersets and foresets of a soft binary relation over $U \times V$.

All of the generalized rough set models use single relation of the universe, which would have limitations for addressing multi-granulation information in many real-world circumstances. Therefore, Qian et al. [19, 20] introduced a multi-granulation rough set for handling this issue. Multi-granulation has gained the attention of numerous scholars worldwide as a hot issue in management science. A brief review of multi-granulation is as follows. Qain et al. [34] proposed the pessimistic multigranultion rough set. Xu et al. [37] presented a generalized multgranulation rough set; Yang et al. [44] generalized multgranulation rough set from crisp to fuzzy, multi-granulation fuzzy rough set presented by Xu et al. [40, 49], Zhan et al. [49, 51] discussed two types of coverings based multigranulation rough sets, Intuitionistic fuzzy multigranulation rough sets were presented by Huang et al. [9], Ali et al. introduced a novel sort of dominance-based MGRS and discussed its use in conflict analysis issues in [1].

Many practical as diseases symptoms and medications used in diagnostics and comprise a diverse universe of objects. To address the rough set issues that occur in the single universe of objects Liu [13], Yan et al. [45] introduced a generalized RS model over dual universes of objects instead of the single universe of objects in respect of developing the relationship between the single-universe and dual universe models. The probabilistic RS over dual universes was established by Ma and Sun [16] to measure the uncertainty of knowledge, Liu et al. [14] discussed the graded RS model over dual universes and its properties, Shabir et al. [23] presented SBr based approximation of a set over dual universes and its application in the reduction of an information system, Zhang et al. [53] presented FRS over dual universes with interval-valued data, FR approximation of a set over dual universes were established by Wu et al. [36]. Sun et al. [32] presented MGRS over dual universes of objects. MGRS over dual universes is a well-structured framework that deals with numerous DM problems. It has become a hot topic in the field of multiple decision problems and attracting a broad spectrum of theoretical and practical studies. Zhang et al. [52] presented PFMGRS over dual universes, which may be used in mergers and acquisitions. MGFRS over dual universes and its application to DM and three ways GDM based on MGF decision theoretic RS over dual universes were presented by Sun et al. [27, 28]. Furthermore, applications in GDM of MG over dual universes can be found in [29, 30]. Zhang et al. [54] presented the FMGRS over dual universes for steam turbine fault diagnosis, Tan et al. [33] demonstrated decision-making with MGRS over dual universes and granulation selection, Sun et al. [31] presented diversified binary relation based FMGRS over dual universes. Recently, Din et al. [5] presented a pessimistic multigranulation roughness of a fuzzy set based on soft binary relations over dual universes and its application to approximate a fuzzy set of a universe in another universe.

Qian et al. [19], presented the notion of MGRS by multi equivalence relations on a universe U. Sun et al. [26], replaced multi equivalence relations by general binary relations. On the other hands, Shabir et al. [22] presented multigranulation roughness based on soft relations, which is optimistic multigranulation of crisp set over two universes to approximate a subset of a universe in another universe. It was a very interesting generalization of multigranulation roughness. But some properties are not satisfied like the lower approximation is not contained in the upper approximation, and the accuracy measure is not defined to find the degree of exactness. To compensate for this issue motivated us to study a novel multigranulation roughness in this paper, we mainly focus on pessimistic multigranulation roughness of set over two universes U and V, to approximate the subset of $N \subseteq U$ in V by using the afterset set of soft binary relations. The paper is organized as follows. In Section 2, we recall some basic definitions of the rough set, multigranulation rough set, soft set, and soft binary relation. In Section 3, we present the pessimistic multigranulation roughness of a set based on two soft binary relations over dual universes and their properties and present the pessimistic multigranulation roughness of a set based on finite number soft binary relations over dual universes. Section 4 illustrates an example and Section 5 concludes the paper.

2. Preliminaries

In this section, we recall some basic notions that will be applied in subsequent sections.

Definition 2.1. *[17] Let U be a finite non-empty set and ρ be an equivalence relation over U. Then $\{v \in U|(u,v) \in \rho\}$ is called an equivalence class of u with respect to ρ, we denote it by $[u]_\rho$. Let $X \subseteq U$. The Pawlak lower and upper approximations of X are defined as*

$$\underline{\rho}(X) = \cup\{[u]_\rho|[u]_\rho \subseteq X\},$$
$$\overline{\rho}(X) = \cup\{[u]_\rho|[u]_\rho \cap X \neq \emptyset\},$$

the pair (U,ρ) is called Pawlak approximation space.

The Pawlak rough set model was transformed by Qian et al. [19,20] to a multigranulation rough set model, where the set approximations are determined by using multiple equivalences relations on the universe.

Definition 2.2. *[19] Let $\hat{\rho}_1, \hat{\rho}_2$ be two equivalence relations on the non-empty universal set U and $X \subseteq U$. Then*

$$\underline{X}_{\hat{\rho}_1+\hat{\rho}_2} = \{u \in U : [u]_{\hat{\rho}_1} \subseteq X \; or \; [u]_{\hat{\rho}_2} \subseteq X\}$$
$$\overline{X}^{\hat{\rho}_1+\hat{\rho}_2} = \{u \in U : [u]_{\hat{\rho}_1} \cap X \neq \emptyset \; and \; [u]_{\hat{\rho}_2} \cap X \neq \emptyset\}$$

are called the lower and upper approximations of X with respect to ρ_1 and ρ_2.

Definition 2.3. *[19] Let $\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_m$ be m equivalence relations on a universal set U and $X \subseteq U$. Then the lower and upper approximations of U are defined as*

$$\underline{X}_{\sum_{i=1}^m \hat{\rho}_i} = \{u \in U : [u]_{\hat{\rho}_i} \subseteq X \; for some \; i, 1 \leq i \leq m\},$$

$$\overline{X}^{\sum_{i=1}^{m}\hat{\rho}_i} = (\underline{X^c}_{\sum_{i=1}^{m}\hat{\rho}_i})^c.$$

Molodtsov (1999) defined a soft set as:

Definition 2.4. *[15] A pair (ρ, A) is called a soft set over U, where ρ is a mapping given by $\rho : A \longmapsto P(U)$, U is a non-empty finite set and A is a subset of E (set of parameters).*
A soft set (ρ_1, A) is a soft subset of a soft set (ρ_2, B) over a common universe U, if $A \subseteq B$ and for all $a \in A, \rho_1(a) \subseteq \rho_2(a)$. Two soft sets over a common universe are equal if they are soft subsets of each other.

Feng et al. (2013) defined soft binary relation on a universe U as follows.

Definition 2.5. *[6] Let (ρ, A) be a soft set over $U \times U$. Then (ρ, A) is called a soft binary relation over U. In fact (ρ, A) is a parameterized collection of binary relations over U, that is, we have a binary relation $\rho(e)$ on U for each parameter $e \in A$.*
We shall denote the collection of all soft binary relations over U by $SBr(U)$.

Li et al. [12], generalized the definition of soft binary relation over a set U to soft binary relation from U to V, as following.

Definition 2.6. *[12] If (ρ, A) is a soft set over $U\times V$, that is $\rho : A \longmapsto P(U\times V)$, then (ρ, A) is said to be a soft binary relation (SB-relation) from U to V. (ρ, A) is a parameterized collection of binary relations from U to V. We shall denote the collection of all soft binary relations from U to V by $SBr(U, V)$.*

Shabir et al. [23] defined lower and upper approximations of a set by using soft binary relations as follows:

Definition 2.7. *[23] If (ρ, A) is a SB-relation from U to V and $X \subseteq V$, then we define two soft sets over U, by*

$$\underline{\rho}^X(e) = \{u \in U : \emptyset \neq u\rho(e) \subseteq X\},$$
$$\overline{\rho}^X(e) = \{u \in U : u\rho(e) \cap X \neq \emptyset\},$$

where $u\rho(e) = \{v \in V : (u, v) \in \rho(e)\}$ for each $e \in A$ and is called afterset of u corresponding to parameter e.
Moreover, $\underline{\rho}^X : A \mapsto P(U)$ and $\overline{\rho}^X : A \mapsto P(U)$ and we say (U, V, ρ) a generalized soft approximation space.
If $X \subseteq U$, then we can define two soft sets over V, by

$$^X\underline{\rho}(e) = \{v \in V : \emptyset \neq \rho(e)v \subseteq X\}$$
$$^X\overline{\rho}(e) = \{v \in V : \rho(e)v \cap X \neq \emptyset\}$$

where $\rho(e)v = \{u \in U : (u, v) \in \rho(e)\}$ for each $e \in A$ and is called foreset of v corresponding to parameter e.
Moreover, $^X\underline{\rho} : A \mapsto P(V)$ and $^X\overline{\rho} : A \mapsto P(V)$.

3. Pessimistic roughness of a set by two soft binary relations

In this section, we discuss the roughness of a set based on soft binary relation over dual universe. We use the aftersets and foresets of a soft binary relations from U to V and approximate subset of V in U and subset of U in V. In this way we get the corresponding soft rough subset of $U(V)$, which is called lower and upper approximations with respect to aftersets and foresets of soft binary relations.

Definition 3.1. *Let $(\rho_1, A), (\rho_2, A)$ be two soft binary relations from universe U to V and $X \subseteq V, A \subseteq E$ (set of parameters). Then we define two soft sets over U by*

$$_p\underline{\rho_1 + \rho_2}^X(a) = \{u \in U \mid \emptyset \neq u\rho_1(e) \subseteq X \text{ and } \emptyset \neq u\rho_2(a) \subseteq X\} \text{ for all } a \in A$$
$$^p\overline{\rho_1 + \rho_2}^X(a) = \{u \in U \mid u\rho_1(a) \cap X \neq \emptyset \text{ or } u\rho_2(e) \cap X \neq \emptyset\} \text{ for all } a \in A$$

called the pessimistic lower and pessimistic upper approximations of X with respect to the aftersets. We denote these soft sets by $(_p\underline{\rho_1 + \rho_2}^X(a), A), (^p\overline{\rho_1 + \rho_2}^X(a), A)$, respectively. The pair $\left((_p\underline{\rho_1 + \rho_2}^X(A), (\overline{_p\rho_1 + \rho_2}^X(A)\right)$ is called the generalized soft rough set approximation of X with respect to aftersets of ρ_1 and ρ_2. The sets $POS_{\rho_1+\rho_2}(X) = {}_p\underline{\rho_1 + \rho_2}^X(A)$, $BN_{\rho_1+\rho_2}(X) = {}^p\overline{\rho_1 + \rho_2}^X(A) - {}_p\underline{\rho_1 + \rho_2}^X(A)$ and $NEG^X_{\rho_1+\rho_2}(A) = U - {}^p\overline{\rho_1 + \rho_2}^X(A)$ are called ρ_1, ρ_2-positive, ρ_1, ρ_2-boundary and ρ_1, ρ_2-negative regions X in U respectively. These regions induced the partition of U with respect to each parameter, which classified the elements of U. If $BN_{\rho_1+\rho_2}(X) = 0$ empty set then we say the corresponding soft subset of U, to $X \subseteq V$ is exact (definable) with respect to ρ_1, ρ_2. Otherwise inexact (Rough).

Definition 3.2. *Let $(\rho_1, A), (\rho_2, A)$ be two soft binary relations from universe U to V and $X \subseteq U, A \subseteq E$(set of parameters). Then we define two soft sets over V by*

$$^X\underline{\rho_1 + \rho_2}_p(a) = \{v \in V \mid \emptyset \neq \rho_1(a)v \subseteq X \text{ and } \emptyset \neq \rho_2(a)v \subseteq X\} \text{ for all } a \in A$$
$$^X\overline{\rho_1 + \rho_2}^p(a) = \{v \in V \mid \rho_1(a)v \cap N \neq \emptyset \text{ or } \rho_2(a)v \cap X \neq \emptyset\} \text{ for all } a \in A$$

called the lower and upper approximations of X with respect to the foresets. We denote these soft sets by $(^X\underline{\rho_1 + \rho_2}_p(a), A), (^X\overline{\rho_1 + \rho_2}^p(a), A)$, respectively. The pair $\left((^X\underline{\rho_1 + \rho_2}_p(A), (^X\overline{\rho_1 + \rho_2}^p(A)\right)$ is called the generalized soft rough set approximation of X with respect to aftersets of ρ_1 and ρ_2. The sets $POS_{\rho_1+\rho_2}(X) = {}^X\underline{\rho_1 + \rho_2}_p(A)$, $BN_{\rho_1+\rho_2}(X) = {}^X\overline{\rho_1 + \rho_2}^p(A) - {}^p\underline{\rho_1 + \rho_2}_p(A)$ and $NEG^X_{\rho_1+\rho_2}(A) = U - {}^X\overline{\rho_1 + \rho_2}^p(A)$ are called ρ_1, ρ_2-positive, ρ_1, ρ_2-boundary and ρ_1, ρ_2-negative regions X in V respectively. These regions induce the partition of V with respect to each parameter, which classified the elements of V. If $BN_{\rho_1+\rho_2}(X) = 0$ empty set then we say the corresponding soft subset of V, to $X \subseteq U$ is exact (definable) with respect to ρ_1, ρ_2. Otherwise inexact (Rough).

To explain the above definitions, we have the following example.

Example 3.1. *Let $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$ and $A = \{a_1, a_2\}$.*
Let $X = \{v_1, v_2\} \subseteq V$, $Y = \{u_1, u_2\} \subseteq U$ and $(\rho_1, A), (\rho_2, A)$ be two soft relations from universe U to V defined by

$$\rho_1(a_1) = \{(u_1, v_1), (u_2, v_1), (u_3, v_2)\}, \qquad \rho_2(a_1) = \{(u_2, v_2), (u_2, v_3), (u_3, v_1)\},$$
$$\rho_1(a_2) = \{(u_2, v_2), (u_3, v_1)\}, \qquad \rho_2(a_2) = \{(u_1, v_3), (u_3, v_2), (u_3, v_3)\}.$$

Then their aftersets and foresets are

$$\begin{array}{llll}
u_1\rho_1(a_1)=\{v_1\}, & u_1\rho_2(a_1)=\emptyset, & \rho_1(a_1)v_1=\{u_1,u_2\}, & \rho_2(a_1)v_1=\{u_3\}\\
u_2\rho_1(a_1)=\{v_1\}, & u_2\rho_2(a_1)=\{v_2,v_3\}, & \rho_1(a_1)v_2=\{u_1\}, & \rho_2(a_1)v_2=\{u_2\}\\
u_3\rho_1(a_1)=\{v_2\}, & u_3\rho_2(a_1)=\{v_1\}, & \rho_1(a_1)v_3=\emptyset, & \rho_2(a_1)v_3=\{u_2\}\\
u_1\rho_1(a_2)=\emptyset, & u_1\rho_2(a_2)=\{v_3\}, & \rho_1(a_2)v_1=\{u_3\}, & \rho_2(a_2)v_1=\emptyset\\
u_2\rho_1(a_2)=\{v_2\}, & u_2\rho_2(a_2)=\emptyset, & \rho_1(a_2)v_2=\{u_2\}, & \rho_2(a_2)v_2=\{u_3\}\\
u_3\rho_1(a_2)=\{v_1\}, & u_3\rho_2(a_2)=\{v_2,v_3\}, & \rho_1(a_2)v_3=\emptyset, & \rho_2(a_2)v_3=\{u_1,u_3\}
\end{array}$$

Then
${}_p\underline{\rho_1+\rho_2}^X(a_1)=\{u_3\}, {}_p\underline{\rho_1+\rho_2}^X(a_2)=\emptyset, {}^p\overline{\rho_1+\rho_2}^X(a_1)=\{u_1,u_2,u_3\}, {}^p\overline{\rho_1+\rho_2}^X(a_2)=\{u_2,u_3\}$
So we get two soft sets over $U, ({}_p\underline{\rho_1+\rho_2}^X(a),A)=\{\{u_3\},\emptyset\}$ *and* $({}^p\overline{\rho_1+\rho_2}^X(a),A)=\{\{u_1,u_2,u_3\},\{u_2,u_3\}\}$.
Similarly,
${}^Y\underline{\rho_1+\rho_2}_p(a_1)=\{v_2\}, {}^Y\underline{\rho_1+\rho_2}_p(a_2)=\emptyset, {}^Y\overline{\rho_1+\rho+2}^p(a_1)=\{v_1,v_2,v_3\}, {}^Y\overline{\rho_1+\rho_2}^p(a_2)=\{v_2,v_3\}$,
So we ge two soft sets over $V, ({}^Y\underline{\rho_1+\rho_2}_p(a),A)=\{\{v_2\},\emptyset\}$ *and* $({}^Y\overline{\rho_1+\rho_2}^p(a),A)=\{\{v_1,v_2,v_3\},\{v_2,v_3\}\}$.

The following proposition shows the relationship between the lower and upper approximations.

Proposition 3.1. *Let* $(\rho_1,A),(\rho_2,A)$ *be two soft binary relations from universe* U *to* V *and* $X\subseteq V$. *Then*

(1) ${}_p\underline{\rho_1+\rho_2}^X(a)\subseteq {}^p\overline{\rho_1+\rho_2}^X(a)$
(2) ${}_p\underline{\rho_1+\rho_2}^{X^c}(a)\subseteq \left({}^p\overline{\rho_1+\rho_2}^X(a)\right)^c$,
(3) ${}^p\overline{\rho_1+\rho_2}^{X^c}(a)\subseteq \left({}_p\underline{\rho_1+\rho_2}^X(a)\right)^c$.

Proof.

(1) *Let* $u\in {}_p\underline{\rho_1+\rho_2}^X(a)$. *Then* $\emptyset\neq u\rho_1(a)\subseteq X$ *and* $\emptyset\neq u\rho_2(a)\subseteq X$, *these imply that* $u\rho_1(a)\cap X\neq\emptyset$ *and* $u\rho_2(a)\cap X\neq\emptyset$. *That is* $u\in {}^p\overline{\rho_1+\rho_2}^X(a)$. *Thus* ${}_p\underline{\rho_1+\rho_2}^X(a)\subseteq {}^p\overline{\rho_1+\rho_2}^X(a)$.
(2) *Let* $u\in {}_p\underline{\rho_1+\rho_2}^{X^c}(a)$. *Then* $\emptyset\neq u\rho_1(a)\subseteq X^c$ *and* $\emptyset\neq u\rho_2(a)\subseteq X^c$. *These imply that* $u\rho_1(a)\cap X=\emptyset$ *and* $u\rho_2(a)\cap X=\emptyset$. *Thus* $u\notin {}^p\overline{\rho_1+\rho_2}^X(a)$, *that is* $u\in({}^p\overline{\xi_1+\rho_2}^X(a))^c$. *Thus* ${}_p\underline{\rho_1+\rho_2}^X(a)\subseteq ({}^p\overline{\rho_1+\rho_2}^X(a))^c$.
(3) *Let* $u\in {}^p\overline{\xi_1+\rho_2}^{X^c}(a)$. *Then* $u\rho_1(a)\cap X^c\neq\emptyset$ *or* $u\rho_2(a)\cap X^c\neq\emptyset$, *these imply that* $u\rho_1(a)\not\subseteq X$ *or* $u\xi_2(a)\not\subseteq X$. *Thus* $u\notin {}_p\underline{\rho_1+\rho_2}^X$, *that is* $u\in({}_p\underline{\xi_1+\rho_2}^X(a))^c$. *Thus* ${}^p\overline{\rho_1+\rho_2}^{X^c}(a)\subseteq({}_p\underline{\xi_1+\rho_2}^X(a))^c$.

To show that the reverse inclusion in the parts of the Proposition 3.1. It is not true that we have the following example.

Example 3.2. *(Continue from Example 3.1) Utilize the* U,V *and* $(\rho_1,A),(\rho_2,A)$ *of Example 3.1. Let* $X=\{v_1,v_2\}\subseteq W, X^c=\{v_3\}$. *Then*

(1) ${}_p\underline{\rho_1+\rho_2}^X(a_1)=\{u_1\}\not\supseteq\{u_1,u_2,u_3\}={}^p\overline{\rho_1+\rho_2}^X(a_1)$.
(2) ${}_p\underline{\rho_1+\rho_2}^{X^c}(a_2)=\emptyset\not\supseteq\{u_3\}=\left({}^p\overline{\rho_1+\rho_2}^X(a_2)\right)^c$.
(3) ${}^p\overline{\rho_1+\rho_2}^{X^c}(a_2)=\{u_2\}\not\supseteq\{u_1,u_2,u_3\}=\left({}^p\overline{\rho_1+\rho_2}^X(a_2)\right)^c$.

Remark 3.1. *If* $u\rho_1(e) \neq \emptyset$ *and* $u\rho_2(e) \neq \emptyset$ *for all* $a \in A$ *and* $u \in U$, *then* ${}_p\underline{\rho_1 + \rho_2}^{X^c} = \left({}^p\overline{\rho_1 + \rho_2}^X\right)^c$ *and* ${}^p\overline{\rho_1 + \rho_2}^{X^c} = \left({}_p\underline{\rho_1 + \rho_2}^X\right)^c$

Proposition 3.2. *Let* $(\rho_1, A), (\rho_2, A)$ *be two soft binary relations from universe* U *to* V *and* $X \subseteq U$. *Then*

(1) ${}^X\underline{\rho_1 + \rho_2}_p(a) \subseteq {}^X\overline{\rho_1 + \rho_2}^p(a)$

(2) ${}^{X^c}\underline{\rho_1 + \rho_2}_p(a) \subseteq \left({}^X\overline{\xi_1 + \rho_2}^p(a)\right)^c$,

(3) ${}^{X^c}\overline{\rho_1 + \rho_2}^p(a) \subseteq \left({}^X\underline{\rho_1 + \rho_2}^p(a)\right)^c$.

Proof. The proof is similar to the proof of Proposition 3.1.

Remark 3.2. *If* $\rho_1(e)v \neq \emptyset$ *and* $\rho_2(e)v \neq \emptyset$ *for all* $a \in A$ *and* $v \in V$, *then* ${}^{X^c}\underline{\rho_1 + \rho_2}_p = \left({}^X\overline{\rho_1 + \rho_2}^p\right)^c$ *and* ${}^{X^c}\overline{\rho_1 + \rho_2}^p = \left({}^X\underline{\rho_1 + \rho_2}_p\right)^c$

Proposition 3.3. *Let* $(\rho_1, A), (\rho_2, A)$ *be two soft binary relations from universe* U *to* V *and* $X \subseteq V$. *Then the following properties hold.*

(1) ${}_p\underline{\rho_1 + \rho_2}^X(a) = \underline{\rho_1}^X(a) \cap \underline{\rho_2}^X(a)$

(2) ${}^p\overline{\rho_1 + \rho_2}^X(a) = \overline{\rho_1}^X(a) \cup \overline{\rho_2}^X(a)$

(3) ${}_p\underline{\rho_1 + \rho_2}^V(a) \subseteq V$ *and* ${}^p\overline{\rho_1 + \rho_2}^V(a) \subseteq V$

(4) ${}_p\underline{\rho_1 + \rho_2}^\emptyset(a) = \emptyset$ *and* ${}^p\overline{\rho_1 + \rho_2}^\emptyset(a) = \emptyset$

(5) ${}_p\underline{\rho_1 + \rho_2}^X(a) = {}_p\underline{\rho_2 + \rho_1}^X(a)$ *and* ${}^p\overline{\rho_1 + \rho_2}^X(a) = {}^p\overline{\rho_2 + \rho_1}^X(a)$

Proof.

(1) *Let* $u \in {}_p\underline{\rho_1 + \rho_2}^X(a)$. *Then* $\emptyset \neq u\rho_1(a) \subseteq X$ *and* $\emptyset \neq u\rho_2(a) \subseteq X$. *These imply that* $u \in \underline{\rho_1}^X(a)$ *and* $u \in \underline{\rho_2}^X(a)$. *That is* $u \in \underline{\rho_1}^X(a) \cap \underline{\rho_2}^X(a)$. *Thus* ${}_p\underline{\rho_1 + \rho_2}^X(a) \subseteq \underline{\rho_1}^X(a) \cap \underline{\rho_2}^X(a)$.
Conversely, let $u \in \underline{\rho_1}^X(a) \cap \underline{\rho_2}^X(a)$. *Then* $u \in \underline{\rho_1}^X(a)$ *and* $u \in \underline{\rho_2}^X(a)$. *Thus* $\emptyset \neq u\rho_1(a) \subseteq X$ *and* $\emptyset \neq u\rho_2(a) \subseteq X$. *These imply that* $u \in {}_p\underline{\rho_1 + \rho_2}^X(a)$. *Thus* $\underline{\rho_1}^X(a) \cap \underline{\rho_2}^X(a) \subseteq {}_p\underline{\rho_1 + \rho_2}^X(a)$.
Hence ${}_p\underline{\rho_1 + \rho_2}^X(a) = \underline{\rho_1}^X(a) \cap \underline{\rho_2}^X(a)$.

(2) *Let* $u \in {}^p\overline{\rho_1 + \rho_2}^X(a)$. *Then* $u\rho_1(a) \cap X \neq \emptyset$ *or* $u\rho_2(a) \cap X \neq \emptyset$. *These imply that* $u \in \overline{\rho_1}^X(a)$ *or* $u \in \overline{\rho_2}^X(a)$, *that is* $u \in (\overline{\rho_1}^X(a) \cup \overline{\rho_2}^X(a))$. *Thus* ${}^p\overline{\rho_1 + \rho_2}^X(a) \subseteq \overline{\rho_1}^X(a) \cup \overline{\rho_2}^X(a)$.
Conversely, let $u \in \overline{\rho_1}^X(a) \cup \overline{\rho_2}^X(a)$. *Then* $u \in \overline{\rho_1}^X(a)$ *or* $u \in \overline{\rho_2}^X(a)$, *that is* $u\rho_1(a) \cap X \neq \emptyset$ *or* $u\rho_2(a) \cap X \neq \emptyset$. *These imply that* $u \in {}^p\overline{\rho_1 + \rho_2}^X(a)$. *Thus* $\overline{\rho_1}^X(a) \cup \overline{\rho_2}^X(a) \subseteq {}^p\overline{\rho_1 + \rho_2}^X(a)$,
Hence ${}^p\overline{\rho_1 + \rho_2}^X(a) = \overline{\rho_1}^X(a) \cup \overline{\rho_2}^X(a)$.

(3) (a) *By definition* ${}_p\underline{\rho_1 + \rho_2}^V(a) = \{u \in U \mid \emptyset \neq u\rho_1(a) \subseteq V$ *and* $\emptyset \neq u\rho_2(a) \subseteq V\} \subseteq U$.
(b) *By definition* ${}^p\overline{\rho_1 + \rho_2}^V(a) = \{u \in U \mid u\rho_1(a) \cap V \neq \emptyset$ *or* $u\rho_2(a) \cap V \neq \emptyset\} \subseteq U$.

(4) (a) *By definition* ${}_p\underline{\rho_1 + \rho_2}^\emptyset(a) = \{u \in U \mid \emptyset \neq u\rho_1(a) \subseteq \emptyset$ *and* $\emptyset \neq u\rho_2(a) \subseteq \emptyset\} = \emptyset$.
(b) *On contrary suppose that* ${}^p\overline{\rho_1 + \rho_2}^\emptyset(a) \neq \emptyset$. *Then there exists* $u \in U$ *such that* $u \in {}^p\overline{\rho_1 + \rho_2}^\emptyset(a)$. *Thus* $u\rho_1(a) \cap \emptyset \neq \emptyset$ *or* $u\rho_2(a) \cap \emptyset \neq \emptyset$ *which is a contradiction.*
Hence ${}^p\overline{\rho_1 + \rho_2}^\emptyset(a) = \emptyset$.

(5) (a) *Obvious.* (b) *Obvious.*

Corollary 3.1. (1) ${}_p\underline{\rho_1 + \rho_2}^V(e) = U$ *if* $u\rho_1(a) \neq \emptyset$ *and* $u\rho_2(a) \neq \emptyset$ *for all* $a \in A$ *and for all* $u \in U$.

(2) ${}^{p}\overline{\rho_1 + \rho_2}^{V}(e) = U$ *if* $u\rho_1(a) \neq \emptyset$ *or* $u\rho_2(a) \neq \emptyset$, *for all* $a \in A$ *and for all* $u \in U$.

Proposition 3.4. *Let* $(\rho_1, A), (\rho_2, A)$ *be two soft binary relations from universe* U *to* V *and* $X \subseteq U$. *Then the following properties hold.*

(1) ${}^{X}\underline{\rho_1 + \rho_2}_{p}(a) = {}^{X}\underline{\rho_1}(a) \cap {}^{X}\underline{\rho_2}(a)$,
(2) ${}^{X}\overline{\rho_1 + \rho_2}^{p}(a) = {}^{X}\overline{\rho_1}(a) \cup {}^{X}\overline{\rho_2}(a)$,
(3) ${}^{U}\underline{\rho_1 + \rho_2}_{p}(a) \subseteq V$ *and* ${}^{U}\overline{\rho_1 + \rho_2}^{p}(a) \subseteq V$
(4) ${}^{\emptyset}\underline{\rho_1 + \rho_2}_{p}(a) = \emptyset$ *and* ${}^{\emptyset}\overline{\rho_1 + \rho_2}_{p}(a) = \emptyset$
(5) ${}^{X}\underline{\rho_1 + \rho_2}_{p}(a) = {}^{X}\underline{\rho_2 + \rho_1}_{p}(a)$ *and* ${}^{X}\overline{\rho_1 + \rho_2}^{p}(a) = {}^{X}\overline{\rho_2 + \rho_1}^{p}(a)$.

Proof.
The proof is similar to the proof of Proposition. 3.3.

Corollary 3.2. (1) ${}^{U}\underline{\rho_1 + \rho_2}_{p}(a) = V$ *if* $\rho_1(a)v \neq \emptyset$ *and* $\rho_2(a)v \neq \emptyset$, *for all* $a \in A$ *and for* $v \in V$.
(2) ${}^{U}\overline{\rho_1 + \rho_2}(a) = V$ *if* $\rho_1(a)v \neq \emptyset$ *or* $\rho_2(a)v \neq \emptyset$, *for all* $a \in A$ *and for all* $v \in V$.

Proposition 3.5. *Let* $(\rho_1, A), (\rho_2, A)$ *be two soft binary relations from universe* U *to* V *and* $X_1 \subseteq V, X_2 \subseteq V$. *Then the following properties hold, for all* $a \in A$.

(1) ${}_{p}\underline{\rho_1 + \rho_2}^{X_1 \cap X_2}(a) = {}_{p}\underline{\rho_1 + \rho_2}^{X_1}(a) \cap {}_{p}\underline{\rho_1 + \rho}^{X_2}(a))$,
(2) ${}^{p}\overline{\rho_1 + \rho_2}^{X_1 \cup X_2}(a) = {}^{p}\overline{\rho_1 + \rho_2}^{X_1}(a) \cup {}^{p}\overline{\rho_1 + \rho_2}^{X_2}(a)$,
(3) $X_1 \subseteq X_2 \Rightarrow \quad {}_{p}\underline{\rho_1 + \rho_2}^{X_1}(a) \subseteq {}_{p}\underline{\rho_1 + \rho_2}^{X_2}(a)$,
(4) $X_1 \subseteq X_2 \Rightarrow \quad {}^{p}\overline{\rho_1 + \rho_2}^{X_1}(a) \subseteq {}^{p}\overline{\rho_1 + \rho_2}^{X_2}(a)$,
(5) ${}^{p}\overline{\rho_1 + \rho_2}^{X_1 \cap X_2}(a) \subseteq {}^{p}\overline{\rho_1 + \rho_2}^{X_1}(a) \cap {}^{p}\overline{\rho_1 + \rho_2}^{X_2}(a)$,
(6) ${}_{p}\underline{\rho_1 + \rho_2}^{X_1 \cup X_2}(a) \supseteq {}_{p}\underline{\rho_1 + \rho_2}^{X_1}(a) \cup {}_{p}\underline{\rho_1 + \rho_2}^{X_2}(a)$.

Proof.

(1) *By part (1) of Proposition 3.3, we have* ${}_{p}\underline{\rho_1 + \rho_2}^{X_1 \cap X_2}(a) = \underline{\rho_1}^{X_1 \cap X_2}(a) \cap \underline{\rho_2}^{X_1 \cap X_2}(a)$
$= \left(\underline{\rho_1}^{X_1}(a) \cap \underline{\rho_1}^{X_2}(a)\right) \cap \left(\underline{\xi_2}^{X_1}(a) \cap \underline{\rho_2}^{X_2}(a)\right) = \left(\underline{\rho_1}^{X_1}(a) \cap \underline{\xi_2}^{X_1}(a)\right) \cap \left(\underline{\rho_1}^{X_2}(a) \cap \underline{\rho_2}^{X_2}(a)\right)$
$= {}_{p}\underline{\rho_1 + \rho_2}^{X_1}(a) \cap {}_{p}\underline{\rho_1 + \rho_2}^{X_2}(a)$.
(2) *By part (2) of Proposition 3.3, we have* ${}^{p}\overline{\rho_1 + \rho_2}^{X_1 \cup X_2}(a) = \overline{\rho_1}^{X_1 \cup X_2}(a) \cup \overline{\rho_2}^{X_1 \cup X_2}(a)$
$= \left(\overline{\rho_1}^{X_1}(a) \cup \overline{\rho_1}^{X_2}(a)\right) \cup \left(\overline{\rho_2}^{X_1}(a) \cup \overline{\rho_2}^{X_2}(a)\right) = \left(\overline{\rho_1}^{X_1}(a) \cup \overline{\rho_2}^{X_1}(a)\right) \cup \left(\overline{\rho_1}^{X_2}(a) \cup \overline{\rho_2}^{X_2}(a)\right)$
$= {}^{p}\overline{\rho_1 + \rho_2}^{X_1} \cup {}^{p}\overline{\rho_1 + \rho_2}^{X_2}(a)$.
(3) *Let* $u \in {}_{p}\underline{\rho_1 + \rho_2}^{X_1}(a)$. *Then* $\emptyset \neq u\rho_1(a) \subseteq X_1$ *and* $\emptyset \neq u\rho_2(a) \subseteq X_1$.
Since $X_1 \subseteq X_2$ *so* $\emptyset \neq u\rho_1(a) \subseteq X_1 \subseteq X_2$ *and* $\emptyset \neq u\rho_2(a) \subseteq X_1 \subseteq X_2$.
Thus $u \in {}_{p}\underline{\rho_1 + \rho_2}^{X_2}(a)$. *Hence* ${}_{p}\underline{\rho_1 + \rho_2}^{X_1}(a) \subseteq {}_{p}\underline{\rho_1 + \rho_2}^{X_2}(a)$.
(4) *Let* $u \in {}^{p}\overline{\xi_1 + \rho_2}^{X_1}(a)$. *Then* $u\rho_1(a) \cap X_1 \neq \emptyset$ *or* $u\xi_2(a) \cap X_1 \neq \emptyset$.
Since $X_1 \subseteq X_2$ *so* $u\rho_1(a) \cap X_2 \neq \emptyset$ *or* $u\rho_2(a) \cap X_2 \neq \emptyset$. *Thus* $u \in {}^{p}\overline{\rho_1 + \rho_2}^{X_2}(a)$. *This implies that* ${}^{p}\overline{\xi_1 + \rho_2}^{X_1}(a) \subseteq {}^{p}\overline{\rho_1 + \rho_2}^{X_2}(a)$.
(5) *Since* $X_1 \supseteq X_1 \cap X_2, X_2 \supseteq X_1 \cap X_2$, *we have from (4)*
$\overline{\rho_1 + \rho_2}^{X_1 \cap X_2}(a) \subseteq \overline{\rho_1 + \xi_2}^{X_1}(a)$ *and* $\overline{\rho_1 + \rho_2}^{X_1 \cap X_2}(a) \subseteq \overline{\rho_1 + \rho_2}^{X_2}(a)$, *these imply that*
$\overline{\rho_1 + \rho_2}^{X_1 \cap X_2}(a) \subseteq \overline{\xi_1 + \rho_2}^{X_1}(a) \cap \underline{\rho_1 + \rho_2}^{X_1}(a)$.

(6) Since $X_1 \subseteq X_1 \cup X_2, X_2 \subseteq X_1 \cup X_2$, we have from (3)
$_p\underline{\rho_1+\rho_2}^{X_1\cup X_2}(a) \supseteq {_p\underline{\xi_1+\rho_2}^{X_1}}(a)$ and $_p\underline{\rho_1+\rho_2}^{X_1\cup X_2}(a) \supseteq {_p\underline{\rho_1+\rho_2}^{X_2}}(a)$, these imply that
$_p\underline{\rho_1+\rho_2}^{X_1\cup X_2}(a) \supseteq {_p\underline{\xi_1+\rho_2}^{X_1}}(a) \cup {_p\underline{\rho_1+\rho_2}^{X_1}}(a)$.

Proposition 3.6. *Let $(\rho_1, A), (\rho_2, A)$ be two soft binary relations from universe U to V and $X_1 \subseteq U, X_2 \subseteq U$. Then the following properties hold, for all $a \in A$.*

(1) ${}^{X_1\cap X_2}\underline{\rho_1+\rho_2}_p(a) = {}^{X_1}\underline{\rho_1+\rho_2}_p(a) \cap {}^{X_2}\underline{\xi_1+\rho}_p(a))$,
(2) ${}^{X_1\cup X_2}\overline{\rho_1+\rho_2}^p(a) = {}^{X_1}\overline{\xi_1+\rho_2}^p(a) \cup {}^{X_2}\overline{\rho_1+\rho_2}^p(a)$,
(3) $X_1 \subseteq X_2 \Rightarrow \quad {}^{X_1}\underline{\xi_1+\rho_2}_p(a) \subseteq {}^{X_2}\underline{\xi_1+\rho_2}_p(a)$,
(4) $X_1 \subseteq X_2 \Rightarrow \quad {}^{X_1}\overline{\rho_1+\xi_2}^p(a) \subseteq {}^{X_2}\overline{\rho_1+\rho_2}^p(a)$,
(5) ${}^{X_1\cap X_2}\overline{\xi_1+\rho_2}^p(a) \subseteq {}^{X_1}\overline{\rho_1+\rho_2}^p(a) \cap {}^{X_2}\overline{\xi_1+\rho_2}^p(a)$,
(6) ${}^{X_1\cup X_2}\underline{\rho_1+\rho_2}_p(a) \supseteq {}^{X_1}\underline{\rho_1+\xi_2}_p(a) \cup {}^{X_2}\underline{\rho_1+\rho_2}_p(a)$.

Proof. The proof is similar to the proof of Proposition 3.5.

Proposition 3.7. *Let $(\rho_1, A), (\rho_2, A)$ be two soft binary relations from non-empty universe U to V with $(\rho_1, A) \subseteq (\rho_2, A)$ and $X \subseteq V$. Then*

(1) $_p\underline{\rho_1+\rho_2}^X(a) = \underline{\rho_2}^X(a)$,
(2) $^p\overline{\rho_1+\rho_2}^X(a) = \overline{\rho_2}^X(a)$.

Proof. Suppose $(\rho_1, A) \subseteq (\rho_2, A)$, for any $u \in U, u\rho_1(a) \subseteq u\rho_2(a)$
for all $a \in A$ therefore, we have that

$$\underline{\rho_2}^X(a) \subseteq \underline{\rho_1}^X(a), \tag{3.1}$$

$$\overline{\rho_1}^X(a) \subseteq \overline{\rho_2}^X(a). \tag{3.2}$$

Thus

(1) $_p\underline{\rho_1+\rho_2}^X(a) = \underline{\rho_1}^X(a) \cap \underline{\rho_2}^X(a)$, that is $\;_p\underline{\rho_1+\rho_2}^X(a) = \underline{\rho_2}^X(a)$ by Eq (3.1).
(2) $^p\overline{\rho_1+\rho_2}^X(a) = \overline{\rho_1}^X(a) \cup \overline{\rho_2}^X(a)$, that is $\;^p\overline{\rho_1+\rho_2}^X(a) = \overline{\rho_2}^X(a)$ by Eq (3.2).

Proposition 3.8. *Let $(\rho_1, A), (\rho_2, A)$ be two soft binary relations from universe U to V, with $(\rho_1, A) \subseteq (\rho_2, A)$ and $X \subseteq U$. Then*

(1) ${}^X\underline{\rho_1+\rho_2}_p(a) = {}^X\underline{\rho_2}(a)$,
(2) ${}^X\overline{\rho_1+\rho_2}^p(a) = {}^X\overline{\rho_2}(a)$.

Proof. The proof is similar to the proof of Proposition 3.7.

Definition 3.3. *Let (ρ_1, A) and (ρ_2, A) be two soft binary relations from non-empty universal set U to V, and $X \subseteq V$. Then the accuracy measure with respect to afterset is defined by*

$$\alpha(\rho_1+\rho_2, X)(a) = \frac{|_p\underline{\rho_1+\rho_2}^X(a)|}{|^p\overline{\rho_1+\rho_2}^X(a)|}$$

Definition 3.4. *Let (ρ_1, A) and (ρ_2, A) be two soft binary relations from non-empty universal set U to V, and $X \subseteq U$. Then the accuracy measure with respect to foreset is defined by*

$$\alpha(X, \rho_1 + \rho_2)(a) = \frac{|{}^X\underline{\rho_1 + \rho_2}_p(a)|}{|{}^X\overline{\rho_1 + \rho_2}^p(a)|}.$$

Where |.| shows the cardinality. Its is obvious that $0 \leq \alpha(X, \rho_1 + \rho_2)(a) \leq 1$. When $\alpha(X, \rho_1 + \rho_2)(a) = 1$, then X is pessimistic multigranulation definable otherwise pessimistic multigranulation undefinable.

3.1. Pessimistic Roughness of a set over multi soft binary relations

In the previous section, we defined the multigranulation roughness of the set by two soft relations. In this subsection, we generalize the concept of the previous section and present the multigranulation roughness of the set by finite numbers of soft binary relations and discuss its basic properties.

Definition 3.5. *Let $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ be n soft relations from a non-empty universal set U to V, and $X \subseteq V$. Then we define two soft sets over U by*

$$\underline{{}_p\sum_{j=1}^{n}\rho_j}^X(a) = \{u \in U \mid \emptyset \neq u\rho_j(a) \subseteq X,\ for\ all\ j = 1, 2, \ldots n\}$$

$$\overline{{}^p\sum_{j=1}^{n}\rho_j}^X(a) = \{u \in U \mid u\rho_j(a) \cap X \neq \emptyset,\ for\ some\ j = 1, 2, \ldots, n\}$$

called the pessimistic lower and pessimistic upper approximations of X with respect to aftersets, we denote these soft sets by
$(\underline{{}_p\sum_{j=1}^{n}\rho_j}^X(a), A), (\overline{{}^p\sum_{j=1}^{n}\rho_j}^X(a), A)$ *respectively.*

Definition 3.6. *Let $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ be n soft relations from a non-empty universal set U to V, and $X \subseteq U$. Then we define two soft sets over V by*

$${}^X\underline{\sum_{j=1}^{n}\rho_j}_p(a) = \{v \in V \mid \emptyset \neq \rho_j(a)v \subseteq X,\ for\ all\ j = 1, 2, \ldots n\}$$

$${}^X\overline{\sum_{j=1}^{n}\rho_j}^p(a) = \{v \in V \mid \rho_j(a)v \cap X \neq \emptyset,\ for\ some\ j = 1, 2, \ldots, n\}$$

called the pessimistic lower and pessimistic upper approximations of X with respect to foresets, we denote these soft sets by
$({}^X\underline{\sum_{j=1}^{n}\rho_j}_p(a), A), ({}^X\overline{\sum_{j=1}^{n}\rho_j}^p(a), A)$ *respectively.*

Proposition 3.9. *Let $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ be n soft relations from a non-empty universal set U to V, and $X \subseteq V$. Then*

(1) $\underline{{}_p\sum_{j=1}^{n}\rho_j}^X(a) \subseteq \overline{{}^p\sum_{j=1}^{n}\rho_j}^X(a)$,

(2) ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X^c}(a) \subseteq \left({}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X}(a)\right)^c$,

(3) ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X^c}(a) \subseteq \left({}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X}(a)\right)^c$.

Proof. The proof is similar to the proof of Proposition 3.1.

Remark 3.3. *If $u\rho_j(a) \neq \emptyset$ for all $a \in A, u \in U$ and for all j. Then* ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X^c} = \left({}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X}\right)^c$ *and* ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X^c} = \left({}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X}\right)^c$

Proposition 3.10. *Let $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ be n soft relations from a non-empty universal set U to V, and $X \subseteq U$. Then*

(1) ${}^{X}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) \subseteq {}^{X}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a)$

(2) ${}^{X^c}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) \subseteq \left({}^{X}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a)\right)^c$,

(3) ${}^{X^c}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a) \subseteq \left({}^{X}\underline{\sum_{j=1}^{n}\rho_j}^{p}(a)\right)^c$.

Proof. The proof is similar to the proof of Proposition 3.1.

Remark 3.4. *If $\rho_j(a)v \neq \emptyset$, for all $a \in A, v \in V$ and for all j. Then* ${}^{X^c}\underline{\sum_{j=1}^{n}\rho_j}_{p} = \left({}^{X}\overline{\sum_{j=1}^{n}\rho_j}^{p}\right)^c$ *and* ${}^{X^c}\overline{\sum_{j=1}^{n}\rho_j}^{p} = \left({}^{X}\underline{\sum_{j=1}^{n}\rho_j}_{p}\right)^c$.

Proposition 3.11. *Let $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ be n soft relations from a non-empty universal set U to V, and $X \subseteq V$. Then the following properties hold.*

(1) ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X}(a) = \cap_{j=1}^{n}\underline{\rho_j}^{X}(a)$

(2) ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X}(a) = \cup_{j=1}^{n}\overline{\rho_j}^{X}(a)$

(3) ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{V}(a) \subseteq U$ *and* ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{V}(a) \subseteq U$

(4) ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{\emptyset}(a) = \emptyset$ *and* ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{\emptyset}(a) = \emptyset$.

Proof. The proof is similar to the proof of Proposition 3.3.

Corollary 3.3. (1) ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{V}(a) = U$ *if $u\rho_j(e) \neq \emptyset$, for all $a \in A$, for all $u \in U$ and for all j.*

(2) ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{V}(a) = U$ *if $u\rho_j(a) \neq \emptyset$ for all $a \in A$, for all $u \in U$ and for some j.*

Proposition 3.12. *Let $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ be n soft relations from a non-empty universal set U to V, and $X \subseteq U$. Then the following properties hold.*

(1) ${}^{X}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) = \cap_{j=1}^{n}{}^{X}\underline{\rho_j}(a)$,

(2) ${}^{X}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a) = \cup_{j=1}^{n}{}^{X}\overline{\rho_j}(a)$,

(3) ${}^{U}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) \subseteq V$ *and* ${}^{U}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a) \subseteq V$

(4) ${}^{\emptyset}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) = \emptyset$ *and* ${}^{\emptyset}\overline{\sum_{j=1}^{n}\rho_j}_{p}(a) = \emptyset$

(5) ${}^{X}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) = {}^{X}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a)$ and ${}^{X}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a) = {}^{X}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a)$.

Proof. *The proof is similar to the proof of Proposition 3.3.*

Corollary 3.4. *(1)* ${}^{U}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) = V$ *if* $\rho_j(a)v \neq \emptyset$, *for all* $a \in A$, *for* $v \in V$ *and for all* j.

(2) ${}^{U}\overline{\sum_{j=1}^{n}\rho_j}(a) = V$ *if* $\rho_1(a)v \neq \emptyset$ *for all* $a \in A$, *for all* $v \in V$, *and for some* j.

Proposition 3.13. *Let* $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ *be* n *soft relations from a non-empty universal set* U *to* V, *and* $X \subseteq V$. *Then the following properties hold, for all* $a \in A$.

(1) ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{\cap_{i=1}^{m}X_i}(a) = \cap_{i=1}^{m}{}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X_i}(a)$,
(2) ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{\cup_{i=1}^{m}X_i}(a) = \cup_{i=1}^{m}{}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X_i}(a)$,
(3) $X_1 \subseteq X_2 \Rightarrow \quad {}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X_1}(a) \subseteq {}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X_2}(a)$,
(4) $X_1 \subseteq X_2 \Rightarrow \quad {}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X_1}(a) \subseteq {}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X_2}(a)$,
(5) ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{\cap_{i=1}^{m}X_i}(a) \subseteq \cap_{i=1}^{m}{}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X_i}(a)$,
(6) ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{\cup X_i}(a) \supseteq \cup_{i=2}^{m}{}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X_i}(a)$.

Proof. *The proof is similar to the proof of Proposition 3.5.*

Proposition 3.14. *Let* $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ *be* n *soft relations from a non-empty universal set* U *to* V, *and* $X \subseteq U$. *Then the following properties hold, for all* $a \in A$.

(1) ${}^{\cap_{i=1}^{m}X_i}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) = \cap_{i=1}^{m}{}^{X_i}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a)$,
(2) ${}^{\cup_{i=1}^{m}X_i}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a) = \cup_{i=1}^{m}{}^{X_i}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a)$,
(3) $X_1 \subseteq X_2 \Rightarrow \quad {}^{X_1}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) \subseteq {}^{X_2}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a)$,
(4) $X_1 \subseteq X_2 \Rightarrow \quad {}^{X_1}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a) \subseteq {}^{X_2}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a)$,
(5) ${}^{\cap_{i=1}^{m}X_i}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a) \subseteq \cap_{i=1}^{m}{}^{X_i}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a)$,
(6) ${}^{\cup_{i=1}^{m}X_i}\underline{\rho_1 + \rho_2}_{p}(a) \supseteq \cup_{i=1}^{m}{}^{X_i}\underline{\rho_1 + \rho_2}_{p}(a)$.

Proof. *The proof is similar to that of Proposition 3.5.*

Proposition 3.15. *Let* $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ *be* n *soft relations from a non-empty universal set* U *to* V, *and* $X \subseteq V$, *with* $(\rho_1, A) \subseteq (\rho_2, A) \subseteq (\rho_3, A) \subseteq \ldots \subseteq (\rho_n, A)$ *Then*

(1) ${}_{p}\underline{\sum_{j=1}^{n}\rho_j}^{X}(a) = \underline{\rho_n}^{X}(a)$,
(2) ${}^{p}\overline{\sum_{j=1}^{n}\rho_j}^{X}(a) = \overline{\rho_n}^{X}(a)$.

Proof. *The proof is similar to the Proposition 3.7.*

Proposition 3.16. *Let* $(\rho_1, A), (\rho_2, A), (\rho_3, A) \ldots\ldots (\rho_n, A)$ *be* n *soft relations from a non-empty universal set* U *to* V, *and* $X \subseteq U$, *with* $(\rho_1, A) \subseteq (\rho_2, A) \subseteq (\rho_3, A) \subseteq \ldots \subseteq (\rho_n, A)$. *Then*

(1) ${}^{X}\underline{\sum_{j=1}^{n}\rho_j}_{p}(a) = {}^{X}\underline{\rho_n}(a)$,
(2) ${}^{X}\overline{\sum_{j=1}^{n}\rho_j}^{p}(a) = {}^{X}\overline{\rho_n}(a)$.

Proof. The proof is similar to that of Proposition 3.7.

Definition 3.7. *Let $(\rho_1, A), (\rho_2, A), (\rho_3, A), \cdots (\rho_n, A)$, be n soft binary relation from non-empty universal set U to V, and $X \subseteq V$. Then the accuracy measure with respect to afterset are defined by*

$$\alpha(\sum_{j=1}^{n} \rho_j, X)(a) = \frac{|\underline{{}_p\sum_{j=1}^{n} \rho_j}^X(a)|}{|\overline{{}^p\sum_{j=1}^{n} \rho_j}^X(a)|}.$$

Definition 3.8. *Let $(\rho_1, A), (\rho_2, A), (\rho_3, A), \cdots (\rho_n, A)$, be n soft binary relation from non-empty universal set U to V, and $X \subseteq U$. Then the accuracy measure with respect to foreset are defined by*

$$\alpha(X, \sum_{j=1}^{n} \rho_j)(a) = \frac{|\underline{{}^X\sum_{j=1}^{n} \rho_j}_p^X(a)|}{|\overline{{}^X\sum_{j=1}^{n} \rho_j}^p(a)|}.$$

4. Example

Example 4.1. *Suppose a University advertises the vacancies of lecturers for their two campuses U and V. The set $U = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8\}$, represent the candidate applied in campus U and the set $V = \{\pi'_1, \pi'_2, \pi'_3, \pi'_4, \pi'_5, \pi'_6, \pi'_7\}$ represent the candidate applied in campus V. Let two teams of experts compare them based on pure Mathematics and applied mathematics, a set of parameters $A = \{p = Pure\ Math, a = Applied\ Math\}$. From these comparisons, we have*

$\rho_1 : A \to P(U \times V)$*, represents the comparison of first-team of experts defined by.*

$$\begin{aligned}\rho_1(p) = &\{(\pi_1, \pi'_2), (\pi_1, \pi'_3), (\pi_2, \pi'_2), (\pi_2, \pi'_5), (\pi_3, \pi'_4), (\pi_3, \pi'_5), (\pi_4, \pi'_1), (\pi_4, \pi'_3), (\pi_5, \pi'_1), (\pi_5, \pi'_6), (\pi_7, \pi'_4), (\pi_7, \pi'_7)\},\\ \rho_1(a) = &\{(\pi_1, \pi'_3), (\pi_1, \pi'_6), (\pi_2, \pi'_1), (\pi_2, \pi'_4), (\pi_3, \pi'_1), (\pi_4, \pi'_5), (\pi_4, \pi'_7), (\pi_5, \pi'_2), (\pi_5, \pi'_7), (\pi_7, \pi'_3), (\pi_7, \pi'_6), (\pi_8, \pi'_1),\\ &(\pi_8, \pi'_7)\},\end{aligned}$$

$\rho_2 : A \to P(U \times V)$ *represents the comparison of second-team of experts defined by*

$$\begin{aligned}\rho_2(p) = &\{(\pi_1, \pi'_2), (\pi_2, \pi'_3), (\pi_2, \pi'_5), (\pi_3, \pi'_4), (\pi_4, \pi'_3), (\pi_4, \pi'_5), (\pi_4, \pi'_6), (\pi_5, \pi'_4), (\pi_6, \pi'_7), (\pi_7, \pi'_3), (\pi_7, \pi'_7)(\pi_8, \pi'_2),\\ &(\pi_8, \pi'_5)\},\\ \rho_2(a) = &\{(\pi_1, \pi'_3), (\pi_1, \pi'_4), (\pi_2, \pi'_3), (\pi_2, \pi'_4), (\pi_2, \pi'_7), (\pi_3, \pi'_1), (\pi_3, \pi'_6), (\pi_4, \pi'_2), (\pi_4, \pi'_4), (\pi_5, \pi'_2), (\pi_6, \pi'_5), (\pi_7, \pi'_6),\\ &(\pi_8, \pi'_1), (\pi_8, \pi'_3)\}.\end{aligned}$$

We get two SBrs from U to $V fromthesecomparisons$. Now the aftersets are

$$\begin{array}{llll} \pi_1\rho_1(p) = \{\pi'_2, \pi'_3\}, & \pi_1\rho_1(a) = \{\pi'_3, \pi'_6\}, & \pi_1\rho_2(p) = \{\pi'_2\}, & \pi_1\rho_2(a) = \{\pi'_3, \pi'_4\} \\ \pi_2\rho_1(p) = \{\pi'_2, \pi'_5\}, & \pi_2\rho_1(a) = \{\pi'_1, \pi'_4\}, & \pi_2\rho_2(p) = \{\pi'_3, \pi'_5\}, & \pi_2\rho_2(a) = \{\pi'_3, \pi'_4, \pi'_7\} \\ \pi_3\rho_1(p) = \{\pi'_4, \pi'_5\}, & \pi_3\rho_1(a) = \{\pi'_1\}, & \pi_3\rho_2(p) = \{\pi'_4\}, & \pi_3\rho_2(a) = \{\pi'_1, \pi'_6\} \\ \pi_4\rho_1(p) = \{\pi'_1, \pi'_3\}, & \pi_4\rho_1(a) = \{\pi'_5, \pi'_7\}, & \pi_4\rho_2(p) = \{\pi'_3, \pi'_5, \pi'_6\}, & \pi_4\rho_2(a) = \{\pi'_2, \pi'_4\} \end{array}$$

$$\begin{array}{llll}
\pi_5\rho_1(p) = \{\pi'_1, \pi'_6\}, & \pi_5\rho_1(a) = \{\pi'_2, \pi'_7\}, & \pi_5\rho_2(p) = \{\pi'_4\}, & \pi_5\rho_2(a) = \{\pi'_2\} \\
\pi_6\rho_1(p) = \emptyset, & \pi_6\rho_1(a) = \emptyset, & \pi_6\rho_2(p) = \{\pi'_7\}, & \pi_6\rho_2(a) = \{\pi'_5\} \\
\pi_7\rho_1(p) = \{\pi'_4, \pi'_7\}, & \pi_7\rho_1(a) = \{\pi'_3, \pi'_6\}, & \pi_7\rho_2(p) = \{\pi'_3, \pi'_7\}, & \pi_7\rho_2(a) = \{\pi'_6\} \\
\pi_8\rho_1(p) = \emptyset, & \pi_8\rho_1(a) = \{\pi'_1, \pi'_7\}, & \pi_8\rho_2(p) = \{\pi'_2, \pi'_5\}, & \pi_8\rho_2(a) = \{\pi'_1\}.
\end{array}$$

And foresets are

$$\begin{array}{llll}
\rho_1(p)\pi'_1 = \{\pi_4, \pi_5\}, & \rho_1(a)\pi'_1 = \{\pi_2, \pi_3, \pi_8\}, & \rho_2(p)\pi'_1 = \emptyset, & \rho_2(a)\pi'_1 = \{\pi_3, \pi_8\} \\
\rho_1(p)\pi'_2 = \{\pi_1, \pi_2\}, & \rho_1(a)\pi'_2 = \{\pi_5\}, & \rho_2(p)\pi'_2 = \{\pi_8\}, & \rho_2(a)\pi'_2 = \{\pi_4, \pi_5\} \\
\rho_1(p)\pi'_3 = \{\pi_1, \pi_4\}, & \rho_1(a)\pi'_3 = \{\pi_7\}, & \rho_2(p)\pi'_3 = \{\pi_2, \pi_4, \pi_7\}, & \rho_2(a)\pi'_3 = \{\pi_1, \pi_2\} \\
\rho_1(p)\pi'_4 = \{\pi_7\}, & \rho_1(a)\pi'_4 = \{\pi_2\}, & \rho_2(p)\pi'_4 = \{\pi_3, \pi_5\}, & \rho_2(a)\pi'_4 = \{\pi_1, \pi_4\} \\
\rho_1(p)\pi'_5 = \{\pi_2, \pi_3\}, & \rho_1(a)\pi'_5 = \{\pi_4\}, & \rho_2(p)\pi'_5 = \{\pi_2, \pi_4, \pi_8\}, & \rho_2(a)\pi'_5 = \{\pi_6\} \\
\rho_1(p)\pi'_6 = \{\pi_5\}, & \rho_1(a)\pi'_6 = \{\pi_1, \pi_7\}, & \rho_2(p)\pi'_6 = \{\pi_4\}, & \rho_2(a)\pi'_6 = \{\pi_3, \pi_7\} \\
\rho_1(p)\pi'_7 = \{\pi_7\}, & \rho_1(a)\pi'_7 = \{\pi_4, \pi_5, \pi_8\}, & \rho_2(p)\pi'_7 = \{\pi_6, \pi_7\}, & \rho_2(a)\pi'_7 = \{\pi_2\}.
\end{array}$$

If the candidates $X = \{\pi'_2, \pi'_3, \pi'_6, \pi'_8\} \subset V$ are recommended for selection in campus V then who will be recommended for selection in campus U?

Now we categorize the candidates who applied on campus U.

$$\begin{array}{ll}
{}_{p}\underline{\rho_1 + \rho_2}^{X}(p) = \{\pi_1\}, & {}^{p}\overline{\rho_1 + \rho_2}^{X}(p) = \{\pi_1, \pi_2, \pi_4, \pi_5, \pi_7, \pi_8\}, \\
{}_{p}\underline{\rho_1 + \rho_2}^{X}(a) = \{\pi_7\} & {}^{p}\overline{\rho_1 + \rho_2}^{X}(a) = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_7\}.
\end{array}$$

In this case, the lower approximation contains all those candidates who applied to campus U recommended by both teams of experts, and the upper approximation contains all those candidates who applied to campus U recommended by one of the teams of experts.

If the candidates $X = \{\pi_1, \pi_2, \pi_4, \pi_7\} \subset U$ recommended for selection in campus U, then who will be recommended for selection in campus V?

Now we categorize the candidates who applied in campus V.

$$\begin{array}{ll}
{}^{X}\underline{\rho_1 + \rho_2}_{p}(p) = \{\pi'_3\}, & {}^{X}\overline{\rho_1 + \rho_2}^{p}(p) = \{\pi'_1, \pi'_2, \pi'_3, \pi'_4, \pi'_5, \pi'_6, \pi'_7\}, \\
{}^{X}\underline{\rho_1 + \rho_2}_{p}(a) = \{\pi'_3, \pi'_4\} & {}^{X}\overline{\rho_1 + \rho_2}^{p}(a) = \{\pi'_1, \pi'_2, \pi'_3, \pi'_4, \pi'_5, \pi'_6, \pi'_7\}.
\end{array}$$

In this case, the lower approximation contains all those candidates who applied to campus V recommended by both teams of experts, and the upper approximation contains all those candidates who applied to campus V recommended by one of the teams of experts.

5. Comparason

In this section, we will analyze comparatively the effectiveness of our method. To deal with incompleteness and vagueness, an MGRS model is proposed in terms of equivalence relations by

Qain et al. [19], which is better than RS. To make the equivalence relation more flexible, the conditions had to be relaxed; Shabir et al. [22] presented the MGRS of a crisp set based on soft binary relations and its application in data classification, Ayub et al. [3] introduced SMGRS which is the particular case of MGRS [22]. Here, we have a new hybrid model APMRS by using multi-soft binary relations. Let us suppose we illustrate the example 4.1. If the candidates $X = \{\pi_1, \pi_2, \pi_4, \pi_7\} \subset U$ in campus U recommended for selection, then who will be recommended for selection in campus V? Now we categorize the candidate who applies in campus V through Shabir et al. [22] with respect to aftersets. ${}^X\underline{\rho_1 + \rho_2}(P) = \{\pi'_2, \pi'_3, \pi'_4, \pi'_6, \pi'_7\}$ and ${}^X\overline{\rho_1 + \rho_2}(P) = \{\pi'_3, \pi'_4, \pi'_5, \pi'_7\}$. That is ${}^X\underline{\rho_1 + \rho_2} \nsubseteq {}^X\overline{\rho_1 + \rho_2}$ and ${}^X\overline{\rho_1 + \rho_2} \nsubseteq {}^X\underline{\rho_1 + \rho_2}$ that is [22] is not parallel with Pawlak Rough set and Qian et al. MGRS. However, our model satisfies this property. Secondly, when ${}^X\underline{\rho_1 + \rho_2} \nsubseteq {}^X\overline{\rho_1 + \rho_2}$ and, ${}^X\overline{\rho_1 + \rho_2} \nsubseteq {}^X\underline{\rho_1 + \rho_2}$, then we are unable to defined accuracy measure, but in our model, we defined accuracy measure. SMGRS [3] is the particular case of [22]; the same issue occurs in [3], which also occurs in [22]. That is why our proposed model is more effective.

6. Conclusions

This article studies a novel pessimistic multigranulation roughness of set over two universes. Firstly, we present the roughness of a set by using the aftersets and foresets of two soft binary relations over dual universes and their essential properties. From this roughness, we got two new soft sets. Then we extended this definition and defined the roughness of a set by using the aftersets and foresets of the finite number of soft binary relations over dual universes. We also investigated some algebraic properties and an example in data classifications to illustrate our proposed pessimistic multigranulation rough set model. In the future, we can apply this model to classify the data in many practical fields like medical sciences, classification of the data of different diseases, classification of economics data, management science, and social sciences data.

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Conflict of interest

The authors declare no conflict of interest.

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