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Research article

Clustering property for quantum Markov chains on the comb graph

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Abstract: Quantum Markov chains (QMCs) on graphs and trees were investigated in connection with many important models arising from quantum statistical mechanics and quantum information. These quantum states generate many important properties such as quantum phase transition and clustering properties. In the present paper, we propose a construction of QMCs associated with an XX-Ising model over the comb graph $\mathbb{N} \rhd_0 \mathbb{Z}$. Mainly, we prove that the QMC associated with the disordered phase, enjoys a clustering property.

Keywords: quantum Markov chain; Ising model; comb graph; clustering property
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1. Introduction

Over the past few decades, quantum Markov chains (QMCs) (see [1,2,7,8,18,21]) have undergone great development through the vast volume of relevant scientific literature in different research areas such as computational physics [42], interacting particle systems [10], quantum spin models [19,20,24], quantum information [15,41], quantum cryptography [21].

In view of the absence of a satisfactory theory of quantum Markovian fields on general graphs, it is quite natural to restrict the study to special class of graphs for which the considered problem is more solvable. Thanks to their hierarchical simplified structure consist a natural candidate. Quantum Markov chains on graphs [3,9,23,44] are multi-dimensional extensions of 1D QMCs. Namely, QMCs on the Cayley trees (CT) have been investigated in connection with quantum phase transitions for Pauli models [4–6, 25, 28–30]. This consists a quantum extension of an increasing number of works on classical Gibbs measures [31, 32]. Types of von Neumann factors associated with QMCs based on Ising and XY type models have been investigated [27, 33]. Moreover, clustering property have been showed for QMCs on the Cayley trees [26]. The structure of quantum Markov states (QMS) on CT have been studied in details [34–37]. However, an extension of Fermi Markov states [17] to trees is still missing.

Recently, the notions of stopping rules and recurrence for QMCs on trees have been introduced [45]. In [13], a bridge between recurrence and phase transition for QMCs on CT have been established. In [39, 40, 46] QMCs on CT were associated with open quantum random walks [11, 14] extending the 1D case [16].

QMCs on Trees have been investigated in connection with interesting phenomena such as quantum phase transition [28], quantum walks [39] and clustering properties [38]. Namely, the clustering analysis [22] play a key role in several areas such as data science [47], image segmentation [43].

In this paper, we show a clustering property for QMCs associated with an XX-Ising model on the comb graph $\mathbb{N} \rhd_o \mathbb{Z}$. Unlike the CT the underlying comb graph is non-homogeneous. Therefore, we construct QMCs based on two different types of hamiltonian. We prove the uniqueness of QMCs associated with the considered model. Notice that the present work extends the clustering property for QMCs on CT [26] to non-homogeneous trees. Moreover, it generalizes a previous work [38] that deals with QMCs on the comb $\mathbb{N} \rhd_o \mathbb{N}$.

The paper is organized as follows: Section 2, is devoted to some preliminaries. In Section 3, we provide a construction of QMCs on the comb graph. In Section 4, we investigate QMCs associated with XX-Ising type model on the comb graph. Moreover, we prove the uniqueness of QMCs associated with the considered model. Section 5 is devoted to the main result which concerns the clustering property of the considered QMC.

2. Preliminaries

Let $G = (L, E)$ be a connected, locally finite, infinite graph, here L denotes the set of vertices and E denotes the vertex set . Each edge $l \in E$ is identified to non ordered pair of vertices (its endpoints) $l =< u, v >=< v, u >$ and E is then identifiable to a part of $L \times L$.

$$E \subset \{\{u, v\} : u, v \in L\}.$$

Let us recall some basic notions on graph theory:

(i) We call nearest-neighbors vertices u and v, and we denote by $u \sim v$, if $l =< u, v >\in E$.

(ii) A path on the graph is a finite list of vertices $u \sim u_1 \sim \cdots \sim u_{d-1} \sim v$.

(iii) The distance $d(u, v), u, v \in L$, on the graph, is defined to be the length of the shortest path joining u to v.

Let $\mathcal{G}^{(1)} = (L^{(1)}, E^{(1)})$ and $\mathcal{G}^{(2)} = (L^{(2)}, E^{(2)})$ be two graphs where $L^{(1)} = \mathbb{N}$ and $L^{(2)} = \mathbb{Z}$ with distinguished vertex $0 \in L^{(2)}$. Let

$$L = \mathbb{N} \times Z,$$

and

$$E=\{\{(u,v),(u',v')\},\quad (u,v),(u',v')\in L,(u,v)\sim(u',v')\},$$

where $(u,v)\sim(u',v')$ if and only if one of the following assertions is satisfied:

(i) $u\sim u'$ and $v=v'=0$.

(ii) $u=u'$ and $v\sim v'$.

The graph $\mathcal{G}=(L,E)$ is the comb product of $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$ with a distinguished vertex $o\in\mathbb{Z}$ denoted by $\mathbb{N}\rhd_0\mathbb{Z}$.

Let us fix a root $o=(0,0)$. Define the sets

$$W_m=\{u\in L: d(u,o)=m\},\quad \Lambda_m=\bigcup_{k=0}^{m}W_k. \tag{2.1}$$

For $u\in L$, define the set of its direct successors

$$\mathcal{S}(u)=\{v\in\Lambda_m^c: u\sim v\}. \tag{2.2}$$

One can see that elements of W_m are on the form

$$u=(k,l),\quad k+|l|=m,$$

where $k=\{0,\dots,m\}$ and $l=\{-m,\dots,m\}$. It follows the enumeration

$$u_{W_m}^{(-m)}=(0,-m),\dots,u_{W_m}^{(0)}=(m,0),\dots,u_{W_m}^{(m)}=(0,m). \tag{2.3}$$

The graph under consideration (see Figure 1) is a tree. There are two types of vertices according to the number of nearest-neighbors (or also the number of direct successors). We distinguish vertices with three direct successors and others with only one.

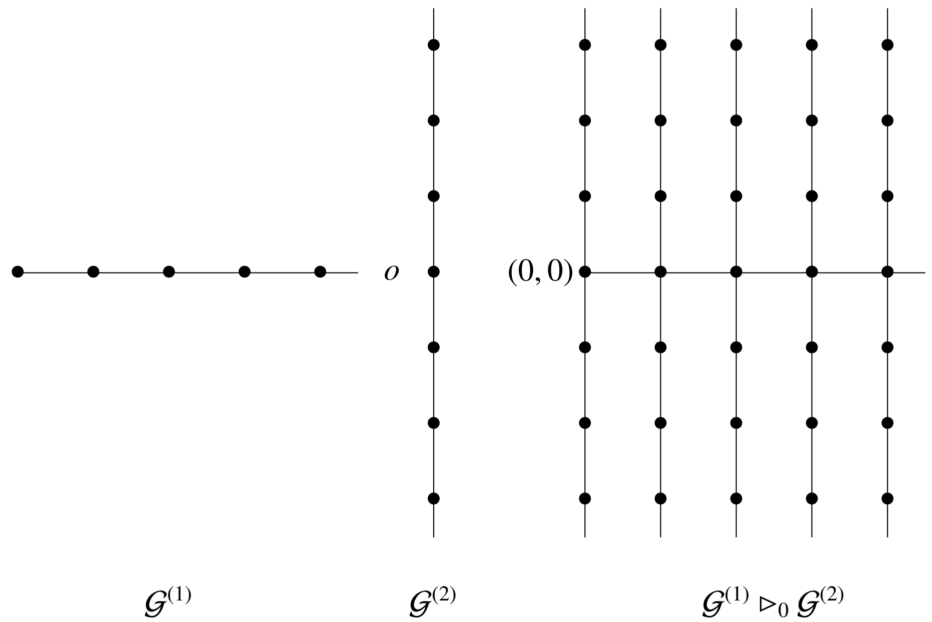

Figure 1. Comb graph: $\mathbb{N}\rhd_0\mathbb{Z}$.

Define

$$L_1 = \{u_1 \in L \,:\, |\mathcal{S}(u_1)| = 1\}, \tag{2.4}$$

and

$$L_3 = \{u_3 \in L \,:\, |\mathcal{S}(u_3)| = 3\}. \tag{2.5}$$

It is clear that $L = L_1 \cup L_3$. Each element $u_3 \in L_3$ has the form $u_3 = (k, 0)$ with $k \in \mathbb{N}$. Its set of direct successors is

$$\mathcal{S}(u_3) = \{u_3 + e_1,\, u_3 \pm e_2\}\,, \tag{2.6}$$

whereas, elements of L_1 have the form $v_1 = (k, l)$ where $l \geq 1$ and

$$\mathcal{S}(v_1) = \begin{cases} \{v_1 + e_2\}, & \text{if } l > 0, \\ \\ \{v_1 - e_2\}, & \text{if } l < 0, \end{cases} \tag{2.7}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Let us define the restriction of the usual addition of the commutative group $\mathbb{Z}^2$ on $comb^+$ as follows: for any two elements $u = (k, l)$ and $v = (k', l')$

$$u \circ v = (k + k', l + l'). \tag{2.8}$$

For these notations, one has

$$u \circ u^{(0)} = u^{(0)} \circ u = u.$$

Operation (2.8) induces on $comb^+$ a semi-group of translation with unit $u^{(0)}$. With the above structure of semi-group, we define the translations $\tau_g : comb^+ \to comb^+$, $g \in L_3$ as follows

$$\tau_g(u) = g + u, \tag{2.9}$$

and $\tau_{(0)} = id$. Let

$$\Lambda_m = \left\{u \in L \,:\, d(u, u^{(0)}) = m\right\}.$$

Thanks to the tree structure, one has

$$\Lambda_{m+1} = \bigcup_{u \in \Lambda_m} \mathcal{S}(u) \quad \text{and} \quad \mathcal{S}(u) \cap S(v) = \emptyset, \quad \forall u \neq v. \tag{2.10}$$

3. Construction of QMCs on the comb $\mathbb{N} \rhd_0 \mathbb{Z}$

Let $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ be three unitary C*–algebras. A completely positive identity preserving linear map E: $\mathcal{A} \to \mathcal{B}$ satisfies

$$E(ca) = cE(a), \quad a \in \mathcal{A}, c \in \mathcal{C}, \tag{3.1}$$

is called *Quasi-conditional expectation* (QCE).

Remark 3.1. Any $K \in \mathcal{A}$, satisfies

$$E_0(K^*K) = \mathrm{I} \tag{3.2}$$

is called E_0-conditional amplitude. where E_0: $\mathcal{A} \to \mathcal{B}$ is a conditional expectation [1]. For any $\mathcal{B} \subseteq \mathcal{A}$ sub–∗–algebra, denote by

$$\mathcal{B}' := \{x \in \mathcal{A} \ : \ xy = yx \ , \ \forall y \in \mathcal{B}\},$$

the commutant of $\mathcal{B}$ in $\mathcal{A}$, If $K \in C'$ then $E_0(K^*(\cdot)K) : \mathcal{A} \to \mathcal{B}$ is a (normalized) QCE w.r.t. the triplet $C \subseteq \mathcal{B} \subseteq \mathcal{A}$.

Remark 3.2. Every NQCE w.r.t. $C \subseteq \mathcal{B} \subseteq \mathcal{A}$ satisfies

$$E(ac) = E(a)c, \qquad a \in \mathcal{A}, c \in C, \tag{3.3}$$

$$E(C' \cap \mathcal{A}) \subseteq C' \cap \mathcal{B}. \tag{3.4}$$

To each vertex $u \in V$ an algebra of observable $\mathcal{B}_u \equiv \mathcal{B}(\mathcal{H}_u)$ is associated, where $\mathcal{H}_u$ is a finite dimensional Hilbert space. Consider the quasi-local algebra

$$\mathcal{B}_L := \bigotimes_{u \in L} \mathcal{B}_u,$$

which is obtained as inductive limit of the net

$$\mathcal{B}_\Lambda := \bigotimes_{u \in \Lambda} \mathcal{B}_u \otimes \mathrm{I}_{\Lambda^c}, \quad \Lambda \subset L, \ |\Lambda| < \infty.$$

where for each $\Lambda' \subset L$, we denote $\mathrm{I}_{\Lambda'}$ the identity of $\mathcal{B}_{\Lambda'}$. See [12] for a systematic study of quasi-local algebras.

Remark 3.3. Starting from any QCE $E_{\Lambda_{m]}}$: $\mathcal{B}_{\Lambda_{m+1]}} \to \mathcal{B}_{\Lambda_{m]}}$ with respect to $\mathcal{B}_{\Lambda_{n-1]}} \subseteq \mathcal{B}_{\Lambda_{n]}} \subseteq \mathcal{B}_{\Lambda_{n+1]}}$ one can derive a transition expectation (TE) from $\mathcal{B}_{\Lambda_{[n,n+1]}}$ into $\mathcal{B}_{\Lambda_n}$ by the next restriction

$$\mathcal{E}_{\Lambda_{[m,m+1]}} : \mathcal{B}_{\Lambda_m} := E_{\Lambda_{m]}}\Big|_{\mathcal{B}_{[m,m+1]}}.$$

Conversely, every TE $\mathcal{E}_{\Lambda_{[m,m+1]}}$: $\mathcal{B}_{\Lambda_{[m,m+1]}} \to \mathcal{B}_{\Lambda_m}$ is extendable to a QCE $E_{\Lambda_{n]}}$ w.r.t. the given triplet in the following way

$$E_{\Lambda_{m]}} := id_{\mathcal{B}_{m-1]}} \otimes \mathcal{E}_{\Lambda_{[m,m+1]}}. \tag{3.5}$$

The reader is referred to [7] for further details about the extendability of transition expectations in a generalized framework, including both the tensor and the Fermi cases.

Definition 3.4. A backward quantum Markov chain (b-QMC) on the algebra $\mathcal{B}_L$ is a triplet $(\rho_0, (\mathcal{E}_{[m,m+1]},), (h_m))$ of positive linear functional ρ_0 on $\mathcal{B}_{(u_0)}$, a sequence of TE $\mathcal{E}_{[m,m+1]} : \mathcal{B}_{\Lambda_{[m,m+1]}} \to \mathcal{B}_{\Lambda_m}$ and a sequence $h_m \in \mathcal{B}^+_{\Lambda_m}$ such that for each $a \in \mathcal{B}_L$, the limit

$$\varphi(a) := \lim_{m \to +\infty} \rho_0(E_{\Lambda_{0]}}(E_{\Lambda_{1]}}(\cdots(E_{\Lambda_{m]}}(a \otimes h_{m+1}))\cdots))) \tag{3.6}$$

exists for the weak-∗-topology and it defines a state φ on the full algebra $\mathcal{B}_L$. In this case the limit state φ is also called QMC. The sequence (h_m) is called sequence of boundary conditions of the QMC.

Thanks to (3.5), one can immediately check that, φ evaluated on the element $a = a_0 \otimes a_1 \otimes \cdots \otimes a_m \in \mathcal{B}_{\Lambda_{m]}}$, $a_j \in \mathcal{B}_{\Lambda_j}$, is provited by the correlations

$$\varphi(a) = \rho_0 \left(\mathcal{E}_{\Lambda_{[0,1]}} \left(a_{\Lambda_0} \otimes \mathcal{E}_{\Lambda_{[1,2]}} \left(a_1 \otimes \cdots \otimes \mathcal{E}_{\Lambda_{[m,m+1]}} (a_m \otimes h_{m+1}) \cdots \right)\right)\right),$$

which highlight the quantum Markov structure.

We are going to construct a state on $\mathcal{B}_{\Lambda_{m]}}$ with initial state $\omega_0 \in \mathcal{B}_{(0),+}$ and boundary conditions $\{h_u \in \mathcal{B}_{u,+} : \ u \in L\}$.

For every $n \in \mathbb{N}$, denote

$$K_{u\vee\mathcal{S}(u)} := \prod_{v\in\mathcal{S}(u)} K_{<u,v>}, \tag{3.7}$$

$$K_{[m,m+1]} := \prod_{u\in\overrightarrow{\Lambda}_m} K_{u\cup S(u)}, \quad 1 \leq m \leq n, \tag{3.8}$$

$$h_n^{1/2} := \prod_{u\in\overrightarrow{\Lambda}_n} h_u^{1/2}, \quad h_n = h_n^{1/2}(h_n^{1/2})^*, \tag{3.9}$$

$$K_n := \omega_0^{1/2} \prod_{m=1}^{n-1} K_{[m,m+1]} h_n^{1/2}, \tag{3.10}$$

$$W_{n]} := K_n^* K_n. \tag{3.11}$$

One can see that $W_{n]}$ is positives.

In the sequel, $\mathrm{Tr}_{n]}$: $\mathcal{B}_L \to \mathcal{B}_{\Lambda_n}$ denotes the (normalized) partial trace i.e. $\mathrm{Tr}_{n]}(\mathrm{I}_{\Lambda_n}) = \mathrm{I}_{\Lambda_n}$, here $\mathrm{I}_{\Lambda_n} = \bigotimes_{u\in\Lambda_n} \mathrm{I}_u$), for any finite part Λ_n.

Let's set a positive functional $\varphi^{(n,b)}_{w_0,h}$ on $\mathcal{B}_{\Lambda_n}$ by

$$\varphi^{(n,b)}_{w_0,h}(a) = \mathrm{Tr}(\mathcal{W}_{n+1]}(a \otimes \mathrm{I}_{W_{n+1}})), \tag{3.12}$$

for each $a \in B_{\Lambda_n}$. Note that, the trace Tr is normalized (i.e. $\mathrm{Tr}(\mathrm{I}_u) = 1$).

To obtain a state $\varphi^{(b)}$ on $\mathcal{B}_L$ satisfying

$$\varphi^{(b)}\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi^{(n,b)}_{w_0,h},$$

we must impose some constrains on the boundary conditions $\{w_0, h\}$ so that the positive functionals $\{\varphi^{(n,b)}_{w_0,h}\}$ satisfy the following compatibility condition, i.e.

$$\varphi^{(n+1,b)}_{w_0,h}\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi^{(n,b)}_{w_0,h}. \tag{3.13}$$

Theorem 3.5. Let $w_0 \in \mathcal{B}_{(0),+}$ and $h = \{h_u \in \mathcal{B}_{u,+}\}_{u\in L}$. If

$$\mathrm{Tr}(\omega_0 h_0) = 1, \tag{3.14}$$

$$\mathrm{Tr}_{u]}\left(K^*_{u\vee S(u)} \mathrm{I}^{(u)} \otimes h^{S(u)} K_{u\vee S(u)}\right) = h^{(u)}, \ \forall u \in L. \tag{3.15}$$

Then the sequebce $\{\varphi^{(m,b)}_{w_0,h}\}$ satisfy condition (3.13). Moreover, there exists a unique b-QMC $\varphi^{(b)}_{w_0,h}$ on $\mathcal{B}_L$ such that

$$\varphi^{(b)}_{w_0,h} = w - \lim_{m\to\infty} \varphi^{(m,b)}_{w_0,h}.$$

Remark 3.6. *Theorem 3.5 extends results of [25, 26, 28, 29, 34] where only considered Bethe lattice or Cayley tree. The first attempt to investigate QMCs on the comb graph was done in [38] by considering* $\mathbb{N} \rhd_0 \mathbb{N}$.

4. QMC associated with XX-Ising type model on the comb graph

In this section, we define the model and study the b-QMC φ associated to the XX-Ising model on the Comb graph $\mathbb{N} \rhd \mathbb{Z}$. Let $\mathcal{B}_u = M_2(\mathbb{C})$, for all $u \in L$. The Pauli spin operators $\sigma_x, \sigma_y, \sigma_z$ are given by

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The shift of an element $a \in M_2(\mathbb{C})$ to the u^{th} component of the infinite tensor product $\mathcal{B}_L = \bigotimes_{x\in L} \mathcal{B}_x$ will be denoted by

$$a^{(u)} := \tau_u(a).$$

Define the nearest neighbors interactions: for each $u_1 \in L_1$, $v \in S(u_1)$,

$$K_{<u_1,v>} := cos(\beta)\mathrm{I}^{(u_1)} \otimes \mathrm{I}^{S(u_1)} - i\sin(\beta)\sigma_x^{(u_1)} \otimes \sigma_x^{S(u_1)}, \ \ \beta > 0, \tag{4.1}$$

and for $u_3 \in L_3, w \in S(u_3)$,

$$K_{<u_3,w>} = \exp\{\beta H_{<u_3,w>}\}, \quad \beta > 0, \tag{4.2}$$

where

$$H_{<u_3,w>} = \frac{1}{2}\left(\mathrm{I}^{(u_3)} \otimes \mathrm{I}^{(w)} + \sigma_z^{(u_3)} \otimes \sigma_z^{(w)}\right). \tag{4.3}$$

A simple calculation leads to

$$K_{<u_3,w>} = K_0\mathrm{I}^{(u_3)} \otimes \mathrm{I}^{(w)} + K_3\sigma_z^{(u_3)} \otimes \sigma_z^{(w)},$$

where

$$K_0 = \frac{\exp(J_0\beta) + 1}{2}, \quad K_3 = \frac{\exp(J_0\beta) - 1}{2}, \ J_0 > 0.$$

One finds: for $u_1 \in L_1$ and $v \in S(u_1)$

$$K_{u_1 \vee S(u_1)} = K_{<u_1,v>} = cos(\beta)\mathrm{I}^{(u_1)} \otimes \mathrm{I}^{(v)} - i\sin(\beta)\sigma_x^{(u_1)} \otimes \sigma_x^{(v)}, \ \ \beta > 0. \tag{4.4}$$

And for $v \in L_3$ (its successors $S(v) = \{v + e_1, v \pm e_2\}$) one finds,

$$\begin{aligned}
K_{v\vee S(v)} &= K_{<v,v+e_1>}K_{<v,v+e_2>}K_{<v,v-e_2>} \\
&= K_0^3\mathrm{I}^{(v)} \otimes \mathrm{I}^{(v+e_1)} \otimes \mathrm{I}^{(v+e_2)} \otimes \mathrm{I}^{(v-e_2)} + K_0^2K_3\sigma_z^{(v)} \otimes \mathrm{I}^{(v+e_1)} \otimes \mathrm{I}^{(v+e_2)} \otimes \sigma_z^{(v-e_2)} \\
&+K_0^2K_3\sigma_z^{(v)} \otimes \mathrm{I}^{(v+e_1)} \otimes \sigma_z^{(v+e_2)} \otimes \mathrm{I}^{(v-e_2)} + K_0K_3^2\mathrm{I}^{(v)} \otimes \mathrm{I}^{(v+e_1)} \otimes \sigma_z^{(v+e_2)} \otimes \sigma_z^{(v-e_2)} \\
&+K_3K_0^2\sigma_z^{(v)} \otimes \sigma_z^{(v+e_1)} \otimes \mathrm{I}^{(v+e_2)} \otimes \mathrm{I}^{(v-e_2)} + K_0K_3^2\mathrm{I}^{(v)} \otimes \sigma_z^{(v+e_1)} \otimes \mathrm{I}^{(v+e_2)} \otimes \sigma_z^{(v-e_2)} \\
&+K_0K_3^2\mathrm{I}^{(v)} \otimes \sigma_z^{(v+e_1)} \otimes \sigma_z^{(v+e_2)} \otimes \mathrm{I}^{(v-e_2)} + K_3^3\sigma_z^{(v)} \otimes \sigma_z^{(v+e_1)} \otimes \sigma_z^{(v+e_2)} \otimes \sigma_z^{(v-e_2)}.
\end{aligned}$$

Recall that, a net $\{h^u\}$ is *translation-invariant* if

$$h^{(u)} = h^{\tau_g(u)}, \quad \forall u, g \in L.$$

This means that

$$h^{(u)} = h^{(v)}, \quad \forall u, v \in L. \tag{4.5}$$

In what follows, we consider only translation-invariant solutions of (3.14), (3.15). Put $h^{(u)} = h$ for all $u \in L$, where

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Theorem 4.1. For the XX-Ising model (4.1), (4.2) there exists a unique b-QMC φ_α with translation-invariant boundary condition h_α satisfying (3.14). Moreover, for each $a \in \mathcal{B}_{\Lambda_{n]}}$ one has

$$\varphi_\alpha^{(b)}(a) = \alpha^{2n+1}\mathrm{Tr}\left(\prod_{j=0}^{n}\prod_{u\in\Lambda_j} K^*_{\{u\}\vee S(u)} a \prod_{j=0}^{n}\prod_{u\in\Lambda_j} K_{\{u\}\vee S(u)}\right). \tag{4.6}$$

Proof. Let $u_1 \in L_1$ and v its unique successor ($S(u_1) = \{v\}$), then (3.15) is reduced to

$$\begin{aligned} h^{(u_1)} &= \mathrm{Tr}_{u_1]}(K^*_{u_1\vee S(u_1)}\mathrm{I}^{(u_1)} \otimes h^{(v)} K_{u_1\vee S(u_1)}) \\ &= \mathrm{Tr}_{u_1]}(\cos^2(\beta)\mathrm{I}^{(u_1)} \otimes h^{(v)} + \sin^2(\beta)\mathrm{I}^{(u_1)} \otimes \sigma_x h^{(v)}\sigma_x \\ &+ i\sin\beta\cos(\beta)(\sigma_x^{(u_1)} \otimes h^{(v)}\sigma_x - \sigma_x^{(u_1)} \otimes \sigma_x h^{(v)})). \end{aligned}$$

One can check that

$$\mathrm{Tr}(\sigma_x h^{(v)}\sigma_x) = \mathrm{Tr}(h^{(v)}) \quad \text{and} \quad \mathrm{Tr}(h^{(v)}\sigma_x) = \mathrm{Tr}(\sigma_x h^{(v)}).$$

Then, we find that

$$h^{(u_1)} = \mathrm{Tr}(h^{(v)})\,\mathrm{I}^{(u_1)}. \tag{4.7}$$

Now for $u_3 \in L_3$ according to the above computation (3.15) becomes

$$h^{(u_3)} = \mathrm{Tr}_{u_3]}(K^*_{\{u_3\vee S(u_3)\}}\mathrm{I}^{(u_3)} \otimes h^{S(u_3)} K_{\{u_3\vee S(u_3)\}}).$$

Since the boundary condition satisfy (4.5), according to (4.7) we have

$$h^{(u_3)} = h^{(u_1)} = h = \mathrm{Tr}(h^{(v)})\mathrm{I} = \alpha\mathrm{I},$$

for some $\alpha > 0$, then (3.15) is reduced to

$$\begin{aligned} h &= \mathrm{Tr}_{u_3]}(K^*_{\{u_3\vee S(u_3)\}}\mathrm{I}^{(u_3)} \otimes h^{S(u_3)} K_{\{u_3\vee S(u_3)\}}) \\ &= \alpha^3\mathrm{Tr}_{u_3]}(K^*_{\{u_3\vee S(u_3)\}} K_{\{u_3\vee S(u_3)\}}) \\ &= \alpha^3(K_0^2 + K_3^2)^3\mathrm{I}. \end{aligned}$$

Therefore, $\alpha = \alpha^3(K_0^2 + K_3^2)^3$ and this is equivalent to

$$\alpha = \frac{1}{(K_0^2 + K_3^2)^{3/2}} = \frac{2^{3/2}}{(e^{2J_0\beta} + 1)^{3/2}}.$$

Hence,

$$h = \alpha \mathrm{I} \tag{4.8}$$

is the unique commune solution of (3.15). The initial state ω_0 can be chosen $\omega_0 = \frac{1}{\alpha}\mathrm{I}$.

From (3.12) one has

$$\begin{aligned}\varphi_\alpha^{(b)}(a) &= \mathrm{Tr}(W_{n]}a) = \mathrm{Tr}(\omega_0 K_{n]}^* a h_n K_{n]}) \\ &= \alpha^{|\Lambda_n|-1}\mathrm{Tr}\left(\prod_{j=0}^{n}\prod_{u\in\Lambda_j} K^*_{\{u\}\vee S(u)} a \prod_{j=0}^{n}\prod_{u\in\Lambda_j} K_{\{u\}\vee S(u)}\right).\end{aligned}$$

Since $|\Lambda_n| = 2n+1$ one gets (4.6). This finishes the proof. □

Remark 4.2. *The QMC φ_α given in Theorem 4.1 is the state associated with the disordered phase of the underlying quantum. This state always exists for some fixed point reasons. Therefore, the existence of phase transition requires a t least one additional state satisfying conditions of under some conditions (see [28, 39]. Notice that, if the boundary condition is non-homogeneous, phenomena of phase transitions may appear even for the above considered model.*

5. Clustering property

A state φ on $\mathcal{B}_L$ is said to enjoy the *clustering property* if for every $a, b \in \mathcal{B}_L$ one has

$$\lim_{|g|\to\infty} \varphi(a\tau_g(b)) = \varphi(a)\varphi(b). \tag{5.1}$$

From Theorem 4.1, there is a unique b-QMC $\varphi_\alpha^{(b)}$ with translation-invariant boundary condition h_α satisfying (3.14). Now, let establish clustering property for this B-QMC $\varphi_\alpha^{(b)}$.

First, Let denote for each $n \in \mathbb{N}^*$

$$\Lambda_n = \{u_{\Lambda_n}^{(-n)}, \cdots, u_{\Lambda_n}^{(-1)}, u_{\Lambda_n}^{(0)}, u_{\Lambda_n}^{(1)}, \cdots, u_{\Lambda_n}^{(n)}\} : \varphi_\alpha^{(b)}\lceil_{\mathcal{B}_{\Lambda_n}} =: \varphi_\alpha^{(n,b)},$$

such that $u_{\Lambda_n}^{(0)} \in L_3$ and $u_{\Lambda_n}^{(i)} \in L_1$, $i = \pm1, \pm2, \cdots \pm n$.

Moreover, to prove Theorem 5.2, we need the following:

Lemma 5.1. Let $u_{m_0} \in \mathcal{B}_{\Lambda_{m_0}}$, for a certain integer m_0, and $f_n \in \mathcal{B}_{\Lambda_n}$ of the form

$$f_n = f \otimes \mathrm{I}_{\Lambda_n \setminus \{u_{\Lambda_n}^{(0)}\}},$$

where $f = f^{(u_{\Lambda_n}^{(0)})}$. Then one has

$$\lim_{n\to\infty} \varphi_\alpha^{(b)}(u_{m_0} \otimes f_n) = \varphi_\alpha^{(b)}(u_{m_0})\varphi_\alpha^{(b)}(f). \tag{5.2}$$

Proof. For $n \geq m_0$, one has

$$\begin{aligned}\varphi_\alpha^{(b)}(u_{m_0} \otimes f) &= \varphi_\alpha^{(b,n)}(u_{m_0} \otimes f) \\ &= \mathrm{Tr}(\omega_0 \mathcal{E}_0 \circ \mathcal{E}_1 \circ \ldots \circ \mathcal{E}_{M_0}(u_{m_0} \otimes \mathcal{E}_{m_0+1}(\mathrm{I}_{\Lambda_{m_0+1}} \otimes \cdots \otimes \mathcal{E}_{n-1}(\mathrm{I}_{\Lambda_{n-1}} \otimes \hat{\mathcal{E}}_n(f \otimes \mathrm{I}_{\Lambda_{n+1}}))\cdots)),\end{aligned}$$

here, as before, $\{w_0 = \frac{1}{\alpha}, h_0 = \alpha \mathrm{I}\}$ is the fixed point of the system with $\alpha = \frac{2^{3/2}}{(e^{2J_0\beta}+1)^{3/2}}$. One finds

$$\begin{aligned}
\hat{\mathcal{E}}_n(f \otimes \mathrm{I}_{\Lambda_{n+1}}) &= \mathrm{Tr}_{n]}(K_{[n,n+1]} h_{n+1}^{1/2} f \otimes \mathrm{I}_{\Lambda_{n+1}} h_{n+1}^{1/2} K_{[n,n+1]}^*) \\
&= \mathrm{Tr}_{n]}(\bigotimes_{u\in\Lambda_n} K_{u\vee S(u)} h_{n+1}^{1/2} f \otimes \mathrm{I}_{\Lambda_{n+1}} h_{n+1}^{1/2} \bigotimes_{u\in\Lambda_n} K_{u\vee S(u)}^*) \\
&= \mathrm{Tr}_{u_{\Lambda_n}^{(0)}]}(K_{u_{\Lambda_n}^{(0)}\vee S(u_{\Lambda_n}^{(0)})} f \otimes h^{(u)} \otimes h^{(u)} \otimes h^{(u)} K_{u_{\Lambda_n}^{(0)}\vee S(u_{\Lambda_n}^{(0)})}^*) \otimes \bigotimes_{u\in\Lambda_n\setminus\{u_{\Lambda_n}^{(0)}\}} \mathrm{Tr}_{u]}(K_{u\vee S(u)} \mathrm{I} \otimes h^{(u)} K_{u\vee S(u)}^*) \\
&= \alpha^3 \Big((K_0^6 + 3K_0^2K_3^4) f + (K_3^6 + 3K_0^4K_3^2)\sigma_z f \sigma_z\Big) \otimes \bigotimes_{x\in\Lambda_n\setminus\{x_{\Lambda_n}^{(0)}\}} h^{(x)} \\
&= \alpha^3 \Big((K_0^6 + 3K_0^2K_3^4) f + (K_3^6 + 3K_0^4K_3^2)\sigma_z f \sigma_z\Big) \otimes \bigotimes_{u\in\Lambda_n\setminus\{u_{\Lambda_n}^{(0)}\}} h^{(u)} \\
&= \alpha^3 g_{u_{\Lambda_n}^{(0)}} \otimes \bigotimes_{u\in\Lambda_n\setminus\{u_{\Lambda_n}^{(0)}\}} h^{(u)},
\end{aligned}$$

where,

$$g_{u_{\Lambda_n}^{(0)}} = (K_0^6 + 3K_0^2K_3^4) f + (K_3^6 + 3K_0^4K_3^2)\sigma_z f \sigma_z.$$

Hence,

$$\begin{aligned}
\mathcal{E}_{n-1}(\mathrm{I}_{\Lambda_{n-1}} \otimes \hat{\mathcal{E}}_n(f \otimes \mathrm{I}_{\Lambda_{n+1}})) &= \alpha^3 \mathrm{Tr}_{u_{\Lambda_{n-1}}^{(0)}]}(K_{u_{\Lambda_{n-1}}^{(0)}\vee S(u_{\Lambda_{n-1}}^{(0)})} \mathrm{I}_{u_{\Lambda_{n-1}}^{(0)}} \otimes g_{u_{\Lambda_n}^{(0)}} \otimes h^{(u)} \otimes h^{(u)} K_{u_{\Lambda_{n-1}}^{(0)}\vee S(u_{\Lambda_n}^{(0)})}^*) \\
&\quad \otimes \bigotimes_{u\in\Lambda_{n-1}\setminus\{u_{\Lambda_{n-1}}^{(0)}\}} \mathrm{Tr}_{u]}(K_{u\vee S(u)} \mathrm{I} \otimes h^{(u)} K_{u\vee S(u)}^*) \\
&= \alpha^3 \Big(\mathrm{Tr}(g)\mathrm{I} + 2\alpha^{2/3} K_0 K_3 \mathrm{Tr}(\sigma_z g)\sigma_z\Big) \otimes \bigotimes_{u\in\Lambda_{n-1}\setminus\{u_{\Lambda_{n-1}}^{(0)}\}} h^{(u)} \\
&= \alpha^3 \mathrm{Tr}(g) \mathrm{I}^{u_{\Lambda_{n-1}}^{(0)}} \otimes \bigotimes_{u\in\Lambda_{n-1}\setminus\{u_{\Lambda_{n-1}}^{(0)}\}} h^{(u)} \quad + 2\alpha^3\alpha^{2/3} K_0 K_3 \mathrm{Tr}(\sigma_z g) \sigma_z^{(u_{\Lambda_{n-1}}^{(0)})} \\
&\quad \otimes \bigotimes_{u\in\Lambda_{n-1}\setminus\{u_{\Lambda_{n-1}}^{(0)}\}} h^{(u)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{E}_{n-2}(\mathrm{I}_{\Lambda_{n-2}} \otimes \mathcal{E}_{n-1}(\mathrm{I}_{\Lambda_{n-1}} \otimes \hat{\mathcal{E}}_n(f \otimes \mathrm{I}_{\Lambda_{n+1}}))) = \alpha^3 \mathrm{Tr}(g) \mathrm{I}_{u_{\Lambda_{n-2}}^{(0)}} \otimes \bigotimes_{u\in\Lambda_{n-1}\setminus\{u_{\Lambda_{n-2}}^{(0)}\}} h^{(u)} \\
&+ 2^2\alpha^3\alpha^{2/3+2/3} K_0^2 K_3^2 \mathrm{Tr}(\sigma_z g) \sigma_z^{(u_{\Lambda_{n-2}}^{(0)})} \otimes \bigotimes_{u\in\Lambda_{n-1}\setminus\{u_{\Lambda_{n-2}}^{(0)}\}} h^{(u)}.
\end{aligned}$$

Now iterating $n - m_0 - 1$ times, we find

$$\begin{aligned}
&\mathcal{E}_{m_0+1}(\mathcal{E}_{m_0+2} \cdots \mathcal{E}_{n-1}(\mathrm{I}_{\Lambda_{n-1}} \otimes \hat{\mathcal{E}}_n(f \otimes \mathrm{I}_{\Lambda_{n+1}})) \cdots)) \\
&= \alpha^3 2^{n-m_0-1} \mathrm{Tr}(g\sigma_z) \alpha^{\frac{2(n-m_0-1)}{3}} K_0^{n-m_0-1} K_3^{n-m_0-1} \sigma_z^{(u_{\Lambda_{m_0+1}}^{(0)})} \otimes \bigotimes_{u\in\Lambda_{n-1}\setminus\{u_{\Lambda_{m_0+1}}^{(0)}\}} h^{(u)} \\
&+ \mathrm{Tr}(g) \alpha^3 \mathrm{I}_{u_{\Lambda_{m_0+1}}^{(0)}} \otimes \bigotimes_{u\in\Lambda_{n-1}\setminus\{u_{\Lambda_{m_0+1}}^{(0)}\}} h^{(u)}.
\end{aligned}$$

Hence, one get

$$\begin{aligned}\varphi_\alpha^{(b)}(u_{m_0}\otimes f) &= \mathrm{Tr}(\omega_0\mathcal{E}_0\circ\mathcal{E}_1\circ\cdots\circ\mathcal{E}_{m_0}(u_{m_0}\otimes \mathrm{I}))\mathrm{Tr}(g)\alpha^2\alpha^{|\Lambda_{m_0+1}|}\\ &\quad +\mathrm{Tr}(\omega_0\mathcal{E}_0\circ\mathcal{E}_1\circ\cdots\circ\mathcal{E}_{m_0}(u_{m_0}\otimes\sigma_z^{(x^{(1)}_{\Lambda_{m_0+1}})}))\mathrm{Tr}(g\sigma_z)\\ &\quad \alpha^2\alpha^{\frac{2(n-m_0-1)}{3}}\alpha^{|\Lambda_{m_0+1}|}2^{n-m_0-1}K_0^{n-m_0}K_3^{n-m_0-1}\\ &= \mathrm{Tr}(\omega_0\mathcal{E}_0\circ\mathcal{E}_1\circ\cdots\circ\hat{\mathcal{E}}_{m_0}(u_{m_0}\otimes \mathrm{I}))\mathrm{Tr}(f)\\ &\quad +\mathrm{Tr}(\omega_0\mathcal{E}_0\circ\mathcal{E}_1\circ\cdots\circ\mathcal{E}_{m_0}(u_{m_0}\otimes\sigma_z^{(x^{(1)}_{\Lambda_{m_0+1}})}))\mathrm{Tr}(g\sigma_z)\\ &\quad \alpha^{(2+\frac{2(n-m_0-1)}{3}+|\Lambda_{m_0+1}|)}2^{n-m_0-1}K_0^{n-m_0}K_3^{n-m_0-1}.\end{aligned}$$

One can see that,

$$2^n(K_0K_3)^n\alpha^{\frac{2n}{3}}=\left(\frac{e^{2J_0\beta}-1}{e^{2J_0\beta}+1}\right)^n.$$

Therefore, by taking the limit $n\to\infty$, we obtain,

$$\begin{aligned}\lim_{n\to\infty}\varphi_\alpha^{(b)}(u_{m_0}\otimes f_n) &= \mathrm{Tr}(\omega_0\mathcal{E}_0\circ\mathcal{E}_1\circ\cdots\circ\hat{\mathcal{E}}_{m_0}(a_{m_0}\otimes \mathrm{I}))\mathrm{Tr}(f)\\ &= \varphi_\alpha^{(b)}(u_{m_0})\varphi_\alpha^{(b)}(f).\end{aligned}$$

Thus, this completes the proof. □

Now we are ready to satate the main result of this paper.

Theorem 5.2. *Let $\varphi_\alpha^{(b)}$ be the b-QMC associated with the XX-Ising model on the comb graph $\mathbb{N}\rhd_0\mathbb{Z}$. Then for each $g\in\mathcal{G}^+$*

$$\lim_{|g|\to+\infty}\varphi_\alpha^{(b)}(a\tau_g(f))=\varphi_\alpha^{(b)}(a)\varphi_\alpha^{(b)}(f), \tag{5.3}$$

for all $a,f\in\mathcal{B}_L$.

Proof. Let $a,f\in\mathcal{B}_{L,loc}$, then $a,f\in\mathcal{B}_{\Lambda_{[0,l_0]}}$ for a certain integer l_0.

Then, let denote

$$u_{l_0}:=a\in\mathcal{B}_{\Lambda_{[0,l_0]}}\quad\text{and}\quad f_{l_0}:=f\in\mathcal{B}_{\Lambda_{[0,l_0]}}.$$

f_{l_0} can be rewritten in the following form

$$f_{l_0}=\bigotimes_{u\in\Lambda_{[0,l_0]}}f_u=\bigotimes_{k=0}^{l_0}f_{\Lambda_k},\quad\text{with}\quad f_{\Lambda_k}=\bigotimes_{u\in\Lambda_k}f_u\in\mathcal{B}_{\Lambda_k}.$$

Furthermore, one can see that

$$\tau_{g_m}(f_{l_0})=\bigotimes_{u\in\Lambda_{[0,l_0]}}f_u^{(u+me_1)}=\bigotimes_{v\in\Lambda_{[m,m+l_0]}}\tilde{f}_v\in\mathcal{B}_{\Lambda_{[m,m+l_0]}},$$

where

$$\tilde{f}_v=\begin{cases}f_{v-me_1}, & \text{if } v-me_1\in\Lambda_{[0,l_0]},\\ 1, & \text{otherwise.}\end{cases}$$

For $k \in [0, l_0]$, $g \in \mathcal{G}^+$ and $b \in \mathcal{B}_{\Lambda_{[k,k+1]}}$ we denote

$$\mathcal{E}^{(\tau_g)}_{[k,k+1]}(\tau_g(b)) := \bigotimes_{v\in\tau_g(\Lambda_k)} \mathrm{Tr}_{v]}\left(K_{\{v\}\vee S(v)}\tau_g(b)K^*_{\{v\}\vee S(v)}\right), \tag{5.4}$$

the τ_g-shift of the transition expectation $\mathcal{E}_{[k,k+1]}$, in fact one can check that

$$\mathcal{E}^{(\tau_g)}_{[k,k+1]}(\tau_g(b_{l_0})) = \tau_g(\mathcal{E}_{[k,k+1]}(b_{l_0})).$$

In light of (5.4), one finds

$$\begin{aligned}\hat{\mathcal{E}}_{[m+l_0,m+l_0+1]}(\tilde{f}_{\Lambda_{m+l_0}}))) &= \left(\bigotimes_{v\in\tau_{gm}(\Lambda_{l_0})} \mathrm{Tr}_{v]}\left(K_{\{v\}\vee S(v)}f_v h_{S(v)}K^*_{\{v\}\vee S(v)}\right)\right)\otimes\left(\bigotimes_{v\in\Lambda_{m+l_0}\setminus\tau_{gm}(\Lambda_{l_0})} h_v\right)\\ &= \alpha^{|\Lambda_{l_0+1}|}\mathcal{E}^{(\tau_{gm})}_{[l_0,l_0+1]}(\tau_{g_m}(f_{\Lambda_{l_0}}))\otimes\left(\bigotimes_{v\in\Lambda_{m+l_0}\setminus\tau_{gn}(\Lambda_{m_0})} h_v\right).\end{aligned}$$

The comb graph $\mathbb{Z} \rhd_0 \mathbb{N}$ satisfies

$$\tau_{g_m}(\Lambda_{k+1}) = \bigcup_{v\in\tau_{gm}(\Lambda_k)} S(v).$$

Therefore,

$$\begin{aligned}&\mathcal{E}_{[m+l_0-1,m+l_0]}(\tilde{f}_{\Lambda_{m+l_0-1}}\otimes\hat{\mathcal{E}}_{[m+l_0,m+l_0+1]}(\tilde{f}_{\Lambda_{m+l_0}})))) \\ &= \alpha^{|\Lambda_{l_0+1}|}\bigotimes_{u\in\tau_{gm}(\Lambda_{l_0-1})}\mathrm{Tr}_{u]}\left(K_{\{u\}\vee S(u)}\left(\tilde{f}_u\otimes\mathcal{E}^{(\tau_{gm})}_{[l_0-1,l_0]}(\tau_{g_m}(f_{\Lambda_{l_0}}))\right)K_{\{u\}\vee S(u)}\right)\otimes\bigotimes_{w\in\Lambda_{m+l_0-1}\setminus\tau_{gm}(\Lambda_{l_0-1})} h_w,\\ &= \alpha^{|\Lambda_{l_0+1}|}\mathcal{E}^{(\tau_{gm})}_{[l_0-1,l_0]}\left(\tau_{g_m}(f_{\Lambda_{l_0-1}})\otimes\mathcal{E}^{(\tau_{gm})}_{[l_0-1,l_0]}\left(\tau_{g_m}(f_{\Lambda_{l_0}})\right)\right)\otimes\bigotimes_{w\in\Lambda_{m+l_0-1}\setminus\tau_{gm}(\Lambda_{l_0-1})} h_w.\end{aligned}$$

An iterative process leads to

$$\begin{aligned}&\mathcal{E}_{[m,m+1]}(\tilde{f}_{\Lambda_m}\otimes\cdots\mathcal{E}_{[m+l_0-1,m+l_0]}(\tilde{f}_{\Lambda_{m+l_0-1}}\otimes\hat{\mathcal{E}}_{[m+l_0,m+l_0]}(\tilde{f}_{m+l_0})))\\ &= \alpha^{|\Lambda_{l_0+1}|}\mathcal{E}^{(\tau_{gm})}_{[0,1]}\left(\tau_{g_m}(f_{\Lambda_0})\otimes\cdots\mathcal{E}^{(\tau_{gm})}_{[l_0-1,l_0]}\left(\tau_{g_m}(f_{\Lambda_{l_0-1}})\otimes\mathcal{E}^{(\tau_{gm})}_{[l_0-1,l_0]}\left(\tau_{g_m}(f_{\Lambda_{l_0}})\right)\right)\right)\otimes\bigotimes_{w\in\Lambda_m\setminus\tau_{gm}(\Lambda_0)} h_w.\end{aligned}$$

Let denote

$$\hat{f}_o := \mathcal{E}_{[0,1]}\left(f_{\Lambda_0}\otimes\cdots\mathcal{E}_{[l_0-1,N_0]}\left(f_{\Lambda_{l_0-1}}\otimes\mathcal{E}_{[l_0-1,l_0]}\left(f_{\Lambda_{l_0}}\otimes h_{l_0+1}\right)\right)\right)\in\mathcal{B}_o.$$

Since $\tau_{g_m}(\Lambda_0) = \{u^{(0)}_{\Lambda_m}\}$ and $h^w = \alpha\mathrm{I}$, for each $w \in L$, one gets

$$\mathcal{E}_{[m,m+1]}(\tilde{f}_{\Lambda_m}\otimes\cdots\mathcal{E}_{[m+N_0-1,m+N_0]}(\tilde{f}_{\Lambda_{m+N_0-1}}\otimes\hat{\mathcal{E}}_{[m+N_0,m+N_0]}(\tilde{f}_{m+N_0}))) = \hat{f}_o^{(u^{(0)}_{\Lambda_m})}\otimes\bigotimes_{w\in\Lambda_m\setminus\{u^{(0)}_{\Lambda_m}\}} h_w.$$

This leads to

$$\varphi^{(b)}_\alpha(u_{l_0}\otimes\tau_{g_m}(f_{l_0}) = \rho_0(\mathcal{E}_{[0,1]}(u_{\Lambda_0}\cdots\mathcal{E}_{[l_0,l_0+1]}(u_{\Lambda_{l_0}}\otimes\mathcal{E}_{[l_0,l_0+1]}(\mathrm{I}_{\Lambda_{l_0+1}}\otimes\cdots$$

$$\mathcal{E}_{[m,m+1]}(\tilde{f}_{\Lambda_m} \otimes \cdots \mathcal{E}_{[m+l_0-1,m+l_0]}(\tilde{f}_{\Lambda_{m+l_0-1}} \otimes \hat{\mathcal{E}}_{[m+l_0,m+l_0]}(\tilde{f}_{m+l_0})))))))$$
$$= \rho_0(\mathcal{E}_{[0,1]}(u_{\Lambda_0} \cdots \mathcal{E}_{[l_0,l_0+1]}(u_{\Lambda_{l_0}} \otimes \mathcal{E}_{[l_0,l_0+1]}(\mathrm{I}_{\Lambda_{l_0+1}} \otimes \cdots \mathcal{E}_{[m,m+1]}(\mathrm{I}_{\Lambda_{m-1}} \otimes (\hat{f}_o^{(u^{(0)}_{\Lambda_m})} \otimes h_{m+1}))))).$$

Therefore, Lemma 5.1 implies that

$$\lim_{m\to+\infty} \varphi_\alpha^{(b)}(a\tau_{g_m}(f)) = \lim_{m\to+\infty} \varphi_\alpha^{(b)}(u_{l_0}\tau_{g_m}(f_{l_0})) = \varphi_\alpha^{(b)}(u_{l_0})\varphi_\alpha^{(b)}(\hat{f}_o) = \varphi_\alpha^{(b)}(a)\varphi_\alpha^{(b)}(f),$$

and this concludes the proof. □

6. Conclusions

We investigate an XX-Ising model on the comb graph $\mathbb{N} \rhd_0 \mathbb{Z}$. Namely, we show the uniqueness of QMC with homogeneous boundary condition associated with the model. Indeed, the considered quantum Markov chain is the one associated with the disordered phase of the system. Our main result concerns a clustering property for this QMC. Notice that, further relevant open problems can be investigated such as the recurrence problem for QMCs on the comb graph, the existence of phase transitions and the QMCs associated with open quantum random walks on the comb graph.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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