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Research article

Differential subordination, superordination results associated with Pascal distribution

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Abstract: This paper aims to study differential subordination and superordination preserving properties for certain analytic univalent functions with in the open unit disk. In the present investigation,we obtain some subordination and superordination results involving Pascal distribution series for certain normalized analytic functions in the open unit disk. Also we estimate the sandwich results for the same class.

Keywords: differential subordination; differential superordination; starlike function; convex function; holomorphic function; Pascal distribution series
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1. Introduction

Let $\mho$ be the family of holomorphic functions in $\Delta = \{z : |z| < 1\}$ and $\mho[t, n]$ be the subclass of $\mho$ involving the functions which can be defined by

$$g(z) = t + t_n z^n + t_{n+1} z^{n+1} + ..., \tag{1.1}$$

let Q be the subclass of $\mho$ involving the function defined by

$$g(z) = z + \sum_{n=2}^{\infty} t_n z^n.$$

Let $\beta, \mathfrak{h} \in \mho$ and consider $\theta(u, v, w, z) : c^3 \times \Delta \to C$. If β and $\theta(\beta(z), z\beta'(z), z^2\beta''(z), z)$ are univalent and if β satisfies the second order superordination,

$$\mathfrak{h}(z) \prec \theta(\beta(z), z\beta'(z), z^2\beta''(z), z), \tag{1.2}$$

then β is a solution of the differential subordination [2]. (If g is subordinate to G, then G is superordinate to g). An holomorphic function α is called a subordinate if $\alpha \prec \beta$ for every β satisfying [2]. A univalent subordinant $\overline{\alpha}$ that satisfies $\alpha \prec \overline{\alpha}$ for all subordinates [2] $\alpha(z)$ is called the best subordinant.

Miller and Mocanu [5] found the conditions on $\mathfrak{h}, \alpha$ and θ it can be given by

$$\mathfrak{h}(z) \prec \theta(\beta(z), z\beta'(z), z^2\beta''(z); z) \Rightarrow \alpha(z) \prec \beta(z). \tag{1.3}$$

For two holomorphic functions

$$\lambda(z) = z + \sum_{n=2}^{\infty} t_n z^n \;\; and \;\; \mu(z) = z + \sum_{n=2}^{\infty} r_n z^n, \quad t_n, r_n \geq 0.$$

The Hadamard product (or) convolution of λ and μ given below

$$(\lambda \star \mu)(z) = z + \sum_{n=2}^{\infty} t_n r_n z^n \;\; = \;\; (\mu \star \lambda)(z). \tag{1.4}$$

A variable X is said to have the Pascal distribution series if it takes the values 0,1,2,3... with the probabilities

$$(1-\alpha)^r, \frac{\alpha r(1-\alpha)^r}{1!} \frac{\alpha^2 r(r+1)(1-\alpha)^r}{2!}, \frac{\alpha^3 r(r+1)(r+2)(1-\alpha)^r}{3!} ...,$$

respectively where α, r are called the parameters and thus

$$P(X = K) = \binom{k+r-1}{r-1} \alpha^k (1-\alpha)^r, k \in 0, 1, 2, 3, ...$$

Many essentially interesting proof techniques involving a power series, whose co-efficients are probabilities of the Pascal distribution series introduced by sheeza et al. [12] that is

$$Q_\alpha^r = z + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} \alpha^{k-1} (1-\alpha)^r z^k, z \in \Delta, (r \geq 1, 0 \leq \alpha \leq 1).$$

The first order differential subordination and superordination which was introduced and studied by Miller, Mocanu and Bulboaca [1, 2, 5]. Also recently studied by various authors for example Magesh and Murugusundaramoorthy [3, 7–9], Magesh et al. [4], and Shanmugam et al. [11] and also obtained sandwich results for various classes of holomorphic functions.

In the present article we determine some sufficient condition for the holomorphic function in Δ to satisfy

$$\alpha_1(z) \prec Q_\alpha^r(z) \prec \alpha_2(z), \tag{1.5}$$

where α_1, α_2 are given univalent functions in Δ with $\alpha_1(0) = 1, \alpha_2(0) = 1$.

2. Preliminary results

To prove our results we need the following lemmas and definitions.

Lemma 2.1. *[10] The function*

$$M(z,n) = t_1(n)z + t_2(n)z^2 + ... \textit{ with } t_1(n) \neq 0 \textit{ for } n \geq 0 \textit{ and } \lim_{n\to\infty} |t_1(n)| = \infty$$

is a subordination chain, if

$$\Re\left\{ z \frac{\frac{\partial M(z,n)}{\partial z}}{\frac{\partial M(z,n)}{\partial n}} \right\} > 0, \quad z \in \Delta, \quad n \geq 0.$$

Definition 2.2. [5] Denote by T, the set of all functions f that are holomorphic and one to one on $\overline{\Delta} - E(f)$ where,

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$, for $\zeta \in \partial\Delta - E(f)$.

Lemma 2.3. *[6] Let α be univalent in the unit disc Δ and ψ and θ be holomorphic in a domain D containing $\alpha(\Delta)$ with $\theta(\omega) \neq 0$ when $\omega \in \alpha(\Delta)$. Set*

$$T(z) = z\alpha'(z)\theta(\alpha(z)) \textit{ and } \mathfrak{h}(z) = \psi(\alpha(z)) + T(z),$$

suppose that
(1) T(z) is starlike univalent in Δ.
(2) $\Re\left\{\frac{z\mathfrak{h}'(z)}{T(z)}\right\} > 0$, for $z \in \Delta$.
If β is holomorphic with $\alpha(0) = \beta(0)$, $\beta(\Delta) \subseteq D$, and

$$\psi(\beta(z)) + z\beta'(z)\theta(\beta(z)) \prec \psi(\alpha(z)) + z\alpha'(z)\theta(\alpha(z)), \tag{2.1}$$

then

$$\beta(z) \prec \alpha(z),$$

and α is th best dominant.

Lemma 2.4. *[2] Let α be convex univalent in the unit disk Δ and υ and ϱ be holomorphic in a domain D containing $\alpha(\Delta)$. Suppose that*
(1) $\Re\left\{\frac{\upsilon'(\alpha(z))}{\varrho(\alpha(z))}\right\} > 0$, for $z \in \Delta$.
(2) $\phi(z) = z\alpha'(z)\varrho(\alpha(z))$ is starlike univalent in Δ.
If $\beta(z) \in \mho[\alpha(0), 1] \cap T$ with $\beta(\Delta) \subseteq D$ and $\upsilon(\alpha(z)) + z\beta'(z)\varrho(\beta(z))$ is univalent in Δ and

$$\upsilon(\alpha(z)) + z\alpha'(z)\varrho(\alpha(z)) \prec \upsilon(\beta(z)) + z\beta'(z)\varrho(\beta(z)), \tag{2.2}$$

then

$$\alpha(z) \prec \beta(z),$$

and α is the best subordinant.

3. Subordination results

To prove our following theorem need to using above Lemma 2.3.

Theorem 3.1. *Let $Q^r_\alpha \in Q$, $\eta_i \in C(i = 1, 2, 3)$, $(\eta_3 \neq 0)$, $\wp \in C$, such that $\wp \neq 0$, α be convex univalent with $\alpha(0) = 1$, and assume that*

$$\Re\left\{\frac{1}{\eta_3} + \frac{1}{(\eta_1 + \eta_2\alpha)\eta_3} + 1 + \frac{z\alpha''}{\alpha'}\left(1 + \frac{\eta_2\alpha}{(\eta_1 + \eta_2\alpha)^2\eta_3}\right) + \frac{\alpha''}{\alpha'}\left(\frac{\eta_1}{(\eta_1 + \eta_2\alpha)^2\eta_3}\right) - z\alpha'\left(\frac{\eta_2}{(\eta_1 + \eta_2\alpha)^2}\eta_3\right)\right\}, \tag{3.1}$$

which is greater than zero, $z \in \Delta$.

If $g \in Q$ satisfies

$$\nabla^{(\eta_i)_1^3}(g; Q^r_\alpha) = \nabla(g, Q^r_\alpha, \eta_1, \eta_2, \eta_3) \prec \alpha(z) + \frac{z\alpha'(z)}{\eta_1 + \eta_2\alpha(z)} + \eta_3 z\alpha'(z), \tag{3.2}$$

where

$$\nabla^{(\eta_i)_1^3}(g; Q^r_\alpha) = \left(\frac{Q^r_\alpha(z)}{z}\right)^{\wp} + \frac{\wp z\alpha(z)(Q'(z)) - \wp\alpha(z)Q(z)}{\left(\eta_1 + \eta_2\left(\frac{Q^r_\alpha(z)}{z}\right)^{\wp}\right)Q(z)} + \eta_3\Big[\wp z\alpha(z)\left(\frac{Q'(z)}{Q(z)}\right) - \wp\alpha(z)\Big], \tag{3.3}$$

then

$$\left(\frac{z + \sum_{k=2}^{\infty}\begin{pmatrix} k & +r & -2 \\ r & -1 & \end{pmatrix}\alpha^{k-1}(1-\alpha)^r z^k}{z}\right)^{\wp} \prec \alpha(z),$$

and α is the best dominant.

Proof. Define the function β by

$$\beta(z) = \left(\frac{z + \sum_{k=2}^{\infty}\begin{pmatrix} k & +r & -2 \\ r & -1 & \end{pmatrix}\alpha^{k-1}(1-\alpha)^r z^k}{z}\right)^{\wp}, (z \in \Delta), \tag{3.4}$$

then the function β is holomorphic in Δ and $\beta(0) = 1$. Therfore, by making use of (3.4), we obtain

$$\begin{aligned}&\left(\frac{Q^r_\alpha(z)}{z}\right)^{\wp} + \frac{\wp z\left(\frac{Q^r_\alpha}{z}\right)^{\wp}(Q'(z)) - \wp\alpha(z)Q(z)}{(\eta_1 + \eta_2)\left(\frac{Q^r_\alpha(z)}{z}\right)^{\wp}Q(z)} + \eta_3\Big[\wp z\alpha(z)\left(\frac{Q'(z)}{Q(z)}\right) - \eta\alpha(z)\Big] \\ &= \beta(z) + \frac{z\beta'(z)}{\eta_1 + \eta_2\beta(z)} + \eta_3 z\beta'(z),\end{aligned} \tag{3.5}$$

by using (3.5) in (3.2) , we have

$$\beta(z) + \frac{z\beta'(z)}{\eta_1 + \eta_2\beta(z)} + \eta_3 z\beta'(z) \prec \alpha(z) + \frac{z\alpha'(z)}{\eta_1 + \eta_2\alpha(z)} + \eta_3 z\alpha'(z). \tag{3.6}$$

By setting

$$\psi(\alpha(z)) = \alpha(z) + \frac{z\alpha'(z)}{\eta_1 + \eta_2\alpha(z)},$$

and

$$\theta(z) = \eta_3 z\alpha'(z).$$

This is easily observed that $\psi(\alpha(z))$, $\theta(z)$ are holomorphic in $c - \{0\}$ and $\theta(z) \neq 0$. Also we see that

$$T(z) = z\alpha'(z)\theta(\alpha(z)) = \eta_3 z\alpha'(z),$$

and

$$\mathfrak{h}(z) = \psi(\alpha(z)) + T(z) = \alpha(z) + \frac{z\alpha'(z)}{\eta_1 + \eta_2\alpha(z)} + \eta_3 z\alpha'(z).$$

Here T(z) is starlike univalent in Δ and we get the result

$$\Re\left\{\frac{z\mathfrak{h}'(z)}{T(z)}\right\} = \Re\left\{\frac{1}{\eta_3} + \frac{1}{(\eta_1 + \eta_2\alpha)\eta_3} + 1 + ...\right\} > 0.$$

Hence the theorem.

By taking

$$\alpha(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B \leq A \leq 1)$$

in Theorem 4.1, we obtain the following corollary.

Corollary 3.2. *Let $\eta_i \in C$ $(i = 1, 2, 3)$, $(\eta_3 \neq 0)$, $\wp \in C$, s.t $\wp \neq 0$ be convex univalent with $\alpha(0) = 1$ and (3.1) hold true. For $g, Q^r_\alpha \in Q$, let $\left(\frac{Q^r_\alpha}{z}\right)^\wp \in H[1, 1] \cap T$ and $\nabla(\eta_i)^3_1(g; Q^r_\alpha)$ defined in (3.3) be univalent in Δ satisfying*

$$\nabla(\eta_i)^3_1(g; Q^r_\alpha) \prec \frac{1 + Az}{1 + Bz} + \frac{z(A - B)}{\eta_1(1 + Bz)^2 + \eta_2(1 + Az)(1 + Bz)} + \eta_3 z\left(\frac{A - B}{(1 + Bz)^2}\right),$$

then

$$\left(\frac{z + \sum_{k=2}^{\infty}\begin{pmatrix} k & +r & -2 \\ r & -1 & \end{pmatrix}\alpha^{k-1}(1 - \alpha)^r z^k}{z}\right)^\wp \prec \frac{1 + Az}{1 + Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

4. Superordination results

Using Lemma 2.4 to prove the following theorem.

Theorem 4.1. *Let $Q^r_\alpha(z) \in Q$, $\eta_i \in C(i = 1, 2, 3)$, $(\eta_3 \neq 0)$, $\wp \in C$, s.t $\wp \neq 0$, α be convex univalent with $\alpha(0) = 1$, and assume that*

$$\Re\left\{\frac{1}{\eta_3} + \frac{1}{\eta_1 + \eta_2\alpha)\eta_3}\right\} \geq 0. \tag{4.1}$$

If $g \in Q$, $Q^r_\alpha(z) \in H$, $[\alpha(0), 1] \bigcap T$, Let $\nabla^{(\eta_i)^3_1}(g; Q^r_\alpha)$ be univalent in Δ and

$$\alpha(z) + \frac{z\alpha'(z)}{\eta_1 + \eta_2\alpha(z)} + \eta_3 z\alpha'(z) \prec \nabla^{(\eta_i)^3_1}(g; Q^r_\alpha), \tag{4.2}$$

where $\nabla^{(\eta_i)^3_1}(g; Q^r_\alpha)$ is given in (3.3), then

$$\alpha(z) \prec \left(\frac{Q^r_\alpha}{z}\right)^\wp,$$

and α is the best subordinant.

Proof. The function β is defined by

$$\beta(z) = \left(\frac{Q^r_\alpha}{z}\right)^\wp, \tag{4.3}$$

simplify above equation, we get

$$\nabla^{(\eta_i)^3_1}(g; Q^r_\alpha) = \beta(z) + \frac{z\beta'(z)}{\eta_1 + \eta_2\beta(z)} + \eta_3 z\beta'(z),$$

then

$$\alpha(z) + \frac{z\alpha'(z)}{\eta_1 + \eta_2\alpha(z)} + \eta_3 z\alpha'(z) \prec \beta(z) + \frac{z\beta'(z)}{\eta_1 + \eta_2\beta(z)} + \eta_3 z\beta'(z).$$

By setting

$$\upsilon(z) = \alpha(z) + \frac{z\alpha'(z)}{\eta_1 + \eta_2\alpha(z)} \quad and \quad \varrho(z) = \eta_3 z\alpha'(z).$$

Here $\upsilon(\alpha(z))$ is holomorphic in C. Also $\varrho(z)$ is holomorphic in $C - \{0\}$ and $\varrho(z) \neq 0$. Consider,

$$\begin{aligned} M(z, n) &= \upsilon(\alpha(z)) + \varrho(\alpha(z))nz\alpha'(z) \\ &= \alpha(z) + \frac{z\alpha'(z)}{\eta_1 + \eta_2\alpha(z)} + \eta_3 nz\alpha'(z) \\ &= t_1(n)z + t_2(n)z + ..., \end{aligned}$$

differentiating the above equation with respect to z and n, we have

$$\begin{aligned} \frac{\partial M(z, n)}{\partial z} &= \alpha'(z) + \frac{(\eta_1 + \eta_2\alpha)[z\alpha'' + \alpha'] - z\alpha'(\eta_2\alpha')}{(\eta_1 + \eta_2\alpha)^2} + \eta_3 nz\alpha'(z) \\ &= t_1(n)z + t_2(n)z + ..., \\ \frac{\partial M(z, n)}{\partial n} &= \eta_3 z\alpha'(z), \end{aligned}$$

and

$$\frac{\partial M(0, n)}{\partial z} = \alpha'(0) + \frac{\eta_1\alpha' + \eta_2\alpha\alpha'}{(\eta_1 + \eta_2\alpha)^2}.$$

From the univalence of α we have $\alpha'(0) \neq 0$ and $\alpha(0) = 1$, it follows that $t_1(n) \neq 0$ for $n \geq 0$ and $\lim_{n\to\infty} |t_1(n)| = \infty$.

A simple compution yields,

$$\Re\left\{z\frac{\frac{\partial M(z,n)}{\partial z}}{\frac{\partial M(z,n)}{\partial n}}\right\}=\Re\left\{\frac{1}{\eta_3}+\frac{1}{(\eta_1+\eta_2\alpha)\eta_3}\right\}.$$

Using the fact that α is convex univalent function in Δ and $\eta_3\neq 0$ we have

$$\Re\left\{z\frac{\frac{\partial M(z,n)}{\partial z}}{\frac{\partial M(z,n)}{\partial n}}\right\}>0, if\ \Re\left\{\frac{1}{\eta_3}+\frac{1}{(\eta_1+\eta_2\alpha)\eta_3}\right\}>0,\ \ z\in\Delta, n\geq 0.$$

Hence the theorem.

By taking

$$\alpha(z)=\frac{1+Az}{1+Bz}\quad(-1\leq B\leq A\leq 1)$$

in Theorem 4.1 we obtain the following corollary.

Corollary 4.2. *Let $\eta_i\in C$ $(i=1,2,3)$, $(\eta_3\neq 0)$, $\wp\in C$, s.t $\wp\neq 0$ be convex univalent with $\alpha(0)=1$ and (4.1) hold true. For $g,Q^r_\alpha\in Q$, let $\left(\frac{Q^r_\alpha}{z}\right)^{\wp}\in H[1,1]\cap T$ and $\nabla(\eta_i)^3_1(g;Q^r_\alpha)$ defined in (3.3) be univalent in Δ satisfying*

$$\frac{1+Az}{1+Bz}+\frac{z(A-B)}{\eta_1(1+Bz)^2+\eta_2(1+Az)(1+Bz)}+\eta_3 z\left(\frac{A-B}{(1+Bz)^2}\right)\prec\nabla(\eta_i)^3_1(g;Q^r_\alpha),$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{z+\sum_{k=2}^{\infty}\begin{pmatrix}k & +r & -2\\ r & -1 & \end{pmatrix}\alpha^{k-1}(1-\alpha)^r z^k}{z}\right)^{\wp},$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

5. Sandwich theorem

To obtain the sandwich results get from combining the subordination results and superordination results

Theorem 5.1. *Let α_1 and α_2 be convex univalent in Δ, $\eta_i\in C$ $(i=1,2,3)$, $(\eta_3\neq 0)$, $\wp\in C$, s.t $\wp\neq 0$ and let α_2 satisfying (3.1) and α_1 satisfying (4.1). For $g,Q^r_\alpha\in Q$, let $\left(\frac{Q^r_\alpha}{z}\right)^{\wp}\in H[1,1]\cap T$ and $\nabla(\eta_i)^3_1(g;Q^r_\alpha)$ defined in (3.3) be univalent in Δ satisfying*

$$\alpha_1(z)+\frac{z\alpha_1'(z)}{\eta_1+\eta_2\alpha_1(z)}+\eta_3 z\alpha_1'(z)\prec\nabla(\eta_i)^3_1,(g;Q^r_\alpha)\prec\alpha_2(z)+\frac{z\alpha_2'(z)}{\eta_1+\eta_2\alpha_2(z)}+\eta_3 z\alpha_2'(z),$$

then

$$\alpha_1(z)\prec\left(\frac{Q^r_\alpha}{z}\right)^{\wp}\prec\alpha_2(z),$$

and α_1,α_2 are respectively best subordinant and best dominant.

Hence the proof of the theorem. By taking

$$\alpha_1(z) = \frac{1 + A_1 z}{1 + B_1 z}, \quad (-1 \le B_1 \le A_1 \le 1)$$

and

$$\alpha_2(z) = \frac{1 + A_2 z}{1 + B_2 z}, \quad (-1 \le B_2 \le A_2 \le 1)$$

in Theorem 5.1, we obtain the following result.

Corollary 5.2. *For $g, Q_\alpha^r \in Q$, let $\left(\frac{Q_\alpha^r}{z}\right)^\wp \in H[1,1] \cap T$ and $\nabla(\eta_i)_1^3(g; Q_\alpha^r)$ defined in (3.3) be univalent in Δ satisfying*

$$\begin{aligned}\frac{1 + A_1 z}{1 + B_1 z} + \frac{z(A_1 - B_1)}{\eta_1(1 + B_1 z)^2 + \eta_2(1 + A_1 z)(1 + B_1 z)} + \eta_3 z\left(\frac{A_1 - B_1}{(1 + B_1 z)^2}\right) &\prec \nabla(\eta_i)_1^3(g; Q_\alpha^r) \\ \prec \frac{1 + A_2 z}{1 + B_2 z} + \frac{z(A_2 - B_2)}{\eta_1(1 + B_2 z)^2 + \eta_2(1 + A_2 z)(1 + B_2 z)} + \eta_3 z\left(\frac{A_2 - B_2}{(1 + B_2 z)^2}\right)&,\end{aligned}$$

then

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \left(\frac{Q_\alpha^r}{z}\right)^\wp \prec \frac{A_2 - B_2}{(1 + B_2 z)^2},$$

and $\frac{1+A_1 z}{1+B_1 z}, \frac{1+A_2 z}{1+B_2 z}$ are respectively the best subordinant and best dominant.

6. Conclusions

This paper deals with the applications of the differential subordination and superordination results involving Pascal distribution series. In addition we found the sandwich results to be in the class of holomorphic functions. Many interesting particular cases of the main theorems are emphazied in the form of corollaries. Furthermore to illustrate the results of application in various classes of analytic function. We anticipate that differential subordination and superordination will be important in several fields related to mathematics, science and technology.

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Conflict of interest

The authors declare no conflicts of interest.

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