Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(4): 7817–7839.
DOI: 10.3934/math.2023393
Received: 23 October 2022
Revised: 05 January 2023
Accepted: 09 January 2023
Published: 31 January 2023

Research article

Solvability and stability analysis of a coupled system involving generalized fractional derivatives

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Abstract: In this article, we investigate the existence of unique maximal and minimal solutions for a coupled differential system in terms of generalized fractional derivative with arbitrary order. The iterative technique of a fixed point operator together with the properties of green's function are used basically. Moreover, we investigate the generalized Ulam-Hyers stability of the solution for the given coupled system. Finally, some examples are given to illustrate the theoretic results.

Keywords: existence and uniqueness; generalized fractional derivative and integral; boundary value problem; minimal and maximal solutions; Ulam-Hyers
Mathematics Subject Classification: 26A33, 34B18, 34B27

1. Introduction

Fractional differential equations (FDE) arise in wide variety of engineering and scientific disciplines as a mathematical modeling of processes and systems in many fields. For details, see [3, 10, 12, 14, 21–25] and the references therein.

The qualitative analysis of FDE are extensively investigated by scientists using different fixed point theorems and other techniques. For more details see [1, 4–8, 11, 13, 16–19, 26] and references therein.

The solvability of the FDE depends mainly on the shape of the given system and the initial or boundary conditions. By mean of fractional calculus we can obtain an integral solution of the given differential system. The fractional calculus are used with some verities due to different shapes of fractional derivatives and integrals. Here in this article, we use some type of generalized fractional derivative that depends on some kind of function and some authors called it as derivative of a function with respect to another function [16, 19].

For the qualitative analysis, almost all researchers are using the basic fixed point theorems such as Banach contraction principle, Schauder fixed point, etc. These theorems are fantastic tools to get the existence and uniqueness of the solution but it they need some strong conditions to satisfy, hence we search for another way to get the existence and uniqueness of the solution that need weaker conditions. The iterative technique is one of such methods that can be applied to obtain the existence and uniqueness of the solution of the main problem. On the other hand, we introduce the idea of maximal and minimal solution for the given problem to justify the variety of mathematical sources. Before going into other similar previous article, we also consider in this article the generalized Ulam-Hyers stability which is important topic for the qualitative analysis of any system.

In [2], Houas and Benbachir studied the fractional problem

$$\begin{cases} D^{\alpha_0}x(t) = F_1(t, x(t), (D^{\alpha_1}, D^{\alpha_2}, ..., D^{\alpha_{n-1}})x(t)), \quad t \in (0, 1), \\ x(0) = x^*, x'(0) = x''(0) = ... = x^{(n-2)} = 0, I^{\beta}x(1) = \lambda I^{\beta}x(\jmath), 0 < \jmath < 1, \end{cases}$$

where ${}^cD^{\alpha_i}, i = 0, 1, 2, ..., n-1$ denote the Caputo FD of order α_i with $n-1 < \alpha_{n-1} < n$, $\lambda \neq 0$ is a real number and f is a given continuous function.

In [4], Wang et al. provided the existence solution to non-zero FDE with boundary values problem (BVP) for a coupled system.

$$\begin{cases} D^{\alpha}u(t) + F_1(t, v) = 0, \text{ in } (0, 1), \\ D^{\beta}v(t) + F_2(t, u) = 0, \text{ in } (0, 1), \\ u(0) = 0, \ u(1) = au(\xi), \\ v(0) = 0, \ v(1) = bv(\xi), \end{cases}$$

where $2 < \alpha, \beta < 3, \ 0 \leq a, b \leq 1, \ \xi \in (0, 1)$ and $f, g \in C([0, 1] \times [0, +\infty), [0, +\infty))$.

In [8], Ali et al. studied the iterative solutions and stability analysis to a CS of FDE.

Whereas, Ali et al. [9] extended their previous work and introduced the below fractional order nonlinear CS withe boundary condition

$$\begin{cases} D^{\alpha}u(t) + F_1(t, v(t)) = 0, \quad t \in [0, 1], \\ D^{\beta}v(t) + F_2(t, u(t)) = 0, \quad t \in [0, 1], \\ u(1) = u'(0) = ... = u^{(n-1)}(0) = 0, \\ v(1) = v'(a) = ... = v^{(n-1)}(0) = 0, \end{cases}$$

where $n = 2, 3, 4, ..., n-1 < \alpha, \beta \leq n, u, v \in C[0, 1]$.

The main purpose of this work is to provide the existence criterion and the Ulam-Hyers (UH) stability analysis of the considered CS

$$\begin{cases} D^{\alpha,\mathfrak{H}}_{a+}u(t) + F_1(t, v(t)) = 0, \quad t \in [a, b], \\ D^{\beta,\mathfrak{H}}_{a+}v(t) + F_2(t, u(t)) = 0, \quad t \in [a, b], \\ u(b) + \lambda_\alpha u(a) = I^{\alpha,\mathfrak{H}}_{a+}F_3(b, v(b)), u'(a) = ... = u^{(n-1)}(a) = 0, \\ v(b) + \lambda_\beta v(a) = I^{\beta,\mathfrak{H}}_{a+}F_4(b, u(b)), v'(a) = ... = v^{(n-1)}(a) = 0, \end{cases} \tag{1.1}$$

where $D^{\alpha,\mathfrak{H}}_{a+}u$, and $D^{\alpha,\mathfrak{H}}_{a+}u$, $n-1 < \alpha, \beta < n$, $n \geq 2$, are the FD of a function u with respect to another function $\mathfrak{H}$, $-1 < \lambda_\alpha, \lambda_\beta \leq 0$, and f, g, h, k are appropriate functions.

In fact, we investigate the existence of unique maximal and minimal solutions for the differential coupled system (1.1) using iterative technique. The so-called green's function will be given and its properties will be discussed in details. The solution of the linear version of the CS (1.1) will be obtained by using the properties of fractional calculus. Moreover, the generalized Ulam-Hyers stability of the solution is also considered. Finally, we present examples to demonstrate consistency to the main results.

The paper is organized as follows. In Section 2, we recall some notions and notation. In the Section 3, we introduce the main results concerning the existence of solution of the problem (1.1). The UH stability of the solution for the fractional CS (1.1) is investigated in Section 4. Finally, in Section 5, we present some applications to the fractional differential coupled system (FDCS) (1.1).

2. Preliminaries

The fractional integrals (FI) and FD of a function u with respect to another function $\mathfrak{H}$ along with their properties will be introduced in this section briefly to be used in our discussion. First of all, we assume that $\mathfrak{H}(t)$ be strictly increasing function on $(a, b]$, having a continuous derivative on (a, b) and $\mathfrak{H}'(t) > 0$ for all $t \in [a, b]$.

Definition 2.1. [3] Let $u \in L[a, b]$ be a real-valued function. The FI of order $\alpha \in \mathbb{R}_+$ for a function u with respect to another function $\mathfrak{H}$ on $[a, b]$ is given by

$$I^{\alpha,\mathfrak{H}}_{a+}u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\mathfrak{H}'(r)u(r)dr}{[\mathfrak{H}(t) - \mathfrak{H}(r)]^{1-\alpha}} \quad a < t,$$

where Γ is the Euler Gamma function, provided that right-hand side is pointwise given on $(0, +\infty)$.

Definition 2.2. [3] The FD of order $\alpha > 0$ for a function u with respect to another function $\mathfrak{H}$ on $[a, T]$ is given by

$$D^{\alpha,\mathfrak{H}}_{a+}u(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{1}{\mathfrak{H}'(t)}\frac{d}{dt})^n \int_a^t \frac{\mathfrak{H}'(r)u(r)dr}{[\mathfrak{H}(t) - \mathfrak{H}(r)]^{\alpha-n+1}} \quad a < t,$$

such that the integral is well-defined on $(0, \infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ stand for the integer part of the real number α.

Lemma 2.3. *[3] Let $\beta > \alpha > 0$. Then the relations*

$$(I^{\alpha,\mathfrak{H}}_{a+}I^{\beta,\mathfrak{H}}_{a+})u(t) = I^{(\alpha+\beta),\mathfrak{H}}_{a+}u(t),$$

$$(D^{\alpha,\mathfrak{H}}_{a+} I^{\beta,\mathfrak{H}}_{a+})u(t) = I^{(\beta-\alpha),\mathfrak{H}}_{a+} u(t),$$
$$(D^{\alpha,\mathfrak{H}}_{a+} I^{\alpha,\mathfrak{H}}_{a+})u(t) = u(t),$$

is hold for (sufficiently good) functions $u(t)$.

Lemma 2.4. *[3] Let $\alpha > 0$, then the FDE*

$$D^{\alpha,\mathfrak{H}}_{a+} u(t) = 0,$$

has the solution in the form of

$$u(t) = \sum_{k=0}^{n-1} \frac{u^{(k)}_{\mathfrak{H}}(a)}{k!} (\mathfrak{H}(t) - \mathfrak{H}(a))^k,$$

where $u^{(k)}_{\mathfrak{H}}(t) = (\frac{1}{\mathfrak{H}'(t)} \frac{d}{dt})^k u(t)$.

Lemma 2.5. *[3] Let $\alpha > 0$, then*

$$I^{\alpha,\mathfrak{H}}_{a+} [D^{\alpha,\mathfrak{H}}_{a+} u(t)] = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}_{\mathfrak{H}}(a)}{k!} (\mathfrak{H}(t) - \mathfrak{H}(a))^k.$$

Next, some definitions and results concerning the minimal ana maximal solutions are recalled.

Definition 2.6. [9] Let $U = C[a, b]$ be the Banach space endowed with norm:
$\| u \| = \max_{t\in[a,T]} |u(t)|$ which satisfies the partial ordering, and let $W = [u_m, u_M]$ with $u_m \leq u_M$ be a set $W \subset U$, and the operator $P : W \longrightarrow U$ is known as increasing function if for each $u_1, u_2 \in W$ and $u_1 \leq u_2$ gives $Pu_1 \leq Pu_2$. The operator P is known as decreasing function if for each $u_1, u_2 \in W$ and $u_1 \leq u_2$ gives $Pu_1 \geq Pu_2$.

Definition 2.7. [9] Suppose I be an identity operator. If $(I - P)u_m \leq 0$, then the function $u_m \in W$ is a minimal solution of $(I - P)u = 0$ and if $(I - P)u_M \geq 0$, then the function $u_M \in W$ is a maximal solution of $(I - P)u = 0$.

Definition 2.8. [20] We say that the subset $W \subset U = C[a, b]$ is uniformly bounded, if $\exists$ a real number $c > 0$ such that $| u(t) | \leq c$ for all u of W and what ever $t \in [a, b]$.

Definition 2.9. [20] We say that the subset $W \subset U = C[a, b]$ is equicontinuous, if $\forall \epsilon > 0$, $\exists \delta > 0$ depending only on ϵ such that for $t_1, t_2 \in [a, b]$ satisfying the inequality $| t_1 - t_2 | \leq \delta$ for all u of W we have $| u(t_1) - u(t_2) | \leq \epsilon$.

Lemma 2.10. *[20] Let U Banach space which satisfies $W \subset U$ and $u_n, u^*_n \in W$ where $u_n \leq u^*_n$, $n \in \mathbb{Z}_+$. If $u_n \longrightarrow u$ and $u^*_n \longrightarrow u^*$, then $u \leq u^*$.*

3. Main results

The solution and analyses of the coupled fractional system (1.1) is the main part of this article. This section devotes for obtaining solutions of such problems. The following assumptions are necessarily to obtain our results:
(I_1) The real-valued functions $f, g, h, k : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the Caratheodory conditions.

(I_2) The functions $F_1(t,v), F_2(t,u), F_3(t,x)$ and $F_4(t,y)$ are increasing in v, u, x and y for every $t \in [a,b]$ respectively.
(I_3) Existence of constants $\Omega_1, \Omega_2, \Omega_3, \Omega_4 > 0$ such that

$$\begin{cases} |F_1(t,v_1(t)) - F_1(t,v_2(t))| \leq \Omega_1 |v_1 - v_2|, \\ |F_2(t,u_1(t)) - F_2(t,u_2(t))| \leq \Omega_2 |u_1 - u_2|, \\ |F_3(t,x_1(t)) - F_3(t,x_2(t))| \leq \Omega_3 |x_1 - x_2|, \\ |F_4(t,y_1(t)) - F_4(t,y_2(t))| \leq \Omega_4 |y_1 - y_2|. \end{cases}$$

(I_4) Existence of constants $C_1, C_2, C_3, C_4 > 0$ such that:

$$| F_1(a,.) |\leq C_1, \quad | F_2(a,.) |\leq C_2, \quad | F_3(a,.) |\leq C_3 \text{ and } | F_4(a,.) |\leq C_4.$$

(I_5) Suppose that $B = \{u \in C([a,b]) :\| u \|\leq R\}$, and $W = [u_m, u_M]$, where $u_m, u_M \in B$,

$$R \geq \frac{C_2(\lambda_\beta + 1)(\Omega_1 + \Omega_3)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) + (C_1 + C_3)\Psi_{\alpha+1}(a)}{1 - \Omega_2(\lambda_\beta + 1)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a)(\Omega_1 + \Omega_3)},$$

and

$$(\Omega_1 + \Omega_3)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) \max\{(\Omega_2 + \Omega_4), \Omega_2(\lambda_\beta + 1)\} < 1.$$

All the constants $M_1, \Omega_2, \Omega_3, M_4, C_1, C_2, C_3, C_4$ will be specified later.

Lemma 3.1. *Let $\rho, \sigma \in C([a,b])$ and $n-1 < \alpha < n$, then the FDE:*

$$\begin{aligned} &D^{\alpha,\mathfrak{H}}_{a+} u(t) + \rho(t) = 0, \\ &u(b) + \lambda_\alpha u(a) = I^{\alpha,\mathfrak{H}}_{a+}\sigma(b), u'(a) = ... = u^{(n-1)}(a) = 0, \lambda_\alpha \in (-1,0], \end{aligned} \tag{3.1}$$

has the solution

$$u(t) = \int_a^b \left[G_\alpha(t,r)\rho(r) + \frac{[\mathfrak{H}(b) - \mathfrak{H}(r)]^{\alpha-1}}{(\lambda_\alpha + 1)\Gamma(\alpha)}\sigma(r) \right] \mathfrak{H}'(r)\, dr, \tag{3.2}$$

where $G_\alpha(t,r)$ is given by

$$G_\alpha(t,r) = \begin{cases} \dfrac{\frac{1}{\lambda_\alpha+1}[\mathfrak{H}(b)-\mathfrak{H}(r)]^{\alpha-1} - [\mathfrak{H}(t)-\mathfrak{H}(r)]^{\alpha-1}}{\Gamma(\alpha)} & a \leq s \leq t \leq b, \\ \dfrac{[\mathfrak{H}(b)-\mathfrak{H}(r)]^{\alpha-1}}{(\lambda_\alpha+1)\Gamma(\alpha)} & a \leq t \leq s \leq b, \end{cases} \tag{3.3}$$

$G_\alpha(t,r)$ is known as Green's function.

Proof. Using Lemma 2.5 to the linear BVP (3.1) yields that

$$u(t) = C_1 + C_2[\mathfrak{H}(t) - \mathfrak{H}(a)] + C_3[\mathfrak{H}(t) - \mathfrak{H}(a)]^2 + ... + C_n[\mathfrak{H}(t) - \mathfrak{H}(a)]^{n-1} - I^{\alpha,\mathfrak{H}}_{a+}\rho(t). \tag{3.4}$$

We use the conditions $u'(a) = u^{(2)}(a) = ... = u^{(n-1)}(a) = 0$, we have $C_2 = C_3 = ... = C_n = 0$. Further, as $u(b) + \lambda_\alpha u(a) = I^{\alpha,\mathfrak{H}}_{a+}\sigma(b)$, then we have

$$C_1 = \frac{1}{\lambda_\alpha + 1} I^{\alpha,\mathfrak{H}}_{a+}\sigma(b) + \frac{1}{\lambda_\alpha + 1} I^{\alpha,\mathfrak{H}}_{a+}\rho(b).$$

Therefore, we deduce from (3.4) that

$$\begin{aligned} u(t) &= \frac{1}{\lambda_\alpha+1}\int_a^b \frac{[\mathfrak{H}(b)-\mathfrak{H}(r)]^{\alpha-1}}{\Gamma(\alpha)}\mathfrak{H}'(r)\sigma(r)dr + \frac{1}{\lambda_\alpha+1}\int_a^b \frac{[\mathfrak{H}(b)-\mathfrak{H}(r)]^{\alpha-1}}{\Gamma(\alpha)}\mathfrak{H}'(r)\rho(r)dr \\ &\quad - \int_a^t \frac{[\mathfrak{H}(t)-\mathfrak{H}(r)]^{\alpha-1}}{\Gamma(\alpha)}\mathfrak{H}'(r)\rho(r)dr. \end{aligned}$$

Hence, we obtain the result (3.3) and the proof is finished. □

We can now obtain the representation of the integral solution of the couple system (1.1).

Lemma 3.2. *The CS (1.1) has the integral representation*

$$\begin{cases} u(t) = \int_a^b \left[G_\alpha(t,r)F_1(r,v(r)) + \Psi_\alpha(r)F_3(r,v(r))\right]\mathfrak{H}'(r)dr, \; t\in[a,T], \\ v(t) = \int_a^b \left[G_\beta(t,r)F_2(r,u(r)) + \Psi_\beta(r)k(r,u(r))\right]\mathfrak{H}'(r)dr, \; t\in[a,T], \end{cases} \tag{3.5}$$

where $\Psi_\alpha(r) = \frac{[\mathfrak{H}(b)-\mathfrak{H}(r)]^{\alpha-1}}{(\lambda_\alpha+1)\Gamma(\alpha)}$, $G_\alpha(t,r)$ *and* $G_\beta(t,r)$ *are given as in (3.3).*

The next are some properties of the green function $G_\alpha(t,r)$ that can easily obtained by its definition.

Lemma 3.3. *The following properties are satisfied for all* $t,s,\tau\in[a,b]$*:*

(1) $0\le G_\alpha(t,r)\le\Psi_\alpha(r)\le\Psi_\alpha(a)$, *and* $0\le G_\beta(t,r)\le\Psi_\beta(r)\le\Psi_\beta(a)$.
(2) $\int_a^b G_\alpha(t,r)dr\le(b-a)\Psi_\alpha(a)$, $\int_a^b G_\alpha(t,r)\mathfrak{H}'(r)dr\le\Psi_{\alpha+1}(a)$, *and* $\int_a^b G_\beta(t,r)\mathfrak{H}'(r)dr\le\Psi_{\beta+1}(a)$.
(3) $\int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr\le\Psi_{\alpha+1}(a)$, *and* $\int_a^b G_\beta(t,r)\mathfrak{H}'(r)dr\le\Psi_{\beta+1}(a)$.
(4) $|G_\alpha(r,\jmath)-G_\alpha(a,\jmath)|\le\frac{(\mathfrak{H}(b)-\mathfrak{H}(\jmath))^{\alpha-1}}{\Gamma(\alpha)}=(\lambda_\alpha+1)\Psi_\alpha(\jmath)$, *where* $G_\alpha(a,\jmath)=\Psi_\alpha(\jmath)$.

The CS (3.5) can be writhen as follows:

$$\begin{cases} u(t) &= \int_a^b \left[G_\alpha(t,r)F_1(r,v(r)) + \Psi_\alpha(r)F_3(r,v(r))\right]\mathfrak{H}'(r)\,dr \\ &= \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath) \\ &\quad + \Psi_\alpha(r)\,F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr, \\ v(t) &= \int_a^b \left[G_\beta(t,r)F_2(r,u(r)) + \Psi_\beta(r)k(r,u(r))\right]\mathfrak{H}'(r)dr \\ &= \int_a^b [G_\beta(t,r)F_2(r, \int_a^b \left[G_\alpha(r,\jmath)F_1(\jmath,v(\jmath)) + \Psi_\alpha(\jmath)h(\jmath,v(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath) \\ &\quad + \Psi_\beta(r)\,k(r, \int_a^b \left[G_\alpha(r,\jmath)F_1(\jmath,v(\jmath)) + \Psi_\alpha(\jmath)h(\jmath,v(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr. \end{cases} \tag{3.6}$$

Remark 3.4. Due to existence of the symmetry between u and v, to obtain the results of the study, it is sufficient to study on u, and in the same way we get the results for v.

We define the operator $P : W \to W$ as

$$P(u(t)) = \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath) + \Psi_\alpha(r)\,F_3(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr. \tag{3.7}$$

In accordance with (3.6) and (3.7), we can obtain

$$[I - P](u(t)) = 0, \quad t \in [a,b]. \tag{3.8}$$

We notice that Eqs (3.6) and (3.8) are the same results that are fixed points of P. As in (I_2), such that, for $u, u^* \in W$ with $u \le u^*$, we get

$$\begin{aligned} P(u(t)) &= \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath) \\ &+ \Psi_\alpha(r)\,F_3(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr \\ &\le \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u^*(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u^*(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath) \\ &+ \Psi_\alpha(r)\,F_3(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u^*(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u^*(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr \\ &= P(u^*(t)), \end{aligned} \tag{3.9}$$

thus P denotes increasing operator.
(I_6) Let that the minimal and maximal results of (3.8) are respectively u_m and $u_M \in W$, and $u_m \le u_M$ on $[a,b]$.

Lemma 3.5. *Let* (I_1)*–*(I_6) *hold. If* $P : W \to W$*, then P is equicontinuous and uniformly bounded.*

Proof. Under assumptions $(I_1) - (I_6)$, let $u_i, u_j \in W \subset U$, where $i, j = 1,2,3,...,n$. The proof consists of two steps.
Step 1: P is equicontinuous. In accordance with the definition of the operator P that is given in (3.7), we have

$$\begin{aligned} &\mid P(u_i(t)) - P(u_j(t)) \mid \\ =&\mid (\int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u_i(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u_i(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath) \\ &+ \Psi_\alpha(r)\,F_3(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u_i(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u_i(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr) \\ &- (\int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u_j(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u_j(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath) \\ &+ \Psi_\alpha(r)\,F_3(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u_j(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u_j(\jmath))\Big]\,\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr) \mid \end{aligned}$$

$$=\left|\int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(F_1(r,\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_i(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_i(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)\right.$$

$$-F_1(r,\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_j(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_j(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath))dr$$

$$+\int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(F_3(r,\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_i(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_i(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)$$

$$\left.-F_3(r,\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_j(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_j(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath))dr\right|$$

$$\leq\int_a^b G_\alpha(t,r)\mathfrak{H}'(r)\mid F_1(r,\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_i(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_i(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)$$

$$-F_1(r,\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_j(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_j(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)\mid dr$$

$$+\int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)\mid F_3(r,\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_i(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_i(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)$$

$$-F_3(r,\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_j(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_j(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)\mid dr$$

$$\leq\Omega_1\int_a^b G_\alpha(t,r)\mathfrak{H}'(r)\mid\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_i(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_i(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_j(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_j(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath\mid dr$$

$$+\Omega_3\int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)\mid\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_i(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_i(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b\left[G_\beta(r,\jmath)F_2(\jmath,u_j(\jmath))+\Psi_\beta(\jmath)k(\jmath,u_j(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath\mid dr$$

$$\leq\Omega_1\int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b G_\beta(r,\jmath)\mid F_2(\jmath,u_i(\jmath))-F_2(\jmath,u_j(\jmath))\mid\mathfrak{H}'(\jmath)d\jmath)dr$$

$$+\Omega_1\int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mid k(\jmath,u_i(\jmath))-k(\jmath,u_j(\jmath))\mid\mathfrak{H}'(\jmath)d\jmath)dr$$

$$+\Omega_3\int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b G_\beta(r,\jmath)\mid F_2(\jmath,u_i(\jmath))-F_2(\jmath,u_j(\jmath))\mid\mathfrak{H}'(\jmath)d\jmath)dr$$

$$+\Omega_3\int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath)\mid k(\jmath,u_i(\jmath))-k(\jmath,u_j(\jmath))\mid d\jmath)dr$$

$$\leq\Omega_1\Omega_2\int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b G_\beta(r,\jmath)\mid u_i(\jmath)-u_j(\jmath)\mid\mathfrak{H}'(\jmath)d\jmath)dr$$

$$+\Omega_1\Omega_4\int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath)\mid u_i(\jmath)-u_j(\jmath)\mid d\jmath)dr$$

$$+\Omega_2\Omega_3\int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b G_\beta(r,\jmath)\mathfrak{H}'(\jmath)\mid u_i(\jmath)-u_j(\jmath)\mid d\jmath)dr$$

$$+ \Omega_3\Omega_4 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath) \mid u_i(\jmath) - u_j(\jmath) \mid d\jmath)dr$$
$$\leq \Omega_1\Omega_2 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b G_\beta(r,\jmath)\mathfrak{H}'(\jmath)d\jmath)dr \parallel u_i - u_j \parallel$$
$$+ \Omega_1\Omega_4 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath)d\jmath)dr \parallel u_i - u_j \parallel$$
$$+ \Omega_2\Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b G_\beta(r,\jmath)\mathfrak{H}'(\jmath)d\jmath)dr \parallel u_i - u_j \parallel$$
$$+ \Omega_3\Omega_4 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath)d\jmath)dr \parallel u_i - u_j \parallel$$
$$\leq \Omega_1\Omega_2\Psi_{\beta+1}(a) \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)dr \parallel u_i - u_j \parallel$$
$$+ \Omega_1\Omega_4\Psi_{\beta+1}(a) \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)dr \parallel u_i - u_j \parallel$$
$$+ \Omega_2\Omega_3\Psi_{\beta+1}(a) \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr \parallel u_i - u_j \parallel$$
$$+ \Omega_3\Omega_4\Psi_{\beta+1}(a) \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr \parallel u_i - u_j \parallel$$
$$\leq \Omega_1\Omega_2\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) \parallel u_i - u_j \parallel$$
$$+ \Omega_1\Omega_4\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) \parallel u_i - u_j \parallel$$
$$+ \Omega_2\Omega_3\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) \parallel u_i - u_j \parallel$$
$$+ \Omega_3\Omega_4\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) \parallel u_i - u_j \parallel$$
$$\leq (\Omega_1 + \Omega_3)(\Omega_2 + \Omega_4)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) \parallel u_i - u_j \parallel$$
$$\leq \Omega \parallel u_i - u_j \parallel,$$

where

$$\Omega = (\Omega_1 + \Omega_3)(\Omega_2 + \Omega_4)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a). \tag{3.10}$$

Step2: P is uniformly bounded. Again by (3.7), we obtain

$$\mid P(u(t)) \mid$$
$$=\mid \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath)$$
$$+ \Psi_\alpha(r)\, F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr \mid$$
$$\leq \int_a^b G_\alpha(t,r)\mathfrak{H}'(r) \mid F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \mid dr$$
$$+ \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r) \mid F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \mid dr$$
$$\leq \int_a^b G_\alpha(t,r)\mathfrak{H}'(r) \mid F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath)$$

$$
\begin{aligned}
&- F_1(a, \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \\
&+ F_1(a, \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \mid dr \\
&+ \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r) \mid F_3(r, \int_a^b \left[G_\beta(r, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \\
&- F_3(a, \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \\
&+ F_3(a, \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \mid dr \\
&\leq \int_a^b G_\alpha(t, r)\mathfrak{H}'(r) \mid F_1(r, \int_a^b \left[G_\beta(r, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \\
&- F_1(a, \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \mid dr \\
&+ \int_a^b G_\alpha(t, r)\mathfrak{H}'(r) \mid F_1(a, \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \mid dr \\
&+ \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r) \mid F_3(r, \int_a^b \left[G_\beta(r, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \\
&- F_3(a, \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \mid dr \\
&+ \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r) \mid F_3(a, \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath) \mid dr \\
&\leq \Omega_1 \int_a^b G_\alpha(t, r)\mathfrak{H}'(r) \mid \int_a^b \left[G_\beta(r, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath \\
&- \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath \mid dr + C_1 \int_a^b G_\alpha(t, r)\mathfrak{H}'(r)dr \\
&+ \Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r) \mid \int_a^b \left[G_\beta(r, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath \\
&- \int_a^b \left[G_\beta(a, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath \mid dr + C_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr \\
&\leq \Omega_1 \int_a^b G_\alpha(t, r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r, \jmath) - G_\beta(a, \jmath) \mid . \mid F_2(\jmath, u(\jmath)) \mid \mathfrak{H}'(\jmath)d\jmath)dr \\
&+ C_1 \int_a^b G_\alpha(t, r)\mathfrak{H}'(r)dr + \Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r, \jmath) - G_\beta(a, \jmath) \mid \\
&\times \mid F_2(\jmath, u(\jmath)) \mid \mathfrak{H}'(\jmath)d\jmath)dr + C_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr \\
&\leq \Omega_1 \int_a^b G_\alpha(t, r)\mathfrak{H}'(r)
\end{aligned}
$$

$$
\begin{aligned}
&\times (\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid\mid F_2(\jmath, u(\jmath)) - g(a,u(a)) + g(a,u(a)) \mid \mathfrak{H}'(\jmath)d\jmath)dr\\
&+ C_1 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)dr + \Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)\\
&\times (\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid\mid F_2(\jmath, u(\jmath)) - g(a,u(a)) + g(a,u(a)) \mid \mathfrak{H}'(\jmath)d\jmath)dr\\
&+ C_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr\\
&\leq \Omega_1 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid\mid F_2(\jmath,u(\jmath)) - F_2(a,u(a)) \mid \mathfrak{H}'(\jmath)d\jmath)dr\\
&+ \Omega_1 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid\mid F_2(a,u(a)) \mid \mathfrak{H}'(\jmath)d\jmath)dr + C_1 \int_a^b G_\alpha(t,r)dr\\
&+ \Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid\mid F_2(\jmath,u(\jmath)) - F_2(a,u(a)) \mid \mathfrak{H}'(\jmath)d\jmath)dr\\
&+ \Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid\mid F_2(a,u(a)) \mid \mathfrak{H}'(\jmath)d\jmath)dr\\
&+ C_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr\\
&\leq \Omega_1\Omega_2 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid\mid u(\jmath) - u(a) \mid \mathfrak{H}'(\jmath)d\jmath)dr\\
&+ \Omega_1 C_2 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid \mathfrak{H}'(\jmath)d\jmath)dr + C_1 \int_a^b G_\alpha(t,r)dr\\
&+ \Omega_2\Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid\mid u(\jmath) - u(a) \mid \mathfrak{H}'(\jmath)d\jmath)dr\\
&+ C_2\Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \mid G_\beta(r,\jmath) - G_\beta(a,\jmath) \mid \mathfrak{H}'(\jmath)d\jmath)dr + C_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr\\
&\leq 2\Omega_1\Omega_2(\lambda_\beta + 1) \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath)d\jmath)dr \parallel u \parallel\\
&+ \Omega_1 C_2(\lambda_\beta + 1) \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath)d\jmath)dr + C_1 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)dr\\
&+ \Omega_2\Omega_3(\lambda_\beta + 1) \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath)d\jmath)dr \parallel u \parallel\\
&+ C_2\Omega_3(\lambda_\beta + 1) \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b \Psi_\beta(\jmath)\mathfrak{H}'(\jmath)d\jmath)dr + C_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)dr\\
&\leq \Omega_1\Omega_2(\lambda_\beta + 1)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) \parallel u \parallel + \Omega_1 C_2(\lambda_\beta + 1)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a)\\
&+ C_1\Psi_{\alpha+1}(a) + \Omega_2\Omega_3(\lambda_\beta + 1)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) \parallel u \parallel\\
&+ C_2\Omega_3(\lambda_\beta + 1)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a) + C_3\Psi_{\alpha+1}(a)\\
&\leq R.
\end{aligned}
$$

This finishes the proof. □

Lemma 3.6. *Let* (I_1)–(I_6) *be hold. Then there exists an iterative convergent sequence of the CS (3.6).*

Proof. The operator P is equicontinuous and uniformly bounded by Lemma 3.5. Thanks to the Ascoli-Arzela theorem, we deduce that compact operator is P. Let the minimal solution of the Eq (3.8) be $u_0 = u_m$. Then, according to condition (I_6), we get $u_0 \leq u_M$. As P is an increasing operator, then we obtain $u_0 \leq Pu_0 \leq Pu_M \leq u_M$, that is, $u_0 \leq u_1 \leq u_M$ on the interval $[a, b]$, where $u_1 = Pu_0$ is an iterative results of Eq (3.8). By utilizing P, we obtain $Pu_0 \leq Pu_1 \leq Pu_M \leq u_M$ that is, $u_1 \leq u_2 \leq u_M$ on $[a, b]$, where $u_2 = Pu_1$. We get a bounded monotone sequence $\{u_n\}$ such that

$$u_0 \leq u_1 \leq u_2 \leq ... \leq u_{n-1} \leq u_n \leq u_M \;\; on \;\; [a, b], \tag{3.11}$$

where $u_n = Pu_{n-1}$ is result of Eq (3.8). As $\{u_n\}$ bounded monotone sequence, $\exists\, u \in W$ so that $u_n \to u$ as $n \to \infty$. Therefore $u = Pu$, is the result of Eq (3.6) given by:

$$\begin{aligned} u(t) = \;\; & \int_a^b [G_\alpha(t, r)F_1(r, \int_a^b \Big[G_\beta(r, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\Big]\, \mathfrak{H}'(\jmath)d\jmath) \\ & + \Psi_\alpha(r)\, F_3(r, \int_a^b \Big[G_\beta(r, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\Big]\, \mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr, \\ & t \in [a, b]. \end{aligned}$$

Hence, in view of (3.10) and (3.11), we get

$$\begin{aligned} \| u_2 - u_1 \| =& \| Pu_1 - Pu_0 \| \leq M \| u_1 - u_0 \|, \\ \| u_3 - u_2 \| =& \| Pu_2 - Pu_1 \| \leq M^2 \| u_1 - u_0 \|, \\ & \cdots \\ \| u_{n+1} - u_n \| =& \| Pu_n - Pu_{n-1} \| \leq M^n \| u_1 - u_0 \| . \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \| u_{m+n} - u_n \| \;\; & \leq \| u_{m+n} - u_{m+n-1} \| + \| u_{m+n-1} - u_{m+n-2} \| +...+ \| u_{n+1} - u_n \| \\ & \leq \Omega^n \frac{1 - \Omega^m}{1 - \Omega} \| u_1 - u_0 \|, \end{aligned} \tag{3.12}$$

for positive integers m and n. The condition $M < 1$, implies that $\| u_{m+n} - u_n \| \to 0$ when $n \to \infty$. Thus $\{u_n\}$ is a Cauchy sequence in W. Let $u^*(t) = \lim_{n\to\infty} u_n(t)$, thus $Pu^* = u^*$. Therefore, if $m \to \infty$ in (3.12), then error estimate for the minimal solution is

$$\| u^* - u_n \| \; \leq \frac{\Omega^n}{1 - \Omega} \| u_1 - u_0 \| .$$

Now, let us choose $u_0^* = u_M$, as the previous steps, we have a sequence $\{u_n^*\}$ so that $u_0^* \geq u_1^* \geq u_2^* \geq ... \geq u_{n-1}^* \geq u_n^* \geq u_m$ on $[a, b]$. This sequence converges to a solution $\bar{u}^*$ of the integral form (3.6). Thus, we may obtain an estimate error of the maximal result that is given by

$$\| u_n^* - \bar{u}^* \| \leq \frac{\Omega^n}{1 - \Omega} \| u_0^* - u_1^* \| .$$

Therefore, by employing Lemma 3.2, the iterative sequences approach for minimal and maximal solutions of (3.5) are:

$$u_n(t) = \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath, u_{n-1}(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u_{n-1}(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath)$$

$$+ \Psi_\alpha(r)\, F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath, u_{n-1}(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u_{n-1}(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr, \qquad n \geq 1.$$

$$v_n(t) = \int_a^b [G_\beta(t,r)F_2(r, \int_a^b [G_\alpha(r,\jmath)F_1(\jmath, v_{n-1}(\jmath)) + \Psi_\alpha(\jmath)h(\jmath, v_{n-1}(\jmath))] \mathfrak{H}'(\jmath)d\jmath)$$

$$+ \Psi_\beta(r)\, k(r, \int_a^b [G_\alpha(r,\jmath)F_1(\jmath, v_{n-1}(\jmath)) + \Psi_\alpha(\jmath)h(\jmath, v_{n-1}(\jmath))] \mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr, \qquad n \geq 1.$$

$$u_n^*(t) = \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath, u_{n-1}^*(\jmath)) + \Psi_\beta(\jmath))k(\jmath, u_{n-1}^*(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath)$$

$$+ \Psi_\alpha(r)\, F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath, u_{n-1}^*(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u_{n-1}^*(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr, \qquad n \geq 1.$$

$$v_n^*(t) = \int_a^b [G_\beta(t,r)F_2(r, \int_a^b [G_\alpha(r,\jmath)F_1(\jmath, v_{n-1}^*(\jmath)) + \Psi_\alpha(\jmath)h(\jmath, v_{n-1}^*(\jmath))] \mathfrak{H}'(\jmath)d\jmath)$$

$$+ \Psi_\beta(r)\, k(r, \int_a^b [G_\alpha(r,\jmath)F_1(\jmath, v_{n-1}^*(\jmath)) + \Psi_\alpha(\jmath)h(\jmath, v_{n-1}^*(\jmath))] \mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr, \qquad n \geq 1.$$

Hence, we have

$$u^*(t) = \lim_{n\to\infty} u_n(t), \quad v^*(t) = \int_a^b \left[G_\beta(t,r)F_2(r, u^*(r)) + \Psi_\beta(r)k(r, u^*(r))\right] \mathfrak{H}'(r)dr,$$

$$\bar{u^*}(t) = \lim_{n\to\infty} u_n^*(t), \quad \bar{v^*}(t) = \int_a^b \left[G_\beta(t,r)F_2(r, \bar{u^*}(r)) + \Psi_\beta(r)k(r, \bar{u^*}(r))\right] \mathfrak{H}'(r)dr.$$

This finishes the proof. □

Theorem 3.7. *Let* (I_1)–(I_6) *be hold. Then the CS* (3.5) *has unique minimal and maximal solutions.*

Proof. We prove firstly that (u^*, v^*) and $(\bar{u^*}, \bar{v^*})$ that are constructed using the iterative sequences u_n and u_n^* in Lemma 3.6 are the minimal and maximal solutions of (3.5). Choose an arbitrary element $w \in W$ with $Pw = w$ and $u_n \leq w \leq u_n^*$. As P is increasing and applying Lemma 3.6, we have $u^*(t) \leq w(t) \leq \bar{u^*}(t)$, and therefore, $u^*(t)$ and $\bar{u^*}(t)$ are the minimal and maximal fixed points of P respectively. Hence (u^*, v^*) and $(\bar{u^*}, \bar{v^*})$ are the minimal and maximal solution of (3.5) respectively. Next, for the concern of uniqueness property of minimal and maximal solutions of the system (3.5), let $u_m, u_M \in W$ be the minimal and maximal solutions of $Pu = u$ respectively. Then $u_m \leq Pu_m, u_M \geq Pu_M$. We use u_m and u_M as initial iterations respectively so that $u_{m_n} \to u_m^*$ and $u_{M_n} \to u_M^*, n \to \infty$. We also have $Pu_m^* = u_m^*, Pu_M^* = u_M^*$. To prove $u_m^* = u^*$, observe that $u_0 \leq u_m^*$ and P is increasing, so we have $u_n = P^n u_0 \leq P^n u_m^*$ for each $n \in \{1, 2, 3, ...\}$. Then $u_0 \leq u_1 \leq u_2 \leq ... \leq u_n \leq ... \leq u_m^*$. Therefore from (3.11) and mathematical induction, it is obvious that $\| u_m^* - u_n \| = \| P^n u_m^* - P^n u_0 \| \leq \Omega^n \| u_m^* - u_0 \| \to 0$ as $n \to \infty$. Therefore $\| u_m^* - u^* \| \to 0$ as $n \to \infty$, which gives $u^* = u_m^*$. using the same procedure, we have $\bar{u^*} = u_M^*$. Therefore, the minimal and maximal solutions of the CS (3.5) are unique. □

4. Generalized UH stability of the solutions of FDE (1.1)

The stability analysis of FD systems is the most important qualitative aspect in order to control such systems. This makes scientists focus on investigations of different types of stability analysis and specially the UH stability for fractional sense. In this section, we give sufficient conditions that make the fractional nonlinear system (1.1) is UH stable on the interval $[a, b]$. For this concern, let us recall the following auxiliary definitions [15].

Definition 4.1. The fractional system (1.1) is said to be UH stable if we can find a real number $C_{f,g} \geq 0$ with the property that, for every $\epsilon = \max\{\epsilon_1, \epsilon_2\} > 0$ with $\epsilon_1 > 0$ and $\epsilon_2 > 0$, and for every solution $(u, v) \in C[a, b] \times C[a, b]$ of the inequality

$$\begin{cases} |D^{\alpha,\mathfrak{H}}_{a+}u(t) + F_1(t, v(t))| \leq \epsilon_1, \quad t \in [a, b], \\ |D^{\beta,\mathfrak{H}}_{a+}v(t) + F_2(t, u(t))| \leq \epsilon_2, \quad t \in [a, b], \end{cases} \tag{4.1}$$

$\exists$ a unique solution $(\bar{u}, \bar{v}) \in C[a, b] \times C[a, b]$ of the proposed BVPs (1.1) with

$$\| (u, v) - (\bar{u}, \bar{v}) \| \leq C_{f,g}\epsilon, \quad t \in [a, b].$$

Definition 4.2. The fractional system (1.1) is said to be generalized UH stable if we can find $\Phi_{f,g} : [0, \infty) \longrightarrow [0, \infty)$ with $\Phi_{f,g}(0) = 0$, with the property that, for every $\epsilon = \max\{\epsilon_1, \epsilon_2\} > 0$ with $\epsilon_1 > 0$ and $\epsilon_2 > 0$, and for every solution $(u, v) \in C[a, b] \times C[a, b]$ of the inequality (4.1), $\exists$ a unique solution $(\bar{u}, \bar{v}) \in C[a, b] \times C[a, b]$ of the proposed BVPs (1.1) with

$$\| (u, v) - (\bar{u}, \bar{v}) \| \leq \Phi_{f,g}(\epsilon).$$

Remark 4.3. The pair $(u, v) \in C[a, b] \times C[a, b]$ is said to satisfy the inequality (4.1) if and only if we can find functions $p, q \in C[a, b]$ depending only on (u, v) respectively, that are satisfying
(i) $|p(t)| \leq \epsilon_1,\ |q(t)| \leq \epsilon_2$, for all $t \in [a, b]$, and
(ii) $D^{\alpha,\mathfrak{H}}_{a+}u(t) + F_1(t, v(t)) = p(t)$, for all $t \in [a, b]$ and $D^{\beta,\mathfrak{H}}_{a+}v(t) + F_2(t, u(t)) = q(t)$ for all $t \in [a, b]$.

Thanks to Remark 4.3 and Lemma 3.1 for $t \in [a, b]$, the considered solution (u, v) of the problem

$$\begin{cases} D^{\alpha,\mathfrak{H}}_{a+}u(t) + F_1(t, v(t)) = p(t), \quad t \in [a, b], \\ D^{\beta,\mathfrak{H}}_{a+}v(t) + F_2(t, u(t)) = q(t), \quad t \in [a, b], \\ u(b) + \lambda_\alpha u(a) = I^{\alpha,\mathfrak{H}}_{a+}F_3(T, v(b)), u'(a) = ... = u^{(n-1)}(a) = 0, \\ v(b) + \lambda_\beta v(a) = I^{\beta,\mathfrak{H}}_{a+}F_4(T, u(b)), v'(a) = ... = v^{(n-1)}(a) = 0, \end{cases}$$

given by

$$\begin{aligned} u(t) = \int_a^b &[G_\alpha(t, r) \\ &\times F_1(r, \int_a^b \left[G_\beta(r, \jmath)F_2(\jmath, u(\jmath)) + \Psi_\beta(\jmath)k(\jmath, u(\jmath))\right] \mathfrak{H}'(\jmath)d\jmath - \int_a^b G_\beta(r, \jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath) \end{aligned}$$

$$
\begin{aligned}
&+\Psi_\alpha(r)\,F_3(r,\int_a^b\Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath))+\Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big]\mathfrak{H}'(\jmath)d\jmath\\
&-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr-\int_a^b G_\alpha(t,r)p(r)\mathfrak{H}'(r)dr,\quad t\in[a,b],\\
v(t)=&\int_a^b[G_\beta(t,r)\\
&\times F_2(r,\int_a^b[G_\alpha(r,\jmath)F_1(\jmath,v(\jmath))+\Psi_\alpha(\jmath)h(\jmath,v(\jmath))]\,\mathfrak{H}'(\jmath)d\jmath-\int_a^b G_\alpha(r,\jmath)p(\jmath)\mathfrak{H}'(\jmath)d\jmath)\\
&+\Psi_\beta(r)\,k(r,\int_a^b[G_\alpha(r,\jmath)F_1(\jmath,v(\jmath))+\Psi_\alpha(\jmath)h(\jmath,v(\jmath))]\,\mathfrak{H}'(\jmath)d\jmath\\
&-\int_a^b G_\alpha(r,\jmath)p(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr-\int_a^b G_\beta(t,r)q(r)\mathfrak{H}'(r)dr,\quad t\in[a,b],
\end{aligned}
\tag{4.2}
$$

which satisfy the following inequalities

$$
\begin{aligned}
&|u(t)-\int_a^b[G_\alpha(t,r)F_1(r,\int_a^b\Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath))+\Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big]\mathfrak{H}'(\jmath)d\jmath\\
&-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)+\Psi_\alpha(r)\,F_3(r,\int_a^b\Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath))+\Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big]\mathfrak{H}'(\jmath)d\jmath\\
&-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr|\le\epsilon_1\Psi_{\alpha+1}(a),
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
&|v(t)-\int_a^b G_\beta(t,r)F_2(r,\int_a^b[G_\alpha(r,\jmath)F_1(\jmath,v(\jmath))+\Psi_\alpha(\jmath)\mathfrak{H}'(\jmath)h(\jmath,v(\jmath))]\,d\jmath-\int_a^b G_\alpha(r,\jmath)p(\jmath)d\jmath)\\
&+\Psi_\beta(r)\mathfrak{H}'(r)\,k(r,\int_a^b[G_\alpha(r,\jmath)F_1(\jmath,v(\jmath))+\Psi_\alpha(\jmath)\mathfrak{H}'(\jmath)h(\jmath,v(\jmath))]\,d\jmath-\int_a^b G_\alpha(r,\jmath)p(\jmath)d\jmath)dr|\\
&\le\epsilon_2\Psi_{\beta+1}(a),
\end{aligned}
\tag{4.4}
$$

for $t\in[a,b]$.

Theorem 4.4. *Let* (I_1)–(I_6) *be hold. Then the solutions of the CS (1.1) are UH stable and also generalized UH stable.*

Proof. Consider (u,v) be any solution of the inequality (4.1), then by Remark 4.3 and Lemma 3.1, the pair (u,v) also satisfies the inequalities (4.2)–(4.4). Now assume that $(\bar{u},\bar{v})$ is the unique solution of the CS (1.1), then

$$
\begin{aligned}
&\|\,\bar{u}-u\,\|\\
\le&\|\,\bar{u}-\int_a^b[G_\alpha(t,r)F_1(r,\int_a^b\Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath))+\Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big]\mathfrak{H}'(\jmath)d\jmath\\
&-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)+\Psi_\alpha(r)
\end{aligned}
$$

$$\times F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr)\parallel$$

$$+\parallel u - \int_a^b G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b [G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath) + \Psi_\alpha(r)$$

$$\times F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr\parallel$$

$$\leq\parallel \bar{u} - \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath) + \Psi_\alpha(r)$$

$$\times F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr)\parallel$$

$$+\parallel u - \int_a^b G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b [G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath) + \Psi_\alpha(r)$$

$$\times F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr\parallel$$

$$\leq\parallel \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,\bar{u}(\jmath)) + \Psi_\beta(\jmath)k(\jmath,\bar{u}(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)$$

$$+\Psi_\alpha(r)\,F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,\bar{u}(\jmath)) + \Psi_\beta(\jmath)k(\jmath,\bar{u}(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr$$

$$-[\int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath) + \Psi_\alpha(r)\,F_3(r, \int_a^b \left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]\mathfrak{H}'(\jmath)d\jmath$$

$$-\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr]\parallel$$

$$+ \parallel u - \int_a^b [G_\alpha(t,r)F_1(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big] \mathfrak{H}'(\jmath)d\jmath$$
$$- \int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath) + \Psi_\alpha(r)\, F_3(r, \int_a^b \Big[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\Big] \mathfrak{H}'(\jmath)d\jmath$$
$$- \int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)]\mathfrak{H}'(r)dr \parallel$$
$$\leq \Omega \parallel \bar{u} - u \parallel + \Omega_1 \int_a^b G_\alpha(t,r)\mathfrak{H}'(r)(\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)dr$$
$$+ \Omega_3 \int_a^b \Psi_\alpha(r)\mathfrak{H}'(r)(\int_a^b G_\beta(r,\jmath)q(\jmath)\mathfrak{H}'(\jmath)d\jmath)dr + \epsilon_1\Psi_{\alpha+1}(a). \tag{4.5}$$

Therefore from inequality (4.5)

$$\begin{aligned} &\parallel \bar{u} - u \parallel \leq \Omega \parallel \bar{u} - u \parallel + (\Omega_1 + \Omega_3)\Psi_{\alpha+1}(a)\Psi_{\beta+1}(a)\epsilon_2 + \epsilon_1\Psi_{\alpha+1}(a), \\ &\parallel \bar{u} - u \parallel \leq \epsilon C_f, \quad C_f = \frac{\Psi_{\alpha+1}(a)[1 + (\Omega_1 + \Omega_3)\Psi_{\beta+1}(a)]}{1 - M} > 0. \end{aligned} \tag{4.6}$$

By the same arguments, it is easy to prove that

$$\parallel \bar{v} - v \parallel \leq \epsilon C_g, \quad C_g = \frac{\Psi_{\beta+1}(a)[1 + (\Omega_2 + \Omega_4)\Psi_{\alpha+1}(a)]}{1 - M} > 0. \tag{4.7}$$

Therefore, from inequalities (4.6) and (4.7), we have

$$\parallel (u,v) - (\bar{u},\bar{v}) \parallel \leq \epsilon \max\{C_f, C_g\} = \epsilon C_{f,g}. \tag{4.8}$$

Therefore the solutions of the CS (1.1) are UH stable. Further, if $\Phi_{f,g}(\epsilon) = \epsilon$, hence (4.8) can be expressed as

$$\parallel (u,v) - (\bar{u},\bar{v}) \parallel \leq C_{f,g}\Phi_{f,g}(\epsilon). \tag{4.9}$$

Therefore, $\Phi_{f,g}(0) = 0$ in (4.9) hold. Thus, results of the CS (1.1) are showed generalized UH stable. This finishes the proof. □

5. Application

It is well-known that the Riemann-Liouville and Hadamard FD are frequently used as applications on fractional systems. Let us firstly recall the following remark.

Remark 5.1. If we take $\mathfrak{H}(t) = t$, then the FI and FD of a function u with respect to another function $\mathfrak{H}$, is just the respective FI and FD due to Riemann-Liouville.

If $\mathfrak{H}(t) = \ln t$, then the FI and FD of a function u with respect to another function $\mathfrak{H}$, is the Hadamard FI and FD.

Example 5.2. Let the FDE is given by:

$$\begin{cases} D_{0+}^{\alpha,(t+1)^2}u(t) + F_1(t,v(t)) = 0, \quad t \in [0,1], \\ D_{0+}^{\beta,(t+1)^2}v(t) + F_2(t,u(t)) = 0, \quad t \in [0,1], \\ u(1) - 0.5u(0) = I_{0+}^{\alpha,(t+1)^2}F_3(1,v(1)), u'(0) = ... = u^{(n-1)}(0) = 0, \\ v(1) - 0.5v(0) = I_{0+}^{\beta,(t+1)^2}k(1,u(1)), v'(0) = ... = v^{(n-1)}(0) = 0, \end{cases} \tag{5.1}$$

where $\mathfrak{H}(t) = (t+1)^2$ is an increasing and positive monotone function on $[0,\ 1]$ and $\mathfrak{H}'(t) = 2(t+1)$ is continuous and $\mathfrak{H}'(t) \neq 0$ for each $t \in [0,\ 1]$.

The FI of a function u with respect to another function $\mathfrak{H}(t) = (t+1)^2$ is given by

$$I_{0+}^{\alpha,(t+1)^2} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{2(s+1)u(r)dr}{[(t+1)^2 - (s+1)^2]^{1-\alpha}}, \quad t > 0.$$

The FD of a function u with respect to another function $\mathfrak{H}(t) = (t+1)^2$ is given by

$$D_{0+}^{\alpha,(t+1)^2} u(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{1}{2(t+1)} \frac{d}{dt})^n \int_0^t \frac{2(s+1)u(r)dr}{[(t+1)^2 - (s+1)^2]^{\alpha-n+1}}, \quad t > 0.$$

Let $U = C([0,1])$, be the Banach space with $\| u \| = \max_{t\in[0,1]} |u(t)|$. Then the CS (5.1) has the integral representation

$$\begin{cases} u(t) = \int_a^b 2(r+1)\left[G_\alpha(t,r)F_1(r,v(r)) + \frac{2[4-(s+1)^2]^{\alpha-1}}{\Gamma(\alpha)} F_3(r,v(r))\right] dr, \ t \in [0,1], \\ v(t) = \int_a^b 2(r+1)\left[G_\beta(t,r)F_2(r,u(r)) + \frac{2[4-(s+1)^2]^{\beta-1}}{\Gamma(\beta)} k(r,u(r))\right] dr, \ t \in [0,1], \end{cases} \tag{5.2}$$

where

$$G_\alpha(t,r) = \begin{cases} \frac{2[4-(s+1)^2]^{\alpha-1} - [(t+1)^2-(r+1)^2]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le r \le t \le 1, \\ \frac{2[4-(r+1)^2]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le r \le 1, \end{cases}$$

and

$$G_\beta(t,r) = \begin{cases} \frac{2[4-(r+1)^2]^{\beta-1} - [(t+1)^2-(r+1)^2]^{\beta-1}}{\Gamma(\beta)}, & 0 \le r \le t \le 1, \\ \frac{2[4-(r+1)^2]^{\beta-1}}{\Gamma(\beta)}, & 0 \le t \le r \le 1. \end{cases}$$

The functions G_α, G_β satisfy the following property:

$$\int_0^1 G_\alpha(t,r)\mathfrak{H}'(r)dr \le \frac{2(3^\alpha)}{\Gamma(\alpha+1)}, \quad \int_0^1 G_\beta(t,r)\mathfrak{H}'(r)dr \le \frac{2(3^\beta)}{\Gamma(\beta+1)}.$$

Also, we have

$$\Psi_\alpha(r) = \frac{2[4-(r+1)^2]^{\alpha-1}}{\Gamma(\alpha)}, \ \Psi_\beta(\jmath) = \frac{2[4-(\jmath+1)^2]^{\beta-1}}{\Gamma(\beta)}.$$

The CS (5.2) can be written as follows

$$\begin{cases} u(t) &= \int_0^1 2(r+1)\left[G_\alpha(t,r)F_1(r,v(r)) + \Psi_\alpha(r)\, F_3(r,v(r))\right] dr \\ &= \int_0^1 2(r+1)[G_\alpha(t,r)F_1(r, \int_0^1 2(\jmath+1)\left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right] d\jmath) \\ &+ \Psi_\alpha(r)F_3(r, \int_0^1 2(\jmath+1)\left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right] d\jmath)]dr, \\ v(t) &= \int_0^1 2(r+1)\left[G_\beta(t,r)F_2(r,u(r)) + \Psi_\beta(r)k(r,u(r))\right] dr \\ &= \int_0^1 2(r+1)[G_\beta(t,r)F_2(r, \int_0^1 2(\jmath+1)\left[G_\alpha(r,\jmath)F_1(\jmath,v(\jmath)) + \Psi_\alpha(\jmath)h(\jmath,v(\jmath))\right] d\jmath) \\ &+ \Psi_\beta(r)k(r, \int_0^1 2(\jmath+1)\left[G_\alpha(r,\jmath)F_1(\jmath,v(\jmath)) + \Psi_\alpha(\jmath)h(\jmath,v(\jmath))\right] d\jmath)]dr. \end{cases} \tag{5.3}$$

We define $P : W \to W$ where $W \subset U = C([0,1])$ is an operator given by

$$\begin{aligned} P(u(t)) &= \int_0^1 2(r+1)[G_\alpha(t,r)F_1(r, \int_0^1 2(\jmath+1)\left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]d\jmath) \\ &+ \Psi_\alpha(r)F_3(r, \int_0^1 2(\jmath+1)\left[G_\beta(r,\jmath)F_2(\jmath,u(\jmath)) + \Psi_\beta(\jmath)k(\jmath,u(\jmath))\right]d\jmath)]dr. \end{aligned} \tag{5.4}$$

Then P is an increasing operator that satisfies

$$\mid P(u_i(t)) - P(u_j(t)) \mid \le \frac{4(3^{\alpha+\beta})(\Omega_1+\Omega_3)(\Omega_2+\Omega_4)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \parallel u_i - u_j \parallel .$$

Moreover, the operator P is continuous and also satisfies

$$\mid P(u(t)) \mid \le \frac{2(3^{\alpha+\beta})\Omega_2(\Omega_1+\Omega_3)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \parallel u \parallel + \frac{2(3^{\alpha+\beta})C_2(\Omega_1+\Omega_3)}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{2(3^{\alpha})(C_1+C_3)}{\Gamma(\alpha+1)} \le R,$$

then the operator P is bounded.

By Lemma 3.6, if the conditions (I_1)–(I_6) are satisfied, then the solutions of the problem (5.1) can be given iteratively by a convergent sequence which converges to an integral solution. Moreover, using Theorem 3.7, the CS (5.1) has unique minimal and maximal results.

Example 5.3. Consider the following CS:

$$\begin{cases} D_{0+}^{\frac{5}{2},(t+1)^2} u(t) + \frac{t}{10e^t(t+2)^{12}} \frac{|v(t)|}{1+|v(t)|} = 0, \\ D_{0+}^{\frac{8}{3},(t+1)^2} v(t) + \frac{t}{20(t+3)^8} \frac{|u(t)|}{1+u^2(t)} = 0, v \\ u(1) - 0.5u(0) = I_{0+}^{\frac{5}{2},(t+1)^2} F_3(1, v(1)), u'(0) = ... = u^{(n-1)}(0) = 0, \\ v(1) - 0.5v(0) = I_{0+}^{\frac{8}{3},(t+1)^2} k(1, u(1)), v'(0) = ... = v^{(n-1)}(0) = 0, \end{cases} \tag{5.5}$$

with

$$\alpha = \frac{5}{2},\ \beta = \frac{8}{3},\ a = 0,\ T = 1,$$

$$F_1(t, v(t)) = \frac{t}{10e^t(t+2)^{12}} \frac{|v(t)|}{1+|v(t)|},\ F_2(t, u(t)) = \frac{t}{20(t+3)^8} \frac{|u(t)|}{1+u^2(t)}$$

$$F_3(t, v(t)) = \frac{\sqrt{t}}{10^2(t+2)^{10}} \frac{e^{v(t)}}{1+e^{v(t)}},\ F_4(t, u(t)) = \frac{\sqrt{t^3}}{20(t+3)^8} \frac{e^{u(t)}}{1+e^{u(t)}}.$$

We can check easily that all the assumptions of Theorem 3.7 are satisfied with

$$n = 3,\ \Omega_1 = 2.44 \times 10^{-5},\ \Omega_2 = 7.62 \times 10^{-6},\ \Omega_3 = 6.63 \times 10^{-6},\ \Omega_4 = 5.17 \times 10^{-6},$$

$$\Psi_{3.5}(0) = 9.38117,\ \Psi_{\frac{11}{3}}(0) = 24.8851,\ M = 9.265 \times 10^{-8} < 1.$$

Then by Theorem 3.7, the CS (5.5) has unique minimal and maximal solutions respectively.

The iterative sequences for the approximation of minimal and maximal solutions (u_n, v_n) and (u_n^*, v_n^*) respectively are:

$$\begin{cases} u_n(t) = \int_0^1 2(r+1)\left[G_\alpha(t,r)F_1(r, v_{n-1}(r)) + \Psi_\alpha(r)\, F_3(r, v_{n-1}(r))\right] dr \\ v_n(t) = \int_0^1 2(r+1)\left[G_\beta(t,r)F_2(r, u_{n-1}(r)) + \Psi_\beta(r)\, F_4(r, u_{n-1}(r))\right] dr, \end{cases} \tag{5.6}$$

and

$$\begin{cases} u_n^*(t) = \int_0^1 2(r+1)\left[G_\alpha(t,r)F_1(r, v_{n-1}^*(r)) + \Psi_\alpha(r)\, F_3(r, v_{n-1}^*(r))\right] dr, \\ v_n^*(t) = \int_0^1 2(r+1)\left[G_\beta(t,r)F_2(r, u_{n-1}^*(r)) + \Psi_\beta(r)\, F_4(r, u_{n-1}^*(r))\right] dr. \end{cases} \tag{5.7}$$

Hence, we have

$$u(t) = \lim_{n\to\infty} u_n(t), \quad v(t) = \int_0^1 2(r+1)\left[G_\beta(t,r)F_2(r, u(r)) + \Psi_\beta(r)k(r, u(r))\right] dr,$$

and

$$u^*(t) = \lim_{n\to\infty} u_n^*(t), \quad v^*(t) = \int_0^1 2(r+1)\left[G_\beta(t,r)F_2(r, u^*(r)) + \Psi_\beta(r)k(r, u^*(r))\right] dr.$$

Let $(u_0, v_0) = (-0.01, -0.01)$, and $(u_0^*, v_0^*) = (0.01, 0.01)$ is the minimal and maximal results of the system (5.5) respectively, therefore, the corresponding maximum error estimates ($n = 3$) are given as:

$$\| u - u_3 \| \quad \leq \frac{\Omega^3}{1-\Omega} \| u_1 - u_0 \| \leq 7.95 \times 10^{-22} \times \sup_{t\in[0,1]} |u_1(t) + 0.01| \leq 7.95 \times 10^{-24},$$

$$\| u^* - u_3^* \| \quad \leq \frac{\Omega^3}{1-\Omega} \| u_0^* - u_1^* \| \leq 7.95 \times 10^{-22} \times \sup_{t\in[0,1]} |0.01 - u_1^*(t)| \leq 7.95 \times 10^{-24}.$$

Since $M < 1$, the CS (5.5) is UH stable and then generalized UH stable.

6. Conclusions

In this paper, we investigate the existence of unique maximal and minimal solutions for a differential coupled system involving some generalized fractional derivative of arbitrary order using iterative technique. Moreover, the generalized Ulam-Hyers stability of the solution is also considered. We present examples to demonstrate consistency to the main results.

Actually, by using green's function and iterative technique, we considered a generalized fractional system of arbitrary order together with a new type of nonperiodic boundary integral conditions. This might be a novel approach that will provide substantial potential for developing more new ideas in this field.

The results of this paper can be extended to multiple system of fractional equations with nonlocal integrodifferential conditions. Indeed, this fractional system of multiple equations along with new boundary conditions can be considered and discussed. Finally, the results of this paper can be extended to by using proportional fractional derivative or conformable definitions. We leave such investigations of these topics as future work for interested readers.

Acknowledgments

The authors Aziz Khan and Thabet Abdeljawad would like to thank Prince Sultan University for paying the APC and the support through TAS research lab. This work was accomplished and reached the final steps through the sabbatical leave of Professor Mohammed M. Matar when he visited Professor T. Abdeljawad in the Department of Mathematics and Sciences in Prince Sultan University-Riyadh-Saudi Arabia in 2022-2023.

Conflict of interest

The authors declare that they have no conflicts of interests.

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