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Research article

On the exponential sums estimates related to Fourier coefficients of GL_3 Hecke-Maaß forms

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Abstract: Let F be a normlized Hecke-Maaß form for the congruent subgroup $\Gamma_0(N)$ with trivial nebentypus. In this paper, we study the problem of the level aspect estimates for the exponential sum

$$\mathscr{L}_F(\alpha) = \sum_{n\leq X} A_F(n,1)e(n\alpha).$$

As a result, we present an explicit non-trivial bound for the sum $\mathscr{L}_F(\alpha)$ in the case of $N = P$. In addition, we investigate the magnitude for the non-linear exponential sums with the level being explicitly determined from the sup-norm's point of view as well.

Keywords: Maaß forms; exponential sums; Fourier coefficients
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1. Introduction and statement of the main results

The estimation of the exponential sums with multiplicative coefficients is an ancient but vital topic in number theory. To be specific, let $a(n)$ be the coefficients of certain L-functions, or more general coefficients of arithmetic interest. We shall be interested in sums of the type

$$\sum_{n\leq X} a(n)e(n\alpha)$$

for any $\alpha \in \mathbb{R}$ and $X \geq 2$; see e.g., Hardy-Littlewood [4] and Montgomery-Vaughan [13] for the history. For any integer $N \geq 1$, let us put

$$\Gamma_0(N) = \left\{ g \in SL_3(\mathbb{Z}) : g \equiv \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \mod N \right\}.$$

Let $F(z)$ be a normlized Hecke-Maaß form of type $\nu = (\nu_1, \nu_2)$ for the congruent subgroup $\Gamma_0(N)$ with trivial nebentypus, which has a Fourier-Whittaker expansion with the Fourier coefficients $A_F(m, n)$. The Fourier coefficients of F and that of its contragredient $\widetilde{F}$ are related by $A_F(m, n) = A_{\widetilde{F}}(n, m)$ for any $(mn, N) = 1$, with $A_F(1, 1) = 1$. See, e.g., [15, Section 2] for definition and backgrounds. One important problem in number theory is to obtain a uniform non-trivial bound for the sum

$$\mathscr{L}_F(\alpha) = \sum_{n \le X} A_F(n, 1) e(n\alpha) \tag{1.1}$$

for any $\alpha \in \mathbb{R}$ and $X \ge 2$, which has its own interest and deep implications for Diophantine approximation, the moments of L-values and subconvexity etc; see e.g., [10, 12] for relevant descriptions and heuristics. It might be traced back to the work of the pioneering work of Miller [12], who considered the level one forms and gave the well-known estimate that, for any $\varepsilon > 0$,

$$\mathscr{L}_F(\alpha) \ll X^{\frac{3}{4}+\varepsilon} \tag{1.2}$$

uniformly in $\alpha \in \mathbb{R}$, but with the implied constant depending on F. Godber [2] has considered the situation where F is self-dual and arises as the symmetric square lift of a holomorphic newform f of weight $\kappa \in 2\mathbb{Z}_{>0}$ and trivial level, and showed that

$$\mathscr{L}_F(\alpha) \ll_\varepsilon X^{\frac{3}{4}+\varepsilon} \kappa^{\frac{1}{2}+\varepsilon} \tag{1.3}$$

for any $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Li-Young [11], on the other hand, considered the Maaß case, given the explicit dependence on the spectral parameter of the form F; for any $GL_2(\mathbb{Z})$- Maaß form of spectral parameter $t_f = \sqrt{\lambda - 1/4}$ with λ being the Laplacian eigenvalue, they obtained that

$$\mathscr{L}_F(\alpha) \ll_\varepsilon X^{\frac{3}{4}+\varepsilon} |t_f|^{\frac{2}{3}+\varepsilon},$$

and the exponent of t_f can be improved to $1/2$, if the Ramanujan conjecture is assumed. Later, their results were extended to the difficult case of general Maaß forms by Li [10], who showed that, for any tempered Maaß form F for $SL_3(\mathbb{Z})$, there holds that

$$\mathscr{L}_F(\alpha) \ll_\varepsilon X^{\frac{3}{4}+\varepsilon} \lambda_F^{\frac{5}{12}+\varepsilon},$$

and the exponent of λ_F can be improved to $1/4$, if the Ramanujan conjecture is assumed. Here, $\lambda_F = \prod_{i=1}^{3}(1 + |\alpha_i|)$, and the Langlands parameters $\alpha_1 = -\nu_1 - 2\nu_2 + 1$, $\alpha_2 = -\nu_1 + \nu_2$ and $\alpha_3 = 2\nu_2 + \nu_2 - 1$. The explicit dependence problem involving the spectral parameter was therefore solved completely in the case of $N = 1$.

In this note, we will be interested in a different point of view, where the level parameters of the forms are varying at different rate; we are dedicated to investigating the level aspect non-trivial bounds for $\mathscr{L}_F(\alpha)$. An open problem in number theory is that if one could establish a uniform non-trivial bound for the individual sum $\mathscr{L}_F(\alpha)$ with the level parameter associated to the form F explicitly determined; it seems that there are no accounts for this cause in the current literature. We are able to prove:

Theorem 1.1. *Let P be a prime. Let F be a normalized Hecke-Maaß form for $\Gamma_0(P)$ with trivial nebentypus. Then, for any $\alpha \in \mathbb{R}$ and $X \ge 2$, we have*

$$\mathscr{L}_F(\alpha) \ll X^{\frac{3}{4}+\varepsilon} P^{\frac{1}{2}+\varepsilon}, \tag{1.4}$$

where the implied constant merely depends on ε and the Langlands parameters α_i, $1 \le i \le 3$.

Another motivation in this paper is to investigate the bounds for non-linear exponential sums, which, however, is known to has significant implications for many problems in number theory (see e.g., [7, Appendix C] for relevant heuristic descriptions). Particularly, we obtain:

Theorem 1.2. *Let F be a Hecke-Maaß form for $SL_3(\mathbb{Z})$ underlying the symmetric square lift of a GL_2-newform of square-free level N. For any $\alpha \in \mathbb{R}^+$ and $0 < \beta < 2\alpha^{-1}X^{2-2\alpha}$, we then have*

$$\sum_{n \leq X} A_F(n,1)e(\beta n^\alpha) \ll \beta^{\frac{3}{2}} X^{\frac{1+3\alpha}{2}+\varepsilon} N^{\frac{1}{4}+\varepsilon}, \tag{1.5}$$

where the implied constant depends on ε and F.

Remark 1.1. *It would be interesting to prove an analogue of Theorem 1.1 for a Hecke-Maaß form with the level being any integer $N \geq 2$. A key point, however, is that, when the additive twist colludes with the level, i.e., $(N/(N,q),q) \neq 1$ in (2.9) below, the establishment of the level aspect Voronoĭ formula is rather tricky for the GL_3 forms, even for the special case where the forms arise as the symmetric square lifts from GL_2. In principle, Corbett's formula [1, Theorem 1.1] can cover this, but it requires a non-trivial analysis of the p-adic Bessel transforms which, however, becomes more involved in this case. See [1, 5] for relevant details.*

Remark 1.2. *Notice that, just lately Kumar-Mallesham-Singh [9] obtained the order $X^{3/4+9\alpha/28+\varepsilon}$ for the general GL_3 Hecke-Maaß forms of level one. Our result $X^{(1+3\alpha)/2+\varepsilon}$, however, is exhibited to be a strengthened upper-bound whenever $\alpha < 7/33$; the dependence of the level, on the other hand, is explicitly determined as well.*

Notations. *Throughout the paper, ε always denotes an arbitrarily small positive constant which might not be the same at each occurrence. $n \sim X$ means that $X/2 < n \leq X$. We also follow the notational convention that $e(x) = \exp(2\pi i x)$ and $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$. As usual, we denote by $S(m,n;c)$ the Kloosterman sum which is given in the following way $S(m,n;c) = \sum^*_{x \bmod c} e((m\overline{x}+nx)/c)$ for any positive integers m,n and c, where $*$ indicates that the summation is restricted to $(x,c) = 1$, and $\overline{x}$ is the inverse of x modulo c.*

2. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 after describing some preliminaries.

2.1. Voronoĭ formula

Let w be a compactly supported smooth function on $(0,\infty)$ and $\widetilde{w}$ be its Mellin transform. For any $\rho = 0, 1$, define

$$\gamma_\rho(s) = \prod_{j=1}^{3} \frac{\Gamma_\mathbb{R}(1+s+\alpha_j+\rho)}{\Gamma_\mathbb{R}(-s-\alpha_j+\rho)} \tag{2.1}$$

and set

$$\gamma_\pm(s) = \frac{1}{2}\left(\gamma_0(s) \mp i\gamma_1(s)\right).$$

Now, let

$$\Omega_w^{\pm}(x) = \frac{1}{2\pi i}\int_{(\sigma)} x^{-s}\gamma_{\pm}(s)\widetilde{w}(-s)\mathrm{d}s, \tag{2.2}$$

where $\sigma > -1 + \max\{-\Re\alpha_1, -\Re\alpha_2, -\Re\alpha_3\}$. We have the following Voronoĭ formula for Hecke-Maaß forms of prime level P (see [15]).

Lemma 2.1. *Let $w(x), A_F(m,n)$ be as before. Let $a, \overline{a}, q \in \mathbb{Z}$ with $q \neq 0$, $(a,q) = 1$ and $a\overline{a} \equiv 1 \bmod q$. Set $P^* = P/(P,q)$ and assume that $(P^*, q) = 1$. We then have*

$$\begin{aligned}\sum_{n\in\mathbb{Z}_{\neq 0}} &\frac{A_F(n,1)}{|n|} e\left(\frac{an}{q}\right) w\left(\frac{n}{X}\right)\\ &= \frac{\varepsilon(F) q\sqrt{P^*}}{X} \sum_{m\in\mathbb{Z}_{\neq 0}} \sum_{d|q} \frac{A_F(d,m)}{|md|} S\left(\overline{aP^*}, m; q/d\right) \Omega_w^{\pm}\left(\frac{md^2X}{q^3P^*}\right),\end{aligned} \tag{2.3}$$

where $\varepsilon(F)$ is a complex number of modulus one (depending on F).

2.2. A Fourier-Mellin transform

In the course of the paper, we shall evaluate the asymptotic expansions of the resulting Bessel-transforms after the Voronoĭ. In particular, we shall have a need of the following results concerning the Fourier-Mellin transforms of smooth functions; see e.g., [2, 11].

Lemma 2.2. *Let h be a smooth function, compactly supported on $[1/2, 5/2]$ with bounded derivatives. For any $t, \gamma \in \mathbb{R}$, define*

$$I = \int_0^{\infty} h(x) e(\gamma x) x^{it-1}\mathrm{d}x.$$

Then, if $|t| \geq 1$ and $|\gamma| \geq 1$, one has the following asymptotic formula that

$$I = \sqrt{2\pi}h\left(-\frac{t}{\gamma}\right)|t|^{-\frac{1}{2}} e\left(\frac{t(\log|t| - \log|\gamma| - 1)}{2\pi} + \frac{\mathrm{sgn}(\gamma)}{8}\right) + O\left(|t|^{-\frac{3}{2}}\right). \tag{2.4}$$

Moreover, for any $\gamma \in \mathbb{R}$, one has the crude bound

$$I \ll \left(1 + \frac{|t|}{1 + |\gamma|^{1+\varepsilon}}\right)^{-A} \tag{2.5}$$

for any $\varepsilon > 0$ and sufficiently large $A \in \mathbb{Z}_{>0}$.

2.3. Gamma functions and the stationary phase

For the applications after a while, we shall need the following estimations involving the Gamma functions and the stationary phase method concerning the exponential integrals.

Lemma 2.3. *We have the asymptotic formula*

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + \sum_{m=1}^{M} \frac{c_m}{s^{2m-1}} + O\left(\frac{1}{|s|^{2M}}\right) \tag{2.6}$$

for any $M \in \mathbb{Z}_{>0}$ and some constants $c_m \in \mathbb{C}$. Let $0 < \delta < \pi$ be a fixed number. Write $s = \sigma + it$. In the sector $\arg s < \pi - \delta$*, we then have*

$$\Gamma(s) = \sqrt{2\pi} \exp\left(\left(s - \frac{1}{2}\right)\log s - s\right)\left(1 + O\left(\frac{1}{|s|}\right)\right).$$

Particularly, in the vertical strip $0 \le A_1 \le \sigma \le A_2$ *and* $|t| > 1$*, one has*

$$\Gamma(s) = \sqrt{2\pi}\, t^{s-\frac{1}{2}} \exp\left(-\frac{\pi t}{2} - it + \frac{\pi}{2}\left(\sigma - \frac{1}{2}\right)i\right)\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{2.7}$$

and

$$|\Gamma(s)| = \sqrt{2\pi}\, t^{\sigma-\frac{1}{2}} \exp\left(-\frac{\pi|t|}{2}\right)\left(1 + O\left(\frac{1}{|t|}\right)\right).$$

Lemma 2.4. *For any $a, b \in \mathbb{R}$ with $b < a$, let f, g be two smooth real valued functions on* $[b, a]$*. Then, for any $r \in \mathbb{Z}_{>0}$, we have*

$$\int_b^a g(x)e(f(x))\mathrm{d}x \ll \frac{Var(g)}{\min|f^{(r)}(x)|^{\frac{1}{r}}}.$$

Here $Var(g) = \sup V(g;T)$ is the total variation of g on $[b, a]$, where T is any division of the interval $[b, a]$, i.e., $T : b = x_0 < x_1 < \cdots < x_{n-1} < x_n = a$, and $V(g;T) = \sum_{i=1}^{n}|g(x_i) - g(x_{i-1})|$.

2.4. *Proof of the main theorem*

This subsection is devoted to the proof of Theorem 1.1. We introduce a non-oscillating smooth function $V(n)$ supported on $[1/2, 5/2]$ with the values in $[0, 1]$, which equals 1 on $(1, 2]$, and satisfies that $V^{(j)} \ll_j 1$ for any $j \in \mathbb{Z}_{\ge 0}$. By the unsmoothing procedure (see e.g., [11, Section 8]), it suffices to estimate

$$\mathscr{L}_F^*(\alpha) = \sum_{n \in \mathbb{Z}_{>0}} V\left(\frac{n}{X}\right)A_F(n, 1)e(n\alpha).$$

Fix $Q \ge 1$. By Dirichlet's theorem on Diophantine Approximation, one might write

$$\alpha = \frac{l}{q} + \gamma, \; |\gamma| \le \frac{1}{qQ} \tag{2.8}$$

for some $l, q \in \mathbb{Z}$, with $(l, q) = 1$ and $1 \le q \le Q$. We thus re-write $\mathscr{L}_F^*$ as

$$\mathscr{L}_F^*(\alpha) = \sum_{n \in \mathbb{Z}_{>0}} V^\flat\left(\frac{n}{X}\right)A_F(n, 1)e\left(\frac{ln}{q}\right) \tag{2.9}$$

with $V^\flat(x) = V(x)e(\gamma xX)$.

In what follows, our tactic is to apply the Voronoĭ formula, Lemma 2.1. We shall restrict to $Q \le X^{2/3-\varepsilon}P^{1/3}$, and artificially assume that $\gamma > 0$; taking conjugates, if necessary, we get the opposite situation. Notice that, whenever $(q, P) \ne 1$, the Voronoĭ formula can still be put into use. Indeed, it turns out that our analysis in this paper still goes through with a slight modification, which indicates the less importance of this case, as far as the final contribution is concerned. As such, we shall proceed to present our analysis merely in the co-prime situation. Appealing to the Voronoĭ formula in (2.3), one finds $\mathscr{L}_F^*$ is (essentially) converted into

$$\sqrt{P}q \sum_{\substack{\pm,n,d:\\ n,d\in\mathbb{Z}_{>0}:\\ d|q}} \frac{A_F(d,n)}{dn} S(\overline{lP}, \pm n; q/d)\, \Omega^{\pm}_{V^\flat}\left(\frac{nd^2X}{q^3P}\right), \tag{2.10}$$

where the Bessel-transform $\Omega^{\pm}_{V^\flat}$ is defined as in (2.2). Notice that, by Lemma 2.2, it follows that

$$\widetilde{V^\flat}(-s) \ll \left(1 + \frac{|t|}{1 + |\gamma X|^{1+\varepsilon}}\right)^{-A}$$

for any sufficiently large $A \in \mathbb{Z}_{>0}$. We now turn to the evaluations of the Gamma factors of $\gamma_{\pm}(s)$. By (2.6) in Lemma 2.3, one verifies that, for any $C \in \mathbb{R}$ such that $C > \max\{-1-\sigma, \sigma\}$,

$$\frac{\Gamma_{\mathbb{R}}(1+\sigma+it+C)}{\Gamma_{\mathbb{R}}(-\sigma-it+C)} \ll \left(\frac{\sqrt{C^2+t^2}}{2}\right)^{\sigma+\frac{1}{2}} \left(1 + \frac{P_1(C,t)}{C^2+t^2}\right), \tag{2.11}$$

up to a polynomial $P_1(C,t)$ of degree one in two variables C, t. Upon recalling (2.1), this, in turn, yields

$$\gamma_{\pm}(s) \ll (1+|t|^3)^{\sigma+\frac{1}{2}},$$

where $s = \sigma + it$, and the implied constant depends on σ and the Langlands parameters α_i, $1 \leq i \leq 3$. We might thus find out the following upper-bound for $\Omega^{\pm}_{V^\flat}$:

$$\begin{aligned}\Omega^{\pm}_{V^\flat}(y) &\ll \int_{-\infty}^{\infty} y^{-\sigma}(1+|t|^3)^{\sigma+\frac{1}{2}} \left(1 + \frac{|t|}{1+|\gamma X|^{1+\varepsilon}}\right)^{-A} \mathrm{d}t \\ &\ll \left(1 + |\gamma X|^{\frac{5}{2}+\varepsilon}\right)\left(\frac{1+|\gamma X|^{3+\varepsilon}}{|y|}\right)^{\sigma}.\end{aligned} \tag{2.12}$$

It is clear that

$$|y| \ll 1 + (\gamma X)^{3+\varepsilon}, \tag{2.13}$$

upon taking σ sufficiently large. Notice that, the estimate $\ll 1 + |\gamma X|^{5/2+\varepsilon}$ above fails to provide a non-trivial bound for $\mathscr{L}_F^*(\alpha)$. We shall have to proceed to refine the analysis by distinguishing two scenarios:

Case I. $\gamma X \ll X^{\varepsilon}$. One finds, in this case, q is relatively large such that $q > X^{1-\varepsilon}/Q$. From (2.12), it is clear that $\Omega^{\pm}_{V^\flat}(y) \ll X^{\varepsilon}$. Exploiting the Weil bound for individual Kloosterman sums thus shows that the expression in (2.10) is dominated by

$$\ll X^{\varepsilon} \sqrt{P} q^{\frac{3}{2}} \sum_{\substack{\pm,n,d:\\ n,d\in\mathbb{Z}_{>0}:\\ nd^2 \ll \mathfrak{H}_{(n,d)}\\ d|q}} \frac{|A_F(d,n)| \sqrt{(n, q/d)}}{d^{\frac{3}{2}} n} \ll \sqrt{P} q^{\frac{3}{2}+\varepsilon} \tag{2.14}$$

with $\mathfrak{H}_{(n,d)} = q^3PX^{-1+\varepsilon} \gg 1$, upon noticing the hierarchy that $X^{1-\varepsilon}Q^{-1} > X^{1/3+\varepsilon}P^{-1/3}$ by the restriction on the parameter Q.

Case II. $\gamma X \gg X^{\varepsilon}$. Now, we come to coping with the case where γ is suitably large such that $\gamma > X^{-1+\varepsilon}$. One sees that, by (2.4),

$$\widetilde{V^{\flat}}(-s) \asymp V\left(\frac{t}{\gamma X}\right)|t|^{-\frac{1}{2}}e\left(\frac{-t(\log|t| - \log\gamma - 1)}{2\pi}\right).$$

On the other hand, it follows from (2.7) that, for any $C \in \mathbb{R}$ with $C > \max\{-1-\sigma, \sigma\}$ and $|t| > 1$, there instead holds the following asymptotic formula

$$\frac{\Gamma_{\mathbb{R}}(1+\sigma+it+C)}{\Gamma_{\mathbb{R}}(-\sigma-it+C)} \asymp |t|^{\sigma+\frac{1}{2}}e\left(\frac{t(\log|t| - \log 2\pi - 1)}{2\pi}\right), \tag{2.15}$$

which in turn, implies that

$$\gamma_{\pm}(s) \asymp t|^{3\sigma+\frac{3}{2}}e\left(\frac{3t(\log|t| - \log 2\pi - 1)}{2\pi}\right)$$

with $|t| > 1$. Recall (2.2). Shifting the line $\Re s = \sigma$ to $\Re s = -1/2$ eventually allows us to estimate $\Omega^{\pm}_{V^{\flat}}$ as

$$\Omega^{\pm}_{V^{\flat}}(y) \asymp \sqrt{|y|}\int_{-\infty}^{\infty}|y|^{-it}\, e\left(\frac{2t\log|t| - t(3\log 2\pi - \log\gamma + 2)}{2\pi}\right) \times V\left(\frac{t}{\gamma X}\right)|t|^{-\frac{1}{2}}\mathrm{d}t. \tag{2.16}$$

At the moment, if one defines

$$f(t) = 2t\log t - t\left(3\log 2|y|\pi - \log\gamma + 2\right),$$

we see that $|f''(t)| \gg t^{-1-\varepsilon}$ for any $\varepsilon > 0$. It thus follows that the right-hand side of (2.16) is controlled by

$$\ll_{\varepsilon} \sqrt{|y|}\, P^{\varepsilon} \ll_{\varepsilon} (\gamma X)^{\frac{3}{2}+\varepsilon},$$

upon making use of Lemma 2.4 with $r = 2$, and recalling (2.13). Having this in hand, an adaptation of the procedure as in Case I enable us to find (2.10) is bounded by

$$\ll \sqrt{P}q^{\frac{3}{2}} \sum_{\substack{\pm,n,d:\\ n,d\in\mathbb{Z}_{>0}:\\ nd^2 \ll \mathfrak{I}_{(n,d)}\\ d|q}} \frac{|A_F(d,n)|\sqrt{(n,q/d)}\,(\gamma X)^{\frac{3}{2}+\varepsilon}}{d^{\frac{3}{2}}n} \ll \frac{\sqrt{P}X^{\frac{3}{2}+\varepsilon}}{Q^{\frac{3}{2}+\varepsilon}},$$

upon recalling (2.8). Here, $\mathfrak{I}_{(n,d)} = q^3PX^{-1}(\gamma X)^{3+\varepsilon}$ which is $\gg 1$. One collects the bound in (2.14); this implies that totally

$$\mathscr{L}^{*}_{F}(\alpha) \ll \sqrt{P}q^{\frac{3}{2}+\varepsilon} + \frac{\sqrt{P}X^{\frac{3}{2}+\varepsilon}}{Q^{\frac{3}{2}+\varepsilon}},$$

which gives the desired quantity as in (1.4), upon taking $Q = X^{1/2+\varepsilon}$.

3. Proof of Theorem 1.2

In the final section of this paper, we are left with the proof of Theorem 1.2.

3.1. Auxiliary lemmas

To start with, we shall elaborate some preliminaries in the following lemmas:

Lemma 3.1. *For any $z = x + iy$ belonging to the complex upper half-plan and any GL_2-newform f of square-free level N, there hold that*

$$|f(x+iy)| \ll y^{-\frac{1}{2}} N^{\varepsilon}$$

and

$$\|f(z)\|_{\infty} \ll N^{\frac{1}{4}+\varepsilon}.$$

Proof. See [14, Section 2] and [8, Corollary 1.8], respectively. □

Lemma 3.2. *For any $a, b \in \mathbb{R}$ with $a \leq b$, let $h(t)$ be a real function, which satisfies that $0 < \Lambda \leq h'' \leq \vartheta\Lambda$ on $[a, b]$ for certain constant $\vartheta > 0$. Then, we have*

$$\sum_{a \leq n \leq b} e(h(n)) \ll \vartheta\sqrt{\Lambda}(b-a) + \frac{1}{\sqrt{\Lambda}},$$

where the implied constant is absolute.

Proof. See e.g., [6, Corollary 8.13]. □

Lemma 3.3. *For any $\alpha \in \mathbb{R}$ and $\ell \in \mathbb{R}^+$ satisfying that $0 < \alpha < 1$ and $0 < \ell < 2\alpha^{-1}X^{2-\alpha}$, one then has*

$$\int_0^1 \left|\sum_{n\sim X} e\left(\beta n^2 - \ell n^{\alpha}\right)\right| \mathrm{d}\beta \ll \frac{\ell^{\frac{3}{2}}}{X^{2-\frac{3\alpha}{2}-\varepsilon}} + \frac{\sqrt{\ell}}{X^{1-\frac{\alpha}{2}-\varepsilon}} + \frac{1}{X^{1-\varepsilon}} \tag{3.1}$$

for any $\varepsilon > 0$.

Proof. For any $t \in \mathbb{R}^+$, set $\rho(t) = \beta t^2 - \ell t^{\alpha}$. One might find the stationary phase point occurs at

$$n_0 = \left(\frac{\alpha\ell}{2\beta}\right)^{\frac{1}{2-\alpha}},$$

which reflects that β is located in the neighborhood of $\alpha\ell/(2X^{2-\alpha})$. Thus, in the transition range

$$\left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1-\varepsilon}} \leq \beta \leq \left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1+\varepsilon}}, \tag{3.2}$$

it can be verified that $2\beta(1 + (1-\alpha)\beta^{-\varepsilon}) \leq \rho''(t) \leq 2\beta(1 + (1-\alpha)\beta^{\varepsilon})$. By invoking Lemma 3.2, one finds that

$$\sum_{n\sim X} e\left(\beta n^2 - \ell n^{\alpha}\right) \ll \sqrt{\beta}X^{1+\varepsilon} + \frac{X^{\varepsilon}}{\sqrt{\beta}}. \tag{3.3}$$

This shows that the integral on the left-hand side of (3.1) in the segment (3.2) is controlled by

$$\ll \int_{\left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1-\varepsilon}}}^{\left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1+\varepsilon}}} \left(\sqrt{\beta}X^{1+\varepsilon} + \frac{X^{\varepsilon}}{\sqrt{\beta}}\right)\mathrm{d}\beta \ll \frac{\ell^{\frac{3}{2}}}{X^{2-\frac{3\alpha}{2}-\varepsilon}} + \frac{\sqrt{\ell}}{X^{1-\frac{\alpha}{2}-\varepsilon}}. \tag{3.4}$$

While, in the complementary ranges $0 < \beta < \left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1-\varepsilon}}$ and $\left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1+\varepsilon}} < \beta \leq 1$, by Lemma 2.4 with $r = 1$, it follows that the n-sum on the left-hand side of (3.1) can be estimated as

$$\ll \sup_{n\sim X} \frac{1}{|2n\beta - \alpha\ell n^{\alpha-1}|} \ll \frac{1}{|2\beta X - \alpha\ell X^{\alpha-1}|}.$$

One may thus quickly verify the following inequalities that

$$\int_0^{\left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1-\varepsilon}}} \left|\sum_{n\sim X} e\left(\beta n^2 - \ell n^{\alpha}\right)\right| \mathrm{d}\beta \ll \int_0^{\left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1-\varepsilon}}} \frac{X^{1-\alpha}}{\ell}\mathrm{d}\beta \ll \frac{1}{X^{1-\varepsilon}},$$

$$\int_{\left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1+\varepsilon}}}^{1} \left|\sum_{n\sim X} e\left(\beta n^2 - \ell n^{\alpha}\right)\right| \mathrm{d}\beta \ll \int_{\left(\frac{\alpha\ell}{2X^{2-\alpha}}\right)^{\frac{1}{1+\varepsilon}}}^{1} \frac{1}{\beta X}\mathrm{d}\beta \ll \frac{1}{X^{1-\varepsilon}}.$$

Finally, the assertion of the lemma follows by combining with these two upper-bounds and (3.4). □

3.2. Completing the proof of Theorem 1.2

In this part, based on the lemmas above, we are ready to complete the proof of Theorem 1.2. Akin to [3, (45)], one finds that, for any $\varepsilon > 0$,

$$\frac{\lambda_f(n^2)}{n^{1+2\varepsilon}} = \frac{1}{\Gamma\left(\frac{\kappa}{2} + \varepsilon\right)} \int_0^1 e(-n^2\beta) \int_0^{\infty} y^{\varepsilon} f(\beta + iy)\frac{\mathrm{d}y}{y}\mathrm{d}\beta,$$

if f is a holomorphic newform of weight $\kappa \in 2\mathbb{Z}_{>0}$. It thus turns out that, for any $\ell \in \mathbb{R}^+$,

$$\sum_{n\sim X} \frac{\lambda_f(n^2)e(\ell n^{\alpha})}{n^{1+2\varepsilon}} = \frac{1}{\Gamma\left(\frac{\kappa}{2} + \varepsilon\right)} \int_0^1 \sum_{n\sim X} e(\ell n^{\alpha} - n^2\beta) \int_0^{\infty} y^{\varepsilon} f(\beta + iy)\frac{\mathrm{d}y}{y}\mathrm{d}\beta; \tag{3.5}$$

one instead has

$$\begin{aligned} &\frac{\pi^{\frac{1}{2}+\varepsilon}}{4\Phi\left(\frac{1}{2} + \varepsilon, it_f\right)} \int_0^1 \sum_{n\sim X} e\left(\ell n^{\alpha} - n^2\beta\right) \\ &\qquad\qquad \times \int_0^{\infty} y^{\varepsilon}\left(f(\beta + iy) \pm f(-\beta + iy)\right)\frac{\mathrm{d}y}{y}\mathrm{d}\beta \end{aligned} \tag{3.6}$$

on the right-hand side of (3.5), if f is a Maaß newform of spetrcal parameter t_f. Here, Φ is defined as in [3, (17)], which is of independence of the parameter X and the level N.

We shall now merely consider the expression in (3.5); an entire analogous fashion gives the same magnitude as that for (3.6). By Lemma 3.1, it can be verified that

$$\int_0^{\infty} y^{\varepsilon} f(\beta + iy)\frac{\mathrm{d}y}{y} \ll \int_0^1 y^{-1+\varepsilon}\|f\|_{\infty}\,\mathrm{d}y + N^{\varepsilon}\int_1^{\infty} y^{-\frac{3}{2}+\varepsilon}\mathrm{d}y \ll N^{\frac{1}{4}+\varepsilon}.$$

Recall Lemma 3.3. Upon incorporating the estimate (3.1) into (3.5), we are thus allowed to arrive at

$$\sum_{n\sim X}\lambda_f(n^2)e(\ell n^\alpha)\ll N^{\frac14+\varepsilon}X^\varepsilon\left(\frac{\ell^{\frac32}}{X^{1-\frac{3\alpha}{2}}}+\sqrt{\ell}X^{\frac{\alpha}{2}}+1\right)$$

for any $0<\ell<2\alpha^{-1}X^{2-\alpha}$. From this, it can be inferable that, whenever $0<\beta<2\alpha^{-1}X^{2-2\alpha}$, one has

$$\begin{aligned}&\sum_{n\sim X}A_F(n,1)e(\beta n^\alpha)\\&=\sum_{\sqrt{X}<h\leq\sqrt{2X}}\ \sum_{m\leq X/h^2}\lambda_f(m^2)e\Big(\beta h^{2\alpha}m^\alpha\Big)\\&\ll N^{\frac14+\varepsilon}X^\varepsilon\sum_{\sqrt{X}<h\leq\sqrt{2X}}\Big(\beta^{\frac32}X^{3\alpha-1}h^{2-3\alpha}+\sqrt{\beta}X^\alpha h^{-\alpha}+1\Big)\\&\ll\beta^{\frac32}X^{\frac{1+3\alpha}{2}+\varepsilon}N^{\frac14+\varepsilon}\end{aligned}$$

which directly leads to (1.5).

4. Conclusions

In this paper, we investigate the exponentials sums involving Fourier coefficients of GL_3 Hecke-Maaß forms in the level aspect, and attain a non-trivial explicit bound for the first time. As remarked before, one might not easily circumvent the issue that the additive twist colludes with the level to establish a version of the level aspect Voronoĭ formula, on account of the extra complexities of the calculations of the resulting p-adic Bessel transforms. However, in the special case where the forms arise as the symmetric square lifts from GL_2, this can be done by brute force. See [5] or the discussions on Mathoverflow (*URL: https://mathoverflow.net/questions/337721/voronoi-formula-for-the-symmetric-l-function-with-level-n?r=SearchResults*) for relevant heuristics. On the other hand, in this paper, we achieve a sharp bound for the non-linear exponential sums compared with Kumar-Mallesham-Singh's result, whenever α is suitably small such that $\alpha<7/33$. This essentially benefits from the feature that the Hecke-Maaß form arises as the symmetric square lift of a GL_2 newform.

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Conflict of interest

The author declares that he has no conflicts of interest.

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