Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 5080–5098.
DOI:10.3934/math.2023255
Received: 27 September 2022
Revised: 18 November 2022
Accepted: 05 December 2022
Published: 13 December 2022

Research article

Fixed point approach to solve nonlinear fractional differential equations in orthogonal $\mathcal{F}$-metric spaces

Abdullah Eqal Al-Mazrooei[1] **and Jamshaid Ahmad**[2,*]

[1] Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics, University of Jeddah, P. O. Box 80327, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: jkhan@uj.edu.sa.

Abstract: In this paper, we introduce the notion of a generalized (α,$\Theta_{\mathcal{F}}$)-contraction in the context of an orthogonal $\mathcal{F}$-complete metric space and obtain some new fixed point results for this newly introduced contraction. A nontrivial example is also provided to satisfy the validity of the established results. As consequences of our obtained results, we derive the leading results in [*Fixed Point Theory Appl.*, 2015, 185, 2015] and [*Symmetry*, 2020, 12, 832]. As an application, we investigate the existence and uniqueness of the solution for a nonlinear fractional differential equation.

Keywords: orthogonal set; orthogonal $\mathcal{F}$-metric space; generalized $(\alpha, \Theta_{\mathcal{F}})$-contraction; fixed point; nonlinear fractional differential equation
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

In different fields of pure and applied sciences, the analysis of metric spaces has played an essential role. We can discover various effective and impressive applications of metric spaces in different fields of sciences such as mathematics, computer science, medicine, economics, physics and biology [1–3]. Various researchers generalized, improved and extended the notion of metric spaces [4] to b-metric spaces of Czerwik [5], generalized metric spaces of Branciari [6], $\mathcal{F}$-metric spaces of Jleli et al. [7], orthogonal metric spaces of Gordji et al. [8], orthogonal $\mathcal{F}$-metric spaces of Kanwal et al. [9] and others.

The famous extensions of the concept of metric spaces have been done by Bakhtin [10] and were formally defined by Czerwik [5] in 1993. Czerwik [5] gave the idea of a b-metric space, which broadens the notion of a metric space by improving the triangle inequality metric axiom by putting a constant $s \geq 1$ multiplied on the right-hand side, and it is one of the enormous applied extensions for metric spaces. Khamsi et al. [11] reintroduced this notion under the name metric-type and proved some fixed point results in this newly introduced space. In [6], Branciari gave the notion of a rectangular metric space and generalized the classical metric space by replacing the triangle inequality with a more general inequality that is called a rectangular inequality. This inequality involves distances of four points. In 2018, Jleli et al. [7] gave an important extension of metric space, b-metric space and rectangular metric space which is known as $\mathcal{F}$-metric space. Subsequently, Kanwal et al. [9] initiated the theory of orthogonal $\mathcal{F}$-metric spaces and established some common fixed point results. For more details in this direction, we refer the readers to [12–25].

On the other hand, Stefan Banach [26] introduced the concept of a contraction in the background of complete metric spaces and proved a fixed point result which is known as the Banach contraction principle. It has been extremely convenient in numerous fields, such as optimization problems, differential equations, control theory and many other fields. A number of research articles in this field have been devoted to the generalization and improvement of this result in different directions (see [27, 28]).

In 2014, Jleli et al. [29] generalized the concept of a contraction by introducing a new type of contraction named Θ-contraction in the framework of generalized metric spaces. Ahmad et al. [30] replaced the third condition of a Θ-contraction with a weaker condition and established some theorems in the setting of metric spaces. Later on, Hussain et al. [31] extended this notion of Θ-contraction by adding a general condition in it and obtained some fixed point results. For more characteristics in this way, we refer the readers to [32–36].

Motivated and inspired by the results of Kanwal et al. [9] and Hussain et al. [31], we give the concept of a generalized $(\alpha, \Theta_{\mathcal{F}})$-contraction in the context of an orthogonal $\mathcal{F}$-complete metric space and obtain contemporary common fixed point theorems which enable us to show the uniqueness and existence of the solution for a fractional differential equation.

2. Preliminaries

Frechet [4] introduced the concept of a metric space in this way.

A metric on $\aleph \neq \emptyset$ is a mapping $\nu : \aleph \times \aleph \to [0, +\infty)$ satisfying these properties:

(i) $\nu(\rho, \varsigma) \geq 0$, and $\nu(\rho, \varsigma) = 0$ if and only if $\rho = \varsigma$,
(ii) $\nu(\rho, \varsigma) = \nu(\varsigma, \rho)$,
(iii) $\nu(\rho, \omega) \leq \nu(\rho, \omega) + \nu(\omega, \varsigma)$,

for all $\rho, \varsigma \in \aleph$. If ν is a metric, then $(\aleph, \nu)$ is called a metric space.

In 1993, Czerwik [5] gave the notion of a b-metric by take into consideration this assertion instead of the triangular inequality:

For all $\rho, \varsigma \in \aleph$ and $b \geq 1$,

$$\nu(\rho, \omega) \leq b \left[\nu(\rho, \omega) + \nu(\omega, \varsigma)\right].$$

In 2017, Gordji et al. [8] introduced the notion of the orthogonal set (O-set).

Definition 1. *A non empty $\aleph$ is called an orthogonal set if there exists a binary relation $\perp \subseteq \aleph \times \aleph$ satisfying the condition*

$$\textit{there exists } \rho_0[(\textit{for all } \varsigma \in \aleph, \varsigma \perp \rho_0) \quad \textit{or} \quad (\textit{for all } \varsigma \in \aleph, \rho_0 \perp \varsigma)],$$

Furthermore, ρ_0 is called an orthogonal point. We denote this O-set by $(\aleph,\perp)$.

Gordji et al. [8] considered $\aleph$ as an O-set in metric space and gave the concept of an orthogonal metric space.

Definition 2. *(see [8]) Let $(\aleph,\perp)$ be an O-set. A sequence $\{\rho_n\}$ is said to be an orthogonal sequence if*

$$(\text{for all } n \in \mathbb{N}, \rho_n \perp \rho_{n+1}) \ \text{ or } (\text{for all } n \in \mathbb{N}, \rho_{n+1} \perp \rho_n).$$

We denote an orthogonal sequence by O-sequence.

Recently, Jleli et al. [7] introduced the notion of an $\mathcal{F}$-metric space in such manner.

Let $\mathcal{F}$ be the family of mappings $\xi : (0, +\infty) \to \mathbb{R}$ satisfying

$(\mathcal{F}_1)$ $0 < \rho_1 < \rho_2 \Rightarrow \xi(\rho_1) \leq \xi(\rho_2)$.

$(\mathcal{F}_2)$ For all $\{\rho_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \rho_n = 0$ if and only if $\lim_{n\to\infty} \xi(\rho_n) = -\infty$.

Definition 3. *(see [7]) Let $\aleph \neq \emptyset$ and $\nu : \aleph\times\aleph \to [0, +\infty)$. Assume that there exists $(\xi, \alpha) \in \mathcal{F}\times[0, +\infty)$ such that*

(D_1) $(\rho, \varsigma) \in \aleph \times \aleph$, $\nu(\rho, \varsigma) = 0$ if and only if $\rho = \varsigma$.

(D_2) $\nu(\rho, \varsigma) = \nu(\varsigma, \rho)$, for all $\rho, \varsigma \in \aleph$.

(D_3) For all $(\rho, \varsigma) \in \aleph \times \aleph$, and $(\rho_i)_{i=1}^N \subset \aleph$, with $(\rho_1, \rho_N) = (\rho, \varsigma)$, we have

$$\nu(\rho, \varsigma) > 0 \Rightarrow \xi(\nu(\rho, \varsigma)) \leq \xi(\sum_{i=1}^{N-1} \nu(\rho_i, \rho_{i+1})) + \alpha.$$

for all $N \geq 2$. Then, $(\aleph, \nu)$ is named as an $\mathcal{F}$-metric space.

Example 1. *(see [7]) Let $\aleph = \mathbb{R}$. Then, $\nu : \aleph \times \aleph \to [0, +\infty)$ is an $\mathcal{F}$-metric defined by*

$$\nu(\rho, \varsigma) = \begin{cases} (\rho - \varsigma)^2 & \text{if } (\rho, \varsigma) \in [0, 3] \times [0, 3], \\ |\rho - \varsigma| & \text{if } (\rho, \varsigma) \notin [0, 3] \times [0, 3], \end{cases}$$

with $\xi(t) = \ln(t)$ and $\alpha = \ln(3)$.

Definition 4. *(see [7]) Let $(\aleph, \nu)$ be an $\mathcal{F}$-metric space.*

(i) A sequence $\{\rho_n\} \subseteq \aleph$ is said to be $\mathcal{F}$-convergent if

$$\lim_{n\to\infty} \nu(\rho_n, \rho) = 0.$$

(ii) A sequence $\{\rho_n\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{n,m\to\infty} \nu(\rho_n, \rho_m) = 0.$$

Subsequently, Kanwal et al. [9] integrated both notions of O-set and $\mathcal{F}$-metric space and gave the concept of an orthogonal $\mathcal{F}$-metric space ($O\mathcal{F}$-metric space) as follows.

Definition 5. *(see [9]) Let $(\aleph,\perp)$ be an O-set and $\nu : \aleph \times \aleph \to [0, +\infty)$ be an $\mathcal{F}$-metric; then, $(\aleph, \perp, \nu)$ is named as an orthogonal $\mathcal{F}$-metric space ($O\mathcal{F}$-metric space).*

Example 2. *(see [9]) Let* $\aleph = [0,1]$*. Define* $\mathcal{F}$*-metric* ν *by*

$$\nu(\rho,\varsigma) = \begin{cases} e^{(|\rho-\varsigma|)}, & \text{if } \rho \neq \varsigma, \\ 0, & \text{if } \rho = \varsigma, \end{cases}$$

for all $\rho,\varsigma \in \aleph$, $\xi(t) = -\frac{1}{t}$, $t > 0$, *and* $\alpha = 1$. *Define* $\rho\perp\varsigma$ *if* $\rho\varsigma \leq \rho$ *or* $\rho\varsigma \leq \varsigma$. *Then, for all* $\rho \in \aleph$, $0\perp\varsigma$, *so* $(\aleph,\perp)$ *is an O-set. Then,* $(\aleph,\nu,\perp)$ *is an* $O\mathcal{F}$*-metric space.*

Definition 6. *(see [9]) Let* $(\aleph,\nu,\perp)$ *be an* $O\mathcal{F}$*-metric space, and* $\Re : (\aleph,\nu,\perp) \to (\aleph,\nu,\perp)$*. Then,* $\Re$ *is called* $\perp$*-continuous at* $\rho \in \aleph$ *if for each O-sequence* $\{\rho_n\}$ *in* $\aleph$ *if* $\rho_n \to \rho$*, then* $\Re\rho_n \to \Re\rho$*. Also,* $\Re$ *is* $\perp$*-continuous on* $\aleph$ *if* $\Re$ *is* $\perp$*-continuous at each* $\rho \in \aleph$*.*

Definition 7. *(see [9]) If every Cauchy O-sequence in* $O\mathcal{F}$*-metric space* $(\aleph,\nu,\perp)$ *is* $\mathcal{F}$*-convergent, then* $(\aleph,\nu,\perp)$ *is called O-*$\mathcal{F}$*-complete.*

Samet et al. [22] began the thought of being α-admissible in this manner.

Definition 8. *A mapping* $\Re : \aleph \times \aleph \to [1,\infty)$ *is said to be* α*-admissible if*

$$\alpha(\rho,\varsigma) \geq 1 \quad \textit{implies} \quad \alpha(\Re\rho,\Re\varsigma) \geq 1.$$

Ramezani [23] gave the notion of being orthogonally α-admissible in this way.

Definition 9. *A mapping* $\Re : \aleph \times \aleph \to [1,\infty)$ *is called an orthogonally* α*-admissible if*

$$\rho\perp\varsigma \;\; \textit{and} \;\; \alpha(\rho,\varsigma) \geq 1 \quad \textit{implies} \;\; \alpha(\Re\rho,\Re\varsigma) \geq 1.$$

Recently, Ahmad et al. [12] gave the following property for an orthogonally α-admissible mapping: ($\sqsupset$) $\alpha(\rho,\varsigma) \geq 1$ for any $\rho,\varsigma \in \{\rho^* \in \aleph : \rho^* = \Re\rho^*\}$ and $\rho\perp\varsigma$.

In 2014, Jleli and Samet [29] started a state of the art contraction which is called a Θ-contraction along these lines.

Definition 10. *Let* $\Theta : \mathbb{R}^+ \to [1,\infty)$ *be a function such that*

($\dagger_1$) $\Theta(\rho) < \Theta(\varsigma)$ *for* $\rho < \varsigma$;
($\dagger_2$) for all $\{\rho_n\} \subseteq [0,+\infty)$, $\lim_{n\to\infty}(\rho_n) = 0 \Leftrightarrow \lim_{n\to\infty}\Theta(\rho_n) = 1$;
($\dagger_3$) there exist $0 < h < 1$ *and* $0 < \sigma \leq +\infty$ *such that* $\lim_{\rho\to 0^+}\frac{\Theta(\rho)-1}{\rho^h} = \sigma$.

A mapping $\Re : (\aleph,\nu) \to (\aleph,\nu)$ *is called a* Θ*-contraction if there exist some constant* $\varrho \in (0,1)$ *and a mapping* $\Theta : \mathbb{R}^+ \to [1,\infty)$ *satisfying ($\dagger_1$)–($\dagger_3$) such that*

$$\nu(\Re\rho,\Re\varsigma) > 0 \Longrightarrow +\Theta(\nu(\Re\rho,\Re\varsigma)) \leq [\Theta(\nu(\rho,\varsigma))]^{\varrho}$$

for all $\rho,\varsigma \in \aleph$*. They proved a result for such contraction in this way.*

Theorem 1. *(see [29]) If the mapping* $\Re$ *is a* Θ*-contraction on a complete metric space* $(\aleph,\nu)$*, then there exists* $\rho^* \in \aleph$ *such that* $\rho^* = \Re\rho^*$*.*

Later on, Hussain et al. [31] added another condition,

($\dagger_4$) $\Theta(\rho+\varsigma) \leq \Theta(\rho)\Theta(\varsigma)$,

and extended the above result of Jleli and Samet [7] in complete metric spaces. To be steady with Hussain et al. [31], we represent by Ψ the family of all functions $\Theta : \mathbb{R}^+ \to (1,\infty)$ satisfying ($\dagger_1$)–($\dagger_4$).

3. Results and discussions

We define the notion of a generalized $(\alpha,\Theta_{\mathcal{F}})$-contraction as follows:

Definition 11. *Let $(\aleph, \nu, \perp)$ be an $\mathcal{OF}$-metric space. A mapping $\mathfrak{R} : \aleph \to \aleph$ is called a generalized $(\alpha,\Theta_{\mathcal{F}})$-contraction if there exist $\Theta \in \Psi$, $\alpha : \aleph\times\aleph \longrightarrow [g, \infty)$ and nonnegative real numbers $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ with $\varrho_1 + \varrho_2 + \varrho_3 + 2\varrho_4 < 1$ such that*

$$\begin{aligned} \textit{for all}\, \rho, \varsigma \;\in\; & \aleph, \rho \perp \varsigma, \quad \nu(\mathfrak{R}\rho, \mathfrak{R}\varsigma) \neq 0 \implies \alpha(\rho, \varsigma)\,\Theta(\nu(\mathfrak{R}\rho, \mathfrak{R}\varsigma)) \leq [\Theta(\nu(\rho, \varsigma))]^{\varrho_1} \cdot \\ & \cdot [\Theta(\nu(\rho, \mathfrak{R}\rho))]^{\varrho_2} \cdot [\Theta(\nu(\varsigma, \mathfrak{R}\varsigma))]^{\varrho_3} \cdot [\Theta(\nu(\rho, \mathfrak{R}\varsigma) + \nu(\varsigma, \mathfrak{R}\rho))]^{\varrho_4}. \end{aligned} \tag{3.1}$$

Theorem 2. *Let $(\aleph, \nu, \perp)$ be an O-complete $\mathcal{OF}$-metric space, and $\mathfrak{R} : \aleph \to \aleph$ is $\perp$-continuous, $\perp$-preserving, orthogonally α-admissible and a generalized $(\alpha,\Theta_{\mathcal{F}})$-contraction. If there exists $\rho_0 \in \aleph$ such that $\rho_0 \perp \mathfrak{R}\rho_0$ and $\alpha(\rho_0, \mathfrak{R}\rho_0) \geq 1$, then there exists $\rho^* \in \aleph$ such that $\mathfrak{R}\rho^* = \rho^*$. Moreover, if $\aleph$ has the property ($\sqsupset$), then ρ^* is unique.*

Proof. Let there exist $\rho_0 \in \aleph$ such that $\rho_0 \perp \mathfrak{R}\rho_0$ and $\alpha(\rho_0, \mathfrak{R}\rho_0) \geq 1$, and define the sequence $\{\rho_n\}$ as

$$\rho_1 = \mathfrak{R}\rho_0, \cdots, \rho_{n+1} = \mathfrak{R}\rho_n = \mathfrak{R}^{n+1}\rho_0,$$

for all $n \geq 0$. Now, using the orthogonal α-admissibility of $\mathfrak{R}$, we have

$$\alpha(\mathfrak{R}\rho_n, \mathfrak{R}\rho_{n+1}) \geq 1,$$

for all $n \geq 0$. If $\rho_n = \rho_{n+1}$, for any $n \in \mathbb{N} \cup \{0\}$, then clearly ρ_n is a fixed point of $\mathfrak{R}$. Now, we assume that $\rho_n \neq \rho_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$. Thus, we get

$$\nu(\mathfrak{R}\rho_{n-1}, \mathfrak{R}\rho_n) = \nu(\rho_n, \rho_{n+1}) > 0,$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathfrak{R}$ is $\perp$-preserving, so we have

$$\rho_n \perp \rho_{n+1} \text{ or } \rho_{n+1} \perp \rho_n$$

for all $n \in \mathbb{N} \cup \{0\}$. This means that $\{\rho_n\}$ is an O-sequence. Hence, we presume that

$$0 < \nu(\rho_n, \mathfrak{R}\rho_n) = \nu(\mathfrak{R}\rho_{n-1}, \mathfrak{R}\rho_n), \tag{3.2}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (3.1), (3.2) and ($\dagger_4$), we get

$$\begin{aligned} 1 &< \Theta(\nu(\rho_n, \rho_{n+1})) = \Theta(\nu(\mathfrak{R}\rho_{n-1}, \mathfrak{R}\rho_n)) \\ &\leq \alpha(\rho_{n-1}, \rho_n)\,\Theta(\nu(\mathfrak{R}\rho_{n-1}, \mathfrak{R}\rho_n)) \\ &\leq [\Theta(\nu(\rho_{n-1}, \rho_n))]^{\varrho_1} \cdot [\Theta(\nu(\rho_{n-1}, \mathfrak{R}\rho_{n-1}))]^{\varrho_2} \\ &\quad \cdot [\Theta(\nu(\rho_n, \mathfrak{R}\rho_n))]^{\varrho_3} \cdot [\Theta(\nu(\rho_{n-1}, \mathfrak{R}\rho_n) + \nu(\rho_n, \mathfrak{R}\rho_{n-1}))]^{\varrho_4} \\ &= [\Theta(\nu(\rho_{n-1}, \rho_n))]^{\varrho_1} \cdot [\Theta(\nu(\rho_{n-1}, \rho_n))]^{\varrho_2} \\ &\quad \cdot [\Theta(\nu(\rho_n, \rho_{n+1}))]^{\varrho_3} \cdot [\Theta(\nu(\rho_{n-1}, \rho_{n+1}) + \nu(\rho_n, \rho_n))]^{\varrho_4} \\ &= [\Theta(\nu(\rho_{n-1}, \rho_n))]^{\varrho_1} \cdot [\Theta(\nu(\rho_{n-1}, \rho_n))]^{\varrho_2} \end{aligned}$$

$$
\begin{aligned}
& \cdot [\Theta(\nu(\rho_n,\rho_{n+1}))]^{\varrho_3} \cdot [\Theta(\nu(\rho_{n-1},\rho_{n+1}))]^{\varrho_4} \\
\leq \ & [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho_1} \cdot [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho_2} \\
& \cdot [\Theta(\nu(\rho_n,\rho_{n+1}))]^{\varrho_3} \cdot [\Theta(\nu(\rho_{n-1},\rho_n)+\nu(\rho_n,\rho_{n+1}))]^{\varrho_4} \\
\leq \ & [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho_1} \cdot [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho_2} \\
& \cdot [\Theta(\nu(\rho_n,\rho_{n+1}))]^{\varrho_3} \cdot [\Theta(\nu(\rho_{n-1},\rho_n))\,\Theta(\nu(\rho_n,\rho_{n+1}))]^{\varrho_4} \\
\leq \ & [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho_1} \cdot [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho_2} \\
& \cdot [\Theta(\nu(\rho_n,\rho_{n+1}))]^{\varrho_3} \cdot [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho_4} \cdot [\Theta(\nu(\rho_n,\rho_{n+1}))]^{\varrho_4}
\end{aligned}
$$

which implies

$$
1 < \Theta(\nu(\rho_n,\rho_{n+1})) \leq [\Theta(\nu(\rho_{n-1},\rho_n))]^{\frac{\varrho_1+\varrho_2+\varrho_4}{1-\varrho_3-\varrho_4}} = [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho} \tag{3.3}
$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\varrho = \frac{\varrho_1+\varrho_2+\varrho_4}{1-\varrho_3-\varrho_4} < 1$. Consequently, we have

$$
\begin{aligned}
1 \ &< \ \Theta(\nu(\rho_n,\rho_{n+1})) \leq [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho} \\
&\leq \ [\Theta(\nu(\rho_{n-1},\rho_n))]^{\varrho^2} \\
&\leq \\
& \quad . \\
& \quad . \\
& \quad . \\
&\leq \ [\Theta(\nu(\rho_0,\rho_1))]^{\varrho^n}
\end{aligned} \tag{3.4}
$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, taking $n \to \infty$ and by ($\dagger_2$), we get

$$
\lim_{n\to\infty} \Theta(\nu(\rho_n,\rho_{n+1})) = 1 \ \Leftrightarrow \lim_{n\to\infty} \nu(\rho_n,\rho_{n+1}) = 0. \tag{3.5}
$$

By ($\dagger_3$), there exist $0 < h < 1$ and $l \in (0,\infty]$ such that

$$
\lim_{n\to\infty} \frac{\Theta(\nu(\rho_n,\rho_{n+1})) - 1}{\nu(\rho_n,\rho_{n+1})^h} = l. \tag{3.6}
$$

Let $l < \infty$, and then we take $\beta = \frac{l}{2} > 0$. By definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$
\left|\frac{\Theta(\nu(\rho_n,\rho_{n+1})) - 1}{\nu(\rho_n,\rho_{n+1})^h} - l\right| \leq \beta
$$

for all $n > n_1$. It yields

$$
\frac{\Theta(\nu(\rho_n,\rho_{n+1})) - 1}{\nu(\rho_n,\rho_{n+1})^h} \geq l - \beta = \frac{l}{2} = \beta.
$$

Then,

$$
n\nu(\rho_n,\rho_{n+1})^h \leq \gamma n[\Theta(\nu(\rho_n,\rho_{n+1})) - 1] \tag{3.7}
$$

for all $n > n_1$, where $\gamma = \frac{1}{\beta}$. Now, when $l = \infty$, suppose that $\beta > 0$. By definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$
\beta \leq \frac{\Theta(\nu(\rho_n,\rho_{n+1})) - 1}{\nu(\rho_n,\rho_{n+1})^h}.
$$

It yields
$$n\nu(\rho_n,\rho_{n+1})^h \leq \gamma n[\Theta\left(\nu(\rho_n,\rho_{n+1})\right) - 1].$$
Therefore, in all ways, there exists $\gamma > 0$ and $n_1 \in \mathbb{N}$ such that
$$n\nu(\rho_n,\rho_{n+1})^h \leq \gamma n[\Theta\left(\nu(\mathfrak{R}\rho_{n-1},\mathfrak{R}\rho_n)\right) - 1] \tag{3.8}$$
for all $n > n_1$. Hence, by (3.4) and (3.8), we have
$$n\nu(\rho_n,\rho_{n+1})^h \leq \gamma n([(\Theta(\nu(\rho_0,\rho_1))]^{\varrho^n} - 1).$$
Taking $n \to \infty$, we have
$$\lim_{n\to\infty} n\nu(\rho_n,\rho_{n+1})^h = 0.$$
Hence, there exists $n_2 \in \mathbb{N}$ such that
$$\nu(\rho_n,\rho_{n+1}) \leq \frac{1}{n^{1/h}} \tag{3.9}$$
for all $n > n_2$. This yields
$$\sum_{i=n}^{m-1} \nu(\rho_i,\rho_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/h}}$$
for all $m > n$. As $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ converges, there exists $n_2 \in \mathbb{N}$ such that
$$0 < \sum_{i=n}^{m-1} \frac{1}{i^{1/h}} < \sum_{i=n}^{\infty} \frac{1}{i^{1/h}} < \delta, \tag{3.10}$$
for $n > n_2$. Hence, by (3.10) and $(\mathcal{F}_1)$, we get
$$\xi\left(\sum_{i=n}^{m-1} \nu(\rho_i,\rho_{i+1})\right) \leq \xi\left(\sum_{i=n}^{\infty} \frac{1}{i^{1/h}}\right) < \xi\left(\epsilon\right) - a, \tag{3.11}$$
$m > n \geq n_2$. Using (D_3) and (3.11), we get
$$\nu(\rho_n,\rho_m) > 0,\ m > n \geq n_2 \Longrightarrow \xi\left(\nu(\rho_n,\rho_m)\right) \leq \xi\left(\sum_{i=n}^{m-1} \nu(\rho_i,\rho_{i+1})\right) + a < \xi\left(\epsilon\right),$$
which, from $(\mathcal{F}_1)$, gives that
$$\nu(\rho_n,\rho_m) < \epsilon,$$
for all $m > n \geq n_2$. Therefore, $\{\rho_n\}$ is a Cauchy O-sequence in $(\aleph,\perp,\nu)$. Now, since $(\aleph,\perp,\nu)$ is O-complete, there exists $\rho^* \in \aleph$ such that $\lim_{n\to\infty}\rho_n \to \rho^*$. Now, we show that $\rho^* = \mathfrak{R}\rho^*$. Since $\mathfrak{R}$ is $\perp$-continuous, we have $\mathfrak{R}\rho_n \to \mathfrak{R}\rho^*$ as $n \to \infty$. Thus,
$$\mathfrak{R}\rho^* = \lim_{n\to\infty} \mathfrak{R}\rho_n = \lim_{n\to\infty} \rho_{n+1} = \rho^*.$$
Now, we suppose that $\rho^/ = \mathfrak{R}\rho^/$ is another fixed point of $\mathfrak{R}$ such that $\rho^/ \neq \rho^*$. From ($\sqsupset$), we have $\rho^*\perp\rho^/$ or $\rho^/ \perp \rho^*$ and $\alpha\left(\rho^*,\rho^/\right) \geq 1$. Thus, from (3.1), we have
$$\Theta\left(\nu(\rho^*,\rho^/)\right) \ = \ \Theta\left(\nu(\mathfrak{R}\rho^*,\mathfrak{R}\rho^/)\right) \leq \alpha(\rho^/,\rho^*)\Theta\left(\nu(\mathfrak{R}\rho^*,\mathfrak{R}\rho^/)\right)$$

$$
\begin{aligned}
&\leq \left[\Theta\left(\nu(\rho^{/},\rho^{*})\right)\right]^{\varrho_1}\cdot\left[\Theta\left(\nu(\rho^{/},\Re\rho^{/})\right)\right]^{\varrho_2}\\
&\quad\cdot\left[\Theta\left(\nu(\rho^{*},\Re\rho^{*})\right)\right]^{\varrho_3}\cdot\left[\Theta\left(\nu(\rho^{/},\Re\rho^{*})+\nu(\rho^{*},\Re\rho^{/})\right)\right]^{\varrho_4}\\
&= \left[\Theta\left(\nu(\rho^{/},\rho^{*})\right)\right]^{\varrho_1}\cdot\left[\Theta\left(\nu(\rho^{/},\rho^{/})\right)\right]^{\varrho_2}\\
&\quad\cdot\left[\Theta\left(\nu(\rho^{*},\rho^{*})\right)\right]^{\varrho_3}\cdot\left[\Theta\left(\nu(\rho^{/},\rho^{*})+\nu(\rho^{*},\rho^{/})\right)\right]^{\varrho_4}\\
&\leq \left[\Theta\left(\nu(\rho^{/},\rho^{*})\right)\right]^{\varrho_1}\cdot\left[\Theta\left(\nu(\rho^{/},\rho^{*})\cdot\Theta\left(\nu(\rho^{*},\rho^{/})\right)\right)\right]^{\varrho_4}\\
&= \left[\Theta\left(\nu(\rho^{/},\rho^{*})\right)\right]^{\varrho_1}\cdot\left[\Theta\left(\nu(\rho^{/},\rho^{*})\right)\right]^{\varrho_4}\cdot\left[\Theta\left(\nu(\rho^{/},\rho^{*})\right)\right]^{\varrho_4}\\
&= \left[\Theta\left(\nu(\rho^{/},\rho^{*})\right)\right]^{\varrho_1+2\varrho_4}
\end{aligned}
$$

which implies that

$$\Theta\left(\nu(\rho^{*},\rho^{/})\right)\leq\left[\Theta\left(\nu(\rho^{/},\rho^{*})\right)\right]^{\varrho_1+2\varrho_4}<\Theta\left(\nu(\rho^{/},\rho^{*})\right)$$

which is a contradiction because $\varrho_1+2\varrho_4<1$. Thus $\rho^{/}=\rho^{*}$. Hence the fixed point is unique. □

Corollary 1. *Let* $(\aleph,\nu,\perp)$ *be an* O*-complete* $\mathcal{OF}$*-metric space, and* $\Re:(\aleph,\nu,\perp)\to(\aleph,\nu,\perp)$ *is* $\perp$*-continuous and* $\perp$*-preserving. Suppose there exist* $\Theta\in\Psi$ *and nonnegative real numbers* $\varrho_1,\varrho_2,\varrho_3,\varrho_4$ *with* $\varrho_1+\varrho_2+\varrho_3+2\varrho_4<1$ *such that for all*

$$
\begin{aligned}
\rho,\varsigma \in \aleph,\rho\perp\varsigma,\quad \nu(\Re\rho,\Re\varsigma)\neq 0 \implies \Theta\left(\nu(\Re\rho,\Re\varsigma)\right)\leq\left[\Theta\left(\nu(\rho,\varsigma)\right)\right]^{\varrho_1}.\\
\cdot\left[\Theta\left(\nu(\rho,\Re\rho)\right)\right]^{\varrho_2}\cdot\left[\Theta\left(\nu(\varsigma,\Re\varsigma)\right)\right]^{\varrho_3}\cdot\left[\Theta\left(\nu(\rho,\Re\varsigma)+\nu(\varsigma,\Re\rho)\right)\right]^{\varrho_4}.
\end{aligned}
$$

Then, there exists a unique point $\rho^{*}\in\aleph$ *such that* $\Re\rho^{*}=\rho^{*}$.

Proof. Take $\alpha:\aleph\times\aleph\longrightarrow[1,\infty)$ by $\alpha(\rho,\varsigma)=1$, for all $\rho,\varsigma\in\aleph$ in Theorem 2. □

Corollary 2. *Let* $(\aleph,\nu,\perp)$ *be an* O*-complete* $\mathcal{OF}$*-metric space, and* $\Re:(\aleph,\nu,\perp)\to(\aleph,\nu,\perp)$ *is* $\perp$*-continuous,* $\perp$*-preserving and orthogonally* α*-admissible. Suppose there exist* $\Theta\in\Psi$, $\alpha:\aleph\times\aleph\longrightarrow[1,\infty)$ *and some nonnegative real number* $\varrho\in(0,1)$ *such that*

$$\nu(\Re\rho,\Re\varsigma)\neq 0 \implies \alpha(\rho,\varsigma)\,\Theta\left(\nu(\Re\rho,\Re\varsigma)\right)\leq\left[\Theta\left(\nu(\rho,\varsigma)\right)\right]^{\varrho}$$

for all $\rho,\varsigma\in\aleph,\rho\perp\varsigma$. *If there exists* $\rho_0\in\aleph$ *such that* $\rho_0\perp\Re\rho_0$ *and* $\alpha(\rho_0,\Re\rho_0)\geq 1$, *then, there exists* $\rho^{*}\in\aleph$ *such that* $\Re\rho^{*}=\rho^{*}$. *Moreover, if* $\aleph$ *has the property* ($\sqsupset$), *then* ρ^{*} *is unique.*

Proof. Take $\varrho_1=\varrho<1$ and $\varrho_2=\varrho_3=\varrho_4=0$ in Theorem 2. □

Corollary 3. *(see [9]) Let* $(\aleph,\nu,\perp)$ *be an* O*-complete* $\mathcal{OF}$*-metric space, and* $\Re:(\aleph,\nu,\perp)\to(\aleph,\nu,\perp)$ *is* $\perp$*-continuous and* $\perp$*-preserving. Suppose there exist* $\Theta\in\Psi$ *and some nonnegative real number* $\varrho\in(0,1)$ *such that*

$$\text{for all }\rho,\varsigma\in\aleph,\rho\perp\varsigma,\quad \nu(\Re\rho,\Re\varsigma)\neq 0 \implies \Theta\left(\nu(\Re\rho,\Re\varsigma)\right)\leq\left[\Theta\left(\nu(\rho,\varsigma)\right)\right]^{\varrho} \tag{3.12}$$

holds; then, there exists a unique point $\rho^{*}\in\aleph$ *such that* $\Re\rho^{*}=\rho^{*}$.

Proof. Take $\alpha:\aleph\times\aleph\longrightarrow[1,\infty)$ by $\alpha(\rho,\varsigma)=1$, for all $\rho,\varsigma\in\aleph$ in Corollary 2. □

Example 3. *Define the sequence* $\{\rho_n\}$ *as follows:*

$$\rho_1 = \ln(1),$$
$$\rho_2 = \ln(1+5),$$
$$.$$
$$.$$
$$.$$
$$\rho_n = \ln(1+5+9+...+(4n-3)) = \ln\left(n(2n-1)\right)$$

for all $n \geq 1$. *Let* $\aleph = \{\rho_n : n \in \mathbb{N}\}$ *be provided with* $\nu : \aleph \times \aleph \to [0,+\infty)$, *defined by*

$$\nu(\rho,\varsigma) = \begin{cases} e^{|\rho-\varsigma|}, \; if \; \rho \neq \varsigma, \\ 0, \; if \; \rho = \varsigma, \end{cases}$$

with $\xi(t) = \frac{-1}{t}$ *and* $a = 1$. *For all* $\rho_n, \rho_m \in \aleph$, *define* $\rho_n \perp \rho_m$ *if and only if* $(m \geq 2 \wedge n = 1)$. *Thus,* $(\aleph, \nu, \perp)$ *is an* O*-complete* $O\mathcal{F}$*-metric space. Define* $\Re : (\aleph, \nu, \perp) \to (\aleph, \nu, \perp)$ *by*

$$\Re(\rho_n) = \begin{cases} \rho_1, & if\, n = 1, \\ \rho_{n-1}, & if\, n > 1, \end{cases}$$

and $\alpha : \aleph \times \aleph \to [g,+\infty)$ *by*

$$\alpha(\rho_n,\rho_m) = \begin{cases} 1, if\, \rho_n \neq \rho_m, \\ 0, \; if\, \rho_n = \rho_m. \end{cases}$$

Clearly,

$$\lim_{n\longrightarrow\infty} \frac{\nu(\Re(\rho_n), \Re(\rho_1))}{\nu(\rho_n,\rho_1)} = 1,$$

and then $\Re$ *is not a contraction.*

It is very simple to show that $\Re$ is $\perp$-preserving and $\perp$-continuous. Define $\Theta : (0,\infty) \to \mathbb{R}^+$ by

$$\Theta(t) = e^{\sqrt{te^t}}, \; t > 0.$$

Then, $\Theta \in \Psi$. Now, we prove that $\Re$ is a generalized (α,Θ)-contraction, i.e.,

$$\nu(\Re(\rho_n), \Re(\rho_m)) \neq 0 \Longrightarrow e^{\sqrt{\nu(\Re(\rho_n),\Re(\rho_m))e^{\nu(\Re(\rho_n),\Re(\rho_m))}}} \leq \left[e^{\sqrt{\nu(\rho_n,\rho_m)e^{\nu(\rho_n,\rho_m)}}}\right]^{\varrho}$$

for some $\varrho \in (0,1)$. The above condition is equivalent to

$$\nu(\Re(\rho_n), \Re(\rho_m)) \neq 0 \Longrightarrow \nu(\Re(\rho_n), \Re(\rho_m))e^{\nu(\Re(\rho_n),\Re(\rho_m))} \leq \varrho^2 \nu(\rho_n,\rho_m)e^{\nu(\rho_n,\rho_m)}.$$

So, we have to check that

$$\nu(\Re(\rho_n), \Re(\rho_m)) \neq 0 \Longrightarrow \frac{\nu(\Re(\rho_n), \Re(\rho_m))}{\nu(\rho_n,\rho_m)} e^{\nu(\Re(\rho_n),\Re(\rho_m))-\nu(\rho_n,\rho_m)} \leq \varrho^2.$$

For $m \in \mathbb{N}$, and $m \geq 2$, we get

$$\begin{aligned} \nu(\Re(\rho_m), \Re(\rho_1)) \neq 0 &\Longrightarrow \frac{\nu(\Re(\rho_m), \Re(\rho_1))}{\nu(\rho_m, \rho_1)} e^{\nu(\Re(\rho_m),\Re(\rho_1))-\nu(\rho_m,\rho_1)} \leq \varrho^2 \\ & \frac{\nu(\rho_{m-1}, \rho_1)}{\nu(\rho_m, \rho_1)} e^{\nu(\rho_{m-1},\rho_1)-\nu(\rho_m,\rho_1)} \\ &= \frac{e^{\rho_{m-1}-\rho_1}}{e^{\rho_m-\rho_1}} e^{e^{\rho_{m-1}-\rho_1}-e^{\rho_m-\rho_1}} \\ &= \frac{(m-1)(2m-3)}{m(2m-1)} e^{-4m+3} < e^{-1}. \end{aligned}$$

Thus, the inequality (3.1) is satisfied. Hence, $\Re$ is a generalized (α, Θ)-contraction. Hence, by Theorem 2, $\rho = ln(1)$ is a unique fixed point of $\Re$.

For particular choices of Θ, we get some noteworthy results. If we take $\Theta(t) = e^{\sqrt{t}}$ in (2), we get an extension of Ćirić result [37].

Theorem 3. *Let $(\aleph, \nu, \perp)$ be an O-complete $\mathcal{OF}$-metric space and $\Re : (\aleph, \nu, \perp) \to (\aleph, \nu, \perp)$ is $\perp$-continuous, $\perp$-preserving and orthogonally α-admissible. Suppose that there exists nonnegative real numbers $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ with $\varrho_1 + \varrho_2 + \varrho_3 + 2\varrho_4 < 1$ such that these conditions hold:*

$$\begin{aligned} [\ln(\alpha(\rho, \varsigma))] \sqrt{\nu(\Re\rho, \Re\varsigma)} \leq & \varrho_1 \sqrt{\nu(\rho, \varsigma)} + \varrho_2 \sqrt{\nu(\rho, \Re\rho)} \\ & + \varrho_3 \sqrt{\nu(\varsigma, \Re\varsigma)} + \varrho_4 \sqrt{\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho)}. \end{aligned} \tag{3.13}$$

If there exists $\rho_0 \in \aleph$ such that $\rho_0 \perp \Re\rho_0$ and $\alpha(\rho_0, \Re\rho_0) \geq 1$, then there exists $\rho^* \in \aleph$ such that $\rho^* = \Re\rho^*$. Moreover, if $\aleph$ has the property ($\beth$), then ρ^* is unique.

Remark 1. *If we take the square on both sides, then that condition (3.13) is equivalent to*

$$\begin{aligned} [\ln(\alpha(\rho, \varsigma))]^2 \nu(\Re\rho, \Re\varsigma) \leq & \varrho_1^2 \nu(\rho, \varsigma) + \varrho_2^2 \nu(\rho, \Re\rho) + \varrho_3^2 \nu(\varsigma, \Re\varsigma) + \varrho_4^2 (\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho)) \\ & + 2\varrho_1\varrho_2 \sqrt{\nu(\rho, \varsigma)\nu(\rho, \Re\rho)} + 2\varrho_1\varrho_3 \sqrt{\nu(\rho, \varsigma)\nu(\varsigma, \Re\varsigma)} \\ & + 2\varrho_1\varrho_4 \sqrt{\nu(\rho, \varsigma)(\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho))} \\ & + 2\varrho_2\varrho_3 \sqrt{\nu(\rho, \Re\rho)\nu(\varsigma, \Re\varsigma)} + 2\varrho_2\varrho_4 \sqrt{\nu(\rho, \Re\rho)(\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho))} \\ & + 2\varrho_3\varrho_4 \sqrt{\nu(\varsigma, \Re\varsigma)(\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho))}. \end{aligned}$$

Next, in view of Remark 1, by taking $\varrho_1 = \varrho_4 = 0$ in Theorem 2, we obtain this result, which is an extension of Kannan's result [38].

Theorem 4. *Let $(\aleph, \nu, \perp)$ be an O-complete $\mathcal{OF}$-metric space and $\Re : (\aleph, \nu, \perp) \to (\aleph, \nu, \perp)$ is $\perp$-continuous, $\perp$-preserving and orthogonally α-admissible. Suppose that there exist nonnegative real numbers ϱ_2, ϱ_3 with $0 \leq \varrho_2 + \varrho_3 < 1$ such that these conditions hold: for all $\rho, \varsigma \in \aleph$, $\rho \perp \varsigma$, $\nu(\Re\rho, \Re\varsigma) \neq 0$ implies*

$$[\ln(\alpha(\rho, \varsigma))]^2 \nu(\Re\rho, \Re\varsigma) \leq \varrho_2^2 \nu(\rho, \Re\rho) + \varrho_3^2 \nu(\varsigma, \Re\varsigma) + 2\varrho_2\varrho_3 \sqrt{\nu(\rho, \Re\rho)\nu(\varsigma, \Re\varsigma)}.$$

If there exists $\rho_0 \in \aleph$ *such that* $\rho_0 \perp \Re\rho_0$ *and* $\alpha(\rho_0, \Re\rho_0) \geq 1$, *then, there exists* $\rho^* \in \aleph$ *such that* $\rho^* = \Re\rho^*$. *Moreover, if* $\aleph$ *has the property* ($\sqsupset$), *then* ρ^* *is unique.*

On the other hand, by taking $\varrho_1 = \varrho_2 = \varrho_3 = 0$ *in Theorem 2, we obtain this theorem, which is an expansion of Chatterjea's result [39].*

Theorem 5. *Let* $(\aleph, \nu, \perp)$ *be an O-complete* $O\mathcal{F}$*-metric space and* $\Re : (\aleph, \nu, \perp) \to (\aleph, \nu, \perp)$ *is* $\perp$*-continuous,* $\perp$*-preserving and orthogonally* α*-admissible. Suppose that there exists nonnegative real number* $\varrho_4 \in \left[0, \frac{1}{2}\right)$ *such that for all* $\rho, \varsigma \in \aleph$, $\rho \perp \varsigma$, $\nu(\Re\rho, \Re\varsigma) \neq 0$ *implies*

$$[\ln(\alpha(\rho, \varsigma))]^2 \, \nu(\Re\rho, \Re\varsigma) \leq \varrho_4^2 \left(\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho)\right).$$

If there exists $\rho_0 \in \aleph$ *such that* $\rho_0 \perp \Re\rho_0$ *and* $\alpha(\rho_0, \Re\rho_0) \geq 1$, *then, there exists* $\rho^* \in \aleph$ *such that* $\rho^* = \Re\rho^*$. *Moreover, if* $\aleph$ *has the property* ($\sqsupset$), *then* ρ^* *is unique.*

From Theorem 2, by taking $\varrho_4 = 0$, *we obtain the extension of Reich contraction [40].*

Corollary 4. *Let* $(\aleph, \nu, \perp)$ *be an O-complete* $O\mathcal{F}$*-metric space and* $\Re : (\aleph, \nu, \perp) \to (\aleph, \nu, \perp)$ *is* $\perp$*-continuous,* $\perp$*-preserving and orthogonally* α*-admissible. Suppose that there exist nonnegative real numbers* $\varrho_1, \varrho_2, \varrho_3$ *with* $0 \leq \varrho_1 + \varrho_2 + \varrho_3 < 1$ *such that for all* $\rho, \varsigma \in \aleph$, $\rho \perp \varsigma$, $\nu(\Re\rho, \Re\varsigma) \neq 0$ *implies*

$$\begin{aligned}
[\ln(\alpha(\rho, \varsigma))]^2 \, \nu(\Re\rho, \Re\varsigma) \leq \; & \varrho_1^2 \nu(\rho, \varsigma) + \varrho_2^2 \nu(\rho, \Re\rho) + \varrho_3^2 \nu(\varsigma, \Re\varsigma) \\
& + 2\varrho_1\varrho_2 \sqrt{\nu(\rho, \varsigma)\nu(\rho, \Re\rho)} + 2\varrho_1\varrho_3 \sqrt{\nu(\rho, \varsigma)\nu(\varsigma, \Re\varsigma)} \\
& + 2\varrho_2\varrho_3 \sqrt{\nu(\rho, \Re\rho)\nu(\varsigma, \Re\varsigma)}.
\end{aligned}$$

If there exists $\rho_0 \in \aleph$ *such that* $\rho_0 \perp \Re\rho_0$ *and* $\alpha(\rho_0, \Re\rho_0) \geq 1$, *then there exists* $\rho^* \in \aleph$ *such that* $\rho^* = \Re\rho^*$. *Moreover, if* $\aleph$ *has the property* ($\sqsupset$), *then* ρ^* *is unique.*

Eventually, if we take $\Theta(t) = e^{\sqrt[n]{t}}$ *in (2), then we derive this corollary.*

Corollary 5. *Let* $(\aleph, \nu, \perp)$ *be an O-complete* $O\mathcal{F}$*-metric space and* $\Re : (\aleph, \nu, \perp) \to (\aleph, \nu, \perp)$ *is* $\perp$*-continuous,* $\perp$*-preserving and orthogonally* α*-admissible. Suppose that there exists nonnegative real numbers* $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ *with* $\varrho_1 + \varrho_2 + \varrho_3 + 2\varrho_4 < 1$ *such that for all* $\rho, \varsigma \in \aleph$, $\rho \perp \varsigma$, $\nu(\Re\rho, \Re\varsigma) \neq 0$ *implies*

$$\ln(\alpha(\rho, \varsigma)) \sqrt[n]{\nu(\Re\rho, \Re\varsigma)} \leq \varrho_1 \sqrt[n]{\nu(\rho, \varsigma)} + \varrho_2 \sqrt[n]{\nu(\rho, \Re\rho)} + \varrho_3 \sqrt[n]{\nu(\varsigma, \Re\varsigma)} + \varrho_4 \sqrt[n]{\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho)}$$

If there exists $\rho_0 \in \aleph$ *such that* $\rho_0 \perp \Re\rho_0$ *and* $\alpha(\rho_0, \Re\rho_0) \geq 1$, *then, there exists* $\rho^* \in \aleph$ *such that* $\rho^* = \Re\rho^*$. *Moreover, if* $\aleph$ *has the property* ($\sqsupset$), *then* ρ^* *is unique.*

4. Consequences

Now, we consider some special cases, where in our result we deduce several well-known fixed point theorems of the existing literature.

Theorem 6. *Let* $(\aleph, \nu)$ *be an* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\Re : \aleph \to \aleph$ *be an* α*-admissible mapping, and there exist* $\Theta \in \Psi$, $\alpha : \aleph \times \aleph \longrightarrow [1, \infty)$ *and nonnegative real numbers* $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ *with* $\varrho_1 + \varrho_2 + \varrho_3 + 2\varrho_4 < 1$ *such that for all*

$$\rho, \varsigma \in \aleph, \nu(\Re\rho, \Re\varsigma) \neq 0 \implies \alpha(\rho, \varsigma)\, \Theta(\nu(\Re\rho, \Re\varsigma)) \leq [\Theta(\nu(\rho, \varsigma))]^{\varrho_1}.$$

$$\cdot \left[\Theta\left(\nu(\rho, \Re\rho)\right)\right]^{\varrho_2} \cdot \left[\Theta\left(\nu(\varsigma, \Re\varsigma)\right)\right]^{\varrho_3} \cdot \left[\Theta\left(\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho)\right)\right]^{\varrho_4} \tag{4.1}$$

for all $\rho, \varsigma \in \aleph$. Assume that there exists $\rho_0 \in \aleph$ such that $\alpha(\rho_0, \Re\rho_0) \geq 1$; then, there exists $\rho^* \in \aleph$ such that $\Re\rho^* = \rho^*$. Moreover, if $\aleph$ has the property ($\sqsupset$), then ρ^* is unique.

Proof. Assume that

$$\rho \perp \varsigma \text{ if and only if } \nu(\Re\rho, \Re\varsigma) \neq 0.$$

Fix $\rho_0 \in \aleph$. Since $\Re$ satisfies the inequality (4.1), for all $\varsigma \in \aleph$, $\rho_0 \perp \varsigma$, it yields that $(\aleph, \perp)$ is an O-set. Then, $(\aleph, \nu)$ is O-complete. It is very obvious to prove that $\Re$ is $\perp$-preserving and $\perp$-continuous. Hence, by result 2, there exists a unique point $\rho^* \in \aleph$ such that $\Re\rho^* = \rho^*$. □

Corollary 6. *Let $(\aleph, \nu)$ be an $\mathcal{F}$-complete $\mathcal{F}$-metric space, $\Re : \aleph \to \aleph$, and have nonnegative real numbers $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ with $\varrho_1 + \varrho_2 + \varrho_3 + 2\varrho_4 < 1$ such that*

$$\text{for all } \rho, \varsigma \in \aleph, \;\; \nu(\Re\rho, \Re\varsigma) \neq 0 \implies \Theta\left(\nu(\Re\rho, \Re\varsigma)\right) \leq \left[\Theta\left(\nu(\rho, \varsigma)\right)\right]^{\varrho_1} \cdot \\ \cdot \left[\Theta\left(\nu(\rho, \Re\rho)\right)\right]^{\varrho_2} \cdot \left[\Theta\left(\nu(\varsigma, \Re\varsigma)\right)\right]^{\varrho_3} \cdot \left[\Theta\left(\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho)\right)\right]^{\varrho_4}$$

for all $\rho, \varsigma \in \aleph$ and $\Theta \in \Psi$; then, there exists a unique point $\rho^* \in \aleph$ such that $\Re\rho^* = \rho^*$.

Proof. Take $\alpha : \aleph \times \aleph \longrightarrow [1, \infty)$ by $\alpha(\rho, \varsigma) = 1$ in Theorem 6. □

Theorem 7. *Let $(\aleph, \nu, \perp)$ be an O-complete metric space, and $\Re : (\aleph, \nu, \perp) \to (\aleph, \nu, \perp)$ is $\perp$-continuous, $\perp$-preserving and orthogonally α-admissible. Assume that there exist $\Theta \in \Psi$, $\alpha : \aleph \times \aleph \longrightarrow [1, \infty)$ and nonnegative real numbers $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ with $\varrho_1 + \varrho_2 + \varrho_3 + 2\varrho_4 < 1$ such that*

$$\text{for all } \rho, \varsigma \in \aleph, \rho \perp \varsigma, \;\; \nu(\Re\rho, \Re\varsigma) \neq 0 \implies \alpha(\rho, \varsigma)\, \Theta\left(\nu(\Re\rho, \Re\varsigma)\right) \leq \left[\Theta\left(\nu(\rho, \varsigma)\right)\right]^{\varrho_1} \cdot \\ \cdot \left[\Theta\left(\nu(\rho, \Re\rho)\right)\right]^{\varrho_2} \cdot \left[\Theta\left(\nu(\varsigma, \Re\varsigma)\right)\right]^{\varrho_3} \cdot \left[\Theta\left(\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho)\right)\right]^{\varrho_4}.$$

If there exists $\rho_0 \in \aleph$ such that $\rho_0 \perp \Re\rho_0$ and $\alpha(\rho_0, \Re\rho_0) \geq 1$, then, there exists $\rho^* \in \aleph$ such that $\Re\rho^* = \rho^*$. Moreover, if $\aleph$ has the property ($\sqsupset$), then ρ^* is unique.

Proof. Take $\xi(t) = \ln(t)$, for $t > 0$ and $\alpha = 1$ in Definition 5, and then $O\mathcal{F}$-metric space reduced to O-metric space. It follows directly from Theorem 2. □

Corollary 7. *(see [31]) Let $(\aleph, \nu)$ be an complete metric space, $\Re : \aleph \to \aleph$ and nonnegative real numbers $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ with $\varrho_1 + \varrho_2 + \varrho_3 + 2\varrho_4 < 1$ such that*

$$\text{for all } \rho, \varsigma \in \aleph, \;\; \nu(\Re\rho, \Re\varsigma) \neq 0 \implies \Theta\left(\nu(\Re\rho, \Re\varsigma)\right) \leq \left[\Theta\left(\nu(\rho, \varsigma)\right)\right]^{\varrho_1} \cdot \\ \cdot \left[\Theta\left(\nu(\rho, \Re\rho)\right)\right]^{\varrho_2} \cdot \left[\Theta\left(\nu(\varsigma, \Re\varsigma)\right)\right]^{\varrho_3} \cdot \left[\Theta\left(\nu(\rho, \Re\varsigma) + \nu(\varsigma, \Re\rho)\right)\right]^{\varrho_4}$$

for all $\rho, \varsigma \in \aleph$ and $\Theta \in \Psi$; then, there exists a unique point $\rho^* \in \aleph$ such that $\Re\rho^* = \rho^*$.

Proof. Assume that

$$\rho \perp \varsigma \text{ if and only if } \nu(\Re\rho, \Re\varsigma) \neq 0.$$

It follows from Theorem 7 by considering $\alpha : \aleph \times \aleph \longrightarrow [1, \infty)$ as $\alpha(\rho, \varsigma) = 1$. □

Remark 2. *By using the remark 1 and equating nonnegative real numbers $\varrho_1, \varrho_2, \varrho_3$ and ϱ_4 to zero appropriately in Theorems 6, 7 and Corollaries 6, 7 one can derive a number of results which are more general results in the context of $\mathcal{F}$-metric spaces, orthogonal metric spaces and metric spaces.*

Remark 3. *By taking the functions $\alpha : \aleph \times \aleph \longrightarrow [1, \infty)$ and $\Theta : \mathbb{R}^+ \to [1, \infty)$ in different ways in above results, one can obtain various results in different generalized metric spaces.*

5. Applications

The field of fractional differential equations has been subjected to a comprehensive evolution of theory and applications ([41–43] and references therein). In the present section, we give an application of result 3 to investigate the existence of a solution for a nonlinear fractional differential equation considered in [9,44,45].

Consider a nonlinear differential equation of fractional order

$$^{C}D^{\eta}(\rho(t)) = f(t, \rho(t)) \tag{5.1}$$

$(0 < t < 1,\ 1 < \eta \leq 2)$ via the integral boundary conditions

$$\rho(0) = 0\,,\ \rho^{/}(0) = I,\ (0 < I < 1),$$

where $^{C}D^{\eta}$ denotes the Caputo fractional derivative of order η defined by

$$^{C}D^{\eta} f(t) = \frac{1}{\Gamma(j-\eta)} \int_{0}^{t} (t-s)^{j-\eta-1} f^{j}(s)ds,$$

$(j - 1 < \eta < j,\ j = [\eta] + 1)$ and f is a continuous mapping. We take

$$\aleph = \{\rho : \rho \in C([0,1], \mathbb{R})\}$$

with supremum norm $\|\rho\|_{\infty} = \sup_{t\in[0,1]} |\rho(t)|$. Thus, $(\aleph, \|\rho\|_{\infty})$ is a Banach space. Remember that

$$I^{\eta} f(t) = \frac{1}{\Gamma(\eta)} \int_{0}^{t} (t-s)^{\eta-1} f(s)ds, \quad \text{with } \eta > 0$$

is a Riemann-Liouville fractional integral.

Lemma 1. *(see [9]) The Banach space $(\aleph, \|\cdot\|_{\infty})$ endowed with the $\mathcal{F}$-metric d defined by*

$$d(\rho, \varsigma) = \|\rho - \varsigma\|_{\infty} = \sup_{t\in[0,1]} |\rho(t) - \varsigma(t)|$$

and orthogonal relation $\rho \perp \varsigma$ if and only if $\rho\varsigma \geq 0$, where $\rho, \varsigma \in \aleph$, is an orthogonal $\mathcal{F}$-metric space.

Theorem 8. *Assume that f is a continuous mapping satisfying*

(i) there exists a constant ϑ such that

$$|f(t, \rho) - f(t, \varsigma)| \leq \vartheta\, |\rho - \varsigma|$$

for $t \in [0,1]$ and for all $\rho, \varsigma \in \aleph$ such that $\rho(t)\varsigma(t) \geq 0$ and with $\vartheta\beta < 1$, where

$$\beta = \frac{1}{\Gamma(\eta+1)} + \frac{2\lambda^{\eta+1}\Gamma(\eta)}{(2-\lambda^{2})\,\Gamma(\eta+1)},$$

(ii) there exists $\mathcal{L} : (\aleph,\perp, d) \to (\aleph,\perp, d)$, which is defined by

$$\begin{aligned}\Re\rho(t) &= \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\rho(s))\,ds \\ &\quad + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} f(m,\rho(m))\,dm\right) ds\end{aligned}$$

for all $\rho, \varsigma \in \aleph$ such that $\rho(t)\varsigma(t) \geq 0$, where $(0 < \lambda < 1)$. Also, $\Re$ is orthogonally α-admissible, and there exists $\rho_0(t) \in (\aleph,\perp, d)$ such that $\rho_0(t) \perp \Re\rho_0(t)$ and $\alpha(\rho_0(t), \Re\rho_0(t)) \geq 1$ Then, (5.1) has a unique solution.

Proof. It is conventional that $\rho \in \aleph$ is a solution of (5.1) iff $\rho \in \aleph$ is a solution of the integral equation

$$\begin{aligned}\rho(t) &= \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\rho(s))\,ds \\ &\quad + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} f(m,\rho(m))\,dm\right) ds.\end{aligned}$$

Then, problem (5.1) is equivalent to finding $\rho \in \aleph$ that is a fixed point of $\Re$. Assume that $\perp \subseteq \aleph \times \aleph$ is defined by

$$\rho\perp\varsigma \text{ if and only if } \rho(t)\varsigma(t) \geq 0$$

for all $t \in [0, 1]$. Then, $\aleph$ is orthogonal under this relation $\perp$, since for $\rho \in \aleph$, there exists $\varsigma(t) = 0$, for all $t \in [0, 1]$ such that $\rho(t)\varsigma(t) = 0$. Now, define $d : \aleph \times \aleph \to [0, +\infty)$ by

$$d(\rho, \varsigma) = \|\rho - \varsigma\|_\infty = \sup_{t\in[0,1]} |\rho(t) - \varsigma(t)|$$

for all $\rho, \varsigma \in \aleph$, and then $(\aleph, d, \perp)$ is a complete $\mathcal{OF}$-metric space. This is quite simple from the definition that $\Re$ is $\perp$-continuous. We first show that $\Re$ is $\perp$-preserving. Let $\rho(t)\perp\varsigma(t)$, for all $t \in [0, 1]$. Now, we have

$$\begin{aligned}\Re\rho(t) &= \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\rho(s))\,ds \\ &\quad + \frac{2t}{(2-\lambda^2)\Gamma(\eta)}\int_0^\lambda \left(\int_0^s (s-m)^{\eta-1} f(m,\rho(m))\,dm\right) ds > 0,\end{aligned}$$

which yields that $\Re\rho(t)\perp\Re\varsigma(t)$, that is, $\Re$ is $\perp$-preserving. Subsequently, for all $t \in [0, 1]$, $\rho(t)\perp\varsigma(t)$, we have

$$\alpha(\rho(t), \varsigma(t)) \left|\Re\rho(t) - \Re\varsigma(t)\right|$$

$$\leq \quad \left|\Re\rho(t) - \Re\varsigma(t)\right| = \left| \begin{array}{c} \frac{1}{\Gamma(\eta)} \int\limits_0^t (t-s)^{\eta-1} f(s,\rho(s))\, ds \\ + \frac{2t}{(2-\lambda^2)\Gamma(\eta)} \int\limits_0^\lambda \left(\int\limits_0^s (s-m)^{\eta-1} f(m,\rho(m))\, dm \right) ds \\ - \frac{1}{\Gamma(\eta)} \int\limits_0^t (t-s)^{\eta-1} f(s,\varsigma(s))\, ds \\ - \frac{2t}{(2-\lambda^2)\Gamma(\eta)} \int\limits_0^\lambda \left(\int\limits_0^s (s-m)^{\eta-1} f(m,\varsigma(m))\, dm \right) ds \end{array} \right|$$

$$\begin{aligned} \leq \quad & \frac{1}{\Gamma(\eta)} \int\limits_0^t |t-s|^{\eta-1} \left| f(s,\rho(s)) - f(s,\varsigma(s)) \right| ds \\ & + \frac{2t}{(2-\lambda^2)\Gamma(\eta)} \int\limits_0^\lambda \left(\int\limits_0^s (s-m)^{\eta-1} \left| f(m,\varsigma(m)) - f(m,\rho(m)) \right| dm \right) ds, \end{aligned}$$

which yields

$$\begin{aligned} & \alpha(\rho(t),\varsigma(t)) \left|\Re\rho(t) - \Re\varsigma(t)\right| \\ \leq \quad & \left(\frac{1}{\Gamma(\eta)} \int\limits_0^t |t-s|^{\eta-1} ds + \frac{2t}{(2-\lambda^2)\Gamma(\eta)} \int\limits_0^\lambda \left(\int\limits_0^s |s-m|^{\eta-1} dm \right) ds \right) \vartheta\beta \|\rho - \varsigma\|_\infty \end{aligned}$$

$$\begin{aligned} = \quad & \left(\frac{1}{\Gamma(\eta+1)} + \frac{2\lambda^{\eta+1}\Gamma(\eta)}{(2-\lambda^2)\Gamma(\eta+1)} \right) \vartheta \|\rho(s) - \varsigma(s)\|_\infty \\ = \quad & \vartheta\beta \|\rho - \varsigma\|_\infty . \end{aligned}$$

Take $\varrho = \vartheta\beta < 1$. By the definition of d, we have

$$\alpha(\rho,\varsigma)\, d(\Re\rho, \Re\varsigma) \leq \varrho d(\rho,\varsigma).$$

Then,

$$e^{\sqrt{\alpha(\rho,\varsigma) d(\Re\rho,\Re\varsigma)}} \leq \left[e^{\sqrt{d(\rho,\varsigma)}} \right]^\varrho,$$

where $\varrho \in (0,1)$. Now, if $\Theta(t) = e^{\sqrt{t}}$, for all $t > 0$, then $\Theta \in \Psi$. Hence, from above,

$$\alpha(\rho,\varsigma)\, \Theta\left(d(\Re\rho, \Re\varsigma)\right) \leq \left[\Theta\left(d(\rho,\varsigma)\right)\right]^\varrho$$

for all $\rho, \varsigma \in \aleph$ and $d(\Re\rho, \Re\varsigma) > 0$. Thus, all the conditions of result 3 are satisfied, and thus Eq (5.1) has a unique solution. □

6. Conclusions

In this work, we defined the notion of a generalized (α,$\Theta_{\mathcal{F}}$)-contraction in the background of an orthogonal $\mathcal{F}$-metric space and proved some certain fixed point results. As outcomes of the leading results, we obtained some fixed point theorems in $\mathcal{F}$-metric spaces and orthogonal metric spaces. Moreover, a non trivial example is also furnished to validate the originality of the obtained results. We investigated the existence and uniqueness of a solution for the fractional differential equation as application of our main results.

For future work, the notion of an $\mathcal{F}$-metric space can be extended to a graphical $\mathcal{F}$-metric space, and the results proved in this article can be extended to multivalued mappings and fuzzy set valued mappings. Moreover, we can solve differential and integral inclusions as applications of fixed point results for multivalued mappings in the setting of $\mathcal{F}$-metric space.

Acknowledgments

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors acknowledge with thanks DSR, KAU for financial support.

Conflict of interest

The authors declare that they have no conflicts of interests.

References

1. M. Bestvina, Real trees in topology, geometry and group theory, arXiv:math/9712210.
2. W. Kirk, Some recent results in metric fixed point theory, *J. Fixed Point Theory Appl.*, **2** (2007), 195–207. http://dx.doi.org/10.1007/s11784-007-0031-8
3. C. Semple, M. Steel, *Phylogenetics*, Oxford: Oxford University Press, 2003.
4. M. Frechet, Sur quelques points du calcul fonctionnel, *Rend. Circ. Matem. Palermo*, **22** (1906), 1–72. http://dx.doi.org/10.1007/BF03018603
5. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, **1** (1993), 5–11.
6. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalizedmetric spaces, *Publ. Math. Debrecen*, **57** (2000), 31–37. http://dx.doi.org/10.5486/PMD.2000.2133
7. M. Jleli, B. Samet, On a new generalization of metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 128. http://dx.doi.org/10.1007/s11784-018-0606-6
8. M. Gordji, D. Rameani, M. De La Sen, Y. Cho, On orthogonal sets and Banach fixed point theorem, *Fixed Point Theory*, **18** (2017), 569–578. http://dx.doi.org/10.24193/fpt-ro.2017.2.45
9. T. Kanwal, A. Hussain, H. Baghani, M. De La Sen, New fixed point theorems in orthogonal $\mathcal{F}$-metric spaces with application to fractional differential equation, *Symmetry*, **12** (2020), 832. http://dx.doi.org/10.3390/sym12050832

10. I. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal*, **30** (1989), 26–37.

11. M. Khamsi, N. Hussain, KKM mappings in metric type spaces, *Nonlinear Anal.-Theor.*, **73** (2010), 3123–3129. http://dx.doi.org/10.1016/j.na.2010.06.084

12. J. Ahmad, A. Al-Rawashdeh, A. Al-Mazrooei, Fixed point results for $(\alpha, \perp_{\mathcal{F}})$-contractions in orthogonal F-metric spaces with applications, *J. Funct. Space.*, **2022** (2022), 8532797. http://dx.doi.org/10.1155/2022/8532797

13. L. Alnaser, D. Lateef, H. Fouad, J. Ahmad, Relation theoretic contraction results in F-metric spaces, *J. Nonlinear Sci. Appl.*, **12** (2019), 337–344. http://dx.doi.org/10.22436/jnsa.012.05.06

14. S. Al-Mezel, J. Ahmad, G. Marino, Fixed point theorems for generalized $(\alpha\beta$-$\psi)$-contractions in F-metric spaces with applications, *Mathematics*, **8** (2020), 584. http://dx.doi.org/10.3390/math8040584

15. M. Alansari, S. Mohammed, A. Azam, Fuzzy fixed point results in F-metric spaces with applications, *J. Funct. Space.*, **2020** (2020), 5142815. http://dx.doi.org/10.1155/2020/5142815

16. D. Lateef, J. Ahmad, Dass and Gupta's Fixed point theorem in F-metric spaces, *J. Nonlinear Sci. Appl.*, **12** (2019), 405–411. http://dx.doi.org/10.22436/jnsa.012.06.06

17. A. Hussain, T. Kanwal, Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results, *T. A. Razmadze Math. In.*, **172** (2018), 481–490. http://dx.doi.org/10.1016/j.trmi.2018.08.006

18. Z. Ahmadi, R. Lashkaripour, H. Baghani, A fixed point problem with constraint inequalities via a contraction in incomplete metric spaces, *Filomat*, **32** (2018), 3365–3379. http://dx.doi.org/10.2298/FIL1809365A

19. H. Baghani, M. Ramezani, Coincidence and fixed points for multivalued mappings in incomplete metric spaces with applications, *Filomat*, **33** (2019), 13–26. http://dx.doi.org/10.2298/FIL1901013B

20. A. Ran, M. Reuring, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Am. Math. Soc.*, **132** (2004), 1435–1443. http://dx.doi.org/10.1090/S0002-9939-03-07220-4

21. K. Javed, H. Aydi, F. Uddin, M. Arshad, On orthogonal partial b-metric spaces with an application, *J. Math.*, **2021** (2021), 6692063. http://dx.doi.org/10.1155/2021/6692063

22. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.-Theor.*, **75** (2012), 2154–2165. http://dx.doi.org/10.1016/j.na.2011.10.014

23. M. Ramezani, Orthogonal metric space and convex contractions, *Int. J. Nonlinear Anal.*, **6** (2015), 127–132. http://dx.doi.org/ 10.22075/IJNAA.2015.261

24. A. Asif, M. Nazam, M. Arshad, S. Kim, F-metric, F-contraction and common fixed point theorems with applications, *Mathematics*, **7** (2019), 586. http://dx.doi.org/10.3390/math7070586

25. G. Mani, A. Gnanaprakasam, N. Kausar, M. Munir, Salahuddin. Orthogonal F-contraction mapping on O-complete metric space with applications, *Int. J. Fuzzy Log. Inte.*, **21** (2021), 243–250. http://dx.doi.org/10.5391/IJFIS.2021.21.3.243

26. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fund. Math.*, **3** (1922), 133–181. http://dx.doi.org/10.4064/fm-3-1-133-181
27. J. Ahmad, A. Al-Rawashdeh, A. Azam, Fixed point results for $\{\alpha, \xi\}$-expansive locally contractive mappings, *J. Inequal. Appl.*, **2014** (2014), 364. http://dx.doi.org/10.1186/1029-242X-2014-364
28. J. Ahmad, A. Al-Rawashdeh, A. Azam, New fixed point theorems for generalized F-contractions in complete metric spaces, *Fixed Point Theory Appl.*, **2015** (2015), 80. http://dx.doi.org/10.1186/s13663-015-0333-2
29. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 38. http://dx.doi.org/10.1186/1029-242X-2014-38
30. J. Ahmad, A. Al-Mazrooei, Y. Cho, Y. Yang, Fixed point results for generalized Θ-contractions, *J. Nonlinear Sci. Appl.*, **10** (2017), 2350–2358. http://dx.doi.org/10.22436/jnsa.010.05.07
31. N. Hussain, V. Parvaneh, B. Samet, C. Vetro, Some fixed point theorems for generalized contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2015** (2015), 185. http://dx.doi.org/10.1186/s13663-015-0433-z
32. N. Hussain, J. Ahmad, New Suzuki-Berinde type fixed point results, *Carpathian J. Math.*, **33** (2017), 59–72. http://dx.doi.org/10.37193/CJM.2017.01.07
33. Z. Li, S. Jiang, Fixed point theorems of JS-quasi-contractions, *Fixed Point Theory Appl.*, **2016** (2016), 40. http://dx.doi.org/10.1186/s13663-016-0526-3
34. F. Vetro, A generalization of Nadler fixed point theorem, *Carpathian J. Math*, **31** (2015), 403–410.
35. H. Ali, H. Isik, H. Aydi, E. Ameer, J. Lee, M. Arshad, On multivalued Suzuki-type Θ-contractions and related applications, *Open Math.*, **18** (2020), 386–399. http://dx.doi.org/10.1515/math-2020-0139
36. E. Ameer, H. Aydi, M. Arshad, A. Hussain, A. Khan, Ćirić type multi-valued α_*-η_*-Θ-contractions on b-meric spaces with applications, *Int. J. Nonlinear Anal.*, **12** (2021), 597–614. http://dx.doi.org/10.22075/IJNAA.2021.4865
37. L. Ćirić, Generalized contractions and fixed-point theorems, *Publ. Inst. Math.*, **12** (1971), 9–26.
38. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71–76.
39. S. Chatterjea, Fixed point theorem, *C. R. Acad. Bulg. Sci.*, **25** (1972), 727–730.
40. S. Reich, Kannan's fixed point theorem, *Bull. Univ. Mat. Italiana*, **4** (1971), 1–11.
41. H. Mohammadi, S. Kumar, S. Rezapour, S. Etemad, A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control, *Chaos Soliton. Fract.*, **144** (2021), 110668. http://dx.doi.org/10.1016/j.chaos.2021.110668
42. S. Kumar, P. Shaw, A. Abdel-Aty, E. Mahmoud, A numerical study on fractional differential equation with population growth model, *Numer. Meth. Part. D. E.*, in press. http://dx.doi.org/10.1002/num.22684
43. P. Shaw, S. Kumar, S. Momani, S. Hadid, Dynamical analysis of fractional plant disease model with curative and preventive treatments, *Chaos, Soliton. Fract.*, **164** (2022), 112705. http://dx.doi.org/10.1016/j.chaos.2022.112705

44. D. Gopal, M. Abbas, D. Patel, C. Vetro, Fixed point of α-type F-contractive mappings with an application to nonlinear fractional differential equation, *Acta Math. Sci.*, **36** (2016), 957–970. http://dx.doi.org/10.1016/S0252-9602(16)30052-2
45. D. Baleanu, S. Rezapour, H. Mohammadi, Some existence results on nonlinear fractional differential equations, *Philos. Trans. A Math. Phys. Eng. Sci.*, **371** (2013), 20120144. http://dx.doi.org/10.1098/rsta.2012.0144