AIMS Mathematics, 8(3): 5060–5079.
DOI: 10.3934/math.2023254
Received: 20 October 2022
Revised: 27 November 2022
Accepted: 28 November 2022
Published: 12 December 2022

http://www.aimspress.com/journal/Math

Research article

Multiple solutions to the double phase problems involving concave-convex nonlinearities

Jae-Myoung Kim[1] **and Yun-Ho Kim**[2,*]

[1] Department of Mathematics Education, Andong National University, Andong 36729, Korea
[2] Department of Mathematics Education, Sangmyung University, Seoul 03016, Korea

* **Correspondence:** Email: kyh1213@smu.ac.kr.

Abstract: This paper is concerned with several existence results of multiple solutions for Schrödinger-type problems involving the double phase operator for the case of a combined effect of concave-convex nonlinearities. The first one is to discuss that our problem has infinitely many large energy solutions. Second, we obtain the existence of a sequence of infinitely many small energy solutions to the given problem. To establish such multiplicity results, we employ the fountain theorem and the dual fountain theorem as the primary tools, respectively. In particular we give the existence result of small energy solutions on a new class of nonlinear term.

Keywords: double phase problems; Musielak-Orlicz-Sobolev spaces; variational methods; multiple solutions
Mathematics Subject Classification: 35B38, 35D30, 35J10, 35J20, 35J62

1. Introduction

In this paper, we are working with existence and multiplicity of solutions for the following double phase problem in $\mathbb{R}^N$:

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v + a(x)|\nabla v|^{q-2}\nabla v) + \mathcal{V}(x)(|v|^{p-2}v + a(x)|v|^{q-2}v) = \lambda\rho(x)\,|v|^{r-2}\,z + h(x,v) \text{ in } \mathbb{R}^N, \quad (1.1)$$

where $N \geq 2$, $1 < p < q < N$, $1 < r < p$, $0 \leq a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $\mathcal{V} : \mathbb{R}^N \to (0,\infty)$ is a potential function satisfying

(V) $\mathcal{V} \in L^1_{loc}(\mathbb{R}^N)$, $ess\,inf_{x\in\mathbb{R}^N}\mathcal{V}(x) > 0$, and $\lim_{|x|\to\infty}\mathcal{V}(x) = +\infty$.

To do this, we assume that

(B1) $1 < r < p < q < \gamma < p^*$;

(B2) $0 \le \rho \in L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\text{meas}\Big\{x \in \mathbb{R}^N : \rho(x) \neq 0\Big\} > 0$ for any γ_0 with $p < \gamma_0 < p^*$;

(H1) there are $s \in (p,\ p^*)$, $0 \le \sigma_1 \in L^{s'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and a positive constant c_1 such that

$$|h(x,t)| \le \sigma_1(x) + c_1 |t|^{\gamma - 1}$$

for all $t \in \mathbb{R}$ and for almost all $x \in \mathbb{R}^N$;

(H2) there exists $\nu > q$ and $M_0 > 0$ such that

$$h(x,t)t - \nu H(x,t) \ge -\beta_0(x)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| \ge M_0$ and for some $\beta_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\beta_0(x) \ge 0$, where $H(x,t) = \int_0^t h(x,s)\,ds$;

(H3) there exist $\nu > q$, $\varrho \ge 0$ and $M_1 > 0$ such that

$$h(x,t)t - \nu H(x,t) \ge -\varrho\, |t|^p - \beta_1(x)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| \ge M_1$ and for some $\beta_1 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\beta_1(x) \ge 0$;

(H4) there exist $C > 0$, $1 < \kappa < p$, $\tau > 1$ with $p \le \tau' \kappa \le p^*$ and a positive function $\xi \in L^\tau(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$\liminf_{|t| \to 0} \frac{h(x,t)}{\xi(x)\, |t|^{\kappa - 2}\, t} \ge C$$

uniformly for almost all $x \in \mathbb{R}^N$.

Remark 1.1. *It is clear that the condition* (H3) *is weaker than* (H2)*, which was initially provided by the paper [31]. If we consider the function*

$$h(x,\ell) = \sigma(x)\left(\xi(x)\, |\ell|^{\kappa - 2}\, \ell + |\ell|^{p-2}\, \ell + \frac{2}{p} \sin \ell\right)$$

with its primitive function

$$H(x,\ell) = \sigma(x)\left(\frac{\xi(x)}{\kappa}\, |\ell|^\kappa + \frac{1}{p}\, |\ell|^p - \frac{2}{p} \cos \ell + \frac{2}{p}\right),$$

where $\sigma \in C(\mathbb{R}^N, \mathbb{R})$ *with* $0 < \inf_{x \in \mathbb{R}^N} \sigma(x) \le \sup_{x \in \mathbb{R}^N} \sigma(x) < \infty$*, and* κ, ξ *are given in* (H4)*, then it is obvious that this example satisfies the condition* (H3) *but not* (H2)*. Also the conditions* (H1) *and* (H4) *are satisfied.*

The double phase operator, which is the natural generalization of the p-Laplace operator, has been extensively studied by many researchers. The interest in variational problems with double phase operator is founded on their popularity in diverse fields of mathematical physics, such as plasma physics, biophysics and chemical reactions, strongly anisotropic materials, Lavrentiev's phenomenon, etc.; see [47, 48]. With regard to regularity theory for double phase functionals, we would like to mention a series of notable papers by Mingione et al. [4–6, 12–14]. Also, we refer to the works of Bahrouni-Rădulescu-Repovš [3], Byun-Oh [9], Colasuonno-Squassina [11], Crespo Blanco-Gasiński-Harjulehto-Winkert [15], Gasiński-Winkert [18, 19], Kim-Kim-Oh-Zeng [27], Liu-Dai [33], Papageorgiou-Rădulescu-Repovš [36, 37], Perera-Squassina [38], Ragusa-Tachikawa [39], Zhang-Rădulescu [46], Zeng-Bai-Gasiński-Winkert [44, 45].

The goal of this paper is to provide several existence results of multiple solutions for Schrödinger-type problems involving the double phase operator for the case of a combined effect of concave–convex nonlinearities. The first one is to discuss that problem (1.1) has an infinitely many large energy solutions (see Theorem 2.14). Second, we obtain the existence of a sequence of infinitely many small energy solutions to problem (1.1) (see Theorem 2.21). To get such multiplicity results, we employ the fountain theorem and the dual fountain theorem as the primary tools, respectively. The present paper is motivated by the work of Stegliński [41]. The author obtained such multiplicity results for the double phase problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) + \mathcal{V}(x)(|u|^{p-2}u + a(x)|u|^{q-2}u) = h(x, u) \text{ in } \mathbb{R}^N.$$

Here, the Carathéodory function $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ fulfills the condition (H4) and the following assumption:

(MS) There is a positive function $\eta \in L^1(\mathbb{R}^N)$ such that

$$\ell h(x, \ell) - qH(x, \ell) \le \varsigma h(x, \varsigma) - qH(x, \varsigma) + \eta(x)$$

for any $x \in \mathbb{R}^N$, $0 < \ell < \varsigma$ or $\varsigma < \ell < 0$,

which is first provided by Miyagaki and Souto [34]. However, it is clear that the example in Remark 1.1 does not satisfy the condition (MS). Let us consider the function

$$f(x, \ell) = \sigma(x)\left(\xi(x)\,|\ell|^{\kappa-2}\,\ell + |\ell|^{q-2}\,\ell \ln(1 + |\ell|) + \frac{|\ell|^{q-1}\,\ell}{1 + |\ell|}\right)$$

with its primitive function

$$F(x, \ell) = \sigma(x)\left(\frac{\xi(x)}{\kappa}\,|\ell|^{\kappa} + \frac{1}{q}\,|\ell|^{q} \ln(1 + |\ell|)\right)$$

for all $\ell \in \mathbb{R}$ and $1 < \kappa < p < q$ for all $x \in \mathbb{R}^N$, where σ is given in Remark 1.1. Then, this example fulfills the condition (MS) but not (H3). Such existence results of multiple solutions to double phase problems are particularly motivated by the contributions in recent studies [1, 10, 20–23, 25, 26, 29–32, 40, 42], and the references therein. However our proof of the existence of a sequence of small energy solutions is slightly different from those of previous related studies [10, 21, 25, 30, 32, 42, 43]. Roughly speaking, in view of [10, 21, 25, 30] the conditions on the nonlinear term h near zero and at infinity (see (H5) and (2.21), which will be specified later) play an important role in verifying assumptions in the dual fountain theorem, but we ensure them when (H5) is not assumed and (2.21) is replaced by (H4); see Remark 2.20 for more details and the difference from the papers [32, 42, 43]. For this reason, on a new class of nonlinear term h we give the existence result of small energy solutions via applying the dual fountain theorem. As far as we are, although this work is inspired by the papers [10, 27], and many authors have an interest in the investigation of elliptic problems with double phase operator, this paper is the first effort to develop the multiplicity results for the concave-convex-type double phase problems because we assert our results on a new class of nonlinear term h. The main difficulty for establishing our results under various conditions on the convex term h is to ensure the Cerami compactness condition of the energy functional corresponding to (1.1). To overcome this, we assume the fact that the potential function V is coercive.

The outline of this paper is as follows. We present some necessary preliminary knowledge of function spaces which we will use along the paper. Next, we provide the variational framework related to problem (1.1), and then we obtain various existence results of infinitely many nontrivial solutions to the double phase equations with concave-convex type nonlinearities under appropriate conditions on h.

2. Preliminaries

In this section, we briefly demonstrate some definitions and essential properties of Musielak-Orlicz-Sobolev space. For a deeper treatment of these spaces, we refer to [11, 16, 24, 35].

The functions $\mathcal{H} : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ and $\mathcal{H}_\mathcal{V} : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ are defined as

$$\mathcal{H}(x,t) := t^p + a(x)t^q, \quad \mathcal{H}_\mathcal{V}(x,t) := \mathcal{V}(x)(t^p + a(x)t^q), \tag{2.1}$$

for almost all $x \in \mathbb{R}^N$ and for any $t \in [0,\infty)$, with $1 < p < q$, $0 \le a \in L^1(\mathbb{R}^N)$ and $\mathcal{V} : \mathbb{R}^N \to \mathbb{R}$. Define the Musielak-Orlicz space $L^\mathcal{H}(\mathbb{R}^N)$ as

$$L^\mathcal{H}(\mathbb{R}^N) := \left\{ z : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \ \varrho_\mathcal{H}(z) < \infty \right\},$$

induced by the Luxemburg norm

$$|z|_\mathcal{H} := \inf\left\{ \lambda > 0 : \ \varrho_\mathcal{H}\left(x, \left|\frac{z}{\lambda}\right|\right) \le 1 \right\},$$

where $\varrho_\mathcal{H}$ denotes the $\mathcal{H}$-modular function with

$$\varrho_\mathcal{H}(z) := \int_{\mathbb{R}^N} \mathcal{H}(x,|z|)dx = \int_{\mathbb{R}^N} (|z|^p + a(x)|z|^q)\, dx. \tag{2.2}$$

If we replace in the above definition $\mathcal{H}$ by $\mathcal{H}_\mathcal{V}$, we obtain the definition of the Musielak-Orlicz space $(L_{\mathcal{H}_\mathcal{V}}(\mathbb{R}^N), |\cdot|_{\mathcal{H}_\mathcal{V}})$, i.e.,

$$L_{\mathcal{H}_\mathcal{V}}(\mathbb{R}^N) := \left\{ z : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \ \varrho_\mathcal{V}^\mathcal{H}(z) < \infty \right\},$$

induced by the Luxemburg norm

$$|z|_{\mathcal{H}_\mathcal{V}} := \inf\left\{ \lambda > 0 : \ \varrho_\mathcal{V}^\mathcal{H}\left(x, \left|\frac{z}{\lambda}\right|\right) \le 1 \right\},$$

where $\varrho_\mathcal{V}^\mathcal{H}$ denotes the $\mathcal{H}_\mathcal{V}$-modular function as

$$\varrho_\mathcal{V}^\mathcal{H}(z) := \int_{\mathbb{R}^N} \mathcal{H}_\mathcal{V}(x,|z|)dx = \int_{\mathbb{R}^N} \mathcal{V}(x)\,(|z|^p + a(x)|z|^q)\, dx. \tag{2.3}$$

By [24, 41], the space $L^\mathcal{H}(\mathbb{R}^N)$ and $L_{\mathcal{H}_\mathcal{V}}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Lemma 2.1. ([41]) *For* $\varrho_{\mathcal{V}}^{\mathcal{H}}(z)$ *given in* (2.3) *and* $z \in L_{\mathcal{H}_{\mathcal{V}}}\left(\mathbb{R}^N\right)$*, we have the following*

(i) for $z \neq 0, |z|_{\mathcal{H}_{\mathcal{V}}} = \lambda$ *iff* $\varrho_{\mathcal{V}}^{\mathcal{H}}(\frac{z}{\lambda}) = 1$;

(ii) $|z|_{\mathcal{H}_{\mathcal{V}}} < 1(= 1; > 1)$ *iff* $\varrho_{\mathcal{V}}^{\mathcal{H}}(z) < 1(= 1; > 1)$;

(iii) if $|z|_{\mathcal{H}_{\mathcal{V}}} > 1$*, then* $|z|_{\mathcal{V},\mathcal{H}}^{p} \leq \varrho_{\mathcal{V}}^{\mathcal{H}}(z) \leq |z|_{\mathcal{H}_{\mathcal{V}}}^{q}$;

(iv) if $|z|_{\mathcal{H}_{\mathcal{V}}} < 1$*, then* $|z|_{\mathcal{H}_{\mathcal{V}}}^{q} \leq \varrho_{\mathcal{V}}^{\mathcal{H}}(z) \leq |z|_{\mathcal{H}_{\mathcal{V}}}^{p}$.

Also, an analogous results hold for $\varrho_{\mathcal{H}}(u)$ *given in* (2.2) *and* $\|\cdot\|_{\mathcal{H}}$.

The weighted Musielak-Orlicz-Sobolev space $W_{\mathcal{V}}^{1,\mathcal{H}}(\mathbb{R}^N)$ is defined by

$$W_{\mathcal{V}}^{1,\mathcal{H}}(\mathbb{R}^N) = \{z \in L_{\mathcal{H}_{\mathcal{V}}}(\mathbb{R}^N) : |\nabla z| \in L^{\mathcal{H}}(\mathbb{R}^N)\},$$

and it is equipped with the norm

$$|z| = |\nabla z|_{\mathcal{H}} + |z|_{\mathcal{H}_{\mathcal{V}}}.$$

Note that $W_{\mathcal{V}}^{1,\mathcal{H}}(\mathbb{R}^N)$ is a separable reflexive Banach space; see [28]. In what follows, the notation $E \hookrightarrow F$ means that the space E is *continuously* imbedded into the space F, while $E \hookrightarrow\hookrightarrow F$ means that E is *compactly* imbedded into F.

Lemma 2.2. ([41]) *The following embeddings hold:*

(i) $L_{\mathcal{H}_{\mathcal{V}}}(\mathbb{R}^N) \hookrightarrow L^{\mathcal{H}}(\mathbb{R}^N)$;

(ii) $W_{\mathcal{V}}^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow L^{\tau}(\mathbb{R}^N)$ *for* $\tau \in [p, p^*]$;

(iii) $W_{\mathcal{V}}^{1,\mathcal{H}}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{\tau}(\mathbb{R}^N)$ *for* $\tau \in [p, p^*)$.

Lemma 2.3. ([41]) *Let*

$$\varphi(z) := \int_{\mathbb{R}^N} (|\nabla z|^p + a(x)|\nabla z|^q)\, dx + \int_{\mathbb{R}^N} \mathcal{V}(x)\, (|z|^p + a(x)|z|^q)\, dx. \tag{2.4}$$

The following properties hold:

(i) $\varphi(z) \leq |z|^p + |z|^q$ *for all* $z \in W_{\mathcal{V}}^{1,\mathcal{H}}(\mathbb{R}^N)$;

(ii) If $|z| \leq 1$*, then* $2^{1-q}|z|^q \leq \varphi(z) \leq |z|^p$;

(iii) If $|z| \geq 1$*, then* $2^{-p}|z|^p \leq \varphi(z) \leq 2|z|^q$.

Let us define the functional $\Phi : \mathfrak{X} := W_{\mathcal{V}}^{1,\mathcal{H}}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi(v) = \int_{\mathbb{R}^N} \left(\frac{1}{p}|\nabla v|^p + \frac{a(x)}{q}|\nabla v|^q\right) dx + \int_{\mathbb{R}^N} \mathcal{V}(x) \left(\frac{1}{p}|v|^p + \frac{a(x)}{q}|v|^q\right) dx.$$

Then, it is easy to check that $\Phi \in C^1(\mathfrak{X}, \mathbb{R})$, and double-phase operator $-\mathrm{div}(|\nabla v|^{p-2}\nabla v + a(x)|\nabla v|^{q-2}\nabla v)$ is the derivative operator of Φ in the weak sense. We define $\Phi' : \mathfrak{X} \to \mathfrak{X}^*$ with

$$\langle \Phi'(v), w\rangle = \int_{\mathbb{R}^N} (|\nabla v|^{p-2}\nabla v \cdot \nabla w + a(x)|\nabla v|^{q-2}\nabla v \cdot \nabla w) + \int_{\mathbb{R}^N} \mathcal{V}(x)(|v|^{p-2}vw + a(x)|v|^{q-2}vw)\, dx,$$

for all $w, v \in \mathfrak{X}$. Here, $\mathfrak{X}^*$ denotes the dual space of $\mathfrak{X}$, and $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathfrak{X}$ and $\mathfrak{X}^*$.

Lemma 2.4. ([41]) *Let the assumption* (V) *hold. Then, we have the following*

(i) $\Phi' : \mathfrak{X} \to \mathfrak{X}^*$ *is a bounded, continuous and strictly monotone operator;*

(ii) $\Phi' : \mathfrak{X} \to \mathfrak{X}^*$ *is a mapping of type* (S_+), *i.e., if* $v_n \rightharpoonup v$ *in* $\mathfrak{X}$ *and*

$$\limsup_{n\to\infty} \langle \Phi'(v_n) - \Phi'(v), v_n - v \rangle \leq 0,$$

then $v_n \to v$ *in* $\mathfrak{X}$;

(iii) $\Phi' : \mathfrak{X} \to \mathfrak{X}^*$ *is a homeomorphism.*

Definition 2.5. *We say that* $v \in \mathfrak{X}$ *is a weak solution of problem* (1.1) *if*

$$\int_{\mathbb{R}^N} \left(|\nabla v|^{p-2} \nabla v \cdot \nabla u + a(x) |\nabla v|^{q-2} \nabla v \cdot \nabla u \right) dx + \int_{\mathbb{R}^N} \mathcal{V}(x)(|v|^{p-2} vu + a(x)|v|^{q-2} vu)\, dx$$

$$= \lambda \int_{\mathbb{R}^N} \rho(x) |v|^{r-2} vu\, dx + \int_{\mathbb{R}^N} h(x,v)u\, dx,$$

for any $u \in \mathfrak{X}$.

Let us define the functional $\Psi_\lambda : \mathfrak{X} \to \mathbb{R}$ by

$$\Psi_\lambda(v) = \frac{\lambda}{r} \int_{\mathbb{R}^N} \rho(x) |v|^r \, dx + \int_{\mathbb{R}^N} H(x,v)\, dx.$$

Then, it is easy to show that $\Psi_\lambda \in C^1(\mathfrak{X}, \mathbb{R})$, and its Fréchet derivative is

$$\left\langle \Psi'_\lambda(v), w \right\rangle = \lambda \int_{\mathbb{R}^N} \rho(x)|v|^{r-2} vw\, dx + \int_{\mathbb{R}^N} h(x,v)w\, dx$$

for any $v, w \in \mathfrak{X}$; see [41]. Next, we define the functional $\mathcal{E}_\lambda : \mathfrak{X} \to \mathbb{R}$ by

$$\mathcal{E}_\lambda(v) = \Phi(v) - \Psi_\lambda(v).$$

Then, it follows that the functional $\mathcal{E}_\lambda \in C^1(\mathfrak{X}, \mathbb{R})$, and its Fréchet derivative is

$$\left\langle \mathcal{E}'_\lambda(v), w \right\rangle = \langle \Phi'(v), w \rangle - \left\langle \Psi'_\lambda(v), w \right\rangle \quad \text{for any } v, w \in \mathfrak{X}.$$

Before going to the proofs of our main consequences, we present some useful preliminary assertions.

Lemma 2.6. ([41]) *Assume that* (V), (B1), (B2) *and* (H1) *hold. Then,* Ψ_λ *and* Ψ'_λ *are sequentially weakly strongly continuous.*

Definition 2.7. *Suppose that* $\mathfrak{E}$ *is a real Banach space. We say that the functional* $\mathcal{F}$ *satisfies the Cerami condition ((C)-condition for short) in* $\mathfrak{E}$, *if any (C)-sequence* $\{v_n\} \subset \mathfrak{E}$, *i.e.,* $\{\mathcal{F}(v_n)\}$ *is bounded and* $\|\mathcal{F}'(v_n)\|_{\mathfrak{E}^*}(1 + \|v_n\|) \to 0$ *as* $n \to \infty$, *has a convergent subsequence in* $\mathfrak{E}$.

The following lemmas are the compactness condition for the Palais-Smale type, which plays a crucial role in obtaining our main result. The basic idea of proofs of these consequences follows the analogous arguments as in [26].

Lemma 2.8. *Suppose that* (V), (B1), (B2), (H1) *and* (H2) *hold. Then, the functional* $\mathcal{E}_\lambda$ *ensures the* (C)*-condition for any* $\lambda > 0$.

Proof. Let $\{v_n\}$ be a (C)-sequence in $\mathfrak{X}$, *i.e.*,

$$\sup_{n\in\mathbb{N}} |\mathcal{E}_\lambda(v_n)| \le \mathcal{K}_1 \text{ and } \langle \mathcal{E}'_\lambda(v_n), v_n\rangle = o(1) \to 0, \tag{2.5}$$

as $n \to \infty$, and $\mathcal{K}_1$ is a positive constant. First, we prove that $\{v_n\}$ is bounded in $\mathfrak{X}$. Since $\mathcal{V}(x) \to +\infty$ as $|x| \to \infty$, we have

$$\begin{aligned}\Big(\frac{1}{q}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx - C_1\int_{|v_n|\le M}(|v_n|^p+\sigma_1(x)\,|v_n|+c_1\,|v_n|^\gamma)\,dx \\ \ge \frac{1}{2}\Big(\frac{1}{q}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}\mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx - \mathcal{K}_0,\end{aligned} \tag{2.6}$$

for any positive constant C_1 and some positive constants $\mathcal{K}_0$, where $\mathcal{H}_{\mathcal{V}}(x,t)$ is given in (2.1). Indeed, by Young's inequality we know that

$$\begin{aligned}&\Big(\frac{1}{q}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx - C_1\int_{|v_n|\le M}(|v_n|^p+\sigma_1(x)\,|v_n|+c_1\,|v_n|^\gamma)\,dx \\
&\ge \Big(\frac{1}{q}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx - C_1\int_{|v_n|\le M}\Big(|v_n|^p+\sigma_1^{s'}(x)+|v_n|^s+c_1\,|v_n|^\gamma\Big)\,dx \\
&\ge \frac{1}{2}\Big(\frac{1}{q}-\frac{1}{\nu}\Big)\Big[\int_{\mathbb{R}^N}\mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx+\int_{|v_n|\le M}\mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx\Big] \\
&- C_1\int_{|v_n|\le 1}(|v_n|^p+|v_n|^s+c_1\,|v_n|^\gamma)\,dx \\
&- C_1\int_{1<|v_n|\le M}(|v_n|^p+|v_n|^s+c_1\,|v_n|^\gamma)\,dx - C_1|\sigma_1|_{L^{s'}(\mathbb{R}^N)} \\
&\ge \frac{1}{2}\Big(\frac{1}{q}-\frac{1}{\nu}\Big)\Big[\int_{\mathbb{R}^N}\mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx+\int_{|v_n|\le M}\mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx\Big] \\
&- C_1\,(2+c_1)\int_{|v_n|\le 1}(|v_n|^p+a(x)\,|v_n|^q)\,dx - C_1|\sigma_1|_{L^{s'}(\mathbb{R}^N)} \\
&- C_1\,(1+M^{s-p}+M^{\gamma-p}c_1)\int_{1<|v_n|\le M}(|v_n|^p+a(x)\,|v_n|^q)\,dx \\
&\ge \frac{1}{2}\Big(\frac{1}{q}-\frac{1}{\nu}\Big)\Big[\int_{\mathbb{R}^N}\mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx+\int_{|v_n|\le M}\mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx\Big] \\
&- \widetilde{C}_0\int_{|v_n|\le M}\mathcal{H}(x,|v_n|)\,dx - \widetilde{C}_1,\end{aligned} \tag{2.7}$$

where $\mathcal{H}(x,t)$ is given in (2.1), and

$$\widetilde{C}_0 := C_1 \max\{2+c_1, 2M^{s-p}+M^{\gamma-p}c_1\}.$$

Since $\mathcal{V}(x) \to +\infty$ as $|x| \to \infty$, there is $r_0 > 0$ such that $|x| \ge r_0$ implies $\mathcal{V}(x) \ge \frac{2q\nu\widetilde{C}_0}{\nu-q}$. Then, we know that

$$\mathcal{H}_{\mathcal{V}}(x,|v_n|) \ge \frac{2q\nu\widetilde{C}_0}{\nu-q}\mathcal{H}(x,|v_n|) \tag{2.8}$$

for $|x| \geq r_0$. Set $\mathbb{B}_{r_0} = \{x \in \mathbb{R}^N : |x| < r_0\}$. Then, since $\mathcal{V} \in L^1_{loc}(\mathbb{R}^N)$ and $a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we infer

$$\int_{\{|v_n| \leq M\} \cap \mathbb{B}_{r_0}} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx \leq \widetilde{C}_2 \quad \text{and} \quad \int_{\{|v_n| \leq M\} \cap \mathbb{B}_{r_0}} \mathcal{H}(x, |v_n|)\, dx \leq \widetilde{C}_3$$

for some positive constants $\widetilde{C}_2, \widetilde{C}_3$. This together with (2.7) and (2.8) yields

$$\begin{aligned}
&\Big(\frac{1}{q} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx - C_1 \int_{|v_n| \leq M} (|v_n|^p + \sigma_1(x)\, |v_n| + c_1\, |v_n|^\gamma)\, dx \\
&\geq \frac{\nu - q}{2q\nu} \Big[\int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx + \int_{\{|v_n| \leq M\} \cap \mathbb{B}^c_{r_0}} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx + \int_{\{|v_n| \leq M\} \cap \mathbb{B}_{r_0}} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx \Big] \\
&- \widetilde{C}_0 \left[\int_{\{|v_n| \leq M\} \cap \mathbb{B}^c_{r_0}} \mathcal{H}(x, |v_n|)\, dx + \int_{\{|v_n| \leq M\} \cap \mathbb{B}_{r_0}} \mathcal{H}(x, |v_n|)\, dx \right] - \widetilde{C}_1 \\
&\geq \frac{\nu - q}{2q\nu} \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx + \frac{\nu - q}{2q\nu} \int_{\{|v_n| \leq M\} \cap \mathbb{B}^c_{r_0}} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx \\
&- \widetilde{C}_0 \int_{\{|v_n| \leq M\} \cap \mathbb{B}^c_{r_0}} \mathcal{H}(x, |v_n|)\, dx - \mathcal{K}_0 \\
&\geq \frac{1}{2}\Big(\frac{1}{q} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx - \mathcal{K}_0,
\end{aligned}$$

as claimed. Combining (2.6) with (B1), (B2) and (H1), (H2), one has

$$\begin{aligned}
\mathcal{K}_1 + o(1) &\geq \mathcal{E}_\lambda(v_n) - \frac{1}{\nu} \langle \mathcal{E}'_\lambda(v_n), v_n \rangle \\
&\geq \Big(\frac{1}{q} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v_n|)\, dx + \Big(\frac{1}{q} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx \\
&- \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) \int_{\mathbb{R}^N} \rho(x) |v_n|^r\, dx + \int_{\mathbb{R}^N} \left(\frac{1}{\nu} h(x, v_n) v_n - H(x, v_n)\right) dx \\
&\geq \Big(\frac{1}{q} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v_n|)\, dx + \Big(\frac{1}{q} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx \\
&- \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) \int_{\mathbb{R}^N} \rho(x) |v_n|^r\, dx + \int_{|v_n| > M} \left(\frac{1}{\nu} h(x, v_n) v_n - H(x, v_n)\right) dx \\
&- C_1 \int_{|v_n| \leq M} (|v_n|^p + \sigma_1(x)\, |v_n| + c_1\, |v_n|^\gamma)\, dx \\
&\geq \Big(\frac{1}{q} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v_n|)\, dx + \frac{1}{2}\Big(\frac{1}{q} - \frac{1}{\nu}\Big) \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx \\
&- \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) \int_{\mathbb{R}^N} \rho(x) |v_n|^r\, dx - \frac{1}{\nu} \int_{\mathbb{R}^N} \beta_0(x)\, dx - \mathcal{K}_0 \\
&\geq \frac{1}{2}\Big(\frac{1}{q} - \frac{1}{\nu}\Big) \left(\int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v_n|)\, dx + \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx \right) \\
&- \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) \int_{\mathbb{R}^N} \rho(x) |v_n|^r\, dx - \frac{1}{\nu} \int_{\mathbb{R}^N} \beta_0(x)\, dx - \mathcal{K}_0
\end{aligned}$$

$$\geq \Big(\frac{1}{q}-\frac{1}{\nu}\Big)\frac{1}{q2^{p+1}}\|v_n\|^p - \lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)|\rho|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|v_n\|^r_{L^{\gamma_0}(\mathbb{R}^N)} - \frac{1}{\nu}|\beta_0|_{L^1(\mathbb{R}^N)} - \mathcal{K}_0.$$

Since $p > r > 1$, we assert that the sequence $\{v_n\}$ is bounded in $\mathfrak{X}$, and thus $\{v_n\}$ has a weakly convergent subsequence in $\mathfrak{X}$. Without loss of generality, we suppose that

$$v_n \rightharpoonup v_0 \text{ in } \mathfrak{X} \text{ as } n \to \infty.$$

By Lemma 2.6, we infer that Ψ'_λ is compact, and so $\Psi'_\lambda(v_n) \to \Psi'_\lambda(v_0)$ in $\mathfrak{X}$ as $n \to \infty$. Since $\mathcal{E}'_\lambda(v_n) \to 0$ as $n \to \infty$, we know that

$$\langle \mathcal{E}'_\lambda(v_n), v_n - v_0\rangle \to 0 \text{ and } \langle \mathcal{E}'_\lambda(v_0), v_n - v_0\rangle \to 0,$$

and thus

$$\langle \mathcal{E}'_\lambda(v_n) - \mathcal{E}'_\lambda(z_0), v_n - v_0\rangle \to 0$$

as $n \to \infty$. From this, we have

$$\langle \Phi'(v_n) - \Phi'(v_0), v_n - v_0\rangle = \langle \Psi'_\lambda(v_n) - \Psi'_\lambda(v_0), v_n - v_0\rangle + \langle \mathcal{E}'_\lambda(v_n) - \mathcal{E}'_\lambda(z_0), v_n - v_0\rangle \to 0,$$

namely, $\langle \Phi'(v_n) - \Phi'(v_0), v_n - v_0\rangle \to 0$ as $n \to \infty$. Since $\mathfrak{X}$ is reflexive and Φ' is a mapping of type (S_+) by Lemma 2.4, we assert that

$$v_n \to v_0 \text{ in } \mathfrak{X} \text{ as } n \to \infty.$$

This completes the proof. □

Remark 2.9. *As mentioned in Remark 1.1, condition* (H3) *is weaker than* (H2). *However, to obtain the following compactness condition, we need an additional assumption on the nonlinear term h at infinity.*

Lemma 2.10. *Suppose that* (V), (B1), (B2), (H1) *and* (H3) *hold. In addition,*

(H5) $\lim_{|t|\to\infty} \frac{H(x,t)}{|t|^q} = \infty$ *uniformly for almost all $x \in \mathbb{R}^N$*

holds. Then, the functional $\mathcal{E}_\lambda$ fulfils the (C)*-condition for any $\lambda > 0$.*

Proof. Let $\{v_n\}$ be a (C)-sequence in $\mathfrak{X}$ satisfying (2.5). As in Lemma 2.8, it is sufficient to prove that $\{v_n\}$ is bounded in $\mathfrak{X}$. To this end, suppose to the contrary that $\|v_n\| > 1$ and $\|v_n\| \to \infty$ as $n \to \infty$, and a sequence $\{y_n\}$ is defined by $y_n = v_n/\|v_n\|$. Then, up to a subsequence, still denoted by $\{y_n\}$, we get $y_n \rightharpoonup y_0$ in $\mathfrak{X}$ as $n \to \infty$, and due to Lemma 2.2,

$$y_n \to y_0 \text{ a.e. in } \mathbb{R}^N, \quad y_n \to y_0 \text{ in } L^s(\mathbb{R}^N) \tag{2.9}$$

as $n \to \infty$, for any s with $p \leq s < p^*$. Combining (2.6) with (B1), (B2), (H1) and (H3), one has

$$\begin{aligned}\mathcal{K}_1 + o(1) &\geq \mathcal{E}_\lambda(v_n) - \frac{1}{\nu}\langle \mathcal{E}'_\lambda(v_n), v_n\rangle \\ &\geq \Big(\frac{1}{q}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}\mathcal{H}(x,|\nabla v_n|)\,dx + \frac{1}{2}\Big(\frac{1}{q}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}\mathcal{H}_{\mathcal{V}}(x,|v_n|)\,dx\end{aligned}$$

$$
\begin{aligned}
&- \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) \int_{\mathbb{R}^N} \rho(x)|v_n|^r \, dx - \frac{1}{\nu} \int_{|v_n|>M} (\varrho\, |v_n|^p + \beta_1(x)) \; dx - \mathcal{K}_0 \\
&\geq \frac{1}{2}\Big(\frac{1}{q} - \frac{1}{\nu}\Big)\left(\int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v_n|)\, dx + \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx\right) \\
&- \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) \int_{\mathbb{R}^N} \rho(x)|v_n|^r \, dx - \frac{1}{\nu} \int_{\mathbb{R}^N} (\varrho\, |v_n|^p + \beta_1(x)) \; dx - \mathcal{K}_0 \\
&\geq \frac{1}{2}\Big(\frac{1}{q} - \frac{1}{\nu}\Big)\frac{1}{q2^p}\|v_n\|^p - \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) \int_{\mathbb{R}^N} \rho(x)|v_n|^r \, dx \\
&\qquad\qquad - \frac{1}{\nu} \int_{\mathbb{R}^N} (\varrho\, |v_n|^p + \beta_1(x)) \; dx - \mathcal{K}_0 \\
&\geq \Big(\frac{1}{q} - \frac{1}{\nu}\Big)\frac{1}{q2^{p+1}}\|v_n\|^p - \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} |v_n|^r_{L^{\gamma_0}(\mathbb{R}^N)} \\
&- \frac{\varrho}{\nu}|v_n|^p_{L^p(\mathbb{R}^N)} - \frac{1}{\nu}|\beta_1|_{L^1(\mathbb{R}^N)} - \mathcal{K}_0.
\end{aligned}
$$

Hence, we know that

$$
\begin{aligned}
&\mathcal{K}_1 + o(1) + \lambda \left(\frac{1}{r} - \frac{1}{\nu}\right) |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} |v_n|^r_{L^{\gamma_0}(\mathbb{R}^N)} + \frac{\varrho}{\nu}|v_n|^p_{L^p(\mathbb{R}^N)} + \frac{1}{\nu}|\beta_1|_{L^1(\mathbb{R}^N)} + \mathcal{K}_0 \\
&\geq \Big(\frac{1}{q} - \frac{1}{\nu}\Big)\frac{1}{q2^{p+1}}\|v_n\|^p.
\end{aligned}
$$

Dividing this by $\left(\frac{1}{q} - \frac{1}{\nu}\right)\frac{1}{q2^{p+1}}\|v_n\|^p$ and then taking the limit supremum of this inequality as $n \to \infty$, we have

$$
1 \leq \frac{\varrho}{\left(\frac{1}{q} - \frac{1}{\nu}\right)\frac{\nu}{q2^{p+1}}} \limsup_{n\to\infty} |y_n|^p_{L^p(\mathbb{R}^N)} = \frac{\varrho}{\left(\frac{1}{q} - \frac{1}{\nu}\right)\frac{\nu}{q2^{p+1}}} |y_0|^p_{L^p(\mathbb{R}^N)}. \tag{2.10}
$$

Hence, it follows from (2.10) that $y_0 \neq 0$.

By Lemma 2.3 and the assumption (B2), we have

$$
\begin{aligned}
\mathcal{E}_\lambda(v_n) &\geq \frac{1}{q}\Big(\int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v_n|)\, dx + \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx\Big) \\
&\qquad\qquad - \frac{\lambda}{r} \int_{\mathbb{R}^N} \rho(x)|v_n|^r \, dx - \int_{\mathbb{R}^N} H(x, v_n)\, dx \\
&\geq \frac{1}{q2^p}\|v_n\|^p - \frac{\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} |v_n|^r_{L^{\gamma_0}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} H(x, v_n)\, dx \\
&\geq \frac{1}{q2^p}\|v_n\|^p - C_2\frac{\lambda}{r}\|v_n\|^r - \int_{\mathbb{R}^N} H(x, v_n)\, dx
\end{aligned}
$$

for a positive constant C_2. Since $\mathcal{E}_\lambda(v_n) \leq \mathcal{K}_1$ for all $n \in \mathbb{N}$, $\|v_n\| \to \infty$ as $n \to \infty$, and $r < p$, we assert that

$$
\int_{\mathbb{R}^N} H(x, v_n)\, dx \geq \frac{1}{q2^p}\|v_n\|^p - C_2\frac{\lambda}{r}\|v_n\|^r - \mathcal{E}_\lambda(v_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{2.11}
$$

By Lemma 2.3, we note that

$$
\mathcal{E}_\lambda(v_n) \leq \frac{1}{p}\Big(\int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v_n|)\, dx + \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\, dx\Big)
$$

$$
\begin{aligned}
&-\frac{\lambda}{r}\int_{\mathbb{R}^N}\rho(x)|v_n|^r\,dx-\int_{\mathbb{R}^N}H(x,v_n)\,dx\\
&\le \frac{2}{p}\|v_n\|^q-\int_{\mathbb{R}^N}H(x,v_n)\,dx.
\end{aligned}
$$

So,

$$
\frac{2}{p}\|v_n\|^q \ge \mathcal{E}_\lambda(v_n)+\int_{\mathbb{R}^N}H(x,v_n)\,dx. \tag{2.12}
$$

Owing to assumption (H5), there exists a $\delta > 1$ such that $H(x,t) > |t|^q$ for all $x \in \mathbb{R}^N$ and $|t| > \delta$. Taking into account (H1), we get $|H(x,t)| \le \hat{C}$ for all $(x,t) \in \mathbb{R}^N \times [-t_0, t_0]$ for a constant $\hat{C} > 0$. Therefore, $H(x,t) \ge C_1$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and for some $C_1 \in \mathbb{R}$, and thus

$$
\frac{H(x,v_n)-C_1}{\frac{2}{p}\|v_n\|^q} \ge 0, \tag{2.13}
$$

for all $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Set $A_1 = \left\{x \in \mathbb{R}^N : y_0(x) \neq 0\right\}$. By relation (2.9), we infer that $|v_n(x)| = |y_n(x)|\,\|v_n\| \to \infty$ as $n \to \infty$ for all $x \in A_1$. Thus, by using (H5),

$$
\lim_{n\to\infty}\frac{H(x,v_n)}{\|v_n\|^q}=\lim_{n\to\infty}\frac{H(x,v_n)}{|v_n|^q}|y_n|^q=+\infty, \quad x \in A_1. \tag{2.14}
$$

Hence, we obtain that $\text{meas}(A_1) = 0$. Indeed, if $\text{meas}(A_1) \neq 0$, according to the relations (2.11)–(2.14) and the Fatou lemma, we have

$$
\begin{aligned}
1 &= \liminf_{n\to\infty}\frac{\int_{\mathbb{R}^N}H(x,v_n)\,dx}{\int_{\mathbb{R}^N}H(x,v_n)\,dx+\mathcal{E}_\lambda(v_n)} \ge \liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{H(x,v_n)}{\frac{2}{p}\|v_n\|^q}\,dx\\
&= \liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{H(x,v_n)}{\frac{2}{p}\|v_n\|^q}\,dx-\limsup_{n\to\infty}\int_{\mathbb{R}^N}\frac{C_1}{\frac{2}{p}\|v_n\|^q}\,dx\\
&= \liminf_{n\to\infty}\int_{A_1}\frac{H(x,v_n)-C_1}{\frac{2}{p}\|v_n\|^q}\,dx\\
&\ge \int_{A_1}\liminf_{n\to\infty}\frac{H(x,v_n)-C_1}{\frac{2}{p}\|v_n\|^q}\,dx\\
&= \int_{A_1}\liminf_{n\to\infty}\frac{H(x,v_n)}{\frac{2}{p}\|v_n\|^q}\,dx-\int_{A_1}\limsup_{n\to\infty}\frac{C_1}{\frac{2}{p}\|v_n\|^q}\,dx=\infty,
\end{aligned} \tag{2.15}
$$

which is impossible. Thus, $y_0(x) = 0$ for almost all $x \in \mathbb{R}^N$. Consequently, we yield a contradiction, and thus the sequence $\{v_n\}$ is bounded in $\mathfrak{X}$. The proof is completed. □

Now, we illustrate two existence results of a sequence of infinitely many solutions to the problem (1.1). The primary tools for these consequences are the Fountain theorem in [7] and the Dual Fountain Theorem in [8]. Let $\mathfrak{E}$ be a real reflexive and separable Banach space, and then it is known (see [17, 49]) that there exist $\{e_n\} \subseteq \mathfrak{E}$ and $\{f_n^*\} \subseteq \mathfrak{E}^*$ such that

$$
\mathfrak{E}=\overline{\text{span}\{e_n : n=1,2,\cdots\}}, \quad \mathfrak{E}^*=\overline{\text{span}\{f_n^* : n=1,2,\cdots\}},
$$

and

$$\left\langle f_i^*, e_j \right\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us define $\mathfrak{E}_n = \operatorname{span}\{e_n\}$, $\mathfrak{Y}_k = \bigoplus_{n=1}^{k} \mathfrak{E}_n$, and $\mathfrak{Z}_k = \overline{\bigoplus_{n=k}^{\infty} \mathfrak{E}_n}$.

Lemma 2.11. (Fountain Theorem [7, 25, 43]) *Assume that* $(\mathfrak{E}, \|\cdot\|)$ *is a Banach space, the functional* $\mathcal{F} \in C^1(\mathfrak{E}, \mathbb{R})$ *satisfies the* $(C)_c$*-condition for any* $c > 0$*, and* $\mathcal{F}$ *is even. If for each large enough* $k \in \mathbb{N}$*, there are* $\beta_k > \alpha_k > 0$ *such that*

(1) $\delta_k := \inf\{\mathcal{F}(y) : y \in \mathfrak{Z}_k, \|y\| = \alpha_k\} \to \infty$ *as* $k \to \infty$*,*

(2) $\rho_k := \max\{\mathcal{F}(y) : y \in \mathfrak{Y}_k, \|y\| = \beta_k\} \leq 0$*,*

then $\mathcal{F}$ *has unbounded sequence of critical values, i.e., there is a sequence* $\{y_n\} \subset \mathfrak{E}$ *such that* $\mathcal{F}'(y_n) = 0$ *and* $\mathcal{F}(y_n) \to +\infty$ *as* $n \to +\infty$*.*

Lemma 2.12. *Let us define*

$$\theta_{t,k} = \sup\left\{\int_{\mathbb{R}^N} |u|^t \, dx : u \in \mathfrak{Z}_k, \|u\| \leq 1\right\} \text{ for } t > 1,$$

and

$$\vartheta_k = \max\{\theta_{\gamma_0,k}, \theta_{s,k}, \theta_{\gamma,k}\}. \tag{2.16}$$

Then, $\vartheta_k \to 0$ *as* $k \to \infty$ *(see [25]).*

Lemma 2.13. *Assume that* (V), (B1), (B2), (H1) *and* (H5) *hold. Then, there are* $\beta_k > \alpha_k > 0$ *such that*

(1) $\delta_k := \inf\{\mathcal{E}_\lambda(v) : v \in \mathfrak{Z}_k, \|v\| = \alpha_k\} \to \infty$ *as* $k \to \infty$*,*

(2) $\rho_k := \max\{\mathcal{E}_\lambda(v) : v \in \mathfrak{Y}_k, \|v\| = \beta_k\} \leq 0$*,*

for k large enough.

Proof. The basic idea of the proof is carried out by a similar fashion as in the paper [2] (see also [10]). For convenience to readers, we give the proof. For any $z \in \mathfrak{Z}_k$, suppose that $\|v\| > 1$. From the assumptions (B1) and (B2), (H1) and Lemma 2.3, it follows that

$$\begin{aligned}
\mathcal{E}_\lambda(v) = & \int_{\mathbb{R}^N} \left(\frac{1}{p} |\nabla v|^p + \frac{a(x)}{q} |\nabla v|^q\right) dx + \int_{\mathbb{R}^N} \mathcal{V}(x) \left(\frac{1}{p} |v|^p + \frac{a(x)}{q} |v|^q\right) dx \\
& - \frac{\lambda}{r} \int_{\mathbb{R}^N} \rho(x)|v|^r \, dx - \int_{\mathbb{R}^N} H(x, v) \, dx \\
\geq & \frac{1}{q}\Big(\int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v|) \, dx + \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v|) \, dx\Big) - \frac{\lambda}{r} \int_{\mathbb{R}^N} \rho(x)|v|^r \, dx - \int_{\mathbb{R}^N} H(x, v) \, dx \\
\geq & \frac{1}{q2^p}\|v\|^p - \frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} |v|^r_{L^{\gamma_0}(\mathbb{R}^N)} - |\sigma_1|_{L^{s'}(\mathbb{R}^N)} |v|_{L^s(\mathbb{R}^N)} - \frac{c_1}{\gamma}|v|^\gamma_{L^\gamma(\mathbb{R}^N)} \\
\geq & \frac{1}{q2^p}\|v\|^p - \frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \vartheta_k^r \|v\|^r - |\sigma_1|_{L^{s'}(\mathbb{R}^N)} \vartheta_k \|v\| - \frac{c_1}{\gamma}\vartheta_k^\gamma \|v\|^\gamma \\
\geq & \left(\frac{1}{q2^p} - \frac{\vartheta_k^\gamma c_1}{\gamma}\|v\|^{\gamma - p}\right)\|v\|^p - \frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \vartheta_k^r \|v\|^r - |\sigma_1|_{L^{s'}(\mathbb{R}^N)} \vartheta_k \|v\|.
\end{aligned} \tag{2.17}$$

Since $p < \gamma$, we get

$$\alpha_k = \Big(\frac{q2^{p+1}\vartheta_k^{\gamma}c_1}{\gamma}\Big)^{\frac{1}{p-\gamma}} \to \infty$$

as $k \to \infty$. Hence, if $v \in \mathfrak{Z}_k$ and $|v| = \alpha_k$, then we arrive

$$\mathcal{E}_\lambda(v) \geq \frac{1}{q2^{p+1}}\alpha_k^p - \frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\vartheta_k^r\alpha_k^r - |\sigma_1|_{L^{s'}(\mathbb{R}^N)}\vartheta_k\alpha_k \to \infty \quad \text{as} \quad k \to \infty,$$

which implies (1) because $p > r > 1$ and $\alpha_k \to \infty$, $\vartheta_k \to 0$ as $k \to \infty$.

Now, we show the condition (2). Suppose to the contrary that there is $k \in \mathbb{N}$ such that the condition (2) is not fulfilled. Then, there exists a sequence $\{v_n\}$ in $\mathfrak{Y}_k$ such that

$$|v_n| \to \infty \text{ as } n \to \infty \quad \text{and} \quad \mathcal{E}_\lambda(v_n) \geq 0. \tag{2.18}$$

Let $w_n = v_n/|v_n|$. Since $\dim \mathfrak{Y}_k < \infty$, there is a $w \in \mathfrak{Y}_k \setminus \{0\}$ such that, up to a subsequence still denoted by $\{w_n\}$,

$$|w_n - w| \to 0 \quad \text{and} \quad w_n(x) \to w(x)$$

for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. We claim that $w(x) = 0$ for almost all $x \in \mathbb{R}^N$. If $w(x) \neq 0$, then $|v_n(x)| \to \infty$ for all $x \in \mathbb{R}^N$ as $n \to \infty$. Hence, in accordance with (H5), it follows that

$$\lim_{n\to\infty} \frac{H(x, v_n)}{|v_n|^q} = \lim_{n\to\infty} \frac{H(x, v_n)}{|v_n(x)|^q}\,|w_n(x)|^q = \infty \tag{2.19}$$

for all $x \in \mathcal{B}_1 := \Big\{x \in \mathbb{R}^N : w(x) \neq 0\Big\}$. In the same fashion as in the proof of Lemma 2.10, we can choose a $C_2 \in \mathbb{R}$ such that $H(x, t) \geq C_2$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, and so

$$\frac{H(x, v_n) - C_2}{|v_n|^q} \geq 0$$

for all $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Using (2.19) and the Fatou Lemma, one has

$$\begin{aligned}
\liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{H(x, v_n)}{|v_n|^q}dx &\geq \liminf_{n\to\infty} \int_{\mathcal{B}_1} \frac{H(x, v_n)}{|v_n|^q}dx - \limsup_{n\to\infty} \int_{\mathcal{B}_1} \frac{C_2}{|v_n|^q}dx \\
&= \liminf_{n\to\infty} \int_{\mathcal{B}_1} \frac{H(x, v_n) - C_2}{|v_n|^q}dx \\
&\geq \int_{\mathcal{B}_1} \liminf_{n\to\infty} \frac{H(x, v_n) - C_2}{|v_n|^q}dx \\
&= \int_{\mathcal{B}_1} \liminf_{n\to\infty} \frac{H(x, v_n)}{|v_n|^q}dx - \int_{\mathcal{B}_1} \limsup_{n\to\infty} \frac{C_2}{|v_n|^q}dx.
\end{aligned}$$

Thus, we infer

$$\int_{\mathbb{R}^N} \frac{H(x, v_n)}{|v_n|^q}\,dx \to \infty \quad \text{as } n \to \infty.$$

We may assume that $|v_n| > 1$. Therefore, we have

$$\mathcal{E}_\lambda(v_n) \leq \frac{1}{p}\Big(\int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla v_n|)\,dx + \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |v_n|)\,dx\Big)$$

$$
\begin{aligned}
&- \frac{\lambda}{r}\int_{\mathbb{R}^N}\rho(x)|v_n|^r\,dx - \int_{\mathbb{R}^N} H(x,v_n)\,dx \\
&\le \frac{2}{p}\|v_n\|^q - \int_{\mathbb{R}^N} H(x,v_n)\,dx \\
&\le \|v_n\|^q\left(\frac{2}{q} - \int_{\mathbb{R}^N}\frac{H(x,v_n)}{\|v_n\|^q}\,dx\right) \to -\infty \quad \text{as } n\to\infty,
\end{aligned}
$$

which is a contradiction to (2.18). This completes the proof. □

With the help of Lemma 2.11, we are ready to establish the existence of infinitely many large energy solutions.

Theorem 2.14. *Assume that* (V), (B1), (B2), (H1), (H2) *(resp.* (H3)*) and* (H5) *hold. If* $h(x,-t) = -h(x,t)$ *holds for all* $(x,t)\in\mathbb{R}^N\times\mathbb{R}$*, then, for any* $\lambda > 0$*, the problem* (1.1) *admits a sequence of nontrivial weak solutions* $\{v_n\}$ *in* $\mathfrak{X}$ *such that* $\mathcal{E}_\lambda(v_n)\to\infty$ *as* $n\to\infty$*.*

Proof. Clearly, $\mathcal{E}_\lambda$ is an even functional and ensures the $(C)_c$-condition by Lemma 2.8 (resp. Lemma 2.10). From Lemma 2.13, this assertion is immediately derived from the Fountain theorem. This completes the proof. □

Definition 2.15. Suppose that $(\mathfrak{E},\|\cdot\|)$ is a real separable and reflexive Banach space. We say that $\mathcal{F}$ satisfies the $(C)_c^*$-condition (with respect to $\mathfrak{Y}_n$) if any sequence $\{v_n\}_{n\in\mathbb{N}}\subset\mathfrak{E}$ for which $v_n\in\mathfrak{Y}_n$, for any $n\in\mathbb{N}$,

$$
\mathcal{F}(v_n)\to c \quad\text{and}\quad \|(\mathcal{F}|_{\mathfrak{Y}_n})'(v_n)\|_{\mathfrak{E}^*}(1+\|v_n\|)\to 0 \text{ as } n\to\infty,
$$

possesses a subsequence converging to a critical point of $\mathcal{F}$.

Lemma 2.16. (Dual Fountain Theorem [8, 25]) *Assume that* $(\mathfrak{E},\|\cdot\|)$ *is a Banach space,* $\mathcal{F}\in C^1(\mathfrak{E},\mathbb{R})$ *is an even functional. If there is* $k_0>0$ *such that, for each* $k\ge k_0$*, there exist* $\beta_k>\alpha_k>0$ *such that*

($\mathcal{A}_1$) $\inf\{\mathcal{F}(y): y\in\mathfrak{Z}_k, \|y\|=\beta_k\}\ge 0$,

($\mathcal{A}_2$) $\delta_k := \max\{\mathcal{F}(y): y\in\mathfrak{Y}_k, \|y\|=\alpha_k\}<0$,

($\mathcal{A}_3$) $\phi_k := \inf\{\mathcal{F}(y): y\in\mathfrak{Z}_k, \|y\|\le\beta_k\}\to 0$ *as* $k\to\infty$,

($\mathcal{A}_4$) $\mathcal{F}$ *fulfils the* $(C)_c^*$*-condition for every* $c\in[\phi_{k_0},0)$,

then $\mathcal{F}$ *admits a sequence of negative critical values* $c_n<0$ *satisfying* $c_n\to 0$ *as* $n\to\infty$*.*

From now on, we will check all conditions of the dual fountain theorem.

Lemma 2.17. *Assume that* (V), (B1), (B2), (H1), (H2) *(resp.* (H3) *and* (H5)*) hold. Then, the functional* $\mathcal{E}_\lambda$ *satisfies the* $(C)_c^*$*-condition for any* $\lambda>0$*.*

Proof. Since $\mathfrak{X}$ is a reflexive Banach space, and Φ' and Ψ'_λ are of type (S_+), the proof is almost identical to that in [25]. □

Lemma 2.18. *Assume that* (V), (B1), (B2) *and* (H1) *hold. Then, there is* $k_0>0$*, such that, for each* $k\ge k_0$*, there exists* $\beta_k>0$ *such that*

$$
\inf\{\mathcal{E}_\lambda(v): v\in\mathfrak{Z}_k, \|v\|=\beta_k\}\ge 0.
$$

Proof. From (H1), Lemma 2.3 and the definition of ϑ_k, one has

$$\begin{aligned}\mathcal{E}_\lambda(v) \geq \frac{1}{q}\Big(\int_{\mathbb{R}^N} \mathcal{H}(x,|\nabla v|)\,dx + \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x,|v|)\,dx\Big) \\ - \frac{\lambda}{r}\int_{\mathbb{R}^N} \rho(x)|v|^r\,dx - \int_{\mathbb{R}^N} H(x,v)\,dx \\ \geq \frac{1}{q2^p}\|v\|^p - \frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\vartheta_k^r\|v\|^r - |\sigma_1|_{L^{s'}(\mathbb{R}^N)}\vartheta_k\|v\| - \frac{c_1}{\gamma}\vartheta_k^\gamma\|v\|^\gamma \\ \geq \frac{1}{q2^p}\|v\|^p - \Big(\frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)} + \frac{c_1}{\gamma}\Big)\vartheta_k^r\|v\|^\gamma - |\sigma_1|_{L^{s'}(\mathbb{R}^N)}\vartheta_k\|v\|\end{aligned}$$

for k large enough and $\|v\| \geq 1$. Let us choose

$$\beta_k = \left[\left(\frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)} + \frac{c_1}{\gamma}\right)q2^{p+1}\vartheta_k^r\right]^{\frac{1}{p-2\gamma}}. \tag{2.20}$$

Let $v \in \mathfrak{Z}_k$ with $\|v\| = \beta_k > 1$ for k large enough. Then, there is $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}\mathcal{E}_\lambda(v) &\geq \frac{1}{q2^p}\|v\|^p - \left(\frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)} + \frac{c_1}{\gamma}\right)\vartheta_k^r\|v\|^\gamma - |\sigma_1|_{L^{s'}(\mathbb{R}^N)}\vartheta_k\|v\| \\ &\geq \frac{1}{q2^{p+1}}\beta_k^p - |\sigma_1|_{L^{s'}(\mathbb{R}^N)}\left[\left(\frac{2\lambda}{r}|\rho|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)} + \frac{c_1}{\gamma}\right)q2^{p+1}\right]^{\frac{1}{p-2\gamma}}\vartheta_k^{\frac{r+p-2\gamma}{p-2\gamma}} \geq 0\end{aligned}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, which implies that the conclusion holds since $\lim_{k\to\infty}\beta_k^p = \infty$ and $\vartheta_k \to 0$ as $k \to \infty$. □

Lemma 2.19. *Assume that* (V), (B1), (B2), (H1) *and* (H4) *hold. Then, for each sufficiently large* $k \in \mathbb{N}$, *there exists* $\alpha_k > 0$ *with* $0 < \alpha_k < \beta_k$ *such that*

(1) $\delta_k := \max\{\mathcal{E}_\lambda(v) : v \in \mathfrak{Y}_k, \|v\| = \alpha_k\} < 0$,

(2) $\phi_k := \inf\{\mathcal{E}_\lambda(v) : v \in \mathfrak{Z}_k, \|v\| \leq \beta_k\} \to 0$ *as* $k \to \infty$,

where β_k *is given in Lemma 2.18.*

Proof. (1) Since $\mathfrak{Y}_k$ is finite dimensional, $|\cdot|_{L^\kappa(\xi,\mathbb{R}^N)}$, $|\cdot|_{L^\gamma(\mathbb{R}^N)}$ and $\|\cdot\|$ are equivalent on $\mathfrak{Y}_k$. Then, there exist $\varsigma_{1,k} > 0$ and $\varsigma_{2,k} > 0$ such that

$$\varsigma_{1,k}\|v\| \leq |v|_{L^\kappa(\xi,\mathbb{R}^N)} \text{ and } |v|_{L^\gamma(\mathbb{R}^N)} \leq \varsigma_{2,k}\|v\|$$

for any $v \in \mathfrak{Y}_k$. Let $v \in \mathfrak{Y}_k$ with $\|v\| \leq 1$. From (H1) and (H4), there are $C_1, C_2 > 0$ such that

$$H(x,t) \geq C_1\xi(x)|t|^\kappa - C_2|t|^\gamma$$

for almost all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Then, we have

$$\mathcal{E}_\lambda(v) \leq \frac{2}{p}\|v\|^p - \int_{\mathbb{R}^N} H(x,v)\,dx$$

$$
\begin{aligned}
&\le \frac{2}{p}\|v\|^p - C_1 \int_{\mathbb{R}^N} \xi(x)|v|^\kappa dx + C_2 \int_{\mathbb{R}^N} |v|^\gamma dx \\
&\le \frac{2}{p}\|v\|^p - C_1 |v|_{L^\kappa(\xi,\mathbb{R}^N)} + C_2 |v|_{L^\gamma(\mathbb{R}^N)} \\
&\le \frac{2}{p}\|v\|^p - C_1 \varsigma_{1,k}^\kappa \|v\|^\kappa + C_2 \varsigma_{2,k}^\gamma \|v\|^\gamma.
\end{aligned}
$$

Let $f(s) = \frac{2}{p}s^p - C_1\varsigma_{1,k}^\kappa s^\kappa + C_2\varsigma_{2,k}^\gamma s^\gamma$. Since $\kappa < p < \gamma$, we infer $f(s) < 0$ for all $s \in (0, s_0)$ for sufficiently small $s_0 \in (0, 1)$. Hence, we can find $\alpha_k > 0$ such that $\mathcal{E}_\lambda(v) < 0$ for all $v \in \mathfrak{Y}_k$ with $\|v\| = \alpha_k < s_0$ for k large enough. If necessary, we can change k_0 to a large value, so that $\beta_k > \alpha_k > 0$ and

$$
\delta_k := \max\{\mathcal{E}_\lambda(v) : v \in \mathfrak{Y}_k, \|v\| = \alpha_k\} < 0
$$

for all $k \ge k_0$.

(2) Because $\mathfrak{Y}_k \cap \mathfrak{Z}_k \ne \phi$ and $0 < \alpha_k < \beta_k$, we have $\phi_k \le \delta_k < 0$ for all $k \ge k_0$. For any $v \in \mathfrak{Z}_k$ with $\|v\| = 1$ and $0 < t < \beta_k$, we have

$$
\begin{aligned}
\mathcal{E}_\lambda(tv) &\ge \frac{1}{q}\Big(\int_{\mathbb{R}^N} \mathcal{H}(x, |\nabla tv|)\, dx + \int_{\mathbb{R}^N} \mathcal{H}_{\mathcal{V}}(x, |tv|)\, dx \Big) - \frac{\lambda}{r} \int_{\mathbb{R}^N} \rho(x)|tv|^r\, dx - \int_{\mathbb{R}^N} H(x, tv)\, dx \\
&\ge -\frac{\lambda}{r} \int_{\mathbb{R}^N} \rho(x)|tv|^r\, dx - \int_{\mathbb{R}^N} H(x, tv)\, dx \\
&\ge -\frac{\lambda}{r} |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} |tv|^r_{L^{\gamma_0}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} \sigma_1(x)|tv| dx - \frac{c_1}{\gamma} \int_{\mathbb{R}^N} |tv|^\gamma dx \\
&\ge -\frac{\lambda}{r} |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \beta_k^r |v|^r_{L^{\gamma_0}(\mathbb{R}^N)} - \beta_k \int_{\mathbb{R}^N} \sigma_1(x)|v| dx - \frac{c_1}{\gamma}\beta_k^\gamma \int_{\mathbb{R}^N} |v|^\gamma dx \\
&\ge -\frac{\lambda}{r} |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} \beta_k^r \vartheta_k^r - |\sigma_1|_{L^{s'}(\mathbb{R}^N)} \beta_k \vartheta_k - \frac{c_1}{\gamma}\beta_k^\gamma \vartheta_k^\gamma
\end{aligned}
$$

for k large enough, where ϑ_k and β_k are given in (2.16) and (2.20), respectively. Hence, it follows from the definition of β_k that

$$
\begin{aligned}
0 > \phi_k &\ge -\frac{\lambda |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}}{r} \beta_k^r \vartheta_k^r - |\sigma_1|_{L^{s'}(\mathbb{R}^N)} \beta_k \vartheta_k - \frac{c_1}{\gamma}\beta_k^\gamma \vartheta_k^\gamma \\
&= -\frac{\lambda |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)}}{r} \left[\left(\frac{2\lambda}{r} |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{c_1}{\gamma} \right) q 2^{p+1} \right]^{\frac{r}{p-2\gamma}} \vartheta_k^{\frac{(r+p-2\gamma)r}{p-2\gamma}} \\
&\quad - |\sigma_1|_{L^{s'}(\mathbb{R}^N)} \left[\left(\frac{2\lambda}{r} |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{c_1}{\gamma} \right) q 2^{p+1} \right]^{\frac{1}{p-2\gamma}} \vartheta_k^{\frac{r+p-2\gamma}{p-2\gamma}} \\
&\quad - \frac{c_1}{\gamma} \left[\left(\frac{2\lambda}{r} |\rho|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} + \frac{c_1}{\gamma} \right) q 2^{p+1} \right]^{\frac{\gamma}{p-2\gamma}} \vartheta_k^{\frac{(r+p-2\gamma)\gamma}{p-2\gamma}}.
\end{aligned}
$$

Because $p < p + r < 2\gamma$ and $\vartheta_k \to 0$ as $k \to \infty$, we derive that $\lim_{k\to\infty} \phi_k = 0$. □

Remark 2.20. *In view of [10, 21, 25, 30], the conditions* (H5) *and*

$$
f(x, t) = o(|t|^{q-1}) \text{ as } |t| \to 0 \text{ uniformly for } x \in \mathbb{R}^N, \tag{2.21}
$$

play a decisive role in proving Lemma 2.19. Under these two conditions, the authors in [10, 21, 25, 30] obtained the existence of two sequences $0 < \alpha_k < \beta_k$ *sufficiently large. Unfortunately, by using the same argument as in [21, 25] we cannot show the property (2) in Lemma 2.19 since* $\beta_k \to \infty$ *as* $k \to \infty$*; see [41]. However the authors in [10, 30] overcome this difficulty from new setting for* β_k*. In contrast, the existence of two sequences* $0 < \alpha_k < \beta_k \to 0$ *as* $k \to \infty$ *is obtained in [32, 42, 43] when* (2.21) *is satisfied. On the other hand, we prove Lemma 2.19 when* (H5) *is not assumed, and* (2.21) *is replaced by* (H4)*. For this reason, the proof of Lemma 2.19 is different from that of the papers [10, 21, 25, 30, 32, 42, 43].*

With the aid of Lemmas 2.16 and 2.17, we are in a position to establish our final consequence.

Theorem 2.21. *Assume* (V), (B1), (B2), (H1), (H2) *(resp.* (H3), (H5)*) and* (H4)*. If* $h(x, -t) = -h(x, t)$ *holds for all* $(x, t) \in \mathbb{R}^N \times \mathbb{R}$*, then the problem* (1.1) *admits a sequence of nontrivial weak solutions* $\{v_n\}$ *in* $\mathfrak{X}$ *such that* $\mathcal{E}_\lambda(v_n) \to 0$ *as* $n \to \infty$ *for any* $\lambda > 0$*.*

Proof. Due to Lemma 2.17, we note that the functional $\mathcal{E}_\lambda$ is even and fulfills the $(C)^*_c$-condition for every $c \in [\phi_{k_0}, 0)$. Now, from Lemmas 2.18 and 2.19, we ensure that properties (D_1)–(D_3) in the Dual Fountain Theorem hold. Therefore, problem (1.1) possesses a sequence of weak solutions $\{v_n\}$ with large enough n. The proof is complete. □

3. Conclusions

In this paper, we employ the variational methods to ensure the existence of a sequence of infinitely many energy solutions to Schrödinger-type problems involving the double phase operator. As far as we can see, in these circumstances the present paper is the first effort to develop the multiplicity results of nontrivial weak solutions to the concave-convex-type double phase problems because we derive our results on a new class of nonlinear term. Especially, our proof of the existence of multiple small energy solutions is slightly different from those of previous related works [10, 21, 25, 30, 32, 42, 43].

Acknowledgments

The authors are grateful to the referees for their valuable comments and suggestions for improvement of the paper. Jae-Myoung Kim was supported by a National Research Foundation of Korea Grant funded by the Korean Government (NRF-2020R1C1C1A01006521), and Yun-Ho Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2019R1F1A1057775).

Conflict of interest

The authors declare that they have no competing interests.

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