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Research article

Asymptotic behavior of survival probability for a branching random walk with a barrier

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Abstract: Consider a branching random walk with a mechanism of elimination. We assume that the underlying Galton-Watson process is supercritical, thus the branching random walk has a positive survival probability. A mechanism of elimination, which is called a barrier, is introduced to erase the particles who lie above $ri+\varepsilon i^{\alpha}$ and all their descendants, where i presents the generation of the particles, $\alpha > 1/3, \varepsilon \in \mathbb{R}$ and r is the asymptotic speed of the left-most position of the branching random walk. First we show that the particle system still has a positive survival probability after we introduce the barrier with $\varepsilon > 0$. Moreover, we show that the decay of the probability is faster than $e^{-\beta'\varepsilon^{\beta}}$ as $\varepsilon \downarrow 0$, where β', β are two positive constants depending on the branching random walk and α. The result in the present paper extends a conclusion in Gantert et al. (2011) in some extent. Our proof also works for some time-inhomogeneous cases.

Keywords: branching random walk; barrier; survival probability
Mathematics Subject Classification: 60J80

1. Introduction and results

We consider the branching random walk (BRW) on $\mathbb{R}$. At time 0, an initial ancestor (denoted by ϕ) is located at the origin. At time 1, the ancestor dies and reproduces (including the number and displacement of its children) according to the distribution of a point process L, i.e., ϕ gives birth to $N(\phi)$ children who are located at $\zeta_i(\phi), 1 \le i \le N(\phi)$ ($N(\phi)$ can be 0) and the law of the random vector $(N(\phi), \zeta_i(\phi), 1 \le i \le N(\phi))$ is L. These children (also called particles) consist the first generation. Each of the particles in the first generation reproduces its own children who are thus in the second generation and are positioned (with respect to their parent) according to the same distribution of L. All particles reproduce independently according to the same law L as time goes on. The particle system formed in this way is called a (time-homogeneous) branching random walk. Hence BRW can be viewed as that we attach a displacement information to each particle in a Galton-Watson tree $\mathbf{T}$. For a given particle

$u \in \mathbf{T}$ we write $V(u) \in \mathbb{R}$ for the position of u and $|u|$ for the generation at which u is alive. In the present paper, we focus on the barrier problem of BRW and a more general time-inhomogeneous model. The so-called barrier is in fact a function $f : \mathbb{N} \to \mathbb{R}$. For any realization of the BRW, if a particle u satisfies $V(u) > f(|u|)$, then we remove u and all its descendants. The surviving particles (i.e., which have not been removed) form a new system, which is called a BRW with barrier. For any $i \leq |u|$, we conventionally write u_i for the ancestor of u in generation i. It is evident to see that u is survival if and only if $V(u_i) \leq f(i), \forall i \leq |u|$. Let κ be the log −Laplace transform of L, that is to say

$$\kappa(\theta) := \log \mathbb{E}\left(\sum_{l \in L} e^{-\theta l}\right).$$

Obviously, another equivalent expression of $\kappa(\theta)$ is $\kappa(\theta) = \log \mathbb{E}(\sum_{|u|=1} e^{-\theta V(u)})$. We always assume that

$$\kappa(0) \in (0, \infty), \tag{1.1}$$

which means that the underlying Galton-Watson process is supercritical, i.e., the survival probability of the particle system (BRW) is positive. Under the assumption that there exists $\vartheta > 0$ such that

$$\kappa(\vartheta) = \vartheta\kappa'(\vartheta), \quad \kappa(\vartheta) < +\infty, \tag{1.2}$$

where κ' presents the derivative of κ. Hammersley [1], Kingman [2] and Biggins [3] showed that

$$\lim_{n \to \infty} n^{-1} \min_{u \in \mathbf{T},\ |u|=n} V(u) = -\kappa'(\vartheta), \text{ non-extinction.} \tag{1.3}$$

The above result enlightens the approach for the barrier problem, which is a topic motivated by the parallel simulation, see Lubachevsky et al. [4, 5]. We first introduce some notations before we recall some achieved results on the barrier problem of BRW. On the Galton-Watson tree $\mathbf{T}$ we define a partial order $>$ such that $u > v$ if v is the ancestor of u. We write $u \geq v$ if $u > v$ or $u = v$ (i.e., the particle u is exactly the particle v). Define an infinite path u_∞ through the tree $\mathbf{T}$ as a sequence of particles $(u_i)_{i \in N}$ such that

$$u_0 = \phi, \quad \forall i \in \mathbb{N}, \quad |u_i| = i, \quad u_{i+1} > u_i.$$

We write $\mathcal{T}_\infty$ the collection of the infinite path. Let

$$\rho(\varepsilon, \alpha) := \mathbb{P}(\exists u_\infty \in \mathcal{T}_\infty, \forall i \in \mathbb{N}, V(u_i) \leq \varepsilon i^\alpha - \kappa'(\vartheta) i).$$

Hence one see that $\rho(\varepsilon, \alpha)$ presents the survival probability for the BRW with a barrier

$$f(i) := \varepsilon i^\alpha - \kappa'(\vartheta) i.$$

The first result on the the barrier problem of BRW can be found in Biggins et al. [6]. Under Assumptions (1.1) and (1.2) they claimed that

$$\rho(\varepsilon, 1) > 0 \text{ when } \varepsilon > 0, \quad \rho(\varepsilon, 1) = 0 \text{ when } \varepsilon \leq 0. \tag{1.4}$$

From the view of (1.3), we can have a better understanding on this conclusion. That is to say, when critical slope of the barrier is determined by the first order of $\min_{u \in \mathbf{T},\ |u|=n} V(u)$. Under a slightly stronger assumption, Jaffuel [7] refine the result (1.4). [7] showed that under (1.1), (1.2) and the assumption

$$\exists \delta > 0, \quad \mathbb{E}(N^{1+\delta}(\phi)) < +\infty, \quad \kappa(\vartheta + \delta) < +\infty, \quad \kappa''(\vartheta) \in (0, \infty). \tag{1.5}$$

It is true that

$$\rho(\varepsilon, 1/3) > 0 \text{ when } \varepsilon > a_{\kappa''(\vartheta)}, \quad \rho(\varepsilon, 1/3) = 0 \text{ when } \varepsilon < a_{\kappa''(\vartheta)}, \tag{1.6}$$

where the explicit form of the positive constant $a_{\kappa''(\vartheta)}$ (depending on $\kappa''(\vartheta)$) is obtained. Combining (1.6) with (1.4), we can prove the following statement.

Proposition 1.1. *If* (1.1), (1.2) *and* (1.5) *hold, then* $\rho(\varepsilon, \alpha) > 0$ *when* $\varepsilon > 0, \alpha > 1/3$.

Proof. It is obvious when $\alpha > 1$ since the definition domain of the barrier f is $\mathbb{N}$. Now we deal with the case $\alpha \in (1/3, 1)$. Let $\bar{a}$ be a constant such that $\bar{a} \in (a_{\kappa''(\vartheta)}, \varepsilon)$. Define

$$j := \max_{n \in \mathbb{N}^+} \left\{ (\bar{a})^{\frac{1}{\alpha}} n^{\frac{1}{3\alpha}} - \varepsilon^{\frac{1}{\alpha}} n \right\}.$$

We see j is finite since $\alpha > 1/3$. Choose k large enough such that $k > j\varepsilon^{-\frac{1}{\alpha}}$, which ensures that $\varepsilon(n + k)^\alpha > \bar{a}n^{1/3}, \forall n \in \mathbb{N}^+$. Note that

$$\min_{n \in \mathbb{N}^+} (\varepsilon(n + k)^\alpha - \bar{a}n^{1/3}) > 0,$$

we can find $a_- > 0$ small enough such that

$$\varepsilon(n + k)^\alpha > a_- k + \bar{a}n^{1/3}, \quad \forall n \in \mathbb{N}^+ \text{ and } a_- < \min\{\varepsilon k^{\alpha-1}, \varepsilon\}.$$

Hence it is true that $\varepsilon i^\alpha > a_- i$ for $1 \le i \le k$ and $\varepsilon i^\alpha > a_- k + \bar{a}(i - k)^{1/3}$ for $i > k$. By Markov property we see

$$\begin{aligned}
\rho(\varepsilon, \alpha) &= \mathbb{P}(\exists u \in \mathcal{T}_\infty, \forall i \in \mathbb{N}, V(u_i) \le \varepsilon i^\alpha + ri) \\
&\ge \mathbb{P}(\exists |u| = k, \forall i \le k, V(u_i) \le a_- i + ri) \\
&\times \mathbb{P}(\exists u \in \mathcal{T}_\infty, \forall i \in \mathbb{N}, V(u_i) \le \bar{a}i^{1/3} + a_- k + r(i + k) | V(\phi) = a_- k + rk) \\
&:= P_1 \times P_2.
\end{aligned}$$

(1.4) tells us that $P_1 > 0$ and (1.6) means that $P_2 > 0$, hence we have $\rho(\varepsilon, \alpha) > 0$. □

The decay rate of $\rho(\varepsilon, 1)$ had been obtained in Gantert et al. [8]. When (1.1) and (1.2) hold, [8] obtained the explicit negative constant c such that

$$\overline{\lim_{\varepsilon \downarrow 0}} \sqrt{\varepsilon} \log \rho(\varepsilon, 1) \le c.$$

(We remind that under (1.1), (1.2), (1.5) and some extra assumptions, the lower bound of $\rho(\varepsilon, 1)$ had also been obtained in [8].) In the present paper, we want to extend the upper bound of the rate to some non-linear barrier. It is evident to see that $\rho(\varepsilon, \alpha) = 0, \forall \alpha > 0$ when $\varepsilon = 0$ and $\rho(\varepsilon, \alpha)$ is non-decreasing on ε when the positive constant α is fixed. Combining these two facts with Proposition 1.1, we see that for any given $\alpha > 1/3$, it is reasonable and meaningful to ask the question about the decay rate of $\rho(\varepsilon, \alpha)$ as $\varepsilon \downarrow 0$. In the present paper, we wonder whether the decay rate of $\rho(\varepsilon, \alpha)$ (when $\alpha > 1/3$) as $\varepsilon \downarrow 0$ will be the same as the one of $\rho(\varepsilon, 1)$ (as $\varepsilon \downarrow 0$). Furthermore, if they are different, will the order be different? In other word, we want to investigate the impact of α on the decay rate. Now we give the first result in the present paper.

Theorem 1.1. *If* (1.1) *and* (1.2) *hold, then for* $\alpha > 1$, *we have*

$$\overline{\lim_{\varepsilon \downarrow 0}} \varepsilon^{\frac{1}{3\alpha-1}} \log \rho(\varepsilon, \alpha) \le -\vartheta \left\{ \left(\frac{\pi^2 \kappa''(\vartheta)}{2\alpha^2 \vartheta} \right)^{\alpha} (3\alpha(\alpha-1))^{\alpha-1} \right\}^{\frac{1}{3\alpha-1}} \tag{1.7}$$

and for $\alpha \in (1/3, 1)$, *we have*

$$\overline{\lim_{\varepsilon \downarrow 0}} \varepsilon^{\frac{1}{3\alpha-1}} \log \rho(\varepsilon, \alpha) \le -\frac{\vartheta(3\alpha-1)}{(3\vartheta\alpha)^{\frac{3\alpha}{3\alpha-1}}} \left(\frac{3\pi^2 \vartheta^2 \kappa''(\vartheta)}{2} \right)^{\frac{\alpha}{3\alpha-1}}.$$

Remark 1.1. *We remind that the limit of the right-hand-side of* (1.7) *as* $\alpha \downarrow 1$ *is the exact value of the corresponding one in [8], hence our result can be viewed as an extension for the upper bound part in [8].*

In fact, this asymptotic behavior can be shown for a more general model called a branching random walk with varying environment (BRWve). Let us describe the model as follows. For a sequence of time-inhomogeneous branching random walks $\{(\mathbf{T}^{(n)}, V^{(n)})\}_{n\in\mathbb{N}}$, we only consider the generations from 0 to n in $(\mathbf{T}^{(n)}, V^{(n)})$, where $\mathbf{T}^{(n)}$ presents the (time-inhomogeneous) Galton CWatson tree of the genealogy of this process and $V^{(n)}$ the displacements of the particles in $\mathbf{T}^{(n)}$. Let $\{L_t, t \in [0,1]\}$ be a family of laws of point processes. All particles reproduce independently but the law of reproduce is determined in the following way. For particle $u \in \mathbf{T}^{(n)}, |u| = i < n$, the reproduce law of u is $L_{\frac{i+1}{n}}$. This model has been studied in several papers. Fang and Zeitouni [9] showed that the asymptotic behavior of the maximal displacement $\max_{u\in\mathbf{T}^{(n)},|u|=n} V(u)$ under some special settings (two time intervals) on the reproduction law $\{L_t, t \in [0,1]\}$. Mallein [10] has generalized the result in [9] to more general reproduction law (a sequence of macroscopic time intervals). For a smoothly varying environment, Mallein [11] obtained a new asymptotic behavior of the maximal displacement. However, there is no result on the barrier problem of the BRWve. In the present paper, we want to extend Theorem 1.1 to some BRWve with special settings on the varying environment. Define

$$\kappa_t(\theta) := \log \mathbb{E}\left(\sum_{l\in L_t} e^{-\theta l} \right).$$

Assume that there exists $\vartheta, \nu > 0$ such that for any $s, t \in [0,1]$,

$$\vartheta \kappa_t'(\vartheta) = \kappa_t(\vartheta), \quad \kappa_t(\vartheta) = \kappa_s(\vartheta) < +\infty \tag{1.8}$$

and

$$\sup_{t\in[0,1]} \max\{\kappa_t(\vartheta+\nu),\ \kappa_t(\vartheta-\nu)\} < +\infty. \tag{1.9}$$

Furthermore, we assume that $\kappa_t''(\vartheta)$ satisfies that

$$\kappa_t''(\vartheta) \text{ (as a function of } t\text{) is continuous on } [0,1] \text{ and } \min_{t\in[0,1]} \kappa_t''(\vartheta) > 0. \tag{1.10}$$

Obviously, BRW is a special case of BRWve when the family $\{L_t, t \in [0,1]\}$ is a constant one. In order to deal with the new model (BRWve), from now on we redefine the survival probability $\rho(\varepsilon, \alpha)$ as

$$\rho(\varepsilon, \alpha) := \lim_{n\to+\infty} \min_{k\le n} \mathbb{P}(\exists u \in \mathbf{T}^{(k)} : |u| = k, \forall i \le k, V^{(k)}(u_i) \le \varepsilon i^{\alpha} - \kappa_1'(\vartheta) i).$$

Now we give a generalized version of Theorem 1.1.

Theorem 1.2. *Denote* $\sigma_-^2 := \min_{t\in[0,1]} \vartheta^2\kappa_t''(\vartheta)$, $\gamma_\sigma := \frac{\pi^2\sigma_-^2}{2}$. *If* (1.8), (1.9) *and* (1.10) *hold, then for* $\alpha > 1$, *we have*

$$\overline{\lim_{\varepsilon\downarrow\infty}}\, \varepsilon^{\frac{1}{3\alpha-1}} \log\rho(\varepsilon,\alpha) \le -\vartheta\left\{\left(\frac{\gamma_\sigma}{\alpha^2\vartheta^3}\right)^{\alpha}(3\alpha(\alpha-1))^{\alpha-1}\right\}^{\frac{1}{3\alpha-1}} \tag{1.11}$$

and for $\alpha \in (1/3, 1)$, *we have*

$$\overline{\lim_{\varepsilon\downarrow\infty}}\, \varepsilon^{\frac{1}{3\alpha-1}} \log\rho(\varepsilon,\alpha) \le -\frac{\vartheta(3\alpha-1)}{(3\vartheta\alpha)^{\frac{3\alpha}{3\alpha-1}}}(3\gamma_\sigma)^{\frac{\alpha}{3\alpha-1}}. \tag{1.12}$$

2. Preliminary for the proof

Let us give a sketch of the proof. First we give a decomposition of the survival probability of the BRWve with barrier. Secondly, we transfer BRWve to a triangular array of independent centered random variables by the version of time-inhomogeneous many-to-one formula which has been introduced in [10]. Then the survival probability will be dominated by a series of small deviation probabilities of the triangular array random variables. At last, applying a time-inhomogeneous version of small deviation principle which has been given in [11], the estimate for the upper bound will becomes a extremal problem of some continuous functions.

The many-to-one formula, which is essentially a kind of measure transformation, is a basic tool in the study of the branching random walks, It can be traced down to the early works of Peyrière [12] and Kahane and Peyrière [13]. We refer to Biggins and Kyprianou [14] for more variations of this result. Let $\tau_{n,k}$ be a random measure on $\mathbb{R}$ such that for any $x \in \mathbb{R}$ we have

$$\tau_{n,k}\left((-\infty, x]\right) = \mathbb{E}\left(\sum_{l\in L_{k/n}} 1_{\{l\le x\}} e^{-\vartheta l-\kappa_{k/n}(\vartheta)}\right),$$

For any given n, we introduce a series of independent random variables $\{X_{n,k}\}_{k\in\mathbb{N}^+,k\le n}$ whose distributions are $\{\tau_{n,k}\}_{n,k\in\mathbb{N}^+}$ and define

$$S_k^{(n)} := \sum_{i=1}^{k} X_{n,i}.$$

The following theorem shows the relationship between $S_k^{(n)}$ and the BRWve.

Theorem 2.1. (Mallein [10]) *For any* $n, k \in \mathbb{N}^+$, $k \le n$, *and a measurable function* $f : \mathbb{R}^n \to [0, +\infty)$, *we have*

$$\mathbb{E}\left[\sum_{|u|=n} f(V(u_i), 1\le i\le n)\right] = \mathbb{E}\left[e^{\vartheta S_n + n\kappa_1(\vartheta)} f(S_i, 1\le i\le n)\right].$$

By many-to-one formula, the barrier problem of a BRWve becomes equivalently to the small deviation problem for a time-inhomogeneous random walk.

The small deviation problem is a classic topic which attracts intensive attention for many years. We refer to Mogul'skiĭ [15], Borovkov & Mogul'skiĭ [16], Shao [17] and Lv & Hong [18] as the small deviation principle for sums of independent random variables. In our proof, a time-inhomogeneous version of a small deviation principle which has been given in [11] will be used. We state it as follows.

Theorem 2.2. **(Mallein [11])** *Let $\{\tilde{X}_{n,k}\}_{n,k\in\mathbb{N},k\le n}$ be a triangular array of independent centered random variables. We assume that there exists $\sigma \in C[0,1]$ with $\sigma_- := \min_{s\in[0,1]}\sigma(s) > 0$ and $u > 0$ such that for any $n, k \in \mathbb{N}, k \le n$,*

$$\mathbb{E}(\tilde{X}_{n,k}^2) = \sigma(k/n) \tag{2.1}$$

and

$$\sup_{n,k}\mathbb{E}(e^{u|\tilde{X}_{n,k}|}) < +\infty. \tag{2.2}$$

Set $g, h \in C[0,1]$ and $g(0) < 0 < h(0)$. Denote $\tilde{S}_k^{(n)} := \tilde{S}_0 + \sum_{i=1}^k \tilde{X}_{n,i}$. Then we have

$$\overline{\lim_{n\to+\infty}} \frac{\sup_{x\in\mathbb{R}}\log\mathbb{P}\left(\forall i \le n,\ n^{-1/3}\tilde{S}_i^{(n)} \in \left[g\left(\frac{i}{n}\right), h\left(\frac{i}{n}\right)\right] | \tilde{S}_0 = x\right)}{n^{1/3}} \le -\frac{\pi^2}{2}\int_0^1 \frac{\sigma^2(s)}{(h(s)-g(s))^2}ds.$$

This conclusion extends the main result in [15] to the time-inhomogeneous case.

3. Proof of Theorem 1.2

Recall the barrier function $f(i) := \varepsilon i^\alpha - \kappa_1'(\vartheta)i$, hence $f(i) = \varepsilon i^\alpha - \frac{i\kappa_1(\vartheta)}{\vartheta}$ from (1.8). We define

$$H_{j,n} := \mathbb{P}\left(\begin{array}{l}\exists |u| \in \mathbf{T}^{(n)} : |u| = j, V^{(n)}(u) \le \dfrac{aj^\alpha}{n^{\alpha-1/3}} - \dfrac{j\kappa_1(\vartheta)}{\vartheta} - b(n-j)^{1/3}, \\ \forall i < j, V^{(n)}(u_i) \in \left[\dfrac{ai^\alpha}{n^{\alpha-1/3}} - \dfrac{i\kappa_1(\vartheta)}{\vartheta} - b(n-i)^{1/3}, \dfrac{ai^\alpha}{n^{\alpha-1/3}} - \dfrac{i\kappa_1(\vartheta)}{\vartheta}\right]\end{array}\right),$$

$$H_{*,n} := \mathbb{P}\left(\begin{array}{c}\exists |u| \in \mathbf{T}^{(n)} : |u| = n, \forall i \le n, \\ V(u_i) \in \left[\dfrac{ai^\alpha}{n^{\alpha-1/3}} - \dfrac{i\kappa_1(\vartheta)}{\vartheta} - b(n-i)^{1/3}, \dfrac{ai^\alpha}{n^{\alpha-1/3}} - \dfrac{i\kappa_1(\vartheta)}{\vartheta}\right]\end{array}\right),$$

where the exact value of positive constants a, b will be given later. From the definition of $\rho(\varepsilon, \alpha)$, we see for any $n \in \mathbb{N}$, it is true that

$$\begin{aligned}\rho(an^{1/3-\alpha}, \alpha) &\le \mathbb{P}\left(\exists |u| \in \mathbf{T}^{(n)} : |u| = n, \forall i \le n, V^{(n)}(u_i) \le \frac{ai^\alpha}{n^{\alpha-1/3}} - \frac{i\kappa_1(\vartheta)}{\vartheta}\right) \\ &\le \sum_{j=1}^n H_{j,n} + H_{*,n}.\end{aligned} \tag{3.1}$$

Define $T_i^{(n)} = \vartheta S_i^{(n)} + i\kappa_1(\vartheta)$. By Markov inequality and Theorem 2.2, it is true that

$$H_{j,n} = \mathbb{E}\left(e^{T_j^{(n)}}\mathbf{1}_{\left\{\forall i<j, S_i^{(n)} \in \left[\frac{ai^\alpha}{n^{\alpha-1/3}} - b(n-i)^{1/3} - \frac{i\kappa_1(\vartheta)}{\vartheta},\ \frac{ai^\alpha}{n^{\alpha-1/3}} - \frac{i\kappa_1(\vartheta)}{\vartheta}\right], S_j^{(n)} \le \frac{aj^\alpha}{n^{\alpha-1/3}} - b(n-j)^{1/3} - \frac{j\kappa_1(\vartheta)}{\vartheta}\right\}}\right)$$

$$\le e^{\frac{\vartheta a j^{\alpha}}{n^{\alpha-1/3}} - \vartheta b(n-j)^{1/3}} \mathbb{P}\left(\forall i < j, T_i^{(n)} \in \left[\frac{\vartheta a i^{\alpha}}{n^{\alpha-1/3}} - \vartheta b(n-i)^{1/3}, \frac{\vartheta a i^{\alpha}}{n^{\alpha-1/3}}\right]\right).$$

By the same way we get

$$H_{*,n} \le e^{\vartheta a n^{1/3}} \mathbb{P}\left(\forall i \le n, T_i^{(n)} \in \left[\frac{\vartheta a i^{\alpha}}{n^{\alpha-1/3}} - \vartheta b(n-i)^{1/3}, \frac{\vartheta a i^{\alpha}}{n^{\alpha-1/3}}\right]\right).$$

Note that for any $n \in [Nk, (N+1)k]$, it is true that

$$\frac{\log \rho(a n^{1/3-\alpha}, \alpha)}{n^{1/3}} \le \frac{\log \rho(a(Nk)^{1/3-\alpha}, \alpha)}{((N+1)k)^{1/3}}. \tag{3.2}$$

Hence we have

$$\overline{\lim_{n\to\infty}} \frac{a^{\frac{1}{3\alpha-1}} \log \rho(a n^{1/3-\alpha}, \alpha)}{n^{1/3}} \le \frac{\sqrt[3]{N}}{\sqrt[3]{N+1}} \overline{\lim_{k\to\infty}} \frac{a^{\frac{1}{3\alpha-1}} \log \rho(a(Nk)^{1/3-\alpha}, \alpha)}{\sqrt[3]{Nk}}.$$

Taking $N \to \infty$, from (3.1) we get

$$\begin{aligned} &\overline{\lim_{n\to\infty}} \frac{a^{\frac{1}{3\alpha-1}} \log \rho(a n^{1/3-\alpha}, \alpha)}{n^{1/3}} \\ &\le \overline{\lim_{N\to\infty}}\, \overline{\lim_{k\to\infty}} \frac{a^{\frac{1}{3\alpha-1}} \log \rho(a(Nk)^{1/3-\alpha}, \alpha)}{\sqrt[3]{Nk}} \\ &\le \overline{\lim_{N\to\infty}}\, \overline{\lim_{k\to\infty}} \frac{a^{\frac{1}{3\alpha-1}} \log \left(\sum_{j=1}^{Nk} H_{j,Nk} + H_{*,Nk}\right)}{\sqrt[3]{Nk}}. \end{aligned} \tag{3.3}$$

We observe that

$$\begin{aligned} &\sum_{j=1}^{Nk} H_{j,Nk} + H_{*,Nk} \\ &\le \sum_{l=1}^{N} (k+1) \left(e^{\frac{\vartheta a(lk)^{\alpha}}{(Nk)^{\alpha-1/3}} - \vartheta b(Nk-lk)^{1/3}}\right) \times \\ &\quad \mathbb{P}\left(\forall i \le (l-1)k, T_i^{(n)} \in \left[\frac{\vartheta a i^{\alpha}}{(Nk)^{\alpha-1/3}} - \vartheta b(Nk-i)^{1/3}, \frac{\vartheta a i^{\alpha}}{(Nk)^{\alpha-1/3}}\right]\right). \end{aligned} \tag{3.4}$$

To apply Theorem 2.2, we need to verify that the sequence $\{T_i^{(n)}\}$ satisfies all conditions in Theorem 2.2. According to Theorem 2.1, we see

$$\mathbb{E}(X_{n,i}) = \frac{\mathbb{E}\left(\sum_{l \in L_{i/n}} l e^{-\vartheta l}\right)}{\mathbb{E}\left(\sum_{l \in L_{i/n}} e^{-\vartheta l}\right)} = -\kappa'_{i/n}(\vartheta).$$

We observe that (1.8) and the above equality imply that

$$\mathbb{E}(T_i^{(n)} - T_{i-1}^{(n)}) = \vartheta \mathbb{E}(X_{n,i}) + \kappa_1(\vartheta) = -\vartheta \kappa'_{i/n}(\vartheta) + \kappa_1(\vartheta) = 0,$$

thus we see $\mathbb{E}(T_i^{(n)}) = 0, \forall i, n \in \mathbb{N}$. Moreover, Theorem 2.1 tells that

$$\begin{aligned}\kappa''_{i/n}(\vartheta) &= \frac{\mathbb{E}\left(\sum_{l\in L_{i/n}} l^2 e^{-\vartheta l}\right)\mathbb{E}\left(\sum_{l\in L_{i/n}} e^{-\vartheta l}\right) - \left[\mathbb{E}\left(\sum_{l\in L_{i/n}} l e^{-\vartheta l}\right)\right]^2}{\mathbb{E}\left(\sum_{l\in L_{i/n}} e^{-\vartheta l}\right)^2}\\ &= \mathbb{E}(X_{n,i}^2) - (\mathbb{E}(X_{n,i}))^2.\end{aligned}$$

Hence we have

$$Var(T_i^{(n)} - T_{i-1}^{(n)}) = \vartheta^2 Var(S_i^{(n)} - S_{i-1}^{(n)}) = \vartheta^2 Var(X_{n,i}) = \vartheta^2 \kappa''_{i/n}(\vartheta),$$

where Var presents the variation, that is to say, (1.10) ensures that $\{T_i^{(n)}\}$ meets (2.1). Next we check (2.2). Note that

$$\mathbb{E}\left(e^{u|T_i^{(n)} - T_{i-1}^{(n)}|}\right) \le e^{u\kappa_1(\vartheta)}\mathbb{E}\left(e^{u\vartheta|X_{n,i}|}\right)$$

and

$$\mathbb{E}\left(e^{u\vartheta|X_{n,i}|}\right) \le \mathbb{E}\left(e^{u\vartheta X_{n,i}}\right) + \mathbb{E}\left(e^{-u\vartheta X_{n,i}}\right) = \frac{\kappa_{i/n}(\vartheta(1-u))}{\kappa_{i/n}(\vartheta)} + \frac{\kappa_{i/n}(\vartheta(1+u))}{\kappa_{i/n}(\vartheta)}.$$

Therefore, (1.9) ensures that $\{T_i^{(n)}\}$ meets (2.2). We rewrite the probability in (3.4) as

$$\begin{aligned}&\mathbb{P}\left(\forall i \le (l-1)k, T_i^{(n)} \in \left[\frac{\vartheta a i^\alpha}{(Nk)^{\alpha-1/3}} - \vartheta b (Nk-i)^{1/3}, \frac{\vartheta a i^\alpha}{(Nk)^{\alpha-1/3}}\right]\right)\\ =&\mathbb{P}\left(\forall i \le (l-1)k, \frac{T_i^{(n)}}{[(l-1)k]^{1/3}} - \left(\frac{l-1}{N}\right)^{2/3}\frac{\vartheta a i}{(l-1)k} \in \left[-\vartheta b \sqrt[3]{\frac{N}{l-1} - \frac{i}{(l-1)k}}, 0\right]\right).\end{aligned}$$

Let $(l-1)k$ play the role as n in Theorem 2.2, from (3.3) and (3.4) we get

$$\begin{aligned}&\overline{\lim_{n\to\infty}}(an^{1/3-\alpha})^{\frac{1}{3\alpha-1}} \log \varrho(an^{1/3-\alpha}, \alpha)\\ =& a^{\frac{1}{3\alpha-1}} \overline{\lim_{n\to\infty}} \frac{\log \rho(an^{1/3-\alpha}, \alpha)}{n^{1/3}}\\ \le& \overline{\lim_{N\to\infty}} \max_{1\le l\le N}\left(\vartheta a\left(\frac{l}{N}\right)^\alpha - \vartheta b \sqrt[3]{1 - \frac{l}{N}} - \frac{\pi^2}{2b^2}\sqrt[3]{\frac{l-1}{N}}\int_0^1 \kappa''_x(\vartheta)\left(\frac{N}{l-1} - x\right)^{-\frac{2}{3}} dx\right).\end{aligned} \tag{3.5}$$

Recall the definition of σ_- and define $\gamma_\sigma := \frac{\pi^2\sigma_-^2}{2}$, we get

$$\begin{aligned}&\overline{\lim_{n\to\infty}} n^{-\frac{1}{3}} \log \rho(an^{1/3-\alpha}, \alpha)\\ \le& \overline{\lim_{N\to\infty}} \max_{1\le l\le N}\left(\vartheta a\left(\frac{l}{N}\right)^\alpha - \vartheta b\sqrt[3]{1 - \frac{l}{N}} - \frac{3\gamma_\sigma}{\vartheta^2 b^2}\left(1 - \sqrt[3]{1 - \frac{l-1}{N}}\right)\right)\\ \le& \sup_{x\in[0,1]} \varphi(x),\end{aligned} \tag{3.6}$$

where $\varphi(x) := \vartheta a x^\alpha + \left(\frac{3\gamma_\sigma}{\vartheta^2 b^2} - \vartheta b\right)\sqrt[3]{1-x} - \frac{3\gamma_\sigma}{\vartheta^2 b^2}, x \in [0, 1]$. Because of the monotonicity of $\rho(\varepsilon, \alpha)$ on ε, by a similar argument as (3.2)–(3.3) we can see that for any $a > 0$,

$$\overline{\lim_{\varepsilon\downarrow 0}} \varepsilon^{\frac{1}{3\alpha-1}} \log \rho(\varepsilon, \alpha) = \overline{\lim_{n\to+\infty}} \frac{a^{\frac{1}{3\alpha-1}} \log \rho(an^{1/3-\alpha}, \alpha)}{n^{1/3}}. \tag{3.7}$$

By the light of (3.6) and (3.7), next we need to consider how to take the value of a, b to get the minmum of $\sup_{x\in[0,1]}\varphi(x)$.

(i) For the case $\alpha > 1$, we let positive constants a, b satisfy that

$$\vartheta a\alpha + \frac{\vartheta b}{3} - \frac{\gamma_\sigma}{\vartheta^2 b^2} = 0 \text{ and } \vartheta b - \frac{3\gamma_\sigma}{\vartheta^2 b^2} \le 0. \tag{3.8}$$

Note that

$$\varphi'(x) = \alpha a\vartheta x^{\alpha-1} + \frac{1}{3}\left(\vartheta b - \frac{3\gamma_\sigma}{\vartheta^2 b^2}\right)\frac{1}{(1-x)^{2/3}}.$$

(3.8) implies that $\max_{x\in[0,1]}\varphi'(x) \le 0$. That is to say,

$$\sup_{x\in[0,1]}\varphi(x) = \varphi(0) = -\vartheta b.$$

Combining (3.5), (3.6) with (3.8) we get

$$\begin{aligned}\overline{\lim_{n\to\infty}}(an^{1/3-\alpha})^{\frac{1}{3\alpha-1}}\log\varrho(an^{1/3-\alpha},\alpha) &\le -\vartheta b a^{\frac{1}{3\alpha-1}} \\ &= -\vartheta b\left(\frac{b}{3\alpha} - \frac{\gamma_\sigma}{\alpha\vartheta^3 b^2}\right)^{\frac{1}{3\alpha-1}} \\ &= -\vartheta\left(b^{3\alpha-3}\left[\frac{\gamma_\sigma}{\alpha\vartheta^3} - \frac{b^3}{3\alpha}\right]\right)^{\frac{1}{3\alpha-1}}.\end{aligned} \tag{3.9}$$

Noting that

$$\frac{\mathrm{d}\left[x^{\alpha-1}\left(\frac{3\gamma_\sigma}{\vartheta^3} - x\right)\right]}{\mathrm{d}x} = x^{\alpha-2}\left[\frac{3\gamma_\sigma}{\vartheta^3}(\alpha-1) - \alpha x\right],$$

hence we choose $b = \left(\frac{3\gamma_\sigma(\alpha-1)}{\alpha\vartheta^3}\right)^{1/3}$, which satisfies the second condition in (3.8) and the last line in (3.9) will take its maximum. Finally, from (3.7) we complete the proof of (1.11).

(ii) Now we consider the case $\alpha \in (\frac{1}{3}, 1)$. Recall that

$$\varphi(x) := \vartheta a x^\alpha + \left(\frac{3\gamma_\sigma}{\vartheta^2 b^2} - \vartheta b\right)\sqrt[3]{1-x} - \frac{3\gamma_\sigma}{\vartheta^2 b^2}.$$

Hence it is true that

$$\sup_{x\in[0,1]}\varphi(x) \le \vartheta a + \max\left\{0, \frac{3\gamma_\sigma}{\vartheta^2 b^2} - \vartheta b\right\} - \frac{3\gamma_\sigma}{\vartheta^2 b^2} \le \vartheta\left(a - \max\left\{b, \frac{3\gamma_\sigma}{\vartheta^3 b^2}\right\}\right).$$

From this point we choose $b = (3\gamma_\sigma)^{1/3}/\vartheta$ such that $\vartheta b - \frac{3\gamma_\sigma}{\vartheta^2 b^2} = 0$, hence it is true that

$$\overline{\lim_{\varepsilon\downarrow 0}}\,\varepsilon^{\frac{1}{3\alpha-1}}\log\rho(\varepsilon) \le -\vartheta a^{\frac{1}{3\alpha-1}}(b-a).$$

By direct caculation we see

$$\frac{\mathrm{d}[a^{\frac{1}{3\alpha-1}}(b-a)]}{\mathrm{d}a} = \frac{b}{3\alpha-1}a^{\frac{2-3\alpha}{3\alpha-1}} - \left(\frac{3\alpha}{3\alpha-1}\right)a^{\frac{1}{3\alpha-1}} = ba^{-1} - 3\alpha,$$

thus the best choice of a is $a := \frac{b}{3\alpha}$. Finally we get

$$\overline{\lim_{\varepsilon\downarrow 0}}\,\varepsilon^{\frac{1}{3\alpha-1}}\log\rho(\varepsilon) \le -\frac{\vartheta(3\alpha-1)}{(3\alpha)^{\frac{3\alpha}{3\alpha-1}}}b^{\frac{3\alpha}{3\alpha-1}} = -\frac{\vartheta(3\alpha-1)}{(3\vartheta\alpha)^{\frac{3\alpha}{3\alpha-1}}}(3\gamma_\sigma)^{\frac{\alpha}{3\alpha-1}},$$

which completes the proof of (1.12). □

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Conflict of interest

The author declares no conflict of interest.

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