AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 5036–5048.
DOI: 10.3934/math.2023252
Received: 08 July 2022
Revised: 19 November 2022
Accepted: 21 November 2022
Published: 12 December 2022

Research article

Translation hypersurfaces of semi-Euclidean spaces with constant scalar curvature

Derya Sağlam*and Cumali Sunar

Kırıkkale University, Faculty of Art and Science, Department of Mathematics, Turkey

* **Correspondence:** Email: deryasaglam@kku.edu.tr.

Abstract: In this paper, we present translation hypersurfaces of semi-Euclidean spaces with zero scalar curvature. In addition, we prove that translation hypersurfaces with constant scalar curvature must have zero scalar curvature in the semi-Euclidean space $\mathbb{R}^{n+1}_q$ for $n \geq 3$.

Keywords: translation hypersurfaces; scalar curvature; semi-Euclidean space
Mathematics Subject Classification: 53A04, 53A25, 53A40

1. Introduction and preliminaries

Translation hypersurfaces are special Monge hypersurfaces. Many studies have been carried out with these hypersurfaces until today [1–11].

In [1], Lima presented a complete description of all translation hypersurfaces with constant scalar curvature in the Euclidean space. In [2], they showed that every minimal translation and homothetical lightlike hypersurface is locally a hyperplane. In [3], the minimal translation hypersurfaces of E^4 were studied. Yang, Zhang and Fu obtained a characterization of a class of minimal translation graphs which are the generalization of minimal translation hypersurfaces in the Euclidean space [4]. In [5], the authors studied a characterization of minimal translation graphs in the semi-Euclidean space. Recently, homothetical and translation lightlike graphs, which are generalizations of homothetical and translation lightlike hypersurfaces were investigated in the semi-Euclidean space $\mathbb{R}^{n+2}_q$ [6]. Moreover Sağlam proved that all homothetical and all translation lightlike graphs are locally hyperplanes and according to this fact, both of these graphs are minimal. In [7], Seo gave a classification of the translation hypersurfaces with constant mean curvature or constant Gauss–Kronecker curvature in the Euclidean space and the Lorentz– Minkowski space. Moreover the author characterized the minimal translation hypersurfaces in the upper half-space model of the hyperbolic space. In 2019, Aydın and Ogrenmis studied translation hypersurfaces generated by translating planar curves and classified the translation hypersurfaces with constant Gauss-Kronecker curvature and constant mean curvature in the

4-dimensional isotropic space [8]. In [9], Ruiz-Hernandez investigated translation hypersurfaces in the (n+1)-dimensional Euclidean space whose Gauss-Kronecker curvature depends on its variables. In [10], Sousa, Lima and Vieira studied the geometry of generalized translation hypersurfaces immersed in Euclidean space equipped with a metric conformal to Euclidean metric and obtained results that characterize such hypersurfaces. In [11], Lima, Santos and Sousa gave a classification of the generalized translation graphs with constant mean curvature or constant Gauss–Kronecker curvature in the Euclidean space.

In the semi-Euclidean space $\mathbb{R}^{n+1}_q$, a translation hypersurface M^n is a semi-Riemannian manifold with codimension 1 given by

$$\psi(x_1,\ldots,x_n) = (x_1,\ldots,x_n, F(x_1,\ldots,x_n)),\quad F(x_1,\ldots,x_n) = \sum_{i=1}^{n} f_i(x_i)$$

where f_1, $f_2,\ldots,f_n$ are smooth functions. Each function f_i depends on the real variable x_i and is different from zero for $1 \leq i \leq n$. Or else it is a hyperplane.

In [1], Lima gave the parameterization of translation hypersurfaces with zero scalar curvature into $\mathbb{R}^{n+1}$ for $n \geq 3$. Moreover they showed that every translation hypersurface with constant scalar curvature must have zero scalar curvature in the Euclidean space $\mathbb{R}^{n+1}$ for $n \geq 3$ and proved the following theorem.

Theorem 1.1. Let M^n be a translation hypersurface of $\mathbb{R}^{n+1}$ given by $\psi = (x_1,\ldots,x_n,F)$ for $n \geq 3$. Then M^n has zero scalar curvature iff it is congruent to the graph of the following functions:

1. $F(x_1,\ldots,x_n) = \sum_{i=1}^{n-1} a_i x_i + f_n(x_n) + b$, on $\mathbb{R}^{n-1} \times J$, for some interval J and $f_n : J \subset \mathbb{R} \to \mathbb{R}$ is a smooth function, which defines, after a suitable linear change of variables, a vertical cylinder.

2. A generalized periodic Enneper hypersurface given by

$$\begin{aligned} F(x_1,\ldots,x_n) &= \sum_{i=1}^{n-3} a_i x_i + \frac{\sqrt{\beta}}{a}\ln\left|\frac{\cos\left(-\frac{ab}{a+b}\sqrt{\beta}x_n + c\right)}{\cos\left(a\sqrt{\beta}x_{n-2} + a_0\right)}\right| \\ &\quad + \frac{\sqrt{\beta}}{b}\ln\left|\frac{\cos\left(-\frac{ab}{a+b}\sqrt{\beta}x_n + c\right)}{\cos\left(b\sqrt{\beta}x_{n-1} + b_0\right)}\right| + d, \end{aligned} \tag{1.1}$$

on $R^{n-3} \times I_1 \times I_2 \times I_3$, where $a, a_1,\ldots,a_{n-3}, b, b_0, c, d$ are real constants with $a, b, a+b \neq 0, \beta = 1 + \sum_{i=1}^{n-3} a_i^2$ and I_1, I_2, I_3 are the open intervals defined, respectively, by the conditions $\left|a\sqrt{\beta}x_{n-2} + a_0\right| < \pi/2$, $\left|b\sqrt{\beta}x_{n-1} + b_0\right| < \pi/2$ and $\left|-\frac{ab}{a+b}\sqrt{\beta}x_n + c\right| < \pi/2$.

In this paper, we obtain the parameterization of translation hypersurfaces with zero scalar curvature into $\mathbb{R}^{n+1}_q$. In addition we prove that translation hypersurfaces with constant scalar curvature must have zero scalar curvature in the semi-Euclidean space $\mathbb{R}^{n+1}_q$ for $n \geq 3$.

2. Translation hypersurfaces of semi-Euclidean spaces with constant scalar curvature

Let M^n be a semi-Riemannian manifold and g_{ij} be the components of the metric tensor of M^n and g^{ij} be inverse of the functions g_{ij} for $1 \leq i, j \leq n$. The Christoffel symbols or the affine connection of

M^n are given by

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{m=1}^{n} g^{km}\left(\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right), \tag{2.1}$$

for $1 \leq i, j, k \leq n$. The Components of the Riemannian curvature tensor R of a semi-Riemannian manifold M^n are given by

$$R_{jkl}^i = \frac{\partial \Gamma_{kj}^i}{\partial x_l} - \frac{\partial \Gamma_{lj}^i}{\partial x_k} + \sum_{m=1}^{n} \Gamma_{lm}^i \Gamma_{kj}^m - \sum_{m=1}^{n} \Gamma_{km}^i \Gamma_{lj}^m, \tag{2.2}$$

for $1 \leq i, j, k, l \leq n$. The Components of the Ricci curvature tensor Ric of a semi-Riemannian manifold M^n are given by

$$R_{ij} = \sum_{m=1}^{n} R_{ijm}^m, \tag{2.3}$$

for $1 \leq i, j \leq n$. The scalar curvature S of a semi-Riemannian manifold M^n are given by

$$S = \sum_{i,j=1}^{n} g^{ij} R_{ij} = \sum_{i,j,k=1}^{n} g^{ij} R_{ijk}^k. \tag{2.4}$$

Theorem 2.1. Let M^n be a $n-$dimensional translation hypersurface of the semi-Euclidean space $\mathbb{R}_q^{n+1}$ with a natural orthonormal basis $\{e_1, \dots e_{n+1}\}$ determined by the following equations

$$\psi(x_1, \dots, x_n) = (x_1, \dots, x_n, F(x_1, \dots, x_n)), \quad F(x_1, \dots, x_n) = \sum_{i=1}^{n} f_i(x_i). \tag{2.5}$$

Then the scalar curvature of M^n given by

$$S = \frac{2}{\left(\varepsilon_{n+1} + \sum\limits_{i=1}^{n} \varepsilon_i f_i'^2\right)^2} \sum_{1 \leq i < j \leq n} \varepsilon_i \varepsilon_j f_i'' f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1 \leq k \leq n \\ k \neq i,j}} \varepsilon_k f_k'^2\right), \tag{2.6}$$

where $\varepsilon_i = \langle e_i, e_i \rangle = \pm 1$ for $1 \leq i \leq n+1$.

Proof. It is easy to check that

$$g_{ij} = \left\langle \psi_i, \psi_j \right\rangle = \begin{cases} \varepsilon_i + \varepsilon_{n+1} f_i'^2, & \text{for } i = j \\ \varepsilon_{n+1} f_i' f_j', & \text{for } i \neq j \end{cases} \tag{2.7}$$

and their inverse

$$g^{ij} = \begin{cases} \varepsilon_i \dfrac{\left(\varepsilon_{n+1} + \sum\limits_{\substack{k=1 \\ k \neq i}}^{n} \varepsilon_k f_k'^2\right)}{Q}, & \text{for } i = j \\ -\dfrac{\varepsilon_i \varepsilon_j f_i' f_j'}{Q}, & \text{for } i \neq j \end{cases} \tag{2.8}$$

with $Q=\varepsilon_{n+1}+\sum\limits_{k=1}^{n}\varepsilon_k f_k'^2$ and $i,j=1,...,n$. By the direct calculation from the equations (2.1)–(2.4), we get (2.6).

Theorem 2.2. Let M^n be a $n-$dimensional translation hypersurface of the semi-Euclidean space $\mathbb{R}^{n+1}_q$ for $n\geq 3$ determined by the following equations

$$\psi(x_1,\ldots,x_n)=(x_1,\ldots,x_n,F(x_1,\ldots,x_n)),\quad F(x_1,\ldots,x_n)=\sum_{i=1}^{n}f_i(x_i).$$

Then M^n has zero scalar curvature iff it is locally a hyperplane or it is parameterized by one of the following functions.

1.

$$F(x_1,\ldots,x_n)=\sum_{i=1}^{n-1}a_ix_i+f_n(x_n)+b, \tag{2.9}$$

on $\mathbb{R}^{n-1}\times I$, for some open interval I, where $a_i,b\in\mathbb{R}$, $1\leq i\leq n-1$ and $f_n:I\subset\mathbb{R}\to\mathbb{R}$ is a smooth function. With a appropiate translation, it is a vertical hypercylinder.

2.

$$F(x_1,\ldots,x_n)=\sum_{i=1}^{n-2}a_ix_i+f_{n-1}(x_{n-1})+f_n(x_n)+b, \tag{2.10}$$

on $\mathbb{R}^{n-2}\times I_1\times I_2$, for some open intervals I_1,I_2, where $a_i,b\in\mathbb{R}$, $1\leq i\leq n-2$ with $\sum_{i=1}^{n-2}\varepsilon_ia_i^2=-\varepsilon_{n+1}$ and $f_{n-1}:I_1\subset\mathbb{R}\to\mathbb{R}$, $f_n:I_2\subset\mathbb{R}\to\mathbb{R}$ are smooth functions.

3. Let $a,a_0,a_1,\ldots,a_{n-3},b,b_0,c_0,d$ be real constants with $a\neq 0,b\neq 0,a+b\neq 0,b-a\neq 0,\beta=\varepsilon_{n+1}+\sum_{i=1}^{n-3}\varepsilon_ia_i^2>0$ and I_1,I_2,I_3,I_4,I_5 be some open intervals defined, respectively, by the conditions $\left|a\sqrt{\beta}x_{n-2}+a_0\right|<\pi/2$, $\left|b\sqrt{\beta}x_{n-1}+b_0\right|<\pi/2$, $\left|\frac{ab}{a+b}\sqrt{\beta}x_n+c_0\right|<\pi/2$, $\left|\frac{ab}{b-a}\sqrt{\beta}x_n+c_0\right|<\pi/2$ and $\left|-\frac{ab}{a+b}\sqrt{\beta}x_n+c_0\right|<\pi/2$.

a. If $\varepsilon_{n-1}\varepsilon_n=1$ and $\varepsilon_{n-2}\varepsilon_n=1$, then

$$\begin{aligned}F(x_1,\ldots,x_n)&=\sum_{i=1}^{n-3}a_ix_i+\frac{1}{a}\ln\left|\frac{\cos\left(\frac{ab}{a+b}\sqrt{\beta}x_n+c_0\right)}{\cos\left(a\sqrt{\beta}x_{n-2}+a_0\right)}\right|\\&\quad+\frac{1}{b}\ln\left|\frac{\cos\left(\frac{ab}{a+b}\sqrt{\beta}x_n+c_0\right)}{\cos\left(b\sqrt{\beta}x_{n-1}+b_0\right)}\right|+d,\end{aligned} \tag{2.11}$$

on $\mathbb{R}^{n-3}\times I_1\times I_2\times I_3$.

b. If $\varepsilon_{n-1}\varepsilon_n=-1$ and $\varepsilon_{n-2}\varepsilon_n=1$, then

$$F(x_1,\ldots,x_n)=\sum_{i=1}^{n-3}a_ix_i+\frac{1}{a}\ln\left|\frac{\cos\left(\frac{ab}{b-a}\sqrt{\beta}x_n+c_0\right)}{\cos\left(a\sqrt{\beta}x_{n-2}+a_0\right)}\right|$$

$$-\frac{1}{b}\ln\left|\cos\left(\frac{ab}{b-a}\sqrt{\beta}x_n+c_0\right)\cos\left(b\sqrt{\beta}x_{n-1}+b_0\right)\right|+d, \tag{2.12}$$

on $\mathbb{R}^{n-3}\times I_1\times I_2\times I_4$.

c. If $\varepsilon_{n-1}\varepsilon_n=1$ and $\varepsilon_{n-2}\varepsilon_n=-1$, then

$$\begin{aligned} F(x_1,\ldots,x_n) &= \sum_{i=1}^{n-3}a_ix_i-\frac{1}{a}\ln\left|\cos\left(\frac{ab}{b-a}\sqrt{\beta}x_n+c_0\right)\cos\left(a\sqrt{\beta}x_{n-2}+a_0\right)\right| \\ &\quad +\frac{1}{b}\ln\left|\frac{\cos\left(\frac{ab}{b-a}\sqrt{\beta}x_n+c_0\right)}{\cos\left(b\sqrt{\beta}x_{n-1}+b_0\right)}\right|+d, \end{aligned} \tag{2.13}$$

on $\mathbb{R}^{n-3}\times I_1\times I_2\times I_4$.

d. If $\varepsilon_{n-1}\varepsilon_n=-1$ and $\varepsilon_{n-2}\varepsilon_n=-1$, then

$$\begin{aligned} F(x_1,\ldots,x_n) &= \sum_{i=1}^{n-3}a_ix_i-\frac{1}{a}\ln\left|\cos\left(\frac{-ab}{a+b}\sqrt{\beta}x_n+c_0\right)\cos\left(a\sqrt{\beta}x_{n-2}+a_0\right)\right| \\ &\quad -\frac{1}{b}\ln\left|\cos\left(\frac{-ab}{a+b}\sqrt{\beta}x_n+c_0\right)\cos\left(b\sqrt{\beta}x_{n-1}+b_0\right)\right|+d, \end{aligned} \tag{2.14}$$

on $\mathbb{R}^{n-3}\times I_1\times I_2\times I_5$.

If $\beta=0$, then M^n is locally a hyperplane.

Proof. From Theorem 1.1, M^n has zero scalar curvature iff

$$\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j f_i^{''}f_j^{''}\left(\varepsilon_{n+1}+\sum_{\substack{1\le k\le n\\ k\ne i,j}}\varepsilon_k f_k^{'2}\right)=0. \tag{2.15}$$

We will examine the proof according to the following cases.

Case 1. Let $\varepsilon_{n+1}+\sum_{\substack{1\le k\le n\\ k\ne i,j}}\varepsilon_k f_k^{'2}=0$ for all $1\le i<j\le n$, then the functions $f_k^{'}$ are constant for all $1\le k\le n$. Consequently M^n is locally a hyperplane.

Case 2. Let $f_i^{''}(x_i)=0$ for all $i=1,\ldots n-1$, then M^n is parameterized by the equation (2.9).

Case 3. Let $f_i^{''}(x_i)=0$ for all $i=1,\ldots n-2$, then $f_i^{'}(x_i)=a_i$, $a_i\in\mathbb{R}$. Also we can rewrite (2.15) by the following equation

$$\varepsilon_{n-1}\varepsilon_n f_{n-1}^{''}f_n^{''}\left(\varepsilon_{n+1}+\sum_{k=1}^{n-2}\varepsilon_k a_k^2\right).$$

According to this equation, we have the following cases:

i. If $f_{n-1}^{''}=0$, corresponding to Case 1.

ii. If $f_n^{''}=0$, corresponding to Case 1.

iii. If $\varepsilon_{n+1}+\sum_{k=1}^{n-2}\varepsilon_k a_k^2=0$, then M^n is parameterized by the equation (2.10).

Case 4. Let $f_i^{''}(x_i) = 0$ for all $i = 1, \dots n-3$, then $f_i^{'}(x_i) = a_i$, $a_i \in \mathbb{R}$. Also we can rewrite (2.15) by the following equation

$$\varepsilon_{n-2}\varepsilon_{n-1}f_{n-2}^{''}f_{n-1}^{''}(\beta + f_n^{'2}) + \varepsilon_{n-2}\varepsilon_n f_{n-2}^{''}f_n^{''}(\beta + f_{n-1}^{'2}) + \varepsilon_{n-1}\varepsilon_n f_{n-1}^{''}f_n^{''}(\beta + f_{n-2}^{'2}) = 0,$$

where $\beta = \varepsilon_{n+1} + \sum_{k=1}^{n-3} \varepsilon_k a_k^2$. If we multiply both sides of the above equation by $\varepsilon_{n-2}\varepsilon_{n-1}\varepsilon_n$, then we obtain

$$\varepsilon_n f_{n-2}^{''}f_{n-1}^{''}(\beta + f_n^{'2}) + \varepsilon_{n-1}f_{n-2}^{''}f_n^{''}(\beta + f_{n-1}^{'2}) + \varepsilon_{n-2}f_{n-1}^{''}f_n^{''}(\beta + f_{n-2}^{'2}) = 0. \tag{2.16}$$

According to the assumption, the functions $f_{n-2}^{''}$, $f_{n-1}^{''}$ and $f_n^{''}$ are different from zero. Also we get $\beta + f_k^{'2} \neq 0$ for $k = n-2, n-1, n$. Hence we rewrite (2.16)

$$\varepsilon_n \frac{f_{n-2}^{''}f_{n-1}^{''}}{(\beta + f_{n-2}^{'2})(\beta + f_{n-1}^{'2})} + \varepsilon_{n-1}\frac{f_{n-2}^{''}f_n^{''}}{(\beta + f_{n-2}^{'2})(\beta + f_n^{'2})} + \varepsilon_{n-2}\frac{f_{n-1}^{''}f_n^{''}}{(\beta + f_{n-1}^{'2})(\beta + f_n^{'2})} = 0. \tag{2.17}$$

Differentiating the equation with respect to x_{n-2} and x_{n-1}, we find

$$\left(\frac{f_{n-2}^{''}}{\beta + f_{n-2}^{'2}}\right)' = 0 \quad \text{or} \quad \left(\frac{f_{n-1}^{''}}{\beta + f_{n-1}^{'2}}\right)' = 0.$$

If $\left(\frac{f_{n-2}^{''}}{\beta + f_{n-2}^{'2}}\right)' = 0$, then there is a constant $a \neq 0$ such that

$$f_{n-2}^{''} = a\left(\beta + f_{n-2}^{'2}\right). \tag{2.18}$$

Substituting this equation into (2.17), we obtain

$$\varepsilon_n \frac{f_{n-1}^{''}}{\beta + f_{n-1}^{'2}}a + \varepsilon_{n-1}\frac{f_n^{''}}{\beta + f_n^{'2}}a + \varepsilon_{n-2}\frac{f_{n-1}^{''}f_n^{''}}{(\beta + f_{n-1}^{'2})(\beta + f_n^{'2})} = 0. \tag{2.19}$$

Differentiating the equation with respect to x_{n-1} and x_n, we find

$$\left(\frac{f_{n-1}^{''}}{\beta + f_{n-1}^{'2}}\right)' = 0 \quad \text{or} \quad \left(\frac{f_n^{''}}{\beta + f_n^{'2}}\right)' = 0.$$

If $\left(\frac{f_{n-1}^{''}}{\beta + f_{n-1}^{'2}}\right)' = 0$, then there is a constant $b \neq 0$ such that

$$f_{n-1}^{''} = b\left(\beta + f_{n-1}^{'2}\right). \tag{2.20}$$

Substituting this equation into (2.19), we obtain

$$\varepsilon_n ab + \frac{f_n^{''}}{\beta + f_n^{'2}}(\varepsilon_{n-1}a + \varepsilon_{n-2}b) = 0. \tag{2.21}$$

Since $ab \neq 0$, from (2.21), then $\varepsilon_{n-1}a + \varepsilon_{n-2}b \neq 0$. If we rearrange the equation, then we get

$$\frac{f_n''}{\beta + f_n'^2} = -\frac{\varepsilon_n ab}{\varepsilon_{n-1}a + \varepsilon_{n-2}b}. \tag{2.22}$$

If we integrate the equations (2.18), (2.20) and (2.22), then we obtain

$$\arctan\left(\frac{f_{n-2}'(x_{n-2})}{\sqrt{\beta}}\right) = a\sqrt{\beta}x_{n-2} + a_0,$$

$$\arctan\left(\frac{f_{n-1}'(x_{n-1})}{\sqrt{\beta}}\right) = a\sqrt{\beta}x_{n-1} + b_0,$$

$$\arctan\left(\frac{f_n'(x_n)}{\sqrt{\beta}}\right) = -\frac{\varepsilon_n ab\sqrt{\beta}}{\varepsilon_{n-1}a + \varepsilon_{n-2}b}x_n + c_0,$$

where a_0, b_0 and c_0 are constants. From these equations, we get

$$f_{n-2}(x_{n-2}) = -\frac{1}{a}\ln\left|\cos\left(a\sqrt{\beta}x_{n-2} + a_0\right)\right| + a_1,$$

$$f_{n-1}(x_{n-1}) = -\frac{1}{b}\ln\left|\cos\left(b\sqrt{\beta}x_{n-1} + b_0\right)\right| + b_1,$$

$$f_n(x_n) = \frac{\varepsilon_{n-1}a + \varepsilon_{n-2}b}{\varepsilon_n ab}\ln\left|\cos\left(\frac{\varepsilon_n ab\sqrt{\beta}}{\varepsilon_{n-1}a + \varepsilon_{n-2}b}x_n + c_0\right)\right| + c_1,$$

where a_1, b_1 and c_1 are constants. Therefore M^n is parameterized by the equation

$$\begin{aligned}\psi(x_1, \ldots, x_n) = \Bigg(x_1, \ldots, x_n, \sum_{i=1}^{n-3} a_i x_i - \frac{1}{a}\ln\left|\cos\left(a\sqrt{\beta}x_{n-2} + a_0\right)\right| \\ - \frac{1}{b}\ln\left|\cos\left(b\sqrt{\beta}x_{n-1} + b_0\right)\right| \\ + \left(\frac{\varepsilon_{n-2}\varepsilon_n}{a} + \frac{\varepsilon_{n-1}\varepsilon_n}{b}\right)\ln\left|\cos\left(\frac{\varepsilon_n ab\sqrt{\beta}}{\varepsilon_{n-1}a + \varepsilon_{n-2}b}x_n + c_0\right)\right| + d\Bigg)\end{aligned} \tag{2.23}$$

where $d = a_1 + b_1 + c_1$ is a constant. According to the values of ε_{n-2}, ε_{n-1} and ε_n, if we rearrange the equation (2.23), then we get the following parameterizations.

i. If $\varepsilon_{n-1}\varepsilon_n = 1$ and $\varepsilon_{n-2}\varepsilon_n = 1$, then the translation hypersurface M^n is given by (2.11).

ii. If $\varepsilon_{n-1}\varepsilon_n = -1$ and $\varepsilon_{n-2}\varepsilon_n = 1$, then the translation hypersurface M^n is given by (2.12).

iii. If $\varepsilon_{n-1}\varepsilon_n = 1$ and $\varepsilon_{n-2}\varepsilon_n = -1$, then the translation hypersurface M^n is given by (2.13).

iv. If $\varepsilon_{n-1}\varepsilon_n = -1$ and $\varepsilon_{n-2}\varepsilon_n = -1$, then the translation hypersurface M^n is given by (2.14).

Case 5. Let $f_i''(x_i) = 0$ for $1 \leq i \leq k \leq n-4$, and $f_j''(x_j) \neq 0$ for any $j > k$. We prove that this is not possible. Also we can rewrite (2.15) for any fixed $l \geq k+1$ by the following equation

$$\sum_{1\leq i<j\leq n} \varepsilon_i\varepsilon_j f_i'' f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\leq m\leq n\\ m\neq i,j}} \varepsilon_m f_m'^2\right) = \varepsilon_l f_l'' \sum_{\substack{k+1\leq j\leq n\\ j\neq l}} \varepsilon_j f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\leq m\leq n\\ m\neq l,j}} \varepsilon_m f_m'^2\right)$$

$$+ \sum_{\substack{k+1\le i<j\le n \\ i,j\ne l}} \varepsilon_i\varepsilon_j f_i^{''} f_j^{''} \left(\varepsilon_{n+1} + \sum_{\substack{1\le m\le n \\ m\ne i,j}} \varepsilon_m f_m^{'2} \right). \tag{2.24}$$

Differentiating the equation (2.24) with respect to x_l, we obtain

$$f_l^{'''} \sum_{\substack{k+1\le j\le n \\ j\ne l}} \varepsilon_j f_j^{''} \left(\varepsilon_{n+1} + \sum_{\substack{1\le m\le n \\ m\ne l,j}} \varepsilon_m f_m^{'2} \right) + 2f_l^{'} f_l^{''} \sum_{\substack{k+1\le i<j\le n \\ i,j\ne l}} \varepsilon_i\varepsilon_j f_i^{''} f_j^{''} = 0. \tag{2.25}$$

According to the equation (2.25), we define

$$A_l = \sum_{\substack{k+1\le j\le n \\ j\ne l}} \varepsilon_j f_j^{''} \left(\varepsilon_{n+1} + \sum_{\substack{1\le m\le n \\ m\ne l,j}} \varepsilon_m f_m^{'2} \right), \quad B_l = \sum_{\substack{k+1\le i<j\le n \\ i,j\ne l}} \varepsilon_i\varepsilon_j f_i^{''} f_j^{''}. \tag{2.26}$$

A_l and B_l are not dependent on x_l. From (2.25) and (2.26), we have

$$A_l f_l^{'''} + 2B_l f_l^{'} f_l^{''} = 0. \tag{2.27}$$

Also there are two cases.

i. Let $A_l = 0$ for $l \ge k+1$. From (2.26), we get

$$\sum_{\substack{k+1\le j\le n \\ j\ne l}} \varepsilon_j f_j^{''} \left(\varepsilon_{n+1} + \sum_{\substack{1\le m\le n \\ m\ne l,j}} \varepsilon_m f_m^{'2} \right) = 0. \tag{2.28}$$

Differentiating the equation (2.28) with respect to x_p for $p \ge k+1$ and $p \ne l$, we find

$$f_p^{'''} \left(\varepsilon_{n+1} + \sum_{\substack{1\le m\le n \\ m\ne l,p}} \varepsilon_m f_m^{'2} \right) + 2f_p^{'} f_p^{''} \sum_{\substack{k+1\le j\le n \\ j\ne l,p}} \varepsilon_j f_j^{''} = 0. \tag{2.29}$$

According to this equation, one must have

$$\varepsilon_{n+1} + \sum_{\substack{1\le m\le n \\ m\ne l,p}} \varepsilon_m f_m^{'2} \ne 0. \tag{2.30}$$

Otherwise the functions $f_m^{'}$ are constant and we conclude that $f_m^{''} = 0$ for $1 \le m \le n$, $m \ne l, p$. This is a contradiction with the assumption in Case 5. Since $A_l = 0$, according to (2.25), we get

$$2\varepsilon_l f_l^{'} f_l^{''} \sum_{\substack{k+1\le i<j\le n \\ i,j\ne l}} \varepsilon_i\varepsilon_j f_i^{''} f_j^{''} = 0. \tag{2.31}$$

Since $\varepsilon_l \neq 0$ and $f_l'' \neq 0$, we have

$$\sum_{\substack{k+1\leq i<j\leq n\\ i,j\neq l}} \varepsilon_i\varepsilon_j f_i'' f_j'' = 0. \tag{2.32}$$

Differentiating the equation (2.32) with respect to x_p for $p \geq k+1$ and $p \neq l$, we obtain

$$f_p''' \sum_{\substack{k+1\leq j\leq n\\ j\neq l,p}} \varepsilon_j f_j'' = 0. \tag{2.33}$$

Differentiating the equation (2.33) with respect to x_q for $q \geq k+1$ and $q \neq l, p$, we find $f_p''' f_q''' = 0$. Therefore, at most one of the indexes $p \geq k+1$ and $p \neq l$ is nonzero, denoted by p. Also we can get $f_p''' \neq 0$ and $f_q''' = 0$ for all $q \geq k+1$ and $q \neq l, p$. From $f_p''' \neq 0$ and the equation (2.33), we have

$$\sum_{\substack{k+1\leq j\leq n\\ j\neq l,p}} \varepsilon_j f_j'' = 0. \tag{2.34}$$

Substituting this equation into (2.29), since $\varepsilon_{n+1} + \sum_{\substack{1\leq m\leq n\\ m\neq l,p}} \varepsilon_m f_m'^2 \neq 0$, we get $f_p''' = 0$. This is a contradiction with $f_p''' \neq 0$. Also we get $f_p''' = 0$ for all $p \geq k+1$ and $p \neq l$. From (2.29), we conclude that

$$\sum_{\substack{k+1\leq j\leq n\\ j\neq l,p}} \varepsilon_j f_j'' = 0, \tag{2.35}$$

for all $p \geq k+1$ and $p \neq l$. The above linear system has unique solution such that $f_j'' = 0$ for all $k+1 \leq j \leq n$ and $j \neq l$. This is a contradiction with the assumption in Case 5. Consequently, if $A_l = 0$, then Case 5 is not possible.

ii. Let $A_l \neq 0$ for $l \geq k+1$. Since $A_l \neq 0$, from (2.27), we get

$$f_l''' + 2\alpha_l f_l' f_l'' = 0, \tag{2.36}$$

where $\alpha_l = \dfrac{B_l}{A_l}$ is a constant for $l \geq k+1$. Substituting this equation into (2.25), we find

$$\alpha_l f_l' f_l'' \sum_{\substack{k+1\leq j\leq n\\ j\neq l}} \varepsilon_j f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\leq m\leq n\\ m\neq l,j}} \varepsilon_m f_m'^2 \right) - f_l' f_l'' \sum_{\substack{k+1\leq i<j\leq n\\ i,j\neq l}} \varepsilon_i\varepsilon_j f_i'' f_j'' = 0.$$

Since $f_l''(x_l) \neq 0$ for $l \geq k+1$, we obtain

$$\alpha_l \sum_{\substack{k+1\leq j\leq n\\ j\neq l}} \varepsilon_j f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\leq m\leq n\\ m\neq l,j}} \varepsilon_m f_m'^2 \right) - \sum_{\substack{k+1\leq i<j\leq n\\ i,j\neq l}} \varepsilon_i\varepsilon_j f_i'' f_j'' = 0. \tag{2.37}$$

Differentiating the equation (2.37) with respect to x_s for $s \geq k+1$ and $s \neq l$, we obtain

$$\alpha_l f_s''' \left(\varepsilon_{n+1} + \sum_{\substack{1\leq m\leq n \\ m\neq l,s}} \varepsilon_m f_m'^2 \right) + 2\alpha_l f_s' f_s'' \sum_{\substack{k+1\leq j\leq n \\ j\neq l,s}} \varepsilon_j f_j'' - f_s''' \sum_{\substack{k+1\leq j\leq n \\ j\neq l,s}} \varepsilon_j f_j'' = 0.$$

From (2.36), $f_s''' + 2\alpha_s f_s' f_s'' = 0$ for $s \geq k+1$. Also we can rewrite the above equation

$$-\alpha_l \alpha_s f_s' f_s'' \left(\varepsilon_{n+1} + \sum_{\substack{1\leq m\leq n \\ m\neq l,s}} \varepsilon_m f_m'^2 \right) + \alpha_l f_s' f_s'' \sum_{\substack{k+1\leq j\leq n \\ j\neq l,s}} \varepsilon_j f_j'' + \alpha_s f_s' f_s'' \sum_{\substack{k+1\leq j\leq n \\ j\neq l,s}} \varepsilon_j f_j'' = 0.$$

Since $f_s''(x_s) \neq 0$ for $s \geq k+1$, we get

$$-\alpha_l \alpha_s \left(\varepsilon_{n+1} + \sum_{\substack{1\leq m\leq n \\ m\neq l,s}} \varepsilon_m f_m'^2 \right) + \alpha_l \sum_{\substack{k+1\leq j\leq n \\ j\neq l,s}} \varepsilon_j f_j'' + \alpha_s \sum_{\substack{k+1\leq j\leq n \\ j\neq l,s}} \varepsilon_j f_j'' = 0. \tag{2.38}$$

Differentiating the equation (2.38) with respect to x_t for $t \geq k+1$ and $t \neq l$ and $t \neq s$, we obtain

$$-2\alpha_l \alpha_s f_t' f_t'' + \alpha_l f_t''' + \alpha_s f_t''' = 0.$$

From (2.36), $f_t''' + 2\alpha_t f_t' f_t'' = 0$ for $t \geq k+1$. Since $f_t''(x_t) \neq 0$ for $t \geq k+1$, we obtain the above equation

$$\alpha_l \alpha_s + \alpha_l \alpha_t + \alpha_s \alpha_t = 0, \tag{2.39}$$

with $t \neq l$, $t \neq s$ and $l \neq s$. From [1], in a similar way to the proof of Theorem 1.2, this equality imply that at most one of the constants α_l is nonzero for $l \geq k+1$. We assume that $\alpha_l = 0$ for $k+1 \leq l \leq n-1$. From (2.36), $f_l''' = 0$, then f_l'' is constant for $k+1 \leq l \leq n-1$. From (2.37), we obtain

$$\sum_{\substack{k+1\leq i<j\leq n \\ i,j\neq l}} \varepsilon_i \varepsilon_j f_i'' f_j'' = 0$$

for $l \neq n$. Therefore f_n'' is constant and so $\alpha_n = 0$. Thus, from (2.37), we get

$$\sum_{\substack{k+1\leq i<j\leq n \\ i,j\neq l}} \varepsilon_i \varepsilon_j f_i'' f_j'' = 0.$$

According to the equality, at most one of the functions f_l'' is nonzero for $k+1 \leq l \leq n$. This is a contradiction with the assumption in Case 5. Consequently, if $A_l \neq 0$, then Case 5 is not possible.

Theorem 2.3. Let M^n be a $n-$dimensional translation hypersurface of the semi-Euclidean space $\mathbb{R}_q^{n+1}$ for $n \geq 3$ determined by the following equations

$$\psi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, F(x_1, \ldots, x_n)), \quad F(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i).$$

Assume further that M^n has constant scalar curvature. Then its constant scalar curvature must be zero.

Proof. We assume that a translation hypersurface M^n has nonzero constant scalar curvature S. From (2.6) the scalar curvature of M^n is given by

$$S = \frac{2}{Q^2} \sum_{1\le i<j\le n} \varepsilon_i\varepsilon_j f_i'' f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\le k\le n\\ k\neq i,j}} \varepsilon_k f_k'^2\right) \tag{2.40}$$

where $Q = \varepsilon_{n+1} + \sum_{i=1}^{n} \varepsilon_i f_i'^2$. Differentiating the equation (2.40) with respect to x_l, we obtain

$$\begin{aligned} 0 \;=\; & \frac{1}{Q^2}\left[f_l''' \sum_{\substack{1\le j\le n\\ j\neq l}} \varepsilon_j f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\le k\le n\\ k\neq l,j}} \varepsilon_k f_k'^2\right) + 2 f_l' f_l'' \sum_{\substack{1\le i<j\le n\\ i,j\neq l}} \varepsilon_i\varepsilon_j f_i'' f_j'' \right] \\ & - \frac{4 f_l' f_l''}{Q^3} \sum_{1\le i<j\le n} \varepsilon_i\varepsilon_j f_i'' f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\le k\le n\\ k\neq i,j}} \varepsilon_k f_k'^2\right). \end{aligned}$$

If we rearrange this equation, then we get

$$2 f_l' f_l'' S = \frac{1}{Q}\left[f_l''' \sum_{\substack{1\le j\le n\\ j\neq l}} \varepsilon_j f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\le k\le n\\ k\neq l,j}} \varepsilon_k f_k'^2\right) + 2 f_l' f_l'' \sum_{\substack{1\le i<j\le n\\ i,j\neq l}} \varepsilon_i\varepsilon_j f_i'' f_j'' \right]. \tag{2.41}$$

Differentiating the equation (2.41) with respect to x_s and $s \neq l$, we find

$$\begin{aligned} 0 \;=\; & \frac{1}{Q}\left[f_l''' f_s''' \left(\varepsilon_{n+1} + \sum_{\substack{1\le k\le n\\ k\neq l,s}} \varepsilon_k f_k'^2\right) + 2 f_l''' f_s' f_s'' \sum_{\substack{1\le j\le n\\ j\neq l,s}} \varepsilon_j f_j'' + 2 f_l' f_l'' f_s''' \sum_{\substack{1\le j\le n\\ j\neq l,s}} \varepsilon_j f_j'' \right] \\ & - \frac{2 f_s' f_s''}{Q^2}\left[f_l''' \sum_{\substack{1\le j\le n\\ j\neq l}} \varepsilon_j f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\le k\le n\\ k\neq l,j}} \varepsilon_k f_k'^2\right) + 2 f_l' f_l'' \sum_{\substack{1\le i<j\le n\\ i,j\neq l}} \varepsilon_i\varepsilon_j f_i'' f_j'' \right]. \end{aligned}$$

From this equation, we get

$$4 f_l' f_l'' f_s' f_s'' S = f_l''' f_s''' \left(\varepsilon_{n+1} + \sum_{\substack{1\le k\le n\\ k\neq l,s}} \varepsilon_k f_k'^2\right) + 2(f_l''' f_s' f_s'' + f_l' f_l'' f_s''') \sum_{\substack{1\le j\le n\\ j\neq l,s}} \varepsilon_j f_j''. \tag{2.42}$$

Differentiating the equation (2.42) with respect to x_t, $t \neq l$ and $t \neq s$, we have

$$f_l''' f_s''' f_t' f_t'' + f_l''' f_t''' f_s' f_s'' + f_s''' f_t''' f_l' f_l'' = 0. \tag{2.43}$$

We assume that $f_l'' f_s'' f_t'' \neq 0$ and $f_l''' = 0$. According to (2.43), we get $f_s''' = 0$ or $f_t''' = 0$. From (2.42), we have $4f_l' f_l'' f_s' f_s'' S = 0$. This contradicts $f_l'' f_s'' f_t'' \neq 0$ and $S \neq 0$. Also $f_l''' \neq 0$ and likewise $f_s''' \neq 0$ and $f_t''' \neq 0$. From $f_l'' f_s'' f_t'' \neq 0$ and (2.43), we find

$$\frac{f_l'''}{f_l' f_l''} \frac{f_s'''}{f_s' f_s''} + \frac{f_l'''}{f_l' f_l''} \frac{f_t'''}{f_t' f_t''} + \frac{f_s'''}{f_s' f_s''} \frac{f_t'''}{f_t' f_t''} = 0. \tag{2.44}$$

From (2.44), we get $f_l''' = \alpha_l f_l' f_l''$, with a nonzero constant α_l. Substituting this equation into (2.41), we find

$$2SQ = \alpha_l \sum_{\substack{1\leq j\leq n \\ j\neq l}} \varepsilon_j f_j'' \left(\varepsilon_{n+1} + \sum_{\substack{1\leq k\leq n \\ k\neq l,j}} \varepsilon_k f_k'^2 \right) + 2 \sum_{\substack{1\leq i<j\leq n \\ i,j\neq l}} \varepsilon_i \varepsilon_j f_i'' f_j''. \tag{2.45}$$

Differentiating the equation (2.45) with respect to x_l, we have $f_l' f_l'' S = 0$. This contradicts $f_l'' f_s'' f_t'' \neq 0$ and $S \neq 0$. Hence, it must be $f_l'' f_s'' f_t'' = 0$. Also, at most two of the functions f_l'' are nonzero for $1 \leq l \leq n$. Without loss of generality, we assume that $f_{n-1}'' \neq 0$, $f_n'' \neq 0$ and $f_l'' = 0$ for $1 \leq l \leq n-2$, then $f_l' = a_l$ for $1 \leq l \leq n-2$ and we arrange (2.6)

$$0 \neq Q^2 S = f_{n-1}'' f_n'' \alpha, \tag{2.46}$$

where $\alpha = 2\varepsilon_{n-1}\varepsilon_n \left(\varepsilon_{n+1} + \sum_{k=1}^{n-2} \varepsilon_k a_k^2 \right)$ is a nonzero constant. Differentiating the equation (2.46) with respect to x_{n-1}, we have

$$0 \neq 4\varepsilon_{n-1} f_{n-1}' f_{n-1}'' QS = f_{n-1}''' f_n'' \alpha. \tag{2.47}$$

Differentiating the equation with respect to x_n, we get

$$0 \neq 8\varepsilon_{n-1}\varepsilon_n f_{n-1}' f_{n-1}'' f_n' f_n'' S = f_{n-1}''' f_n''' \alpha. \tag{2.48}$$

Also, there is a nonzero constant β such that $f_{n-1}''' = \beta f_{n-1}' f_{n-1}'' \neq 0$ and from (2.47)

$$0 \neq 4\varepsilon_{n-1} QS = f_n'' \alpha\beta. \tag{2.49}$$

Differentiating the equation (2.49) with respect to x_{n-1}, we get

$$8 f_{n-1}' f_{n-1}'' S = 0.$$

This is a contradiction with $f_{n-1}'' \neq 0$. Thus the constant scalar curvature must be zero.

3. Conclusions

Translation hypersurfaces are special Monge hypersurfaces defined by the following equations

$$\psi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, F(x_1, \ldots, x_n)), \quad F(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i).$$

In this paper, we obtain the parameterization of translation hypersurfaces with zero scalar curvature into $\mathbb{R}_q^{n+1}$. Moreover we prove that translation hypersurfaces with constant scalar curvature must have zero scalar curvature in the semi-Euclidean space $\mathbb{R}_q^{n+1}$ for $n \geq 3$.

References

1. B. P. Lima, Translation hypersurfaces with constant scalar curvature into the Euclidean spaces, *Israel J. Math.*, **201** (2014), 797–811. https://doi.org/10.1007/s11856-014-1083-2
2. I. V. Woestyne, W. Geomans, Translation and homothetical lightlike hypersurfaces of a semi-Euclidean space, *Kuwait J. Sci.. Eng.,* **2** (2011), 35–42.
3. M. Moruz, M. I. Munteanu, Minimal translation hypersurfaces in E^4, *J. Math. Anal. Appl.*, **439** (2016), 798–812.
4. D. Yang, J. Zhang, Y. Fu, A note on minimal translation graphs in Euclidean space, *Mathematics*, **7** (2019), 1–12.
5. D. Sağlam, Minimal translation graphs in semi-Euclidean space, *AIMS Math.*, **6** (2021), 10207–10221. https://doi.org/10.3934/math.2021591
6. D. Sağlam, Minimal homothetical and translation lightlike graphs in $\mathbb{R}_q^{n+2}$, *AIMS Math.*, **7** (2022), 17198–17209. https://doi.org/10.3934/math.2022946
7. K. Seo, Translation hypersurfaces with constant curvature in space forms, *Osaka J. Math.*, **50** (2013), 631–641.
8. M. A. Aydın, A. O. Ogrenmis, Translation hypersurfaces with constant curvature in 4-dimensional isotropic space, *Int. J. Maps. Math.*, **2** (2019), 108–130.
9. G. Ruiz-Hernández, Translation hypersurfaces whose curvature depends partially on its variables, *J. Math. Anal. Appl.*, **497** (2021), 124913. https://doi.org/10.1016/j.jmaa.2020.124913
10. P. A. Sousa, B. P. Lima, B. V. M. Vieira, Generalized translation hypersurfaces in conformally flat spaces, *J. Geom.*, **113** (2022), 1–13. https://doi.org/10.1007/s00022-022-00638-2
11. B. P. Lima, N. L. Santos, P. A. Sousa, Generalized translation hypersurfaces in Euclidean space, *J. Math. Anal. Appl.,* **470** (2019), 1129–1135. https://doi.org/10.1016/j.jmaa.2018.10.053