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Research article

On stability analysis of a class of three-dimensional system of exponential difference equations

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Abstract: The boundedness character, persistent nature, and asymptotic conduct of non-negative outcomes of the system of three dimensional exponential form of difference equations were studied in this research:

$$\begin{aligned} x_{n+1} &= ax_n + by_{n-1}e^{-x_n}, \\ y_{n+1} &= cy_n + dz_{n-1}e^{-y_n}, \\ z_{n+1} &= ez_n + fx_{n-1}e^{-z_n}, \end{aligned}$$

where $a,\ b,\ c,\ d,\ e$ and f are non-negative real values, and the initial values $x_{-1},\ x_0,\ y_{-1},\ y_0,\ z_{-1},\ z_0$ are non-negative real values.

Keywords: difference equations in exponential form; stability character; periodicity; rate of convergence
Mathematics Subject Classification: 39A20, 39A10, 40A05

1. Introduction

Difference equations have a wide range of applications in math, physics and engineering, as well as in business and other professions. See [1–4] and the references given therein for a list of publications and books on difference equations theory and applications. The qualitative features of difference equations of exponential form have recently attracted a lot of attention [5–8]. The authors of [9] explored the difference equation's boundedness, asymptotic nature, periodicity of the solutions, and the stability of the non-negative equilibrium:

$$u_{n+1} = \alpha u_n + \beta u_{n-1}e^{-u_n}, \ \ n = 0, 1, \cdots ,$$

wherein α and β are non-negative constants, and the initial values $u_{-1}, \ u_0$ are non-negative numbers. Because it comes from models that investigate the amount of litter in perennial grasslands, this equation can be called a biological model. The authors of [10] looked at similar conclusions for a system of difference equations:

$$u_{n+1} = \alpha v_n + \beta u_{n-1}e^{-v_n}, \ v_{n+1} = \gamma u_n + \delta v_{n-1}e^{-u_n},$$

wherein $\alpha, \ \beta, \ \gamma$ and δ are non-negative constants, and the initial values $u_{-1}, \ u_0, \ v_{-1}, \ v_0$ are non-negative numbers. In addition, the researcher examines the character of boundedness, persistence, and asymptotic nature of the non-negative solutions of the subsequent exponential difference equations in [1]:

$$u_{n+1} = \alpha u_n + \beta v_{n-1}e^{-u_n}, \ v_{n+1} = \gamma v_n + \delta u_{n-1}e^{-v_n},$$

wherein $\alpha, \ \beta, \ \gamma$ and δ are non-negative constants, and the initial values $u_{-1}, \ u_0, \ v_{-1}, \ v_0$ are also non-negative numbers. We explore the character of boundedness, persistence, and the convergence rate of the non-negative outcomes of (1.1) to the unique positive equilibrium point of the subsequent exponential difference equations, motivated by the studies mentioned above:

$$\begin{aligned} x_{n+1} &= ax_n + by_{n-1}e^{-x_n}, \\ y_{n+1} &= cy_n + dz_{n-1}e^{-y_n}, \\ z_{n+1} &= ez_n + fx_{n-1}e^{-z_n}, \end{aligned} \tag{1.1}$$

where $a, \ b, \ c, \ d, \ e$ and f are non-negative real numbers, and the initial values $x_{-1}, \ x_0, \ y_{-1}, \ y_0, \ z_{-1}, \ z_0$ are also non-negative real numbers.

2. Existence and uniqueness of a positive equilibrium for (1.1)

We look at the existence and uniqueness of the non-negative equilibrium point of the system (1.1) in first theorem.

Theorem 2.1. *The foregoing claims are valid.*

(i) *Assume that*

$$a, \ b, \ c, \ d, \ e, \ f \in (0, \ 1), \ \theta = \frac{bdf}{(1-a)(1-c)(1-e)} > 1. \tag{2.1}$$

This leads to a unique equilibrium $(\overline{x},\ \overline{y},\ \overline{z})$ *for the system* (1.1). *However,*

$$\begin{aligned} \frac{\ln(\theta) - \overline{y}}{1 + \frac{f}{1-e}} &\le \overline{x} \le \ln\theta, \\ \frac{\ln(\theta) - \overline{z}}{1 + \frac{b}{1-a}} &\le \overline{y} \le \ln\theta, \\ \frac{\ln(\theta) - \overline{x}}{1 + \frac{d}{1-c}} &\le \overline{z} \le \ln\theta. \end{aligned} \tag{2.2}$$

(ii) *Assume that* $a,\ b,\ c,\ d,\ e,\ f$ *are positive real values such that*

$$a,\ b,\ c,\ d,\ e,\ f \in (0,\ 1),\ \theta \le 1. \tag{2.3}$$

The zero equilibrium $(0,\ 0,\ 0)$ *is the unique equilibrium solution of system (1.1).*

Proof. (i) Assume the following set of algebraic equations:

$$\begin{aligned} x &= ax + bye^{-x}, \\ y &= cy + dze^{-y}, \\ z &= ez + fxe^{-z}, \end{aligned}$$

or equivalently,

$$\begin{aligned} (1-a)\,x &= bye^{-x}, \\ (1-c)\,y &= dze^{-y}, \\ (1-e)\,z &= fxe^{-z}. \end{aligned} \tag{2.4}$$

Multiplying Eq (2.4),

$$\begin{aligned} (1-a)(1-c)(1-e)\,xyz &= bdfxyze^{-(x+y+z)}, \\ \frac{(1-a)(1-c)(1-e)}{bdf} &= e^{-(x+y+z)}, \end{aligned}$$

then,

$$\begin{aligned} e^{(x+y+z)} &= \frac{bdf}{(1-a)(1-c)(1-e)}, \\ x+y+z &= \ln\frac{bdf}{(1-a)(1-c)(1-e)}. \end{aligned}$$

Then, from Eq (2.4), if $x \neq 0,\ y \neq 0$ and $z \neq 0$, we get

$$x + y + z = \ln\theta. \tag{2.5}$$

From Eqs (2.4) and (2.5),

$$\frac{(1-a)\,xe^{x}}{b} = y,$$

$$\frac{(1-c)\,ye^{y}}{d}=z,$$
$$\frac{(1-e)\,ze^{z}}{f}=x.$$

Put value of $y=\frac{(1-a)xe^{x}}{b}$ in Eq (2.5), we get

$$x=\frac{\ln(\theta)-z}{1+\frac{1-a}{b}e^{x}}.$$

Now, put value of $\frac{(1-c)ye^{y}}{d}=z$ in Eq (2.5), we get

$$y=\frac{\ln(\theta)-x}{1+\frac{1-c}{d}e^{y}}.$$

Similarly, put $\frac{(1-e)ze^{z}}{f}=x$ in Eq (2.5), we get

$$z=\frac{\ln(\theta)-y}{1+\frac{1-e}{f}e^{z}}.$$

We consider the function:

$$F(x)=x-\frac{\ln(\theta)-z}{1+\frac{1-a}{b}e^{x}}.$$

So, from (2.1), we get that $F(0)<0$ and $\underset{x\to\infty}{Lim}F(x)=\infty$. Then, there exist a $\overline{x}\in(0,\ \infty)$ such that

$$\overline{x}=\frac{\ln(\theta)-\overline{z}}{1+\frac{1-a}{b}e^{\overline{x}}}. \tag{2.6}$$

Similarly, we can prove that there exists a $\overline{y}\in(0,\ \infty)$ and $\overline{z}\in(0,\ \infty)$ such that

$$\overline{y}=\frac{\ln(\theta)-\overline{x}}{1+\frac{1-c}{d}e^{\overline{y}}}, \tag{2.7}$$
$$\overline{z}=\frac{\ln(\theta)-\overline{y}}{1+\frac{1-e}{f}e^{\overline{z}}}. \tag{2.8}$$

To find z from $z=ez+fxe^{-z}$ as

$$z=\frac{f}{1-e}xe^{-z}. \tag{2.9}$$

So, from (2.5),

$$x+y+z=\ln\theta,$$
$$x+y+\frac{f}{1-e}xe^{-z}=\ln\theta,$$
$$x\left[1+\frac{f}{1-e}e^{-z}\right]=\ln(\theta)-y,$$

$$x = \frac{\ln(\theta) - y}{1 + \frac{f}{1-e}e^{-z}}.$$

Now, we will find $\overline{x}$ at $z = 0$:

$$\overline{x} = \frac{\ln(\theta) - \overline{y}}{1 + \frac{f}{1-e}}. \tag{2.10}$$

Similarly, we will prove that

$$\overline{y} = \frac{\ln(\theta) - \overline{z}}{1 + \frac{b}{1-a}}, \tag{2.11}$$

$$\overline{z} = \frac{\ln(\theta) - \overline{x}}{1 + \frac{d}{1-c}}. \tag{2.12}$$

Therefore, from (2.1), (2.5) and combining with Eqs (2.6) and (2.10), we obtained

$$\frac{\ln(\theta) - \overline{y}}{1 + \frac{f}{1-e}} \leq \frac{\ln(\theta) - \overline{z}}{1 + \frac{1-a}{b}e^{\overline{x}}} = \overline{x} \leq \ln\theta.$$

In similar way, we obtained

$$\frac{\ln(\theta) - \overline{z}}{1 + \frac{b}{1-a}} \leq \frac{\ln(\theta) - \overline{x}}{1 + \frac{1-c}{d}e^{\overline{y}}} = \overline{y} \leq \ln\theta,$$

$$\frac{\ln(\theta) - \overline{x}}{1 + \frac{d}{1-c}} \leq \frac{\ln(\theta) - \overline{y}}{1 + \frac{1-e}{f}e^{\overline{z}}} = \overline{z} \leq \ln\theta.$$

And thus (2.2) holds. To demonstrate uniqueness, we suppose that another non-negative equilibrium $(\overline{x}_1, \overline{y}_1, \overline{z}_1)$ of (1.1) exists. We can assume that $\overline{x} < \overline{x}_1$ without losing generality. Then we obtain the following from (2.6):

$$\overline{x} = \frac{\ln\theta - \overline{z}}{1 + \frac{1-a}{b}e^{\overline{x}}} < \overline{x}_1 = \frac{\ln\theta - \overline{z}}{1 + \frac{1-a}{b}e^{\overline{x}_1}},$$

and so $e^{\overline{x}_1} \leq e^{\overline{x}}$, which is a contradiction.

So,

$$\overline{x} = \overline{x}_1,$$

similarly,

$$\overline{y} = \overline{y}_1 \textit{ and } \overline{z} = \overline{z}_1.$$

The proof is now completed. □

Proof. (*ii*) Since (2.3) is still valid, then we can deduce from (2.5) that $x + y + z \leq 0$, implying that $(0, 0, 0)$ is the only non-negative equilibrium point. The proof is now finished. □

3. Boundedness and persistence of the positive solutions of (1.1)

We explore the boundedness and persistence of the non-negative solutions of (1.1) in the next proposition.

Theorem 3.1. *There are valid arguments for the following:*

(i) *Assume that*

$$a,\ b,\ c,\ d,\ e,\ f \in (0,1)\,. \tag{3.1}$$

Then, every positive solution of (1.1) *is bounded.*

(ii) *Suppose that* (3.1) *holds. Suppose also that*

$$\frac{b}{1-a} > 1,\ \frac{d}{1-c} > 1,\ \frac{f}{1-e} > 1. \tag{3.2}$$

Then, every positive solution of (1.1) *is bounded and persists.*

Proof. Suppose that $(x_n,\ y_n,\ z_n)$ be an arbitrarily solution to (1.1).

(i) We will assume M is positive, such that

$$M \geq \max\left\{x_{-1},\ y_{-1},\ z_{-1},\ x_0,\ y_0,\ z_0,\ \ln\left(\frac{1}{1-a}\right),\ \ln\left(\frac{1}{1-c}\right),\ \ln\left(\frac{1}{1-e}\right)\right\}. \tag{3.3}$$

The following function is considered:

$$h(x) = Me^{-x} + ax,\ x \in [0,\ M].$$

There is also that

$$h^{'}(x) = -Me^{-x} + a,\ h^{''}(x) = Me^{-x} > 0.$$

In light of this, it follows

$$h(x) \leq \max\{h(0)\,,\ h(M)\},\ x \in [0,\ M]. \tag{3.4}$$

Furthermore, we can deduce the following from (3.3):

$$h(0) = M,\ h(M) = Me^{-M} + aM < Me^{-\ln\left(\frac{1}{1-a}\right)} + aM = M(1-a) + aM = M. \tag{3.5}$$

From (3.4) and (3.5), we get that

$$h(x) \leq M,\ x \in [0,\ M]. \tag{3.6}$$

As a result of relations (1.1), (3.1), (3.3) and (3.6), it follows that

$$x_1 = ax_0 + by_{-1}e^{-x_0} \leq ax_0 + Me^{-x_0} = h(x_0) \leq M.$$

So, $x_1 \leq M$.

Now, consider the function

$$\begin{aligned} K(y) &= cy + Me^{-y},\ y \in [0,\ M], \\ K'(y) &= c - Me^{-y}, \\ K''(y) &= Me^{-y} > 0. \end{aligned}$$

Therefore, it holds that

$$K(y) \le \max\{K(0), K(M)\},\ y \in [0,\ M]. \tag{3.7}$$

Now, from (3.3),

$$\begin{aligned} &K(0) = M, \\ &K(M) = cM + Me^{-M} < cM + Me^{-\ln\left(\frac{1}{1-c}\right)} = cM + M(1-c) = M. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8),

$$K(y) \le M,\quad y \in [0,\ M].$$

Therefore, relations (1.1), (3.1), (3.3) and (3.8),

$$y_1 = cy_0 + dz_{-1}e^{-y_0} \le cy_0 + Me^{-y_0} = K(y_0) \le M,\ y_1 \le M.$$

Similarly, if

$$g(z) = ez + Me^{-z},$$

then, using the same logic as before, we can show that

$$z_1 \le M.$$

As a result of our inductive reasoning, we can demonstrate

$$\begin{aligned} x_n &\le M,\quad n = 1, 2, 3, \cdots, \\ y_n &\le M,\quad n = 1, 2, 3, \cdots, \\ z_n &\le M,\quad n = 1, 2, 3, \cdots. \end{aligned}$$

So, we conclude from above results $(x_n,\ y_n,\ z_n)$ is bounded. □

Proof. (ii) We can show that $(x_n,\ y_n,\ z_n)$ persists. We look at the numbers for this.

$$R = \ln(b/(1-a)),\ S = \ln(d/(1-c)),\ T = \ln(f/(1-e)). \tag{3.9}$$

Let

$$m = \min\{x_{-1},\ x_0,\ y_{-1},\ y_0,\ z_{-1},\ z_0,\ R,\ S,\ T\}.$$

Then, using (3.2) and arguing as in the proof of (3.1) of [10], we get the following:

If $x_0 \le R$, then,

$$x_1 \ge \min\{x_0,\ y_{-1}\}.$$

In addition, if $x_0 > R,\ y_{-1} \le R$, we take

$$x_1 > y_{-1}.$$

Finally, if $x_0 > R,\ y_{-1} > R$, we get

$$x_1 > R.$$

So, here is what we've got:

$$x_1 \geq m.$$

In a similar manner, we can demonstrate that

$$y_1 \geq m,\ z_1 \geq m.$$

We may prove the following by arguing like we did earlier and using the induction method:

$$x_n \geq m,\ y_n \geq m,\ z_n \geq m.$$

This completes the proof. □

4. Convergence rate

We evaluate the convergence rate of a system (1.1) for all initial values that converge at equilibrium $E(\overline{x},\ \overline{y},\ \overline{z})$ in this segment by existing theory [11]. For various three-dimensional systems, the convergence rate of solutions that converge to an equilibrium has been determined.

Theorem 4.1. *Assume systems* (3.1) *and* (3.2) *hold and*

$$\max\left\{\frac{f}{1-e},\ \frac{d}{1-c},\ \frac{b}{1-a}\right\} < \min\left\{e^{\frac{e}{f}},\ e^{\frac{c}{d}},\ e^{\frac{a}{b}}\right\}. \tag{4.1}$$

Then, each non-negative solution of (1.1) tends to the unique non-negative equilibrium of (1.1).

Proof. Consider $(x_n,\ y_n,\ z_n)$ be an arbitrary solution of (1.1). From Theorem 3.1 we get that

$$\begin{aligned} l_1 &= \underset{n\to\infty}{Lim}\inf x_n > 0, \quad L_1 = \underset{n\to\infty}{Lim}\sup x_n < \infty, \\ l_2 &= \underset{n\to\infty}{Lim}\inf y_n > 0, \quad L_2 = \underset{n\to\infty}{Lim}\sup y_n < \infty, \\ l_3 &= \underset{n\to\infty}{Lim}\inf z_n > 0, \quad L_3 = \underset{n\to\infty}{Lim}\sup z_n < \infty. \end{aligned} \tag{4.2}$$

Then, from (4.2) and for every $\epsilon > 0$, there exist an $n_0(\epsilon)$ such that $n \geq n_0(\epsilon)$,

$$\begin{aligned} l_1 - \epsilon &\leq x_n \leq L_1 + \epsilon, \\ l_2 - \epsilon &\leq y_n \leq L_2 + \epsilon, \\ l_3 - \epsilon &\leq z_n \leq L_3 + \epsilon, \end{aligned} \tag{4.3}$$

and so from (1.1) and (4.3) we have for $n \geq n_0$ that

$$x_{n+2} = ax_{n+1} + by_n e^{-x_{n+1}} \leq ax_{n+1} + b(L_2+\epsilon)e^{-x_{n+1}} = g_{L_2+\epsilon}(x_{n+1}), \tag{4.4}$$

where $g_{L_2+\epsilon}(x) = ax + b(L_2+\epsilon)e^{-x}$.

But we have that

$$\begin{aligned} g^{'}_{L_2+\epsilon}(x) &= a - b(L_2+\epsilon)e^{-x}, \\ g^{''}_{L_2+\epsilon}(x) &= b(L_2+\epsilon)e^{-x} > 0. \end{aligned}$$

Therefore, for $x \in [l_1 - \epsilon,\ L_1 + \epsilon]$, we get that

$$g_{L_2+\epsilon}(x) \le \max\left\{g_{L_2+\epsilon(l_1-\epsilon)},\ g_{L_2+\epsilon(L_1+\epsilon)}\right\}.$$

Then, from (4.4) we take the following:

$$x_{n+2} \le g_{L_2+\epsilon}(x_{n+1}) \le \max\left\{g_{L_2+\epsilon(l_1-\epsilon)},\ g_{L_2+\epsilon(L_1+\epsilon)}\right\},$$

which implies that

$$L_1 \le \max\left\{g_{L_2+\epsilon(l_1-\epsilon)},\ g_{L_2+\epsilon(L_1+\epsilon)}\right\}.$$

So, for $\epsilon \longrightarrow 0$,

$$L_1 \le \max\left\{g_{L_2(l_1)},\ g_{L_2(L_1)}\right\}. \tag{4.5}$$

Similarly, from (1.1) and (4.3) we have for $n \ge n_0$ that

$$y_{n+2} = cy_{n+1} + dz_n e^{-y_{n+1}} \le cy_{n+1} + d(L_3 + \epsilon)e^{-y_{n+1}} = h_{L_3+\epsilon}(y_{n+1}), \tag{4.6}$$

where $h_{L_3+\epsilon}(y) = cy + d(L_3 + \epsilon)e^{-y}$.

But we have that

$$h'_{L_3+\epsilon}(y) = c - d(L_3 + \epsilon)e^{-y},$$
$$h''_{L_3+\epsilon}(y) = d(L_3 + \epsilon)e^{-y} > 0.$$

Therefore, for $y \in [l_2 - \epsilon,\ L_2 + \epsilon]$, we get that

$$h_{L_3+\epsilon}(y) \le \max\left\{h_{L_3+\epsilon}(l_2 - \epsilon),\ h_{L_3+\epsilon}(L_2 + \epsilon)\right\}.$$

Then, from (4.6) we take

$$y_{n+2} \le h_{L_3+\epsilon}(y_{n+1}) \le \max\left\{h_{L_3+\epsilon}(l_2 - \epsilon),\ h_{L_3+\epsilon}(L_2 + \epsilon)\right\},$$

which implies that

$$L_2 \le \max\left\{h_{L_3+\epsilon}(l_2 - \epsilon),\ h_{L_3+\epsilon}(L_2 + \epsilon)\right\}.$$

So, for $\epsilon \longrightarrow 0$,

$$L_2 \le \max\left\{h_{L_3}(l_2),\ h_{L_3}(L_2)\right\}. \tag{4.7}$$

Similarly, from (1.1) and (4.3) we have for $n \ge n_0$ that if

$$K_{L_1+\epsilon}(z) = ez + f(L_1 + \epsilon)e^{-z},$$

we can prove that

$$L_3 \le \max\left\{K_{L_1}(l_3),\ K_{L_1}(L_3)\right\}. \tag{4.8}$$

We claim that

$$l_1 > \ln\left(\frac{bL_2}{a}\right),\ l_2 > \ln\left(\frac{dL_3}{c}\right),\ l_3 > \ln\left(\frac{fL_1}{e}\right). \tag{4.9}$$

Suppose on contrary, that either

$$l_1 \leq \ln\left(\frac{bL_2}{a}\right) \tag{4.10}$$

or

$$l_2 \leq \ln\left(\frac{dL_3}{c}\right) \tag{4.11}$$

or

$$l_3 \leq \ln\left(\frac{fL_1}{e}\right). \tag{4.12}$$

Suppose first that (4.10) valid. Then, since $g'_{L_2}(x) = a - bL_2e^{-x}$, we have that g_{L_2} is a non-increasing for $x \leq \ln\left(\frac{bL_2}{a}\right)$ and consequently we obtained from (4.10) that

$$g_{L_2}(l_1) \leq g_{L_2}(0) = bL_2 < L_2. \tag{4.13}$$

Then, from (4.5) and (4.13) we have that

$$L_1 \leq \max\{L_2,\ g_{L_2}(L_1)\}. \tag{4.14}$$

Since it hold that $h'_{L_3}(y) = c - dL_3e^{-y}$, we conclude that h_{L_3} is non-increasing function for $y \leq \ln\left(\frac{dL_3}{c}\right)$ and non-decreasing for $y \geq \ln\left(\frac{dL_3}{c}\right)$. Then, if $l_2 \geq \ln(\frac{dL_3}{c})$, we have that

$$h_{L_3}(l_2) < h_{L_3}(L_2). \tag{4.15}$$

If $l_2 \leq \ln\left(\frac{dL_3}{c}\right)$, we get that

$$h_{L_3}(l_2) < h_{L_3}(0) = dL_3 < L_3. \tag{4.16}$$

Relations (4.7), (4.15) and (4.16) imply that

$$L_2 \leq \max\{L_3,\ h_{L_3}(L_2)\}. \tag{4.17}$$

Similarly, if (4.12) holds, then,

$$L_3 \leq \max\{L_1,\ K_{L_1}(L_3)\}. \tag{4.18}$$

Suppose now that $L_1 \leq L_2 \leq L_3$. Then, from (4.18), we get that

$$\begin{aligned} L_3 &\leq K_{L_1}(L_3) = eL_3 + fL_1e^{-L_3}, \\ L_3 &\leq eL_3 + fL_3e^{-L_3}, \end{aligned}$$

which implies that

$$L_3 \leq \ln\left(\frac{f}{1-e}\right). \tag{4.19}$$

Since (4.10) holds. We get that

$$l_1 \leq \ln\left(\frac{bL_2}{a}\right),$$

$$\begin{aligned} l_1 &\leq \ln\left(\frac{bL_3}{a}\right), \because L_2 \leq L_3, \\ e^{l_1} &\leq \frac{bL_3}{a}, \\ 1 + l_1 &\leq e^{l_1} \leq \frac{bL_3}{a}, \\ 1 + l_1 &\leq \frac{bL_3}{a}, \end{aligned}$$

and so (4.19) implies that

$$l_1 \leq \frac{bL_3}{a} - 1 \leq \frac{b}{a} \ln\left(\frac{f}{1-e}\right) - 1. \tag{4.20}$$

We get the following from (4.1):

$$\frac{f}{1-e} < e^{\frac{a}{b}},$$

and so

$$\frac{b}{a} \ln\left(\frac{f}{1-e}\right) - 1 < 0.$$

Then, from (4.20) we have that $l_1 < 0$, which is a contradiction. So, (4.10) is not true if $L_1 \leq L_2 \leq L_3$.

Suppose now that $L_1 \leq L_3 \leq L_2$. Then from (4.17) we take that

$$\begin{aligned} L_2 &\leq h_{L_3}(L_2) = cL_2 + dL_3 e^{-L_2}, \\ L_2 &\leq cL_2 + dL_2 e^{-L_2}, \end{aligned}$$

which implies that

$$L_2 \leq \ln\left(\frac{d}{1-c}\right). \tag{4.21}$$

Since (4.10) holds. We get that

$$\begin{aligned} l_1 &\leq \ln\left(\frac{bL_2}{a}\right), \\ e^{l_1} &\leq \frac{bL_2}{a}, \\ 1 + l_1 &\leq e^{l_1} \leq \frac{bL_2}{a}, \\ 1 + l_1 &\leq \frac{bL_2}{a}, \end{aligned}$$

and so (4.21) implies that

$$l_1 \leq \frac{bL_2}{a} - 1 \leq \frac{b}{a} \ln\left(\frac{d}{1-c}\right) - 1. \tag{4.22}$$

Moreover, from (4.1) we get

$$\frac{d}{1-c} < e^{\frac{a}{b}},$$

so,

$$\frac{b}{a}\ln\left(\frac{d}{1-c}\right)-1<0.$$

Then, from (4.22) we have that $l_1 < 0$, which is a contradiction. So, (4.10) is not true if $L_1 \le L_3 \le L_2$.

Now again suppose that $L_2 \le L_3 \le L_1$. Then, from (4.14) we take that

$$\begin{aligned} L_1 &\le g_{L_2}(L_1) = aL_1 + bL_2e^{-L_1}, \\ L_1 &\le aL_1 + bL_1e^{-L_1}, \end{aligned}$$

and so

$$L_1 \le \ln\left(\frac{b}{1-a}\right). \tag{4.23}$$

Since (4.10) holds, we get

$$\begin{aligned} l_1 &\le \ln\left(\frac{bL_2}{a}\right), \\ l_1 &\le \ln\left(\frac{bL_1}{a}\right), \quad \because L_2 \le L_1, \\ e^{l_1} &\le \frac{bL_1}{a}, \\ 1+l_1 &\le e^{l_1} \le \frac{bL_1}{a}, \end{aligned}$$

and so (4.23) implies that

$$l_1 \le \frac{bL_1}{a} - 1 \le \frac{b}{a}\ln\left(\frac{b}{1-a}\right) - 1. \tag{4.24}$$

We get from (4.1) that

$$\frac{b}{1-a} < e^{\frac{a}{b}},$$

and so

$$\frac{b}{a}\ln\left(\frac{b}{1-a}\right)-1<0.$$

Then, from (4.24) we have that $l_1 < 0$, which is a contradiction. So, (4.10) is not true if $L_2 \le L_3 \le L_1$.

Now again suppose that $L_2 \le L_1 \le L_3$. Then, from (4.18) we take that

$$\begin{aligned} L_3 &\le K_{L_1}(L_3) = eL_3 + fL_1e^{-L_3}, \\ L_3 &\le eL_3 + fL_3e^{-L_3}, \end{aligned}$$

and so

$$L_3 \le \ln\left(\frac{f}{1-e}\right). \tag{4.25}$$

Since (4.10) holds. We get

$$l_1 \le \ln\left(\frac{bL_2}{a}\right),$$

$$\begin{aligned} l_1 &\leq \ln\left(\frac{bL_3}{a}\right), \quad \because L_2 \leq L_3, \\ e^{l_1} &\leq \frac{bL_3}{a}, \\ 1 + l_1 &\leq e^{l_1} \leq \frac{bL_3}{a}, \end{aligned}$$

and so (4.25) implies that

$$l_1 \leq \frac{bL_3}{a} - 1 \leq \frac{b}{a}\ln\left(\frac{f}{1-e}\right) - 1. \tag{4.26}$$

We get from (4.1) that

$$\frac{f}{1-e} < e^{\frac{a}{b}},$$

and so

$$\frac{b}{a}\ln\left(\frac{f}{1-e}\right) - 1 < 0.$$

Then, from (4.26) we have that $l_1 < 0$, which is a contradiction. So, (4.10) is not true if $L_2 \leq L_1 \leq L_3$.

Now again suppose that $L_3 \leq L_2 \leq L_1$. Then, from (4.14) we take that

$$\begin{aligned} L_1 &\leq g_{L_2}(L_1) = aL_1 + bL_2 e^{-L_1}, \\ L_1 &\leq aL_1 + bL_1 e^{-L_1}, \end{aligned}$$

and so

$$L_1 \leq \ln\left(\frac{b}{1-a}\right). \tag{4.27}$$

Since (4.10) holds. We get

$$\begin{aligned} l_1 &\leq \ln\left(\frac{bL_2}{a}\right), \\ l_1 &\leq \ln\left(\frac{bL_1}{a}\right), \ \because L_2 \leq L_1, \\ e^{l_1} &\leq \frac{bL_1}{a}, \\ 1 + l_1 &\leq e^{l_1} \leq \frac{bL_1}{a}. \end{aligned}$$

Therefore, (4.27) implies that

$$l_1 \leq \frac{bL_1}{a} - 1 \leq \frac{b}{a}\ln\left(\frac{b}{1-a}\right) - 1. \tag{4.28}$$

We get from (4.1) that

$$\frac{b}{1-a} < e^{\frac{a}{b}},$$

and so

$$\frac{b}{a}\ln\left(\frac{b}{1-a}\right) - 1 < 0.$$

Then, from (4.28) we have that $l_1 < 0$, which is a contradiction. So, (4.10) is not true if $L_3 \leq L_2 \leq L_1$.

Now again suppose that $L_3 \leq L_1 \leq L_2$. Then, from (4.17) we take that

$$\begin{aligned} L_2 &\leq h_{L_3}(L_2) = cL_2 + dL_3 e^{-L_2}, \\ L_2 &\leq cL_2 + dL_2 e^{-L_2}. \end{aligned}$$

Therefore,

$$L_2 \leq \ln\left(\frac{d}{1-c}\right). \tag{4.29}$$

Since (4.10) holds, we get

$$\begin{aligned} l_1 &\leq \ln\left(\frac{bL_2}{a}\right), \\ e^{l_1} &\leq \frac{bL_2}{a}, \\ 1 + l_1 &\leq e^{l_1} \leq \frac{bL_2}{a}, \end{aligned}$$

and so (4.29) implies that

$$l_1 \leq \frac{bL_2}{a} - 1 \leq \frac{b}{a}\ln\left(\frac{d}{1-c}\right) - 1. \tag{4.30}$$

We get from (4.1) that

$$\frac{d}{1-c} < e^{\frac{a}{b}},$$

and so

$$\frac{b}{a}\ln\left(\frac{d}{1-c}\right) - 1 < 0.$$

Then, from (4.30) we have that $l_1 < 0$, which is a contradiction. So, (4.10) is not true if $L_3 \leq L_1 \leq L_2$.

Working in a similar manner and using (4.1), we can prove that (4.11) and (4.12) are not true for each:

$$\begin{aligned} &L_1 \leq L_2 \leq L_3, \\ &L_1 \leq L_3 \leq L_2, \\ &L_2 \leq L_3 \leq L_1, \\ &L_2 \leq L_1 \leq L_3, \\ &L_3 \leq L_2 \leq L_1, \\ &L_3 \leq L_1 \leq L_2. \end{aligned}$$

So relations (4.9) are satisfied.

Since relations (4.9) hold, g_{L_2} is an increasing function for $x \geq \ln\left(\frac{bL_2}{a}\right)$, h_{L_3} is an increasing function for $y \geq \ln\left(\frac{dL_3}{c}\right)$, and K_{L_1} is also an increasing function for $z \geq \ln\left(\frac{fL_1}{e}\right)$. We then obtain

$$g_{L_2}(l_1) \leq g_{L_2}(L_1),$$

$$h_{L_3}(l_2) \le h_{L_3}(L_2),\quad (4.31)$$
$$K_{L_1}(l_3) \le K_{L_1}(L_3).$$

So, from (4.5), (4.7), (4.8) and (4.31) we have that

$$\begin{aligned} L_1 &\le g_{L_2}(L_1), \\ L_2 &\le h_{L_3}(L_2), \\ L_3 &\le K_{L_1}(L_3). \end{aligned} \quad (4.32)$$

Then relations (4.32) imply that

$$\begin{aligned} \frac{(1-a)L_1 e^{L_1}}{b} &\le L_2, \\ \frac{(1-c)L_2 e^{L_2}}{d} &\le L_3, \\ \frac{(1-e)L_3 e^{L_3}}{f} &\le L_1, \end{aligned}$$

as a result of (2.4), we can simply deduce

$$F(L_1) \le 0 = F(\overline{x}).$$

As F is a non-decreasing function, we obtain

$$L_1 \le \overline{x}. \quad (4.33)$$

The following can be proved in a similar way:

$$\begin{aligned} G(L_2) &\le 0 = G(\overline{y}), \\ H(L_3) &\le 0 = H(\overline{z}), \end{aligned}$$

where

$$H(z) = \frac{(1-a)(1-c)(1-e)\, e^{z+s(z)+r(x)}}{bdf} - 1,\ r(x) = \frac{(1-a)\, xe^{x}}{b},\ s(z) = \frac{(1-e)\, ze^{z}}{f}.$$

Due to the fact that G is a non-decreasing function, we obtain

$$L_2 \le \overline{y}. \quad (4.34)$$

Similarly, H is a non-decreasing function, we get

$$L_3 \le \overline{z}. \quad (4.35)$$

We can now demonstrate that

$$\overline{x} < l_1 < L_1,\ \overline{y} < l_2 < L_2,\ \overline{z} < l_3 < L_3. \quad (4.36)$$

We derive the following from (1.1) and (4.3):

$$x_{n+1} \geq ax_n + b(l_2 - \epsilon) e^{-x_n}, \quad n \geq n_o(\epsilon). \tag{4.37}$$

We look at the following function:

$$g_{l_2-\epsilon}(x) = ax + b(l_2 - \epsilon) e^{-x},$$
$$g'_{l_2-\epsilon}(x) = a - b(l_2 - \epsilon) e^{-x}.$$

We have that $g_{l_2-\epsilon}$ is non-decreasing for $x \geq \ln\left(\frac{b(l_2-\epsilon)}{a}\right)$. In addition, since (4.9) valid, then there exists $\epsilon > 0$ such that

$$l_1 - \epsilon > \ln\left(\frac{bL_2}{a}\right) > \ln\left(\frac{b(L_2 - \epsilon)}{a}\right). \tag{4.38}$$

Then, from (4.3) and (4.33) we get

$$x_n \geq l_1 - \epsilon > \ln\left(\frac{b(L_2 - \epsilon)}{a}\right) \geq \ln\left(\frac{b(l_2 - \epsilon)}{a}\right), \ n \geq n_0(\epsilon). \tag{4.39}$$

As a result, relations (4.37) and (4.39) indicate the following:

$$x_{n+1} \geq a(l_1 - \epsilon) + b(l_2 - \epsilon) e^{-(l_1-\epsilon)}, \ n \geq n_0(\epsilon).$$

And so,

$$l_1 \geq a(l_1 - \epsilon) + b(l_2 - \epsilon) e^{-(l_1-\epsilon)}.$$

For $\epsilon \longrightarrow 0$, we get

$$l_1 \geq al_1 + bl_2 e^{-l_1}. \tag{4.40}$$

Similarly, using (1.1) and (4.9) and arguing as above, we get

$$l_2 \geq cl_2 + dl_3 e^{-l_2}, \tag{4.41}$$

and

$$l_3 \geq el_3 + fl_1 e^{-l_3}. \tag{4.42}$$

Therefore, from relations (4.33)–(4.35) and (4.40)–(4.42), we have that

$$l_1 = L_1 = \overline{x}, \ l_2 = L_2 = \overline{y}, \ l_3 = L_3 = \overline{z}.$$

This completes the proof. □

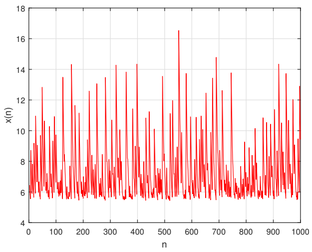
5. Numerical examples

In an effort to our theoretical dialogue, we take into account several interesting numerical examples on this segment. These examples constitute distinct varieties of qualitative conduct of solutions to the system (1.1) of nonlinear difference equations. The first example indicates that positive equilibrium of system (1.1) is unstable with suitable parametric choices. Moreover, from the remaining examples it is clear that unique positive equilibrium point of system (1.1) is globally asymptotically stable with different parametric values. All plots on this segment are drawn with the help of MATLAB.

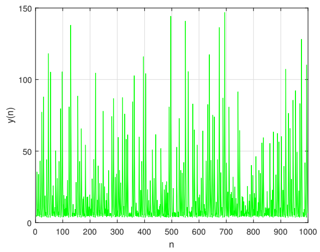
Example 5.1. Let $a = 0.9,\ b = 27,\ c = 0.5,\ d = 94,\ e = 0.3,\ f = 67$. Then the system (1.1) can be written as

$$x_{n+1} = 0.9x_n + 27y_{n-1}e^{-x_n},\ y_{n+1} = 0.5y_n + 94z_{n-1}e^{-y_n},\ z_{n+1} = 0.3z_n + 67x_{n-1}e^{-z_n}, \tag{5.1}$$

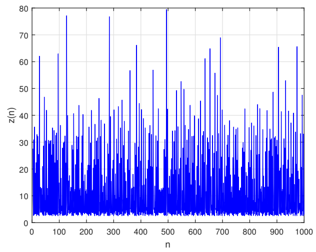
with initial conditions $x_{-1} = 8,\ x_0 = 7,\ y_{-1} = 6,\ y_0 = 5,\ z_{-1} = 4,\ z_0 = 3$. In this case, the positive equilibrium point of the system (5.1) is unstable. Moreover, in Figure 1, the graphs of x_n, y_n and z_n are shown in Figure (1a), (1b) and (1c) respectively, and XY, YZ and ZX attractors of the system (5.1) are shown in Figure (1d), (1e) and (1f) respectively. Also the combined graph of all respective phase portrait of system (5.1) is shown in Figure (1g).



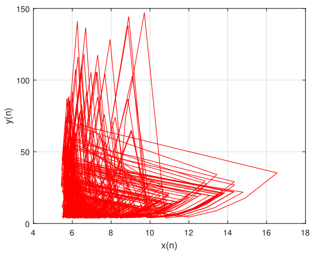
(*a*) Graph of x_n for system (5.1)



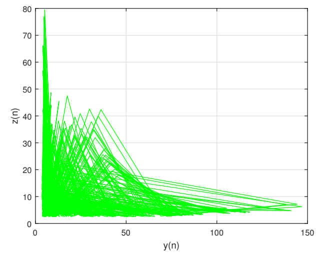
(*b*) Graph of y_n for system (5.1)



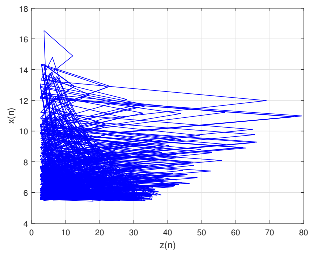
(*c*) Graph of z_n for system (5.1)



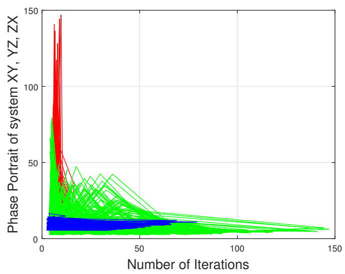
(*d*) XY- Attractor of system (5.1)



(*e*) YZ- Attractor of system (5.1)



(*f*) ZX- Attractor of system (5.1)



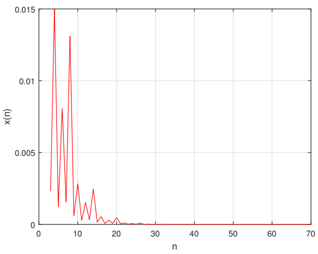
(*g*) Combined graph of attractors of system (5.1)

Figure 1. Shows solution and phase portraits of system (5.1).

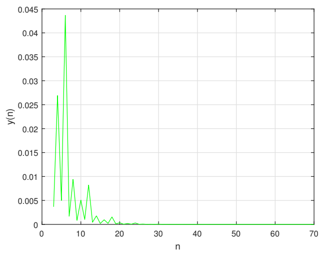
Example 5.2. Let $a = 0.009,\ b = 0.3,\ c = 0.005,\ d = 0.9,\ e = 0.003,\ f = 0.7$. Then the system (1.1) can be written as

$$x_{n+1} = 0.009x_n + 0.y_{n-1}e^{-x_n},\ y_{n+1} = 0.005y_n + 0.9z_{n-1}e^{-y_n},\ z_{n+1} = 0.003z_n + 0.7x_{n-1}e^{-z_n}, \qquad (5.2)$$

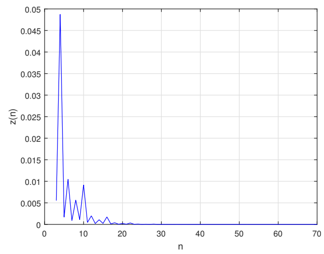
with initial conditions $x_{-1} = 0.008,\ x_0 = 0.007,\ y_{-1} = 0.006,\ y_0 = 0.005,\ z_{-1} = 0.004,\ z_0 = 0.03$. In this case, the positive equilibrium point of the system (5.2) is given by $(\overline{x},\ \overline{y},\ \overline{z}) = (9.404 \times 10^{-11},\ 4.993 \times 10^{-10},\ 1.276 \times 10^{-10})$. Moreover, in Figure 2, the graphs of x_n, y_n and z_n are shown in Figure (2a), (2b) and (2c) respectively, and XY, YZ and ZX attractors of the system (5.1) are shown in Figure (2d), (2e) and (2f) respectively. Also the combined graph of all respective phase portrait of system (5.2) is shown in Figure (2g).



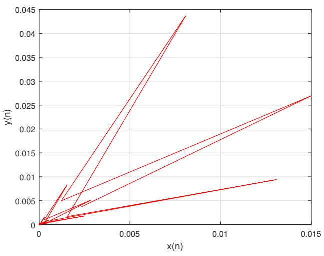
(a) Graph of x_n for system (5.2)



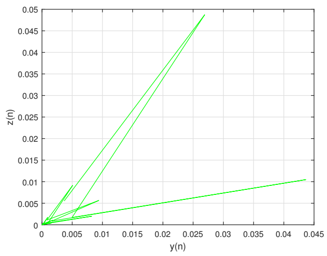
(b) Graph of y_n for system (5.2)



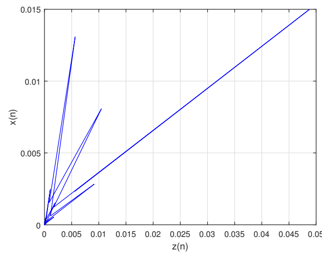
(c) Graph of z_n for system (5.2)



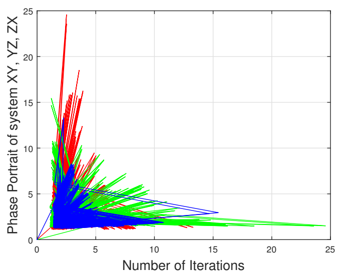
(d) XY- Attractor of system (5.2)



(e) YZ- Attractor of system (5.2)



(f) ZX- Attractor of system (5.2)



(g) Combined graph of attractors of system (5.2)

Figure 2. Shows solution and phase portraits of system (5.2).

6. Conclusions

In this work, we analyze the qualitative behavior of a system of exponential difference equations. Using our model (1.1), we have demonstrated that a positive steady state exists and is unique. We verify the bounds of positive solutions as well as their persistence. We have also established that the positive equilibrium point of system (1.1) under certain parametric conditions is asymptotically stable locally as well as globally. In dynamical structures theory, the goal is to look at a system's global behavior through knowledge of its current state. It is possible to determine what parametric conditions result in these long-term behaviors by determining the possible global behavior of the system. Further, the convergence rate of positive solutions of (1.1) that converge to a unique point of positive equilibrium is determined.

In our future work, we will study some more qualitative properties such as bifurcation analysis, chaos control, and Maximum Lyapunov exponent of the said model. Some interesting numerical simulations with the help of Mathematica presenting bifurcation and chaos control are also part of our future goal.

Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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