Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 5007–5015.
DOI:10.3934/math.2023250
Received: 09 September 2022
Revised: 19 November 2022
Accepted: 01 December 2022
Published: 12 December 2022

Research article

Sharp weak bounds for discrete Hardy operator on discrete central Morrey spaces

Mingquan Wei* **and Xiaoyu Liu**

School of Mathematics and Stastics, Xinyang Normal University, Xinyang, Henan 464000, China

* **Correspondence:** Email: weimingquan11@mails.ucas.ac.cn.

Abstract: In this note, we introduce the discrete (weak) central Morrey spaces, which are central versions of discrete (weak) Morrey spaces. The sharp bounds for discrete Hardy operator from discrete central Morrey spaces to discrete weak central Morrey spaces are proven to be equal to 1. As an application, we obtain the weak version of the well-known discrete Hardy inequality.

Keywords: discrete Hardy operator; sharp bound; discrete central Morrey space; discrete Lebesgue space
Mathematics Subject Classification: 42B20, 42B25, 42B35

1. Introduction

Let f be a locally integrable function on $(0, \infty)$. The classical Hardy operator H is defined by

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)dt,$$

where $x \in (0, \infty)$.

The well-known Hardy inequality yields that for $1 < p < \infty$ and any locally integrable function f on $(0, \infty)$ such that

$$\|f\|_{L^p(0,\infty)} = \left(\int_0^\infty |f(x)|^p dx\right)^{1/p} < \infty,$$

there holds

$$\|H(f)\|_{L^p(0,\infty)} \le \frac{p}{p-1}\|f\|_{L^p(0,\infty)},$$

where the constant $\frac{p}{p-1}$ is optimal, see [1] for the details.

In 1976, the Hardy operator was extended to higher dimension by Faris [2]. Later, Christ and Grafakos [3] obtained a refined definition of the n-dimensional Hardy operator

$$\mathcal{H}(f)(x) = \frac{1}{\Omega_n |x|^n} \int_{|y|<|x|} f(y)dy, \; x \in \mathbb{R}^n \backslash \{0\},$$

where f is a locally measurable function on $\mathbb{R}^n$ and $\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$. Moreover, Christ and Grafakos proved that the operator norm of $\mathcal{H}$ on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ is also $\frac{p}{p-1}$. Since then, the sharp estimates for Hardy-type operators have been built on different function spaces. For example, Lu et al. [4], Wang et al. [5], Wei and Yan [6] established the sharp estimates for product Hardy-type operators on product spaces. Moreover, Fu et al. [7] derived the sharp constants for linear and multilinear Hardy operators on Morrey-type spaces. Other than the Euclid space, the sharp constants for Hardy-type operators were also established on p-adic field [8–13], Heisenberg group [14–16], and q-calculus [17–19].

Note that, in [20], Zhao et al. obtained the sharp weak (p, p) bound for $\mathcal{H}$. We give the result in details for the reader's convenience.

Theorem 1.1. *For* $1 \le p \le \infty$ *and* $f \in L^p(\mathbb{R}^n)$*, we have*

$$\|\mathcal{H}(f)\|_{L^{p,\infty}} \le 1 \cdot \|f\|_{L^p},$$

where the bound 1 *is best possible.*

In the above, for $0 < p < \infty$, the Lebesgue space $L^p(\mathbb{R}^n)$ contains all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty,$$

and the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$ consists of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{1/p} < \infty, \tag{1.1}$$

where $|E|$ denotes the Lebesgue measure of the measurable set $E \subseteq \mathbb{R}^n$. In addition, the space $L^\infty(\mathbb{R}^n)$ is the collection of all measurable functions f on $\mathbb{R}^n$ such that $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| < \infty$. We also denote by $L^{\infty,\infty}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

In fact, when $n = 1$, Theorem 1.1 implies that the sharp constant for H from $L^p(0, \infty)$ to $L^{p,\infty}(0, \infty)$ also equals to 1. Here the space $L^{p,\infty}(0, \infty)$ is defined by (1.1) except that we use $(0, \infty)$ instead of $\mathbb{R}^n$ there.

Recently, much attention has been paid to the sharp weak bounds for Hardy-type operators. For instance, Gao and Zhao [21], Gao et al. [22], Yu and Li [23] gave the sharp weak estimates for Hardy-type operators on (power-weighted) Lebesgue spaces. Moreover, Gao et al. [22] calculated the sharp constants for Hardy operator on central Morrey spaces. We also refer the readers to [24, 25] for the sharp weak bounds for p-adic Hardy-type operators on p-adic Lebesgue spaces.

On the other hand, the discrete version of Hardy inequality is also of great importance. The discrete Hardy inequality (see, for instance, [1, 26]) shows that if $p > 1$ and $\{a_k\}_1^\infty$ is a sequence of nonnegative real numbers, then

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \le \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.2}$$

Moreover, the constant in (1.2) is the best possible one.

Similar to the Lebesgue spaces in Euclidean space, the discrete Lebesgue space $l^p(\mathbb{N})$ (here and hereafter, $\mathbb{N}$ is the collection of all positive integers) consists of all sequences $\vec{a} = \{a_k\}_1^\infty$ of real numbers such that

$$\|\vec{a}\|_{l^p} = \left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p} < \infty \tag{1.3}$$

if $0 < p < \infty$, and for $p = \infty$, we denote by $l^\infty(\mathbb{N})$ the set of all sequences $\vec{a} = \{a_k\}_1^\infty$ of real numbers such that

$$\|\vec{a}\|_{l^\infty} = \sup_{n\in\mathbb{N}}\{|a_n|\} < \infty.$$

We also define the discrete version of Hardy operator $\mathbf{h}$ as

$$\mathbf{h}(\vec{a})(n) = \frac{1}{n}\sum_{k=1}^{n} a_k \tag{1.4}$$

for all senquences $\vec{a} = \{a_k\}_1^\infty$ of real numbers.

Using the notations in (1.3) and (1.4), the discrete Hardy inequality (1.2) can be rewritten as

$$\|\mathbf{h}(\vec{a})\|_{l^p} \leq \frac{p}{p-1}\|\vec{a}\|_{l^p}, \tag{1.5}$$

for all sequences $\vec{a} = \{a_k\}_1^\infty$ such that $\vec{a} \in l^p(\mathbb{N})$. Moreover, the constant in (1.5) is optimal.

In view of the sharp weak estimates for continuous Hardy operator H on Lebesgue spaces, it is natural to ask whether we can obtain the boundedness, or moreover, the sharp constants for discrete Hardy operator $\mathbf{h}$ from discrete Lebesgue spaces to discrete weak Lebesgue spaces. Here the discrete weak Lebesgue space $l^{p,\infty}(\mathbb{N})$ $(1 \leq p < \infty)$ consists of all sequences $\vec{a} = \{a_k\}_1^\infty$ of real numbers such that

$$\|\vec{a}\|_{l^{p,\infty}} = \sup_{\gamma>0}\gamma\ \left(\sharp\{n : |a_n| > \gamma\}\right)^{1/p} < \infty,$$

where $\sharp E$ denotes the number of integers of the set E. In addition, for $p = \infty$, we denote by $l^{\infty,\infty}(\mathbb{N}) = l^\infty(\mathbb{N})$.

In this paper, we will give an affirmative answer to the above question. More generally, we will consider the weak type estimates for discrete Hardy operator in a more general setting. To get a better understanding of our main result in this paper, we need the definition of the weak central Morrey space and its weak version.

The classical Morrey spaces, initially introduced by Morrey [27], play an important role in harmonic analysis and partial differential equations. Moreover, the central version of Morrey spaces are also well used to study the mapping properties of various important integral operators in harmonic analysis. We refer the readers to [28–34] for the studies of Morrey-type spaces. Note that in [35–37], Gunawan et al. initially introduced the discrete version of Morrey spaces and gave some basic properties for these spaces, see also [38] for the mapping properties of discrete Hardy-Littlewood maximal operator and discrete fractional integral operator on discrete Morrey spaces. Inspire by [37], we now give the definition of discrete central Morrey spaces as follows.

Definition 1.1. *Let* $1 \le p < \infty$ *and* $-\frac{1}{p} \le \lambda < 0$*. The discrete central Morrey space* $\dot{m}^{p,\lambda}(\mathbb{N})$ *is defined by*

$$\dot{m}^{p,\lambda}(\mathbb{N}) = \{\vec{a} = \{a_k\}_1^\infty : \|\vec{a}\|_{\dot{m}^{p,\lambda}} < \infty\},$$

where

$$\|\vec{a}\|_{\dot{m}^{p,\lambda}} = \sup_{n\in\mathbb{N}} n^{-(\lambda+1/p)} \left\|\vec{A^n}\right\|_{l^p},$$

in which $\vec{A^n} = \{A_k^n\}_1^\infty$ *is the n-truncation sequence of the sequence* $\vec{a} = \{a_k\}_1^\infty$*, i.e.,*

$$A_k^n = \begin{cases} a_k, & 1 \le k \le n, \\ 0, & k > n. \end{cases} \tag{1.6}$$

By definition, the discrete central Morrey spaces can be viewed as the central version of the discrete Morrey spaces studied in [37]. Similarly, one can define the discrete weak central Morrey space $w\dot{m}^{p,\lambda}(\mathbb{N})$:

Definition 1.2. *Let* $1 \le p < \infty$ *and* $-\frac{1}{p} \le \lambda < 0$*. The discrete weak central Morrey space* $w\dot{m}^{p,\lambda}(\mathbb{N})$ *is defined by*

$$w\dot{m}^{p,\lambda}(\mathbb{N}) = \{\vec{a} = \{a_k\}_1^\infty : \|a\|_{w\dot{m}^{p,\lambda}} < \infty\},$$

where

$$\|\vec{a}\|_{w\dot{m}^{p,\lambda}} = \sup_{n\in\mathbb{N}} n^{-(\lambda+1/p)} \left\|\vec{A^n}\right\|_{l^{p,\infty}},$$

in which $\vec{A^n}$ *is the same as in* (1.6).

Obviously, if we take $\lambda = -1/p$ in Definitions 1.1 and 1.2, then we recover the discrete Lebesgue space $l^p(\mathbb{N})$ and the discrete weak Lebesgue space $l^{p,\infty}(\mathbb{N})$.

In the following section, we will establish the sharp weak estimates for discrete Hardy operator on discrete central Morrey spaces. As an application, we obtain the sharp bounds for discrete Hardy operator from discrete Lebesgue spaces to discrete weak Lebesgue spaces.

2. Main result

The main result in this paper is the following:

Theorem 2.1. *Let* $1 \le p < \infty$ *and* $-\frac{1}{p} \le \lambda < 0$*. The for* $\vec{a} = \{a_k\}_1^\infty \in \dot{m}^{p,\lambda}(\mathbb{N})$*, we have*

$$\|\mathbf{h}(\vec{a})\|_{w\dot{m}^{p,\lambda}} \le 1 \cdot \|\vec{a}\|_{\dot{m}^{p,\lambda}}. \tag{2.1}$$

Moreover, the constant 1 in (2.1) *is the best possible one.*

Proof. By using discrete Hölder's inequality, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} |\mathbf{h}(\vec{a})(n)| \le& \frac{1}{n} \left\|\vec{A^n}\right\|_{l^p} \times n^{1/p'} \\ =& n^{\lambda} n^{-(\lambda+1/p)} \left\|\vec{A^n}\right\|_{l^p} = n^{\lambda} \|f\|_{\dot{m}^{p,\lambda}}, \end{aligned}$$

where $\vec{A^n}$ is recognized as in (1.6) and p' is the conjugate exponent of p, i.e., $1/p + 1/p' = 1$.

In view of $\lambda < 0$, we have

$$\|\mathbf{h}(\vec{a})\|_{w\dot{m}^{p,\lambda}} \le \sup_{n\in\mathbb{N}}\sup_{\gamma>0}\gamma n^{-(\lambda+1/p)}\left(\sharp\{m\in[1,n]: m^{\lambda}\|f\|_{\dot{m}^{p,\lambda}}>\gamma\}\right)^{1/p}$$
$$=\sup_{n\in\mathbb{N}}\sup_{\gamma>0}\gamma n^{-(\lambda+1/p)}\left(\sharp\left\{m\in[1,n]: m<\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}\right\}\right)^{1/p}.$$

Now we devide our discussion into two cases.

Case I. $1\le n\le\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}$. Noting that $\lambda<0$, we have

$$\sup_{\gamma>0}\sup_{1\le n\le\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}}\gamma n^{-(\lambda+1/p)}\left(\sharp\left\{m\in[1,n]: m<\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}\right\}\right)^{1/p}$$
$$\le\sup_{\gamma>0}\sup_{1\le n\le\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}}\gamma n^{-\lambda}\le\|f\|_{\dot{m}^{p,\lambda}}.$$

Csae II. $n>\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}$. Since $\lambda<0$, there holds

$$\sup_{\gamma>0}\sup_{n>\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}}\gamma n^{-(\lambda+1/p)}\left(\sharp\left\{m\in[1,n]: m<\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}\right\}\right)^{1/p}$$
$$\le\sup_{\gamma>0}\sup_{n>\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{1/\lambda}}\gamma n^{-(\lambda+1/p)}\left(\frac{\gamma}{\|f\|_{\dot{m}^{p,\lambda}}}\right)^{\frac{1}{p\lambda}}\le\|f\|_{\dot{m}^{p,\lambda}}.$$

Combining all the estimates for Case I and Case II, we arrive at

$$\|\mathbf{h}(\vec{a})\|_{w\dot{m}^{p,\lambda}}\le 1\cdot\|\vec{a}\|_{\dot{m}^{p,\lambda}}.$$

It remains to show the constant in (2.1) is optimal. To this end, we choose a sequence $\vec{\bar{a}}=\{\bar{a}_k\}_1^{\infty}$ such that

$$\bar{a}_k=\begin{cases}1, & k=1,\\ 0, & k=2,3,\ldots\end{cases}$$

We claim that $\vec{\bar{a}}\in\dot{m}^{p,\lambda}(\mathbb{N})$ and

$$\left\|\vec{\bar{a}}\right\|_{\dot{m}^{p,\lambda}}=1. \tag{2.2}$$

In fact, for any $n\in\mathbb{N}$, we have

$$\left\|\overrightarrow{\bar{A}^n}\right\|_{l^p}^p=\sum_{k=1}^{n}|\bar{a}_k|^p=1,$$

where $\overrightarrow{\bar{A}^n}$ is recognized as in (1.6).

Since $-1/p \le \lambda < 0$, by the definition of $\dot{m}^{p,\lambda}(\mathbb{N})$, we get

$$\left\|\vec{a}\right\|_{\dot{m}^{p,\lambda}} = \sup_{n\in\mathbb{N}} n^{-(\lambda+1/p)} \left\|\overrightarrow{A^n}\right\|_{l^p} = \sup_{n\in\mathbb{N}} n^{-(\lambda+1/p)} = 1.$$

By a simple calculation, one has

$$\mathbf{h}\left(\vec{\vec{a}}\right)(n) = \frac{1}{n}$$

for all $n \in \mathbb{N}$. Therefore,

$$\left\|\mathbf{h}\left(\vec{\vec{a}}\right)\chi_{\{1\}}\right\|_{l^{p,\infty}} = \sup_{0<\gamma<1} \gamma\left(\sharp\{k \in \{1\} : 1/k > \gamma\}\right)^{1/p} = \sup_{0<\gamma<1} \gamma = 1. \tag{2.3}$$

As a consequence of (2.3) and the definition of discrete weak Morrey spaces, we obtain

$$\begin{aligned}\left\|\mathbf{h}\left(\vec{\vec{a}}\right)\right\|_{w\dot{m}^{p,\lambda}} &= \sup_{n\in\mathbb{N}} n^{-(\lambda+1/p)} \left\|\mathbf{h}\left(\vec{\vec{a}}\right)\chi_{[1,n]}\right\|_{l^{p,\infty}}\\ &\ge 1^{-(\lambda+1/p)} \left\|\mathbf{h}\left(\vec{\vec{a}}\right)\chi_{\{1\}}\right\|_{l^{p,\infty}} = 1.\end{aligned} \tag{2.4}$$

Moreover, by using the boundednness of $\mathbf{h}$ on discrete central Morrey spaces proved above, there holds

$$\left\|\mathbf{h}\left(\vec{\vec{a}}\right)\right\|_{w\dot{m}^{p,\lambda}} \le 1 \cdot \left\|\vec{\vec{a}}\right\|_{w\dot{m}^{p,\lambda}} = 1. \tag{2.5}$$

Combining (2.4) with (2.5), we arrive at

$$\left\|\mathbf{h}\left(\vec{\vec{a}}\right)\right\|_{w\dot{m}^{p,\lambda}} = 1,$$

which together with (2.2) yields that

$$\left\|\mathbf{h}\left(\vec{\vec{a}}\right)\right\|_{w\dot{m}^{p,\lambda}} = \left\|\vec{\vec{a}}\right\|_{\dot{m}^{p,\lambda}}.$$

Consequently, the constant 1 in (2.1) is best possible. we are done. □

The following corollary can be seen as the weak version of the well-known discrete Hardy inequality.

Corollary 2.1. *Let* $1 \le p \le \infty$. *The for* $\vec{a} = \{a_k\}_1^\infty \in l^p(\mathbb{N})$*, we have*

$$\|\mathbf{h}(\vec{a})\|_{l^{p,\infty}} \le 1 \cdot \|\vec{a}\|_{l^p}. \tag{2.6}$$

Moreover, the constant 1 in (2.6) *is the best possible one.*

Proof. The case $1 \le p < \infty$ is a direct consequence of Theorem 2.1 by taking $\lambda = -1/p$. For $p = \infty$, it is obvious that $\|\mathbf{h}(\vec{a})\|_{l^\infty} \le 1 \cdot \|\vec{a}\|_{l^\infty}$. On the other hand, by taking $\vec{\vec{a}} = \{\bar{a}_k\}_1^\infty$ such that $\bar{a}_k = 1$ for all $k \in \mathbb{N}$, there holds $\left\|\mathbf{h}\left(\vec{\vec{a}}\right)\right\|_{l^\infty} = \left\|\vec{\vec{a}}\right\|_{l^\infty} = 1$. The proof is finished. □

Conflict of interest

The authors declare no conflicts of interest.

References

1. G. Hardy, Note on a theorem of Hilbert, *Math. Z.*, **6** (1920), 314–317. http://dx.doi.org/10.1007/BF01199965
2. W. Faris, Weak Lebesgue spaces and quantum mechanical binding, *Duke Math. J.*, **43** (1976), 365–373. http://dx.doi.org/10.1215/S0012-7094-76-04332-5
3. M. Christ, L. Grafakos, Best constants for two nonconvolution inequalities, *Proc. Amer. Math. Soc.*, **123** (1995), 1687–1693. http://dx.doi.org/10.2307/2160978
4. S. Lu, D. Yan, F. Zhao, Sharp bounds for Hardy type operators on higher-dimensional product spaces, *J. Inequal. Appl.*, **2013** (2013), 148. http://dx.doi.org/10.1186/1029-242X-2013-148
5. S. Wang, S. Lu, D. Yan, Explicit constants for Hardy's inequality with power weight on n-dimensional product spaces, *Sci. China Math.*, **55** (2012), 2469–2480. http://dx.doi.org/10.1007/s11425-012-4453-4
6. M. Wei, D. Yan, Sharp bounds for Hardy operators on product spaces, *Acta Math. Sci.*, **38** (2018), 441–449. http://dx.doi.org/10.1016/S0252-9602(18)30759-8
7. Z. Fu, L. Grafakos, S. Lu, F. Zhao, Sharp bounds for m-linear Hardy and Hilbert operators, *Houston J. Math.*, **38** (2012), 225–244.
8. T. Batbold, Y. Sawano, G. Tumendemberel, Sharp bounds for certain m-linear integral operators on p-adic function spaces, *Filomat*, **36** (2022), 801–812. http://dx.doi.org/10.2298/FIL2203801B
9. N. Chuong, N. Hong, H. Hung, Bounds of weighted multilinear Hardy-Cesàro operators in p-adic functional spaces, *Front. Math. China*, **13** (2018), 1–24. http://dx.doi.org/10.1007/s11464-017-0677-5
10. Y. Deng, D. Yan, M. Wei, Sharp estimates for m linear p-adic Hardy and Hardy-Littlewood-Pólya operators on p-adic central Morrey spaces, *J. Math. Inequal.*, **15** (2021), 1447–1458. http://dx.doi.org/10.7153/jmi-2021-15-99
11. Z. Fu, Q. Wu, S. Lu, Sharp estimates of p-adic Hardy and Hardy-Littlewood-Pólya operators, *Acta. Math. Sin.-English Ser.*, **29** (2013), 137–150. http://dx.doi.org/10.1007/s10114-012-0695-x
12. H. Hung, The p-adic weighted Hardy-Cesàro operator and an application to discrete Hardy inequalities, *J. Math. Anal. Appl.*, **409** (2014), 868–879. http://dx.doi.org/10.1016/j.jmaa.2013.07.056
13. Q. Wu, Z. Fu, Sharp estimates of m-linear p-adic Hardy and Hardy-Littlewood-Pólya operators, *J. Appl. Math.*, **2011** (2011), 472176. http://dx.doi.org/10.1155/2011/472176
14. J. Chu, Z. Fu, Q. Wu, L^p and BMO bounds for weighted Hardy operators on the Heisenberg group, *J. Inequal. Appl.*, **2016** (2016), 282. http://dx.doi.org/10.1186/s13660-016-1222-x

15. Y. Deng, X. Zhang, D. Yan, M. Wei, Weak and strong estimates for linear and multilinear fractional Hausdorff operators on the Heisenberg group, *Open Math.*, **19** (2021), 316–328. http://dx.doi.org/10.1515/math-2021-0016

16. S. Volosivets, Weighted Hardy and Cesàro operators on Heisenberg group and their norms, *Integr. Transf. Spec. F.*, **28** (2017), 940–952. http://dx.doi.org/10.1080/10652469.2017.1392946

17. D. Fan, F. Zhao, Sharp constant for multivariate Hausdorff q-inequalities, *J. Aust. Math. Soc.*, **106** (2019), 274–286. http://dx.doi.org/10.1017/S1446788718000113

18. J. Guo, F. Zhao, Some q-inequalities for Hausdorff operators, *Front. Math. China*, **12** (2017), 879–889. http://dx.doi.org/10.1007/s11464-017-0622-7

19. L. Maligranda, R. Oinarov, L. Persson, On Hardy q-inequalities, *Czech. Math. J.*, **64** (2014), 659–682. http://dx.doi.org/10.1007/s10587-014-0125-6

20. F. Zhao, Z. Fu, S. Lu, Endpoint estimates for n-dimensional Hardy operators and their commutators, *Sci. China Math.*, **55** (2012), 1977–1990. http://dx.doi.org/10.1007/s11425-012-4465-0

21. G. Gao, F. Zhao, Sharp weak bounds for Hausdorff operators, *Anal. Math.*, **41** (2015), 163–173. http://dx.doi.org/10.1007/s10476-015-0204-4

22. G. Gao, X. Hu, C. Zhang, Sharp weak estimates for Hardy-type operators, *Ann. Funct. Anal.*, **7** (2016), 421–433. http://dx.doi.org/10.1215/20088752-3605447

23. H. Yu, J. Li, Sharp weak bounds for n-dimensional fractional Hardy operators, *Front. Math. China*, **13** (2018), 449–457. http://dx.doi.org/10.1007/s11464-018-0685-0

24. A. Hussain, N. Sarfraz, F. Gurbuz, Sharp weak bounds for p-adic Hardy operators on p-adic linear spaces, *Commun. Fac. Sci. Univ.*, **71** (2022), 919–929. http://dx.doi.org/10.31801/cfsuasmas.1076462

25. N. Sarfraz, F. Gürbüz, Weak and strong boundedness for p-adic fractional Hausdorff operator and its commutator, *Int. J. Nonlin. Sci. Num.*, in press, http://dx.doi.org/10.1515/ijnsns-2020-0290

26. A. Kufner, L. Maligranda, L. Persson, The prehistory of the Hardy inequality, *The American Mathematical Monthly*, **113** (2006), 715–732. http://dx.doi.org/10.1080/00029890.2006.11920356

27. C. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.*, **43** (1938), 126–166. http://dx.doi.org/10.2307/1989904

28. J. Alvárez, J. Lakey, M. Guzmán-Partida, Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures, *Collect. Math.*, **51** (2000), 1–47.

29. Z. Fu, S. Lu, H. Wang, L. Wang, Singular integral operators with rough kernels on central Morrey spaces with variable exponent, *Ann. Acad. Sci. Fenn. M.*, **44** (2019), 505–522. http://dx.doi.org/10.5186/aasfm.2019.4431

30. K. Ho, Atomic decomposition of Hardy-Morrey spaces with variable exponents, *Ann. Acad. Sci. Fenn. M.*, **40** (2015), 31–62. http://dx.doi.org/10.5186/aasfm.2015.4002

31. K. Ho, Singular integral operators with rough kernel on Morrey type spaces, *Stud. Math.*, **244** (2019), 217–243. http://dx.doi.org/10.4064/sm8390-8-2017

32. Y. Sawano, S. Sugano, H. Tanaka, Orlicz-Morrey spaces and fractional operators, *Potential Anal.*, **36** (2012), 517–556. http://dx.doi.org/10.1007/s11118-011-9239-8
33. J. Tao, D. C. Yang, D. Y. Yang, Boundedness and compactness characterizations of Cauchy integral commutators on Morrey spaces, *Math. Method. Appl. Sci.*, **42** (2019), 1631–1651. https://doi.org/http://dx.doi.org/10.1002/mma.5462
34. H. Wang, J. Xu, J. Tan, Boundedness of multilinear singular integrals on central Morrey spaces with variable exponents, *Front. Math. China*, **15** (2020), 1011–1034. http://dx.doi.org/10.1007/s11464-020-0864-7
35. H. Gunawan, D. Hakim, M. Idris, On inclusion properties of discrete Morrey spaces, *Georgian Math. J.*, **29** (2022), 37–44. http://dx.doi.org/10.1515/gmj-2021-2122
36. H. Gunawan, E. Kikianty, Y. Sawano, C. Schwanke, Three geometric constants for Morrey spaces, *Bull. Korean Math. Soc.*, **56** (2019), 1569–1575. http://dx.doi.org/10.4134/BKMS.b190010
37. H. Gunawan, E. Kikianty, C. Schwanke, Discrete Morrey spaces and their inclusion properties, *Math. Nachr.*, **291** (2018), 1283–1296. http://dx.doi.org/10.1002/mana.201700054
38. H. Gunawan, C. Schwanke, The Hardy-Littlewood maximal operator on discrete Morrey spaces, *Mediterr. J. Math.*, **16** (2019), 24. http://dx.doi.org/10.1007/s00009-018-1277-7

AIMS Press