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Research article

Robust stability and passivity analysis for discrete-time neural networks with mixed time-varying delays via a new summation inequality

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Abstract: The summation inequality is essential in creating delay-dependent criteria for discrete-time systems with time-varying delays and developing other delay-dependent standards. This paper uses our rebuilt summation inequality to investigate the robust stability analysis issue for discrete-time neural networks that incorporate interval time-varying leakage and discrete and distributed delays. It is a novelty of this study to consider a new inequality, which makes it less conservative than the well-known Jensen inequality, and use it in the context of discrete-time delay systems. Further stability and passivity criteria are obtained in terms of linear matrix inequalities (LMIs) using the Lyapunov-Krasovskii stability theory, coefficient matrix decomposition technique, mobilization of zero equation, mixed model transformation, and reciprocally convex combination. With the assistance of the LMI Control toolbox in Matlab, numerical examples are provided to demonstrate the validity and efficiency of the theoretical findings of this research.

Keywords: stability analysis; passivity; discrete-time neural networks; interval time-varying delay; Lypunov-Krasovskii theory
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1. Introduction

Nature's vast majority of systems-including the biological nervous system-are dynamic in that external circumstances influence it to have internal memory and behave in a specific manner. The idea of activity development may describe that through time. Time delays, both constant and time-varying, generally exist in dynamic systems, such as chemical process control systems, man-manufacturing systems, cooling systems, hydraulic systems, irrigation channels, metallurgical processes, robotics, and neural networks [1, 6, 8, 10, 11, 13, 14, 19, 22–24, 30–32, 35, 36, 39, 45].

The nonlinear ordinary difference equation in discrete-time state-space form may be used to explain a general class of discrete-time systems. For discrete-time systems, it is possible to employ the nonlinear ordinary difference equation in the discrete-time state-space form to describe them. $x(k+1) = f(x(k), u(k)), y(k) = h(x(k), u(k))$ where $x(k) \in \mathbb{R}^n$ is the internal state vector, $u(k) \in \mathbb{R}^m$ is the control input, and $y(k) \in \mathbb{R}^p$ is the system output. These equations may be obtained from analyzing the dynamical system or process under study. In contrast, others may be derived from discretized or sampled continuous-time dynamics of a nonlinear system under investigation.

The study of the stability analysis of discrete-time systems with time-varying delays has emerged as a popular subject in the area of control theory during the last few years [7, 12, 15, 17, 21, 25–27, 29, 33, 37, 38, 42, 44, 45]. The stability of neural networks is a precondition for solving many engineering issues; it has garnered considerable attention in recent years, and many elegant solutions have been published. [11, 19, 22, 23, 34, 35, 39, 45]. When implementing continuous-time neural networks for computer simulation, for computational or experimental reasons, it is necessary to construct a discrete-time system that is analogous to the continuous-time neural networks that should not be overlooked. [24] points out that discretization cannot consistently maintain the dynamics of the continuous-time counterpart, even for a short sample interval. As a result, it is critical to understand the dynamics of discrete-time neural networks.

Human brain activity, particularly that of neural networks, may be viewed as a very sophisticated parallel computer that is more efficient than any presently existing computer when neural networks are straightforwardly implemented in computers. In the case of neural networks, one of the most crucial characteristics is a temporal delay in the leakage term. When it comes to neural networks, the time delay in the leakage term has a significant impact on their dynamics since the system becomes unstable when there is a delay in reacting to a negative outcome; this causes the system to become unstable [5, 6, 15, 18, 20]. [6] investigated neural networks with a time delay in the leakage term and their findings on the presence and uniqueness of the equilibrium, independent of the time delay and starting circumstances, to determine whether the equilibrium exists. This means that the existence and uniqueness of the equilibrium point are unaffected by the delay in the leakage term. As a result of its importance as a helpful tool for the stability analysis of both linear and nonlinear systems, especially high-order systems, passivity theory, initially proposed in circuit analysis, has drawn a lot of attention and has been extensively investigated. Systems with passive qualities maintain their internal stability. The passivity theory has been widely used in a variety of fields, including signal processing [43], fuzzy control [16], sliding mode control [41], and networked control [4].

The problems of novel delay-range-dependent robust asymptotic stability and passivity criteria for uncertain discrete-time neural networks with interval discrete and distributed time-varying delays are introduced in this paper, which is motivated by earlier discussions. A novel delay-range-dependent stability and passivity analysis is also investigated for uncertain discrete-time neural networks with interval discrete, distributed, and leakage time-varying delays. New delay-range-dependent robust asymptotic stability and passivity criteria in terms of linear matrix inequalities (LMIs) for considered systems are obtained using a class of novel augmented Lyapunov-Krasovskii functionals (LKFs), model transformation, coefficient matrix decomposition technique, reciprocally convex combination, Leibniz-Newton formula, and use of zero equation. Also presented is an improvement in the stability and passivity criterion for discrete-time neural networks with interval time-varying delay dependent on the delay range. Theory may be shown using numerical examples, indicating that it's more effective while being less conservative. The main contributions and highlights of this paper are summarized in the following key points.

(1) The rebuilt summation inequality is used for the robust stability analysis issue for discrete-time neural networks that incorporate interval time-varying leakage, discrete and distributed delays for developing the delay-dependent criteria.

(2) We apply new inequalities to improve the stability criteria, such as Jensen inequality, coefficient matrix decomposition technique, utilization of zero equation, mixed model transformation, and reciprocally convex combination. Using the above new LKFs and the lemmas leads to less conservatism of the obtained results than in published literature, as presented via numerical examples.

(3) We present numerical examples to demonstrate the feasibility and effectiveness of the theorem.

Notations: Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $Z^+ = \{0, 1, 2, 3, ...\}$; N={1,2,3,...}; $\mathbb{R}^{(n\times m)}$ denotes the set of $n \times m$-real matrices; A^T denotes the transpose of the matrix A; A is symmetric if $A = A^T$; I_n is the $n \times n$-identity matrix; matrix A is called semi-positive definite ($A \geq 0$) if $x^T Ax \geq 0$, for all $x \in \mathbb{R}^n$; A is positive definite ($A > 0$) if $x^T Ax > 0$, for all $x \neq 0$; $A > B$ means $A - B > 0(B - A < 0)$; $A \geq B$ means $A - B \geq 0(B - A \leq 0)$; $\rho = \max\{\tau_2, h_2, M\}$; $*$ denotes symmetric terms in a symmetric matrix; $[\star]$ denote the right-side vector in a symmetric quadratic form.

2. Problem formulation and preliminaries

Consider the following uncertain discrete-time neural network with interval time-varying leakage, discrete and distributed delays, as shown in the following system:

$$\left\{\begin{array}{rcl} x(k+1) & = & (A+\Delta A(k))x(k-\tau(k)) + (B+\Delta B(k))f(x(k)) \\ & & +(C+\Delta C(k))g(x(k-h(k))) \\ & & +(D+\Delta D(k))\displaystyle\sum_{i=1}^{M}\delta(i)x(k-i) + w(k), k \in Z^+, \\ z(k) & = & A_z x(k-\tau(k)) + B_z f(x(k)) + C_z g(x(k-h(k))) \\ & & +D_z \displaystyle\sum_{i=1}^{M}\delta(i)x(k-i), k \in N, \\ x(s) & = & \phi(s), \quad s = -\rho, -\rho+1, \ldots, 0, \end{array}\right. \tag{2.1}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ is the system state vector, $z(k)$ is the output vector of neuron network, $w(k)$ is the exogenous disturbance input vector, $A = diag\{a_1, a_2, ..., a_n\}$ is the state feedback coefficient matrix with $|a_i| < 1$, matrices B, C, D, A_z, B_z, C_z and D_z are known real constant matrices with appropriate dimensions, $M \in N$, $\phi(s)$ is the initial condition of system (2.1), $\tau(k)$ represents the leakage delay satisfying

$$0 < \tau_1 \leq \tau(k) \leq \tau_2, \tag{2.2}$$

where τ_1 and τ_2 denote the lower and upper bounds of $\tau(k)$. The time-varying delay $h(k)$ satisfies

$$0 < h_1 \leq h(k) \leq h_2, \tag{2.3}$$

where h_1 and h_2 are known positive integers. There exists a constant $\kappa > 0$ such that function $\delta(i)$ satisfies the following convergence condition

$$\sum_{i=1}^{M} \delta(i) = \kappa < +\infty. \tag{2.4}$$

$\Delta A(k)$, $\Delta B(k), \Delta C(k)$ and $\Delta D(k)$ represent the time-varying parameter uncertainties, and are assumed to satisfy the following linear fractional form

$$[\Delta A(k) \ \ \Delta B(k) \ \ \Delta C(k) \ \ \Delta D(k)] = \Gamma\Delta(k)[H_1 \ \ H_2 \ \ H_3 \ \ H_4], \tag{2.5}$$

where Γ, H_1, H_2, H_3 and H_4 are known real constant matrices with appropriate dimensions. The uncertain matrix $\Delta(k)$ satisfies

$$\Delta(k) = [I - \Omega(k)E]^{-1}\Omega(k), \tag{2.6}$$

and is said to be admissible, where E is a known matrix satisfying

$$I - EE^T > 0, \tag{2.7}$$

and $\Omega(k)$ is an unknown time-varying matrix function satisfying

$$\Omega^T(k)\Omega(k) \leq I. \tag{2.8}$$

Assumption 1. *For $i \in \{1, 2, \ldots, n\}$, the neuron activation functions $f_i(\cdot)$, $g_i(\cdot)$ in system* (2.1) *are continuous and bounded.*

Assumption 2. *For any $s_1, s_2 \in \mathbb{R}, s_1 \neq s_2$, the continuous and bounded activation functions $f_i(\cdot)$ and $g_i(\cdot)$ satisfy*

$$F_i^- \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq F_i^+, \quad i = 1, 2, \ldots, n,$$
$$G_i^- \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq G_i^+, \quad i = 1, 2, \ldots, n,$$

and $f_i(0) = g_i(0) = 0$, where F_i^-, F_i^+, G_i^-, and G_i^+ are known real constants.

Definition 1. *[28] The discrete-time system* (2.1)*, with $\omega(k) = 0$, is said to be robust asymptotically stable if there exists a positive definite scalar function $V(x(k)) : \mathbb{Z}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$ such that*

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0,$$

along the solution of the system (2.1) *for all uncertainties.*

Definition 2. *[36] The discrete-time system* (2.1)*, with* $\omega(k) = 0$ *and* $\Omega(k) = 0$*, is said to be asymptotically stable if there exists a positive definite scalar function* $V(x(k)) : \mathbb{Z}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^+$ *such that*

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0,$$

along the solution of the system (2.1).

Definition 3. *[32] The system* (2.1) *is called passive if there exists a scalar* $\gamma \geq 0$ *such that*

$$2\sum_{i=0}^{k} z^T(i)w(i) \geq -\gamma \sum_{i=0}^{k} w^T(i)w(i),$$

for all $k \in \mathbb{Z}^+$ *and for all solution of* (2.1) *with* $x(0) = 0$ *holds.*

Lemma 1. *[17] Suppose that* $\Delta(k)$ *is given by* (2.6)–(2.8)*. Let* M, S *and* N *be real constant matrices of appropriate dimension with* $M = M^T$*. Then, the inequality*

$$M + S\Delta(k)N + N^T\Delta(k)^T S^T < 0,$$

holds if and only if, for any positive real constant δ*,*

$$\begin{bmatrix} M & S & \delta N^T \\ * & -\delta I & \delta N^T \\ * & * & -\delta I \end{bmatrix} < 0.$$

Lemma 2. *[27] Let* $\gamma_1, \gamma_2, \ldots, \gamma_N : \mathbb{R}^m \mapsto \mathbb{R}$ *have positive values in an open subset* D *of* $\mathbb{R}^m$*. Then, the reciprocally convex combination of* γ_i *over* D *satisfies*

$$\min_{\{\alpha_i | \alpha_i > 0, \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i}\gamma_i(k) = \sum_i \gamma_i(k) + \max_{\epsilon_{i,j}(k)} \sum_{i \neq j} \epsilon_{i,j}(k),$$

subject to

$$\epsilon_{i,j} : \mathbb{R}^m \mapsto \mathbb{R}, \epsilon_{j,i}(k) \triangleq \epsilon_{i,j}(k), \begin{bmatrix} \gamma_i(k) & \epsilon_{i,j}(k) \\ \epsilon_{i,j}(k) & \gamma_j(k) \end{bmatrix} \geq 0.$$

Lemma 3. *The following inequality holds for any* $\alpha \in \mathbb{R}^n$*,* $\beta \in \mathbb{R}^m$*,* $\Xi, Y \in \mathbb{R}^{n\times m}$*,* $X \in \mathbb{R}^{n\times n}$*, and* $Z \in \mathbb{R}^{m\times m}$*,*

$$-2\alpha^T \Xi \beta \leq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^T \begin{bmatrix} X & Y - \Xi \\ * & Z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

where $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \geq 0.$

Lemma 4. *[8] For any positive real constant matrix* $M \in \mathbb{R}^{n\times n}$*,* $M = M^T$*, two constants* $h_2 \geq h_1 > 0$*, such that the following inequalities hold:*

(1) $$\left[\sum_{i=1}^{h_1} x(i)\right]^T M \left[\sum_{i=1}^{h_1} x(i)\right] \leq h_1 \sum_{i=1}^{h_1} x^T(i) M x(i),$$

(2) $$\left[\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} x(j)\right]^T M\left[\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} x(j)\right] \leq \frac{(h_2-h_1)(h_2-h_1+1)}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1} x^T(j)Mx(j),$$

(3) $$\left[\sum_{i=h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} x(j)\right]^T M\left[\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} x(j)\right] \leq \frac{(h_2-h_1)(h_2+h_1+1)}{2}\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} x^T(j)Mx(j).$$

Lemma 5. *[25] For a given positive-definite $n \times n$-matrix $\mathbb{R}$, three given non-negative integers α, β, k satisfying $\alpha < \beta \leq k$, a vector function $x(\cdot) \in \mathbb{R}^n$ and denoting $\Delta x(k) = x(k+1) - x(k)$, we have*

$$\sum_{i=k-\beta}^{k-\alpha-1} \Delta x^T(i)R\Delta x(i) \quad \geq \quad \frac{1}{\beta-\alpha}(\Theta^0_{\alpha,\beta})^T R\Theta^0_{\alpha,\beta} + \frac{3}{\beta-\alpha}(\Theta^1_{\alpha,\beta})^T R\Theta^1_{\alpha,\beta} + \frac{5}{\beta-\alpha}(\Theta^2_{\alpha,\beta})^T R\Theta^2_{\alpha,\beta},$$

where

$$\begin{aligned}
\Theta^0_{\alpha,\beta} &= x(k-\alpha) - x(k-\beta),\\
\Theta^1_{\alpha,\beta} &= x(k-\alpha) + x(k-\beta) - \frac{2}{\beta-\alpha+1}\sum_{i=k-\beta}^{k-\alpha} x(i),\\
\Theta^2_{\alpha,\beta} &= x(k-\alpha) - x(k-\beta) + \frac{6}{\beta-\alpha+1}\sum_{i=k-\beta}^{k-\alpha} x(i) - \frac{12}{(\beta-\alpha+2)(\beta-\alpha+1)}\sum_{i=k-\beta}^{k-\alpha} x(i)\sum_{j=-\beta}^{-\alpha}\sum_{i=k+s}^{k-\alpha} x(i).
\end{aligned}$$

Lemma 6. *Let $\Delta x(k) \in \mathbb{R}^n$ be a vector-valued function with first-order forward difference entries. Then, the following integral inequality holds for any constant matrices $X, M_i \in \mathbb{R}^{n\times n}$, $i = 1, 2, \ldots, 5$ and $h(k)$ is discrete interval time-varying delays with $0 \leq h_1 \leq h(k) \leq h_2$,*

$$\begin{aligned}
&-\sum_{i=k-h_2}^{k-h_1-1} \Delta x^T(i)X\Delta x(i)\\
&\leq \begin{bmatrix} x(k-h_1)\\ x(k-h(k))\\ x(k-h_2)\end{bmatrix}^T \begin{bmatrix} M_1+M_1^T & -M_1^T+M_2 & 0\\ * & M_1+M_1^T-M_2-M_2^T & -M_1^T+M_2\\ * & * & -M_2-M_2^T\end{bmatrix}\begin{bmatrix} x(k-h_1)\\ x(k-h(k))\\ x(k-h_2)\end{bmatrix}\\
&+[h_2-h_1]\begin{bmatrix} x(k-h_1)\\ x(k-h(k))\\ x(k-h_2)\end{bmatrix}^T \begin{bmatrix} M_3 & M_4 & 0\\ * & M_3+M_5 & M_4\\ * & * & M_5\end{bmatrix}\begin{bmatrix} x(k-h_1)\\ x(k-h(k))\\ x(k-h_2)\end{bmatrix},
\end{aligned} \tag{2.9}$$

where

$$\begin{bmatrix} X & M_1 & M_2\\ * & M_3 & M_4\\ * & * & M_5\end{bmatrix} \geq 0.$$

Proof. From the discrete analog of the Newton-Leibniz formula, we obtain

$$0 = x(k-h_1) - x(k-h(k)) - \sum_{i=k-h(k)}^{k-h_1-1} \Delta x(i), \tag{2.10}$$

$$0 = x(k-h(k)) - x(k-h_2) - \sum_{i=k-h_2}^{k-h(k)-1} \Delta x(i). \tag{2.11}$$

For any constant matrices $\Xi_1, \Xi_2 \in \mathbb{R}^{n\times n}$ with zero equation (2.10),

$$\begin{aligned} 0 &= 2[x^T(k-h_1) - x^T(k-h(k)) - \sum_{i=k-h(k)}^{k-h_1-1} \Delta x^T(i)][\Xi_1 x(k-h_1) + \Xi_2 x(k-h(k))] \\ &= \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \begin{bmatrix} \Xi_1 + \Xi_1^T & -\Xi_1^T + \Xi_2 \\ * & -\Xi_2 - \Xi_2^T \end{bmatrix} \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix} \\ &\quad -2\sum_{i=k-h(k)}^{k-h_1-1} \Delta x^T(i) \begin{bmatrix} \Xi_1 & \Xi_2 \end{bmatrix} \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}. \end{aligned} \tag{2.12}$$

Using Lemma 3 with $\alpha = \Delta x(i), \beta = \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}, Y = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$ and $Z = \begin{bmatrix} M_3 & M_4 \\ * & M_5 \end{bmatrix}$, we get

$$\begin{aligned} &-2\sum_{i=k-h(k)}^{k-h_1-1} \Delta x^T(i) \begin{bmatrix} \Xi_1 & \Xi_2 \end{bmatrix} \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix} \\ \leq\ & \sum_{i=k-h(k)}^{k-h_1-1} \begin{bmatrix} \Delta x(i) \\ x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \begin{bmatrix} X & M_1 - \Xi_1 & M_2 - \Xi_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \begin{bmatrix} \Delta x(i) \\ x(k-h_1) \\ x(k-h(k)) \end{bmatrix} \\ \leq\ & \sum_{i=k-h(k)}^{k-h_1-1} \Delta x^T(i) X \Delta x(i) + [h_2 - h_1] \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \begin{bmatrix} M_3 & M_4 \\ * & M_5 \end{bmatrix} \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix} \\ & + \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \left(\begin{bmatrix} M_1 + M_1^T & -M_1^T + M_2 \\ * & -M_2 - M_2^T \end{bmatrix} - \begin{bmatrix} \Xi_1 + \Xi_1^T & -\Xi_1^T + \Xi_2 \\ * & -\Xi_2 - \Xi_2^T \end{bmatrix} \right) \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12), then we obtain

$$\begin{aligned} -\sum_{i=k-h(k)}^{k-h_1-1} \Delta x^T(i) X \Delta x(i) \leq\ & \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \begin{bmatrix} \Xi_1 + \Xi_1^T & -\Xi_1^T + \Xi_2 \\ * & -\Xi_2 - \Xi_2^T \end{bmatrix} \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix} \\ & + \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \left(\begin{bmatrix} M_1 + M_1^T & -M_1^T + M_2 \\ * & -M_2 - M_2^T \end{bmatrix} \right. \\ & \left. - \begin{bmatrix} \Xi_1 + \Xi_1^T & -\Xi_1^T + \Xi_2 \\ * & -\Xi_2 - \Xi_2^T \end{bmatrix} \right) \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix} \\ & + [h_2 - h_1] \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \begin{bmatrix} M_3 & M_4 \\ * & M_5 \end{bmatrix} \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix} \\ =\ & \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \begin{bmatrix} M_1 + M_1^T & -M_1^T + M_2 \\ * & -M_2 - M_2^T \end{bmatrix} \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix} \\ & + [h_2 - h_1] \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}^T \begin{bmatrix} M_3 & M_4 \\ * & M_5 \end{bmatrix} \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \end{bmatrix}. \end{aligned} \tag{2.14}$$

By Eq (2.11), the following equation is true for any constant matrices $\Xi_1, \Xi_2 \in \mathbb{R}^{n\times n}$

$$0 = 2\left[x^T(k-h(k)) - x^T(k-h_2) - \sum_{i=k-h_2}^{k-h(k)-1} \Delta x^T(i)\right]\left[\Xi_1 x(k-h(k)) + \Xi_2 x(k-h_2)\right].$$

Similarly, we have

$$\begin{aligned} -\sum_{i=k-h_2}^{k-h(k)-1} \Delta x^T(i) X \Delta x(i) &\leq \begin{bmatrix} x(k-h(k)) \\ x(k-h_2) \end{bmatrix}^T \begin{bmatrix} M_1 + M_1^T & -M_1^T + M_2 \\ * & -M_2 - M_2^T \end{bmatrix} \begin{bmatrix} x(k-h(k)) \\ x(k-h_2) \end{bmatrix} \\ &\quad + [h_2 - h_1] \begin{bmatrix} x(k-h(k)) \\ x(k-h_2) \end{bmatrix}^T \begin{bmatrix} M_3 & M_4 \\ * & M_5 \end{bmatrix} \begin{bmatrix} x(k-h(k)) \\ x(k-h_2) \end{bmatrix}. \end{aligned} \tag{2.15}$$

Finally, considering (2.14) and (2.15) together, then the summation inequality (2.9) is established. This brings the proof to a conclusion. □

3. Main results

3.1. Stability analysis for discrete-time neural network

This subsection presents a stability analysis of system (2.1) with $\omega(k) = 0$. The LMI based conditions will be derived using Lyapunov technique.

Consider the following neural network with interval leakage delay of the form

$$\begin{cases} x(k+1) = (A + \Delta A(k))x(k-\tau(k)) + (B + \Delta B(k))f(x(k)) \\ \qquad +(C + \Delta C(k))g(x(k-h(k))) \\ \qquad +(D + \Delta D(k)) \sum_{i=1}^{M} \delta(i) x(k-i), k \in Z^+ \\ x(s) = \phi(s), \quad s \in \{-\rho, -\rho+1, \ldots, -1, 0, \}. \end{cases} \tag{3.1}$$

To be more specific, we will present the notations that will be used later

$$\Pi = \left[\Pi_{i,j}\right]_{21\times 21}, \tag{3.2}$$

where $\Pi_{i,j} = \Pi_{j,i}^T$, $i, j = 1, 2, 3, \ldots, 21$,

$$\begin{aligned} \Pi_{1,1} &= P_1 J_1 + P_1 J_2 + J_1^T P_1 + J_2^T P_1 + Q_1^T (A_1 - I) + (A_1^T - I) Q_1 + (h_{12}+1) P_2 \\ &\quad +(\tau_{12}+1)P_3 - 9R_1 - 9R_3 + h_1(L_1 + L_1^T) + h_1^2 L_3 + h_2(M_1 + M_1^T) \\ &\quad +(h_2^2)M_3 + \tau_1(S_1 + S_1^T) + (\tau_1^2)S_3 + \tau_2(T_1 + T_1^T) + (\tau_2^2)T_3 + \xi P_6 - F_1 \Lambda_1, \\ \Pi_{1,2} &= P_1 + J_1^T P_1 + J_2^T P_1 - Q_1^T + (A_1 - I) Q_2, \\ \Pi_{1,3} &= -P_1 J_1 + h_1(-L_1^T + L_2) + (h_1^2) L_4 + h_2(-M_1^T + M_2) + (h_2^2) M_4, \\ \Pi_{1,4} &= 3R_1, \qquad \Pi_{1,6} = -P_1 J_1, \qquad \Pi_{1,7} = -\frac{24}{h_1+1} R_1, \qquad \Pi_{1,8} = -\frac{60}{(h_1+2)(h_1+1)} R_1, \\ \Pi_{1,11} &= -P_1 J_2 + Q_1^T A_2 + (A_1 - I) Q_3 + \tau_1(-S_1^T + S_2) + (\tau_1^2) S_4 + (\tau_2^2) T_4 \\ &\quad +\tau_1(-T_1^T + T_2), \end{aligned}$$

$$
\begin{aligned}
\Pi_{1,12} &= 3R_3, \qquad \Pi_{1,14} = -P_1J_2 - Q_1^TA_1 + (A_1 - I)Q_4,\\
\Pi_{1,15} &= -\frac{24}{\tau_1 - 1}R_3, \qquad \Pi_{1,16} = \frac{60}{(\tau_1 + 2)(\tau_1 + 1)}R_3,\\
\Pi_{1,19} &= Q_1^TB + (A_1 - I)Q_5 + F_2\Lambda_1,\\
\Pi_{1,20} &= Q_1^TC + (A_1 - I)Q_6, \qquad \Pi_{1,21} = Q_1^TD + (A_1 - I)Q_7,\\
\Pi_{2,2} &= P_1 - Q_2^T - Q_2 + (h_2^2)P_4 + (\tau_2^2)P_5 + (h_1^2)R_1 + (h_{12}^2)R_2 + (h_1^2)Z_1 + (h_2^2)Z_2\\
&\quad +(h_{12}^2)Z_3 + (\tau_1^2)R_3 + (\tau_{12}^2)R_4 + (\tau_1^2)Z_4 + (\tau_2^2)Z_5 + (\tau_{12}^2)Z_6,\\
\Pi_{2,3} &= -P_1J_1, \qquad \Pi_{2,6} = -P_1J_1, \qquad \Pi_{2,11} = -P_1J_2 + Q_2^TA_2 - Q_3,\\
\Pi_{2,14} &= -P_1J_2 - Q_2^TA_1 - Q_4, \qquad \Pi_{2,19} = Q_2^TB - Q_5,\\
\Pi_{2,20} &= Q_2^TC - Q_6, \qquad \Pi_{2,21} = Q_2^TD - Q_7,\\
\Pi_{3,3} &= -G_1\Lambda_2 + h_1(L_1 + L_1^T - L_2 - L_2^T) + (h_1^2)(L_3 + L_5) + (h_2^2)(M_3 + M_5)\\
&\quad +h_2(M_1 + M_1^T - M_2 - M_2^T) + h_{12}(N_1 + N_1^T - N_2 - N_2^T) + (h_{12}^2)(N_3 + N_5),\\
\Pi_{3,4} &= h_1(-L_1^T + L_2) + (h_1^2)L_4 + h_{12}(-N_1 + N_2^T) + (h_{12}^2)N_4^T,\\
\Pi_{3,5} &= h_2(-M_1^T + M_2) + (h_2^2)M_4 + h_{12}(-N_1^T + N_2) + (h_{12}^2)N_4, \qquad \Pi_{3,20} = G_2\Lambda_2,\\
\Pi_{4,4} &= -9R_1 - 9R_2 + h_1(-L_2 - L_2^T) + (h_1^2)L_5 + h_{12}(N_1 + N_1^T) + (h_{12}^2)N_3,\\
\Pi_{4,5} &= 3R_2, \qquad \Pi_{4,6} = \frac{36}{h_1 + 1}R_1, \qquad \Pi_{4,7} = -\frac{60}{(h_1 + 2)(h_1 + 1)}R_1,\\
\Pi_{4,8} &= -\frac{24}{h_{12} + 1}R_2, \qquad \Pi_{4,9} = \frac{60}{(h_{12} + 2)(h_{12} + 1)}R_2,\\
\Pi_{5,5} &= -P_2 - 9R_2 + h_2(-M_2 - M_2^T) + (h_2^2)M_5 + h_{12}(-N_2 - N_2^T) + (h_{12}^2)N_5,\\
\Pi_{5,9} &= \frac{36}{h_{12} + 1}R_2, \qquad \Pi_{5,10} = -\frac{60}{(h_{12} + 2)(h_{12} + 1)}R_2,\\
\Pi_{6,6} &= -P_4, \qquad \Pi_{7,7} = -\frac{192}{(h_1 + 1)^2}R_1, \qquad \Pi_{7,8} = \frac{360}{(h_1 + 2)(h_1 + 1)^2}R_1,\\
\Pi_{8,8} &= -\frac{720}{(h_1 + 2)^2(h_1 + 1)^2}R_1, \qquad \Pi_{9,9} = -\frac{192}{(h_{12} + 1)^2}R_2,\\
\Pi_{9,10} &= \frac{360}{(h_{12} + 2)(h_{12} + 1)^2}R_2, \qquad \Pi_{10,10} = -\frac{720}{(h_{12} + 2)^2(h_{12} + 1)^2}R_2,\\
\Pi_{11,11} &= Q_3^TA_2 + A_2Q_3 + \tau_1(S_1 + S_1^T - S_2 - S_2^T) + (\tau_1^2)(S_3 + S_5)\\
&\quad +\tau_2(T_1 + T_1^T - T_2 - T_2^T) + (\tau_2^2)(T_3 + T_5) + (\tau_{12}^2)(U_3 + U_5)\\
&\quad +\tau_{12}(U_1 + U_1^T - U_2 - U_2^T),\\
\Pi_{11,12} &= \tau_1(-S_1^T + S_2) + (\tau_1^2)S_4 + \tau_{12}(-U_1 + U_2^T) + (\tau_{12}^2)U_4^T,\\
\Pi_{11,13} &= \tau_2(-T_1^T + T_2)(\tau_2^2)T_4 + \tau_{12}(-U_1^T + U_2) + (\tau_{12}^2)U_4,\\
\Pi_{11,14} &= -Q_3^TA_1 + A_2Q_4, \qquad \Pi_{11,19} = Q_3^TB + A_2Q_5,\\
\Pi_{11,20} &= Q_3^TC + A_2Q_6, \qquad \Pi_{11,21} = Q_3^TD + A_2Q_7,\\
\Pi_{12,12} &= -9R_3 - 9R_4 + \tau_1(-S_2 - S_2^T) + (\tau_1^2)S_5 + \tau_{12}(-U_1 - U_1^T) + (\tau_{12}^2)U_3,\\
\Pi_{12,13} &= 3R_4, \qquad \Pi_{12,15} = \frac{36}{(\tau_1 + 1)}R_3, \qquad \Pi_{12,16} = -\frac{60}{(\tau_1 + 2)(\tau_1 + 1)})R_3,
\end{aligned}
$$

$$\begin{aligned}
\Pi_{12,17} &= -\frac{24}{\tau_{12}+1}R_4, \qquad \Pi_{12,18} = \frac{60}{(\tau_{12}+2)(\tau_{12}+1)}R_4,\\
\Pi_{13,13} &= -P_3 - 9R_4 + \tau_2(-T_2 - T_2^T) + (\tau_2^2)T_5 + \tau_{12}(-U_2 - U_2^T) + (\tau_{12}^2)U_5,\\
\Pi_{13,17} &= \frac{36}{(\tau_{12}+1)(\tau_{12}+1)}R_4, \qquad \Pi_{13,18} = -\frac{60}{(\tau_{12}+2)(\tau_{12}+1)}R_4,\\
\Pi_{14,14} &= -Q_4^TA_1 - A_1Q_4 - P_5, \qquad \Pi_{14,19} = Q_4^TB - A_1Q_5,\\
\Pi_{14,20} &= Q_4^TC - A_1Q_6, \qquad \Pi_{14,21} = Q_4^TD - A_1Q_7,\\
\Pi_{15,15} &= -\frac{192}{(\tau_1+1)^2}R_3, \qquad \Pi_{15,16} = \frac{360}{(\tau_1+2)(\tau_1+1)^2}R_3,\\
\Pi_{16,16} &= -\frac{720}{(\tau_1+2)^2(\tau_1+1)^2}R_3, \qquad \Pi_{17,17} = -\frac{192}{(\tau_{12}+1)^2}R_4,\\
\Pi_{17,18} &= \frac{360}{(\tau_{12}+2)(\tau_{12}+1)^2}R_4, \qquad \Pi_{18,18} = -\frac{720}{(\tau_{12}+2)^2(\tau_{12}+1)^2}R_4,\\
\Pi_{19,19} &= Q_5^TB + BQ_5 - \Lambda_1, \qquad \Pi_{19,20} = Q_5^TC + BQ_6,\\
\Pi_{19,21} &= Q_5^TD + BQ_7, \qquad \Pi_{20,20} = Q_6^TC + CQ_6 - \Lambda_2,\\
\Pi_{20,21} &= Q_5^TD + BQ_7, \qquad \Pi_{21,21} = Q_7^TD + DQ_7 - \xi^{-1}P_6,
\end{aligned}$$

and others are equal to zero.

First of all, we examine the discrete-time neural network of the type with interval time-varying discrete, leakage, and distributed delays of the form

$$\begin{cases} x(k+1) = Ax(k-\tau(k)) + Bf(x(k)) + Cg(x(k-h(k))) + D\sum_{i=1}^{M}\delta(i)x(k-i), k \in Z^+,\\ x(s) = \phi(s), \quad s \in \{-\rho, -\rho+1, \ldots, -1, 0, \}. \end{cases} \tag{3.3}$$

Theorem 1. *The system* (3.3) *is asymptotically stable, if there exist positive definite symmetric matrices* P_i, Q_j, R_k, Z_i*,* $i = 1,2,3,\ldots,6$*,* $j = 1,2,3,\ldots,7$*,* $k = 1,2,3,4$*, and any appropriate dimensional matrices* Λ_1, Λ_2*, satisfying the following LMIs*

$$\Pi < 0, \tag{3.4}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2\\ * & L_3 & L_4\\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.5}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2\\ * & M_3 & M_4\\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.6}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2\\ * & N_3 & N_4\\ * & * & N_5 \end{bmatrix} \geq 0, \tag{3.7}$$

$$\begin{bmatrix} Z_4 & S_1 & S_2\\ * & S_3 & S_4\\ * & * & S_5 \end{bmatrix} \geq 0, \tag{3.8}$$

$$\begin{bmatrix} Z_5 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & T_5 \end{bmatrix} \geq 0, \tag{3.9}$$

$$\begin{bmatrix} Z_6 & U_1 & U_2 \\ * & U_3 & U_4 \\ * & * & U_5 \end{bmatrix} \geq 0. \tag{3.10}$$

Proof. We begin by demonstrating the asymptotic stability of the system (3.3) under the constraints of the theorem. Let us partition the constant matrix A into its components

$$A = A_1 + A_2, \tag{3.11}$$

where $A_1, A_2 \in \mathbb{R}^{n\times n}$ are real constant matrices in order to improve the bounds of the discrete delay.

After that, we rewrite the system (3.3) using transformation method, so we achieve the following equivalents form:

$$\begin{aligned} x(k+1) &= x(k) + y(k), && (3.12)\\ y(k) &= (A_1 - I)x(k) + A_2 x(k-\tau(k)) - A_1 \sum_{i=k-\tau(k)}^{k-1} y(i) + Bf(x(k)) \\ &\quad + Cg(x(k-h(k))) + D\sum_{i=1}^{M}\delta(i)x(k-i). && (3.13) \end{aligned}$$

Design and implement the following Lyapunov-Krasovskii functional as follows:

$$V(k) = \sum_{i=1}^{10} V_i(k), \tag{3.14}$$

where

$$\begin{aligned} V_1(k) &= x^T(k)P_1x(k),\\ V_2(k) &= \sum_{i=k-h_2}^{k-1} x^T(i)P_2x(i) + \sum_{i=-h_2+1}^{-h_1}\sum_{j=k+i}^{k-1} x^T(j)P_2x(j),\\ V_3(k) &= h_2\sum_{i=-h_2+1}^{0}\sum_{j=k+i-1}^{k-1} y^T(j)P_4y(j),\\ V_4(k) &= h_1\sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1} y^T(j)R_1y(j) + (h_2-h_1)\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} y^T(j)R_2y(j),\\ V_5(k) &= h_1\sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1} y^T(j)Z_1y(j) + h_2\sum_{i=-h_2}^{-1}\sum_{j=k+i}^{k-1} y^T(j)Z_2y(j)\\ &\quad +(h_2-h_1)\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} y^T(j)Z_3y(j), \end{aligned}$$

$$\begin{aligned}
V_6(k) &= \sum_{i=1}^{M} \delta(i) \sum_{j=k-i}^{k-1} x^T(j) P_6 x(j), \\
V_7(k) &= \sum_{i=k-\tau_2}^{k-1} x^T(i) P_3 x(i) + \sum_{i=-\tau_2+1}^{-\tau_1} \sum_{j=k+i}^{k-1} x^T(j) P_3 x(j), \\
V_8(k) &= \tau_2 \sum_{i=-\tau_2+1}^{0} \sum_{j=k+i-1}^{k-1} y^T(j) P_5 y(j), \\
V_9(k) &= \tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i}^{k-1} y^T(j) R_3 y(j) + (\tau_2 - \tau_1) \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} y^T(j) R_4 y(j), \\
V_{10}(k) &= \tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i}^{k-1} y^T(j) Z_4 y(j) + \tau_2 \sum_{i=-\tau_2}^{-1} \sum_{j=k+i}^{k-1} y^T(j) Z_5 y(j) \\
&\quad +(\tau_2 - \tau_1) \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} y^T(j) Z_6 y(j).
\end{aligned}$$

Evaluating the forward difference of $V_i(k)(i = 1, 2, \ldots, 10)$, along the trajectory of system (3.3) is given by

$$\Delta V(k) = \sum_{i=1}^{10} \Delta V_i(k). \tag{3.15}$$

Let us define for $i = 1, 2, \ldots, 10$,

$$\Delta V_i(k) = V_i(k+1) - V_i(k),$$

where

$$\begin{aligned}
\Delta V_1(k) &= [x(k) + y(k)]^T P_1 [x(k) + y(k)] - x^T(k) P_1 x(k) \\
&\quad + \Bigg[2x^T(k) Q_1^T + 2y^T(k) Q_2^T + 2x^T(k - \tau(k)) Q_3^T + 2 \sum_{i=k-\tau(k)}^{k-1} y^T(i) Q_4^T \\
&\quad + 2f(x(k))^T Q_5^T + 2g(x(k - h(k)))^T Q_6^T + 2 \left(\sum_{i=1}^{+\infty} \delta(i) x(k-i) \right)^T Q_7^T \Bigg] \\
&\quad \times \Bigg[-y(k) + (A_1 - I) x(k) + A_2 x(k - \tau(k)) - A_1 \sum_{i=k-\tau(k)}^{k-1} y(i) \\
&\quad + Bf(x(k)) + Cg(x(k - h(k))) + D \sum_{i=1}^{+\infty} \delta(i) x(k-i) \Bigg], \qquad (3.16) \\
\Delta V_2(k) &= x^T(k) P_2 x(k) - x^T(k - h_2) P_2 x(k - h_2) \\
&\quad + \sum_{i=-h_2+1}^{-h_1} \left[x^T(k) P_2 x(k) - x^T(k+i) P_2 x(k+i) \right]
\end{aligned}$$

$$
\begin{aligned}
&= (h_{12}+1)x^T(k)P_2x(k) - x^T(k-h_2)P_2x(k-h_2) - \sum_{i=k-h_2+1}^{k-h_1} x^T(i)P_2x(i) \\
&\leq (h_{12}+1)x^T(k)P_2x(k) - x^T(k-h_2)P_2x(k-h_2). \qquad (3.17)
\end{aligned}
$$

Based on Lemma 4, the forward difference of $V_3(k)$ is calculated as

$$
\begin{aligned}
\Delta V_3(k) &= h_2 \sum_{i=-h_2+1}^{0} \left[y^T(k)P_4y(k) - y^T(k+i-1)P_4y(k+i-1) \right] \\
&\leq h_2^2 y^T(k)P_4y(k) - \sum_{i=k-h_2}^{k-1} y^T(i)P_4 \sum_{i=k-h_2}^{k-1} y(i) \\
&\leq h_2^2 y^T(k)P_4y(k) - \sum_{i=k-h(k)}^{k-1} y^T(i)P_4 \sum_{i=k-h(k)}^{k-1} y(i). \qquad (3.18) \\
\Delta V_4(k) &= h_1^2 y^T(k)R_1y(k) + h_{12}^2 y^T(k)R_2y(k) \\
&\quad -h_1 \sum_{i=k-h_1}^{k-1} y^T(i)R_1y(i) - h_{12} \sum_{i=k-h_2}^{k-h_1-1} y^T(i)R_2y(i). \qquad (3.19) \\
\Delta V_5(k) &= h_1 \sum_{i=-h_1}^{-1} \left[y^T(k)Z_1y(k) - y^T(k+i)Z_1y(k+i) \right] \\
&\quad +h_2 \sum_{i=-h_2}^{-1} \left[y^T(k)Z_2y(k) - y^T(k+i)Z_2y(k+i) \right] \\
&\quad +h_{12} \sum_{i=-h_2}^{-h_1-1} \left[y^T(k)Z_3y(k) - y^T(k+i)Z_3y(k+i) \right] \\
&= h_1^2 y^T(k)Z_1y(k) - h_1 \sum_{i=k-h_1}^{k-1} y^T(i)Z_1y(i) \\
&\quad +h_2^2 y^T(k)Z_2y(k) - h_2 \sum_{i=k-h_2}^{k-1} y^T(i)Z_2y(i) \\
&\quad +h_{12}^2 y^T(k)Z_3y(k) - h_{12} \sum_{i=k-h_2}^{k-h_1-1} y^T(i)Z_3y(i). \qquad (3.20) \\
\Delta V_6(k) &\leq x^T(k)\left(\xi P_6\right)x(k) - \left[\sum_{i=1}^{M} \delta(i)x(k-i) \right]^T \left(\frac{1}{\xi} P_6 \right) \left[\sum_{i=1}^{M} \delta(i)x(k-i) \right]. \qquad (3.21) \\
\Delta V_7(k) &= x^T(k)P_3x(k) - x^T(k-\tau_2)P_3x(k-\tau_2) \\
&\quad + \sum_{i=-\tau_2+1}^{-\tau_1} \left[x^T(k)P_3x(k) - x^T(k+i)P_3x(k+i) \right] \\
&\leq (\tau_{12}+1)x^T(k)P_3x(k) - x^T(k-\tau_2)P_3x(k-\tau_2). \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_8(k) &\leq \tau_2^2 y^T(k)P_5 y(k) - \sum_{i=k-\tau_2}^{k-1} y^T(i)P_5 \sum_{i=k-\tau_2}^{k-1} y(i) \\
&\leq \tau_2^2 y^T(k)P_5 y(k) - \sum_{i=k-\tau(k)}^{k-1} y^T(i)P_5 \sum_{i=k-\tau(k)}^{k-1} y(i). \qquad (3.23) \\
\Delta V_9(k) &= \tau_1^2 y^T(k)R_3 y(k) + \tau_{12}^2 y^T(k)R_4 y(k) \\
&\quad -\tau_1 \sum_{i=k-\tau_1}^{k-1} y^T(i)R_3 y(i) - \tau_{12} \sum_{i=k-\tau_2}^{k-\tau_1-1} y^T(i)R_4 y(i). \qquad (3.24) \\
\Delta V_{10}(k) &= \tau_1^2 y^T(k)Z_4 y(k) - \tau_1 \sum_{i=k-\tau_1}^{k-1} y^T(i)Z_4 y(i) \\
&\quad +\tau_2^2 y^T(k)Z_5 y(k) - \tau_2 \sum_{i=k-\tau_2}^{k-1} y^T(i)Z_5 y(i) \\
&\quad +\tau_{12}^2 y^T(k)Z_6 y(k) - \tau_{12} \sum_{i=k-\tau_2}^{k-\tau_1-1} y^T(i)Z_6 y(i). \qquad (3.25)
\end{aligned}
$$

By Lemma 5, four terms from $\Delta V_4(k)$ and $\Delta V_9(k)$ can each be driven as

$$
\begin{aligned}
\sum_{i=k-h_1}^{k-1} y^T(i)R_1 y(i) &\geq \frac{1}{h_1}[x(k) - x(k-h_1)]^T R_1 [x(k) - x(k-h_1)] \\
&\quad + \frac{3}{h_1}\left[x(k) + x(k-h_1) - \frac{2}{h_1+1}\sum_{i=k-h_1}^{k} x(i)\right]^T R_1[\star] \\
&\quad + \frac{5}{h_1}\left[x(k) - x(k-h_1) + \frac{6}{h_1+1}\sum_{i=k-h_1}^{k} x(i) \right.\\
&\quad \left. - \frac{12}{(h_1+2)(h_1+1)}\sum_{i=-h_1}^{0}\sum_{j=k+i}^{k} x(i)\right]^T R_1[\star], \\
\sum_{i=k-h_2}^{k-h_1-1} y^T(i)R_2 y(i) &\geq \frac{1}{h_{12}}[x(k-h_1) - x(k-h_2)]^T R_2[\star] \\
&\quad + \frac{3}{h_{12}}\left[x(k-h_1) + x(k-h_2) - \frac{2}{h_{12}+1}\sum_{i=k-h_2}^{k-h_1} x(i)\right]^T R_2[\star] \\
&\quad + \frac{5}{h_{12}}\left[x(k-h_1) - x(k-h_2) + \frac{6}{h_{12}+1}\sum_{i=k-h_2}^{k-h_1} x(i) \right.\\
&\quad \left. - \frac{12}{(h_{12}+2)(h_{12}+1)}\sum_{i=-h_2}^{-h_1}\sum_{j=k+i}^{k-k_1} x(i)\right]^T R_2[\star],
\end{aligned}
$$

$$\begin{aligned}\sum_{i=k-\tau_1}^{k-1} y^T(i)R_3y(i) &\geq \frac{1}{\tau_1}[x(k)-x(k-\tau_1)]^T R_3[x(k)-x(k-\tau_1)]\\ &+\frac{3}{\tau_1}\left[x(k)+x(k-\tau_1)-\frac{2}{\tau_1+1}\sum_{i=k-\tau_1}^{k}x(i)\right]^T R_3[\star]\\ &+\frac{5}{\tau_1}\left[x(k)-x(k-\tau_1)+\frac{6}{\tau_1+1}\sum_{i=k-\tau_1}^{k}x(i)\right.\\ &\left.-\frac{12}{(\tau_1+2)(\tau_1+1)}\sum_{i=-\tau_1}^{0}\sum_{j=k+i}^{k}x(i)\right]^T R_3[\star],\\ \sum_{i=k-\tau_2}^{k-\tau_1-1} y^T(i)R_4y(i) &\geq \frac{1}{\tau_{12}}[x(k-\tau_1)-x(k-\tau_2)]^T R_4[\star]\\ &+\frac{3}{\tau_{12}}\left[x(k-\tau_1)+x(k-\tau_2)-\frac{2}{\tau_{12}+1}\sum_{i=k-\tau_2}^{k-\tau_1}x(i)\right]^T R_4[\star]\\ &+\frac{5}{\tau_{12}}\left[x(k-\tau_1)-x(k-\tau_2)+\frac{6}{\tau_{12}+1}\sum_{i=k-\tau_2}^{k-\tau_1}x(i)\right.\\ &\left.-\frac{12}{(\tau_{12}+2)(\tau_{12}+1)}\sum_{i=-\tau_2}^{-\tau_1}\sum_{j=k+i}^{k-k_1}x(i)\right]^T R_4[\star].\end{aligned}$$

By Lemma 6, six terms from $\Delta V_5(k)$ and $\Delta V_{10}(k)$ can each be driven as

$$\begin{aligned}&-\sum_{i=k-h_1}^{k-1} y^T(i)Z_1y(i)\\ \leq\ &\begin{bmatrix}x(k)\\ x(k-h(k))\\ x(k-h_1)\end{bmatrix}^T\begin{bmatrix}L_1+L_1^T & -L_1^T+L_2 & 0\\ * & L_1+L_1^T-L_2-L_2^T & -L_1^T+L_2\\ * & * & -L_2-L_2^T\end{bmatrix}[\star]\\ &+h_1\begin{bmatrix}x(k)\\ x(k-h(k))\\ x(k-h_1)\end{bmatrix}^T\begin{bmatrix}L_3 & L_4 & 0\\ * & L_3+L_5 & L_4\\ * & * & L_5\end{bmatrix}[\star],\end{aligned}$$

$$\begin{aligned}&-\sum_{i=k-h_1}^{k-1} y^T(i)Z_2y(i)\\ \leq\ &\begin{bmatrix}x(k)\\ x(k-h(k))\\ x(k-h_2)\end{bmatrix}^T\begin{bmatrix}M_1+M_1^T & -M_1^T+M_2 & 0\\ * & M_1+M_1^T-M_2-M_2^T & -M_1^T+M_2\\ * & * & -M_2-M_2^T\end{bmatrix}[\star]\\ &+h_1\begin{bmatrix}x(k)\\ x(k-h(k))\\ x(k-h_2)\end{bmatrix}^T\begin{bmatrix}M_3 & M_4 & 0\\ * & M_3+M_5 & M_4\\ * & * & M_5\end{bmatrix}[\star],\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=k-h_2}^{k-h_1-1} y^T(i)Z_3y(i) \\
&\leq \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \\ x(k-h_2) \end{bmatrix}^T \begin{bmatrix} N_1+N_1^T & -N_1^T+N_2 & 0 \\ * & N_1+N_1^T-N_2-N_2^T & -N_1^T+N_2 \\ * & * & -N_2-N_2^T \end{bmatrix} [\star] \\
&\quad +h_1 \begin{bmatrix} x(k-h_1) \\ x(k-h(k)) \\ x(k-h_2) \end{bmatrix}^T \begin{bmatrix} N_3 & N_4 & 0 \\ * & N_3+N_5 & N_4 \\ * & * & N_5 \end{bmatrix} [\star], \\
&-\sum_{i=k-h_1}^{k-1} y^T(i)Z_4y(i) \\
&\leq \begin{bmatrix} x(k) \\ x(k-\tau(k)) \\ x(k-\tau_1) \end{bmatrix}^T \begin{bmatrix} S_1+S_1^T & -S_1^T+S_2 & 0 \\ * & S_1+S_1^T-S_2-S_2^T & -S_1^T+S_2 \\ * & * & -S_2-S_2^T \end{bmatrix} [\star] \\
&\quad +h_1 \begin{bmatrix} x(k) \\ x(k-\tau(k)) \\ x(k-\tau_1) \end{bmatrix}^T \begin{bmatrix} S_3 & S_4 & 0 \\ * & S_3+S_5 & S_4 \\ * & * & S_5 \end{bmatrix} [\star],
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=k-h_1}^{k-1} y^T(i)Z_5y(i) \\
&\leq \begin{bmatrix} x(k) \\ x(k-\tau(k)) \\ x(k-\tau_2) \end{bmatrix}^T \begin{bmatrix} T_1+T_1^T & -T_1^T+T_2 & 0 \\ * & T_1+T_1^T-T_2-T_2^T & -T_1^T+T_2 \\ * & * & -T_2-T_2^T \end{bmatrix} [\star] \\
&\quad +h_1 \begin{bmatrix} x(k) \\ x(k-\tau(k)) \\ x(k-\tau_2) \end{bmatrix}^T \begin{bmatrix} T_3 & T_4 & 0 \\ * & T_3+T_5 & T_4 \\ * & * & T_5 \end{bmatrix} [\star], \\
&-\sum_{i=k-h_2}^{k-h_1-1} y^T(i)Z_6y(i) \\
&\leq \begin{bmatrix} x(k-\tau_1) \\ x(k-\tau(k)) \\ x(k-\tau_2) \end{bmatrix}^T \begin{bmatrix} U_1+U_1^T & -U_1^T+U_2 & 0 \\ * & U_1+U_1^T-U_2-U_2^T & -U_1^T+U_2 \\ * & * & -U_2-U_2^T \end{bmatrix} [\star] \\
&\quad +h_1 \begin{bmatrix} x(k-\tau_1) \\ x(k-\tau(k)) \\ x(k-\tau_2) \end{bmatrix}^T \begin{bmatrix} U_3 & U_4 & 0 \\ * & U_3+U_5 & U_4 \\ * & * & U_5 \end{bmatrix} [\star].
\end{aligned}
$$

From Assumption 2, we have

$$
\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} F_1\Lambda_1^T & -F_2\Lambda_1 \\ -F_2\Lambda_1 & \Lambda_1 \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \leq 0, \tag{3.26}
$$

and

$$\begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix}^T \begin{bmatrix} G_1\Lambda_2^T & -G_2\Lambda_2 \\ -G_2\Lambda_2 & \Lambda_2 \end{bmatrix} \begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix} \leq 0, \tag{3.27}$$

where

$$\begin{aligned}
\Lambda_1 &= diag\{\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1n}\}, \\
\Lambda_2 &= diag\{\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2n}\}, \\
F_1 &= diag\{F_1^+F_1, F_2^+F_2, \ldots, F_n^+F_n\}, \\
F_2 &= diag\left\{\frac{F_1^+ + F_1^-}{2}, \frac{F_2^+ + F_2^-}{2}, \ldots, \frac{F_n^+ + F_n^-}{2}\right\}, \\
G_1 &= diag\{G_1^+G_1, G_2^+G_2, \ldots, G_n^+G_n\}, \\
G_2 &= diag\left\{\frac{G_1^+ + G_1^-}{2}, \frac{G_2^+ + G_2^-}{2}, \ldots, \frac{G_n^+ + G_n^-}{2}\right\},
\end{aligned}$$

where $\lambda_{1i}, \lambda_{2i}, F_i^-, F_i^+, G_i^-$, and G_i^+ (i=1,2,...,n) are known real constants.

According to (3.12)–(3.27), it is straightforward to see that

$$\Delta V(k) \leq \xi^T(k)\Pi\xi(k), \tag{3.28}$$

where $\xi(k) = \Big[x^T(k),\ y^T(k),\ x^T(k-h(k)),\ x^T(k-h_1),\ x^T(k-h_2),\ \sum_{i=k-h(k)}^{k-1} y^T(i),\ \sum_{i=k-h_1}^{k} x^T(i),$
$\sum_{i=-h_1}^{0}\sum_{j=k+i}^{k} x(i),\ \sum_{i=k-h_{12}}^{k} x^T(i),\ \sum_{i=-h_2}^{0}\sum_{j=k+i}^{k-h_1} x^T(i),\ x^T(k-\tau(k)),\ x^T(k-\tau_1),\ x^T(k-\tau_2),$
$\sum_{i=k-\tau(k)}^{k-1} y^T(i),\ \sum_{i=k-\tau_1}^{k} x^T(i),\ \sum_{i=-\tau_1}^{0}\sum_{j=k+i}^{k} x^T(i),\ \sum_{i=k-\tau_2}^{k} x^T(i),\ \sum_{i=-\tau_2}^{0}\ \sum_{j=k+i}^{k-\tau_1} x^T(i),\ f^T(x(k)),$
$g^T(x(k-h(k))),\ \left(\sum_{i=1}^{M}\delta(i)x(k-i)\right)^T \Big]^T$, and Π is defined in (3.2). From (3.4)–(3.10), system (3.3) is asymptotically stable, as defined in Definition 2. The theorem is now complete in its proof. □

If leakage delay term disappears, that is $\tau(k) = 0$, the neural networks system (3.3) reduces to

$$\begin{cases} x(k+1) = Ax(k) + Bf(x(k)) + Cg(x(k-h(k))) + D\sum_{i=1}^{M}\delta(i)x(k-i), \\ x(s) = \phi(s), \quad s \in \{-h_2, -h_2+1, \ldots, -1, 0, \}. \end{cases} \tag{3.29}$$

The delay-dependent stability criterion for the system in (3.29) can be directly deduced from Theorem 1.

We introduce the following notations for later use

$$\bar{\Pi} = \left[\bar{\Pi}_{i,j}\right]_{13\times 13}, \tag{3.30}$$

where $\bar{\Pi}_{i,j} = \bar{\Pi}_{j,i}^T = \Pi_{i,j}$, $i, j = 1, 2, 3, \ldots, 10, 19, 20, 21$, and it is presented in the following theorem.

Theorem 2. *The system* (3.29) *is asymptotically stable, if there exist positive definite symmetric matrices* $P_i, Q_j, R_k, Z_l,\ i = 1,2,4 =,6,\ j = 1,2,3,\ldots,6,\ k = 1,2, l = 1,2,3$ *and any appropriate dimensional matrices* Λ_1, Λ_2, *satisfying the following LMIs*

$$\bar{\Pi} < 0, \tag{3.31}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2 \\ * & L_3 & L_4 \\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.32}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.33}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2 \\ * & N_3 & N_4 \\ * & * & N_5 \end{bmatrix} \geq 0. \tag{3.34}$$

Proof. Based on the same method as Theorem 1, but for this estimation we do not decompose matrix A, thereby rewriting the system (3.29) with model transformation method as the following descriptor system

$$x(k+1) = x(k) + y(k), \tag{3.35}$$

$$y(k) = (A - I)x(k) + Bf(x(k)) + Cg(x(k-h(k))) + D\sum_{i=1}^{M}\delta(i)x(k-i). \tag{3.36}$$

Construct the following Lyapunov-Krasovskii functional as

$$V(k) = \sum_{i=1}^{6} V_i(k), \tag{3.37}$$

where

$$\begin{aligned}
V_1(k) &= x^T(k)P_1x(k),\\
V_2(k) &= \sum_{i=k-h_2}^{k-1} x^T(i)P_2x(i) + \sum_{i=-h_2+1}^{-h_1}\sum_{j=k+i}^{k-1} x^T(j)P_2x(j),\\
V_3(k) &= h_2\sum_{i=-h_2+1}^{0}\sum_{j=k+i-1}^{k-1} y^T(j)P_4y(j),\\
V_4(k) &= h_1\sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1} y^T(j)R_1y(j) + (h_2-h_1)\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} y^T(j)R_2y(j),\\
V_5(k) &= h_1\sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1} y^T(j)Z_1y(j) + h_2\sum_{i=-h_2}^{-1}\sum_{j=k+i}^{k-1} y^T(j)Z_2y(j)\\
&\quad +(h_2-h_1)\sum_{i=-h_2}^{-h_1-1}\sum_{j=k+i}^{k-1} y^T(j)Z_3y(j),
\end{aligned}$$

$$V_6(k) = \sum_{i=1}^{M} \delta(i) \sum_{j=k-i}^{k-1} x^T(j) P_6 x(j).$$

When the forward difference of $V(k)$ is calculated, it is defined as

$$\Delta V(k) = \sum_{i=1}^{6} \Delta V_i(k). \tag{3.38}$$

Let us define for $i = 1, 2, \ldots, 6$,

$$\Delta V_i(k) = V_i(k+1) - V_i(k).$$

We can estimate $V_1(k)$ as follows.

$$\begin{aligned} \Delta V_1(k) = & \left[x(k) + y(k)\right]^T P_1 \left[x(k) + y(k)\right] - x^T(k) P_1 x(k) \\ & + \Bigg[2x^T(k) Q_1^T + 2y^T(k) Q_2^T + 2f(x(k))^T Q_4^T + 2g(x(k-h(k)))^T Q_5^T \\ & + 2\left(\sum_{i=1}^{M} \delta(i) x(k-i)\right)^T Q_6^T\Bigg] \Bigg[- y(k) + (A - I)x(k) + Bf(x(k)) \\ & + Cg(x(k-h(k))) + D \sum_{i=1}^{M} \delta(i) x(k-i)\Bigg]. \end{aligned} \tag{3.39}$$

The proof after this step is omitted since it is analogous to the derivation of the Theorem 1. □

When $D = 0$, the neural networks system (3.3) becomes

$$\begin{cases} x(k+1) = Ax(k-\tau(k)) + Bf(x(k)) + Cg(x(k-h(k))), k \in Z^+, \\ x(s) = \phi(s), \quad s \in \{-\rho, -\rho+1, \ldots, -1, 0, \}. \end{cases} \tag{3.40}$$

The delay-dependent stability criterion for the system in (3.40) can be directly deduced from Theorem 1.

We introduce the following notations for later use

$$\hat{\Pi} = \left[\hat{\Pi}_{i,j}\right]_{20\times 20}, \tag{3.41}$$

where $\hat{\Pi}_{i,j} = \hat{\Pi}_{j,i}^T = \Pi_{i,j}$, $i, j = 1, 2, 3, \ldots, 20$, and it is presented in the following corollary.

Corollary 1. *For given integers h_1, h_2 satisfying $0 < h_1 \leq h_2$, system (3.40) is asymptotically stable for $0 < h_1 \leq h(k) \leq h_2$, if there exist positive definite matrices P_i, Q_j, R_k, Z_i, $i = 1, 2, 3, \ldots, 6$, $j = 1, 2, 3, \ldots, 6$, $k = 1, 2, 3, 4$, and any appropriate dimensional matrices Λ_1, Λ_2, satisfying the following LMIs, satisfying the following LMIs.*

$$\hat{\Pi} < 0, \tag{3.42}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2 \\ * & L_3 & L_4 \\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.43}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.44}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2 \\ * & N_3 & N_4 \\ * & * & N_5 \end{bmatrix} \geq 0, \tag{3.45}$$

$$\begin{bmatrix} Z_4 & S_1 & S_2 \\ * & S_3 & S_4 \\ * & * & S_5 \end{bmatrix} \geq 0, \tag{3.46}$$

$$\begin{bmatrix} Z_5 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & T_5 \end{bmatrix} \geq 0, \tag{3.47}$$

$$\begin{bmatrix} Z_6 & U_1 & U_2 \\ * & U_3 & U_4 \\ * & * & U_5 \end{bmatrix} \geq 0. \tag{3.48}$$

Proof. The proof has been skipped since it is almost identical to the derivation of Theorem 1 except that the matrix D is not included in the proof. □

If leakage delay term disappears and $D = 0$, the neural networks (3.29) becomes

$$\begin{cases} x(k+1) = Ax(k) + Bf(x(k)) + Cg(x(k-h(k))), k \in Z^+ \\ x(s) = \phi(s), \quad s \in \{-h_2, -h_2+1, \ldots, -1, 0, \}. \end{cases} \tag{3.49}$$

The delay-dependent stability criterion for the system in (3.49) can be directly deduced from Theorem 2.

We introduce the following notations for later use

$$\hat{\hat{\Pi}} = \left[\hat{\hat{\Pi}}_{i,j}\right]_{12\times 12}, \tag{3.50}$$

where $\hat{\hat{\Pi}}_{i,j} = \hat{\hat{\Pi}}^T_{j,i} = \Pi_{i,j}$, $i, j = 1, 2, 3, \ldots, 10, 19, 20$, and it is presented in the following corollary.

Corollary 2. *For given integers h_1, h_2 satisfying $0 < h_1 \leq h_2$, system* (3.49) *is asymptotically stable for $0 < h_1 \leq h(k) \leq h_2$, if there exist positive definite matrices P_i, Q_j, R_k, Z_l, $i = 1, 2, 4, 6$, $j = 1, 2, 3, \ldots, 6$, $k = 1, 2$, $l = 1, 2, 3$ and any appropriate dimensional matrices Λ_1, Λ_2, satisfying the following LMIs*

$$\hat{\hat{\Pi}} < 0, \tag{3.51}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2 \\ * & L_3 & L_4 \\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.52}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.53}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2 \\ * & N_3 & N_4 \\ * & * & N_5 \end{bmatrix} \geq 0. \tag{3.54}$$

Proof. The proof is removed since it is comparable to the derivation of Theorem 2 without D and hence does not need to be included. □

For system (2.1), we derive robust asymptotic stability using Theorem 1 and the following notations, which will come in handy later.

$$\begin{aligned} {S_n}^T &= [\Gamma^T Q_1 \quad \Gamma^T Q_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \Gamma^T Q_3 \quad 0 \quad 0 \\ &\quad \Gamma^T Q_4 \quad 0 \quad 0 \quad 0 \quad 0 \quad \Gamma^T Q_5 \quad \Gamma^T Q_6 \quad \Gamma^T Q_7 \quad], \end{aligned} \tag{3.55}$$

$$\begin{aligned} N_n &= [H_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -H_1 \quad 0 \\ &\quad 0 \quad 0 \quad 0 \quad H_2 \quad H_3 \quad H_4]. \end{aligned} \tag{3.56}$$

Theorem 3. *The system* (3.1) *is robustly asymptotically stable, if there exist positive definite symmetric matrices P_i, Q_j, R_k, $i = 1, 2, \ldots, 9$, $j = 1, 2, \ldots, 5$, $k = 1, 2, \ldots, 8$, any appropriate dimensional matrices J, T_1, T_2, S_l, J_m, K_m, M_m, N_m, $l = 1, 2, \ldots, 4$, $m = 1, 2, 3$ and any positive real constant δ satisfying the following LMIs*

$$\begin{bmatrix} \Pi & S_n & \delta {N_n}^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{3.57}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2 \\ * & L_3 & L_4 \\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.58}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.59}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2 \\ * & N_3 & N_4 \\ * & * & N_5 \end{bmatrix} \geq 0, \tag{3.60}$$

$$\begin{bmatrix} Z_4 & S_1 & S_2 \\ * & S_3 & S_4 \\ * & * & S_5 \end{bmatrix} \geq 0, \tag{3.61}$$

$$\begin{bmatrix} Z_5 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & T_5 \end{bmatrix} \geq 0, \tag{3.62}$$

$$\begin{bmatrix} Z_6 & U_1 & U_2 \\ * & U_3 & U_4 \\ * & * & U_5 \end{bmatrix} \geq 0. \tag{3.63}$$

Proof. Together with LMIs of Theorem 1, by replacing A_1, B, C and D in (3.4) with $A_1 + \Delta A(k)$, $B + \Delta B(k)$, $C + \Delta C(k)$ and $D + \Delta D(k)$ in (2.5), respectively. Then, we find that condition (3.57) is equivalent to the following condition

$$\Pi + S_n\Delta(k)N_n + N_n{}^T\Delta(k)^T S_n{}^T < 0. \tag{3.64}$$

By using Lemma 1, we obtain that (3.64) is equivalent to the LMIs as follows

$$\begin{bmatrix} \Pi & S_n & \delta N_n{}^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{3.65}$$

where δ is a positive real constant. From Theorem 1 and conditions (3.57)–(3.63), it follows from Definition 1 that system (3.1) is robustly asymptotically stable. This completes the proof of the theorem. □

If $D = 0$, then system (3.1) reduces to the following system

$$\begin{cases} x(k+1) &= (A + \Delta A(k))x(k-\tau(k)) + (B + \Delta B(k))f(x(k)) \\ & \quad +(C + \Delta C(k))g(x(k-h(k))) \\ x(k) &= \phi(k), \quad k = -\rho, -\rho+1, \dots, 0. \end{cases} \tag{3.66}$$

The delay-dependent stability criteria for the system in (3.66) can be directly deduced from Theorem 3.

We introduce the following notations for later use

$$\begin{aligned} \hat{S_n}^T &= [\Gamma^T Q_1 \quad \Gamma^T Q_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \Gamma^T Q_3 \quad 0 \quad 0 \\ &\quad \Gamma^T Q_4 \quad 0 \quad 0 \quad 0 \quad 0 \quad \Gamma^T Q_5 \quad \Gamma^T Q_6 \quad], & (3.67) \\ \hat{N}_n &= [H_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -H_1 \quad 0 \\ &\quad 0 \quad 0 \quad 0 \quad H_2 \quad H_3]. & (3.68) \end{aligned}$$

and it is presented in the following corollary.

Corollary 3. *The system* (3.66) *is robustly asymptotically stable, if there exist positive definite symmetric matrices* P_i, Q_j, R_k, $i = 1, 2, \dots, 9$, $j = 1, 2, \dots, 4, k = 1, 2, \dots, 8$, *any appropriate dimensional matrices* J, T_1, T_2, S_l, J_m, K_m, M_m, N_m, $l = 1, 2, \dots, 4$, $m = 1, 2, 3$ *and any positive real constant* δ *such that the following LMIs hold*

$$\begin{bmatrix} \hat{\Pi} & \hat{S} & \delta \hat{N}^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{3.69}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2 \\ * & L_3 & L_4 \\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.70}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.71}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2 \\ * & N_3 & N_4 \\ * & * & N_5 \end{bmatrix} \geq 0, \tag{3.72}$$

$$\begin{bmatrix} Z_4 & S_1 & S_2 \\ * & S_3 & S_4 \\ * & * & S_5 \end{bmatrix} \geq 0, \tag{3.73}$$

$$\begin{bmatrix} Z_5 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & T_5 \end{bmatrix} \geq 0, \tag{3.74}$$

$$\begin{bmatrix} Z_6 & U_1 & U_2 \\ * & U_3 & U_4 \\ * & * & U_5 \end{bmatrix} \geq 0. \tag{3.75}$$

Proof. Together with LMI results of Corollary 1, by replacing A_1 and B in (3.42) with $A_1 + \Delta A(k)$ and $B + \Delta B(k)$ in (2.5), respectively. Then, we find that condition (3.69) is equivalent to the following condition

$$\hat{\Pi} + \hat{S}\Delta(k)\hat{N} + \hat{N}^T\Delta(k)^T\hat{S}^T < 0. \tag{3.76}$$

By using Lemma 1, we obtain that (3.76) is equivalent to the LMI as follows

$$\begin{bmatrix} \hat{\Pi} & \hat{S} & \delta\hat{N}^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{3.77}$$

where δ is a positive real constant. From Corollary 1 and conditions (3.69)–(3.75), system (3.66) is robustly asymptotically stable. The proof is completed. □

3.2. Passivity analysis for discrete-time neural network

This subsection focuses on the robust passivity analysis of the uncertain linear discrete-time system with interval discrete and distributed time-varying delays (2.1). The LMI-based conditions will be derived using the Lyapunov technique.

First and foremost, we introduce the following notations for later use

$$S_{n0}^T = [S_n^T \quad 0], \qquad N_{n0} = [N_n \quad 0], \qquad \breve{\Pi} = \left[\breve{\Pi}_{i,j}\right]_{22\times 22},$$

where $\breve{\Pi}_{i,j} = \breve{\Pi}_{j,i}^T = \Pi_{i,j}$, $i, j = 1, 2, 3, \ldots, 22$,

$$\begin{aligned}
&\breve{\Pi}_{1,22} = Q_8^T(A_1 - I) + Q_1 && \breve{\Pi}_{2,22} = -Q_8^T + Q_2,\\
&\breve{\Pi}_{11,22} = -A_z^T + Q_8^T A_2 + Q_3, && \breve{\Pi}_{14,22} = -Q_8^T A_1 + Q_4,\\
&\breve{\Pi}_{19,22} = -B_z^T + Q_8^T B + Q_5, && \breve{\Pi}_{20,22} = -C_z^T + Q_8^T C + Q_6,\\
&\breve{\Pi}_{21,22} = -D_z^T + Q_8^T D + Q_7, && \breve{\Pi}_{22,22} = -\gamma I + Q_8^T + Q_8,
\end{aligned}$$

and others are equal to zero.

Theorem 4. *The system* (2.1) *is robustly passive, if there exist positive definite symmetric matrices* $P_i, Q_j, R_k,\ i = 1,2,\ldots,10, j = 1,2,\ldots,6, k = 1,2,\ldots,8,$ *any appropriate dimensional matrices* $J, T_1, T_2, S_l,\ J_m, K_m, M_m, N_m,\ l = 1,2,\ldots,4, m = 1,2,3$ *and any positive real constant* δ, γ *satisfying the following LMIs*

$$\begin{bmatrix} \breve{\Pi} & S_{n0} & \delta N_{n0}^T \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{3.78}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2 \\ * & L_3 & L_4 \\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.79}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.80}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2 \\ * & N_3 & N_4 \\ * & * & N_5 \end{bmatrix} \geq 0, \tag{3.81}$$

$$\begin{bmatrix} Z_4 & S_1 & S_2 \\ * & S_3 & S_4 \\ * & * & S_5 \end{bmatrix} \geq 0, \tag{3.82}$$

$$\begin{bmatrix} Z_5 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & T_5 \end{bmatrix} \geq 0, \tag{3.83}$$

$$\begin{bmatrix} Z_6 & U_1 & U_2 \\ * & U_3 & U_4 \\ * & * & U_5 \end{bmatrix} \geq 0. \tag{3.84}$$

Proof. The proof follows from Theorem 1 and Theorem 3 by choosing the Lyapunov-Krasovskii functional (3.14) and the forward differences in (3.16)–(3.27) with (2.1)–(2.8) and conditions (3.78)–(3.84), it follows that

$$\Delta V(k) + \left(-2z^T(k)w(k) - \gamma w^T(k)w(k)\right) \leq 0. \tag{3.85}$$

Given a positive integer l and summing both sides of (3.85) from 0 to l with respect to k results in

$$\sum_{k=0}^{l} \Delta V(k) + \sum_{k=0}^{l} \left(-2z^T(k)w(k) - \gamma w^T(k)w(k)\right) \leq 0,$$

$$V(l+1) - V(0) - 2\sum_{k=0}^{l} z^T(k)w(k) - \gamma \sum_{k=0}^{l} w^T(k)w(k) \leq 0.$$

Under the zero condition, we have

$$-\gamma \sum_{k=0}^{l} w^T(k)w(k) \leq 2\sum_{k=0}^{l} z^T(k)w(k). \tag{3.86}$$

Therefore from (3.86), it is easy to get the inequality in Definition 3. Hence it can conclude that the system (2.1) is robustly passive. The proof of this theorem is completed. □

If $D = D_z = 0$, then system (2.1) reduces to the following system

$$\begin{cases} x(k+1) &= (A + \Delta A(k))x(k-\tau(k)) + (B + \Delta B(k))f(x(k)) \\ & \quad +(C + \Delta C(k))g(x(k-h(k))) + w(k), \\ z(k) &= A_z x(k-\tau(k)) + B_z f(x(k)) + C_z g(x(k-h(k))) \\ x(k) &= \phi(k), \quad k = -\rho, -\rho+1, \ldots, 0. \end{cases} \tag{3.87}$$

The delay-dependent passivity criterion for the system in (3.87) can be directly deduced from Theorem 4. We introduce the following notations for later use

$$\hat{S}_{n0}^{\,T} = [\hat{S}_n^{\,T} \quad 0], \qquad \hat{N}_{n0} = [\hat{N}_n \quad 0], \qquad \tilde{\Pi} = \left[\tilde{\Pi}_{i,j}\right]_{21\times 21},$$

where $\tilde{\Pi}_{i,j} = \tilde{\Pi}_{j,i}^T = \hat{\Pi}_{i,j}$, $i, j = 1, 2, 3, \ldots, 21$,

$$\begin{aligned} &\tilde{\Pi}_{1,21} = Q_8^T(A_1 - I) + Q_1 \qquad \tilde{\Pi}_{2,21} = -Q_8^T + Q_2 \\ &\tilde{\Pi}_{11,21} = -A_z^T + Q_8^T A_2 + Q_3, \qquad \tilde{\Pi}_{14,21} = -Q_8^T A_1 + Q_4, \\ &\tilde{\Pi}_{19,21} = -B_z^T + Q_8^T B + Q_5, \qquad \tilde{\Pi}_{20,21} = -C_z^T + Q_8^T C + Q_6, \\ &\tilde{\Pi}_{21,21} = -\gamma I + Q_8^T + Q_8, \end{aligned}$$

and others are equal to zero.

Corollary 4. *The system* (3.87) *is robustly passive if there exist positive definite symmetric matrices* P_i, Q_j, R_k, $i = 1, 2, \ldots, 9$, $j = 1, 2, \ldots, 4, 6$, $k = 1, 2, \ldots, 8$, *any appropriate dimensional matrices* $J, T_1, T_2, S_l, J_m, K_m, M_m, N_m$, $l = 1, 2, \ldots, 4$, $m = 1, 2, 3$ *and any positive real constant* δ, γ *such that the following LMIs hold*

$$\begin{bmatrix} \tilde{\Pi} & \hat{S}_{n0} & \delta \hat{N}_{n0}^{\,T} \\ * & -\delta I & \delta E^T \\ * & * & -\delta I \end{bmatrix} < 0, \tag{3.88}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2 \\ * & L_3 & L_4 \\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.89}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.90}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2 \\ * & N_3 & N_4 \\ * & * & N_5 \end{bmatrix} \geq 0, \tag{3.91}$$

$$\begin{bmatrix} Z_4 & S_1 & S_2 \\ * & S_3 & S_4 \\ * & * & S_5 \end{bmatrix} \geq 0, \tag{3.92}$$

$$\begin{bmatrix} Z_5 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & T_5 \end{bmatrix} \geq 0, \tag{3.93}$$

$$\begin{bmatrix} Z_6 & U_1 & U_2 \\ * & U_3 & U_4 \\ * & * & U_5 \end{bmatrix} \geq 0. \tag{3.94}$$

Proof. The proof is omitted since it is analogous to the derivation of Theorem 4 with Definition 3. □

If If leakage delay term disappears and $D = D_z = 0$, then system (2.1) reduces to the following system

$$\begin{cases} x(k+1) = Ax(k-\tau(k)) + Bf(x(k)) + Cg(x(k-h(k))) + w(k), \\ z(k) = A_z x(k-\tau(k)) + B_z f(x(k)) + C_z g(x(k-h(k))) \\ x(k) = \phi(k), \quad k = -\rho, -\rho+1, \ldots, 0. \end{cases} \tag{3.95}$$

The delay-dependent passivity criterion for the system in (3.95) can be directly deduced from Theorem 4. We introduce the following notations for later use

$$\tilde{\Pi} = \left[\tilde{\Pi}_{i,j}\right]_{21\times 21},$$

where $\tilde{\Pi}_{i,j} = \tilde{\Pi}_{j,i}^T = \hat{\Pi}_{i,j}$, $i, j = 1, 2, 3, \ldots, 21$,

$$\begin{aligned} &\tilde{\Pi}_{1,21} = Q_8^T(A_1 - I) + Q_1 && \tilde{\Pi}_{2,21} = -Q_8^T + Q_2 \\ &\tilde{\Pi}_{11,21} = -A_z^T + Q_8^T A_2 + Q_3, && \tilde{\Pi}_{14,21} = -Q_8^T A_1 + Q_4, \\ &\tilde{\Pi}_{19,21} = -B_z^T + Q_8^T B + Q_5, && \tilde{\Pi}_{20,21} = -C_z^T + Q_8^T C + Q_6, \\ &\tilde{\Pi}_{21,21} = -\gamma I + Q_8^T + Q_8, \end{aligned}$$

and others are equal to zero.

Corollary 5. *The system* (3.95) *is passive, if there exist positive definite symmetric matrices* P_i, Q_j, R_k, $i = 1, 2, \ldots, 9$, $j = 1, 2, \ldots, 4, 6$, $k = 1, 2, \ldots, 8$, *any appropriate dimensional matrices* J, T_1, T_2, S_l, J_m, K_m, M_m, N_m, $l = 1, 2, \ldots, 4$, $m = 1, 2, 3$ *and any positive real constant* γ *such that the following LMIs hold*

$$\tilde{\Pi} < 0, \tag{3.96}$$

$$\begin{bmatrix} Z_1 & L_1 & L_2 \\ * & L_3 & L_4 \\ * & * & L_5 \end{bmatrix} \geq 0, \tag{3.97}$$

$$\begin{bmatrix} Z_2 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geq 0, \tag{3.98}$$

$$\begin{bmatrix} Z_3 & N_1 & N_2 \\ * & N_3 & N_4 \\ * & * & N_5 \end{bmatrix} \geq 0, \tag{3.99}$$

$$\begin{bmatrix} Z_4 & S_1 & S_2 \\ * & S_3 & S_4 \\ * & * & S_5 \end{bmatrix} \geq 0, \tag{3.100}$$

$$\begin{bmatrix} Z_5 & T_1 & T_2 \\ * & T_3 & T_4 \\ * & * & T_5 \end{bmatrix} \geq 0, \tag{3.101}$$

$$\begin{bmatrix} Z_6 & U_1 & U_2 \\ * & U_3 & U_4 \\ * & * & U_5 \end{bmatrix} \geq 0. \tag{3.102}$$

Proof. As with the derivation of Theorem 4 with Definition 3, the proof is skipped here for simplicity's sake. □

Remark 1. *The problem of new delay-range-dependent asymptotic stability criteria for uncertain discrete-time neural networks with interval discrete, distributed, and leakage time-varying delays (Theorems 1–3, Corollarys 1–3) is studied. Moreover, new delay-range-dependent passivity criteria are also investigated for uncertain discrete-time neural networks with interval discrete, distributed, and leakage time-varying delays (Theorem 4, Corollarys 5 and 6). We use a mixed techniques such as new inequalities, Jensen inequality, coefficient matrix decomposition technique, utilization of zero equation, mixed model transformation, and reciprocally convex combination. Using the above new LKFs and the lemmas leads to less conservatism of the obtained results than in published literature, as presented via numerical examples.*

3.3. Numerical examples for discrete-time neural network

In this part, we will provide numerical examples that will illustrate the efficacy and application of the techniques that are being discussed.

Example 1. *Illustrate the effectiveness of the proposed stability criterion (Theorem 1) for the discrete-time system subjected to norm-bounded uncertainties* (3.3) *with parameters as follows*

$$A_1 = A_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.3 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0 \\ 0 & -0.1 & -0.4 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.4 & 0.2 & -0.1 \\ 0 & 0.2 & 0.3 \\ -0.1 & 0 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} -0.2 & 0.1 & 0 \\ 0.2 & 0.3 & 0.2 \\ 0 & -0.2 & 0.2 \end{bmatrix}.$$

For the activation functions

$$F_1 = G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_2 = G_2 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

In addition, we choose $2 \leq \tau(k) \leq 8$. Then, by using the MATLAB LMI Toolbox, we solve LMI (3.4)–(3.10)*, and the corresponding values of the permissible upper limits of h_2 for a range of h_1 values from 4 to 20 are also computed and given in Table 1 as follows:*

Table 1. Upper bounds of time delay h_2 for different h_1 for Example 1.

h_1	4	5	6	7	8	9	10	12	15	20
Theorem2	65	64	64	63	62	62	60	58	56	54

Example 2. *Consider system* (3.29) *with Theorem 2 and the following parameters as*

$$A_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.3 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0 \\ 0 & -0.1 & -0.4 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.4 & 0.2 & -0.1 \\ 0 & 0.2 & 0.3 \\ -0.1 & 0 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} -0.2 & 0.1 & 0 \\ 0.2 & 0.3 & 0.2 \\ 0 & -0.2 & 0.2 \end{bmatrix}.$$

For the activation functions

$$F_1 = G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_2 = G_2 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

Then, by using the MATLAB LMI Toolbox, we solve LMI (3.31)–(3.34)*, and the corresponding values of the permissible upper limits of h_2 for a range of h_1 values from 4 to 20 are also computed and given in Table 2 as follows:*

Table 2. Upper bounds of time delay h_2 for different h_1 for Example 2.

h_1	4	5	6	7	8	9	10	12	15	20
Theorem2	86	86	87	87	87	87	88	88	90	92

Example 3. *Consider system* (3.49) *with Corollary 2 and the following parameters*

$$A_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.45 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.45 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, \qquad C = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}.$$

For the activation functions

$$F_1 = G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad F_2 = G_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

Table 3 lists the findings of the maximum delay limits for different h_2 for system (3.49)*. A comparison of current outcomes with those from the past may be seen in Table 3. Table 3 shows that our findings for this particular case provide higher upper limits to the time delay than those in the references [10, 13, 14, 30, 31].*

Table 3. Upper bounds of time delay h_2 for different h_1 for Example 3.

Methods\h_1 =	6	8	10	15	20
Theorem 1 [13]	20	21	21	23	26
Theorem 2 [14]	19	20	21	24	27
Corollary 1 [31]	20	20	21	24	27
Theorem 1 [10]	21	21	22	24	27
Theorem 2 [10]	20	21	22	24	27
Corollary 3.1 [30]	20	22	24	29	34
Corollary 2	23	24	25	30	35

Example 4. *Illustrate the effectiveness of the proposed robust stability criterion (Corollary 3) for the uncertain discrete-time system subjected to norm-bounded uncertainties, consider the following system*

$$x(k+1) = \begin{bmatrix} 0.8+\alpha(k) & 0 \\ 0 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} x(k-h(k)),$$

where $|\alpha(k)| < \alpha$. *The uncertain system can be expressed in the form of* (3.66) *with the following parameters*

$$A_1 = A_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.45 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_2 = H_3 = E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For given interval $[h_1, h_2]$*, the values of* α *such that the robust asymptotic stability of this system are listed in Table 4. From the table, it is clear that the proposed robust stability criterion accommodates a higher perturbation bound for a given delay range than [3, 7, 29, 40] without losing stability.*

Table 4. Upper delay bounds of $\alpha(k)$ for different $[h_1, h_2]$ for Example 4.

$[h_1, h_2]$	[2,7]	[3,9]	[5,10]	[6,12]	[10,15]
Gao and Chen [3]	0.190	0.145	0.131	0.090	0.065
Huang and Fenh [7]	0.192	0.154	0.142	0.114	1.102
Ramakrishnan and Ray [29]	0.195	0.165	0.154	0.131	1.112
Wang et al. [40]	0.205	0.172	0.161	0.138	-
Corollary 3	0.210	0.179	0.168	0.141	1.126

Example 5. *Illustrate the effectiveness of the proposed stability criterion (Theorem 4) for the discrete-time system subjected to norm-bounded uncertainties* (2.1) *with parameters as follows*

$$A_1 = A_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.3 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0 \\ 0 & -0.1 & -0.4 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2 & 0.1 & -0.1 \\ 0 & 0.1 & 0 \\ -0.1 & 0 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} -0.1 & -0.1 & 0 \\ 0.1 & 0.1 & 0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix},$$

$$H_1 = H_2 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \qquad H_3 = H4 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_z = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.08 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \qquad B_z = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$C_z = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \qquad D_z = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

For the activation functions

$$F_1 = G_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \qquad F_2 = G_2 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

In addition, we choose $2 \leq \tau(k) \leq 8$ *and* $\delta(i) = 1, i = 1, 2, ..., M$. *Using the MATHLAB tools to solve LMIs* (3.78)–(3.84)*, and the corresponding values of the permissible upper limits of* h_2 *for a range of* h_1 *values from 4 to 20, we are also computed and given in Table 5 as follows:*

Table 5. Upper bounds of time delay h_2 for different h_1 for Example 1.

h_1	4	5	6	7	8	9	10	12	15	20
Theorem4	30	30	28	28	27	27	26	25	24	22

Remark 2. *An important property in discrete delayed system theory is stability which applies to analyzing properties of passivity of various discrete delayed systems. In recent years, stability and passivity properties have also been related to the different discrete delayed systems [3, 7, 10, 13, 14, 29–31, 40]. Moreover, in this work, we use refined inequality and mixed techniques to improve the stability and passivity criteria. By applying the abovementioned methods, we obtain less conservative results than the others [3, 7, 10, 13, 14, 29–31, 40].*

4. Conclusions

This paper explored discrete-time neural networks with mixed interval time-varying delays for asymptotic stability and passivity. It has also examined how discrete-time neural networks with time interval variations have resilient asymptotic stability and passivity analysis. The study was carried out using a technique that incorporated the enhanced Lyapunov-Krasovskii functional, mixed model transformation, decomposition approach of the coefficient matrix, and usage of zero equations. A novel set of delays-range-dependent robust asymptotic stability criteria was developed and constructed using LMIs. We can demonstrate numerically that our criteria are less conservative than those found in the current literature. Another numerical example has been provided to show the applicability of the discoveries that have been proposed.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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