Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 4964–4972.
DOI: 10.3934/math.2023248
Received: 10 September 2022
Revised: 13 November 2022
Accepted: 20 November 2022
Published: 12 December 2022

Research article

Pseudo subalgebras and pseudo filters in pseudo BE-algebras

Sun Shin Ahn[1,*], **Young Joo Seo**[2] **and Young Bae Jun**[3]

[1] Department of Mathematics Education, Dongguk University, Seoul 04620, Korea
[2] Research Institute for Natural Sciences, Department of Mathematics, Hanyang University, Seoul 04763, Korea
[3] Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

* **Correspondence:** Email: sunshine@dongguk.edu.

Abstract: As a generalization of BE-algebras, the pseudo BE-algebra was introduced by Borzooei et al., and the notions of pseudo subalgebras and pseudo filters in pseudo BE-algebras were defined, and some related properties were investigated. In order to further study pseudo subalgebras and pseudo filters in pseudo BE-algebras, concepts of pseudo atom, atomic pseudo BE-algebra, and atomic pseudo filter are introduced and related studies are conducted. The conditions under which pseudo-filters can be created using a nonempty set, and the conditions under which non-unit elements can be pseudo-atoms are explored. Characterization of atomic pseudo BE-algebra is discussed, and conditions are provided under which pseudo subalgebra can be pseudo filters. The relationship between a set of pseudo atoms and a pseudo subalgebra is considered, and the conditions under which a pseudo filter can be atomic are found.

Keywords: (atomic) pseudo BE-algebra; pseudo subalgebra; (atomic) pseudo filter; pseudo atom
Mathematics Subject Classification: 03G25, 06F35

1. Introduction

BCK-algebras and BCI-algebras were introduced by Iséki [8] as algebras induced by Meredith's implicational logics BCK and BCI. Kim et al. were defined BE-algebras in [10] as a generalization of BCK-algebras. Since then, they have been studied intensively by many authors (see [1,2,5,11–16]). Pseudo-BCI-algebras were introduced by Dudek and Jun [7] as a non-commutative generalization of BCI-algebras, and Jun et al. [9] studied pseudo-BCI ideals of pseudo-BCI algebras. The pseudo BE-algebra was introduced as a generalization of BE-algebras by Borzooei et al. [3], and has been intensively studied by several authors (see [4, 6, 17]). In particular, the notions of pseudo subalgebras and pseudo filters were defined, and investigated some related properties in [3].

The purpose of this paper is to further study pseudo subalgebras and pseudo filters in pseudo BE-algebras. To this end, we introduce the concepts of pseudo atom, atomic pseudo BE-algebra, and atomic pseudo filter to conduct related research. We note that every pseudo filter is a pseudo subalgebra, but the converse may not be true. So we provide a condition for a pseudo subalgebra to be apseudo filter. We explore the conditions under which we can make a pseudo filter using a nonempty set. We explore conditions for the non-unit element to be a pseudo atom. We discuss the characterization of atomic pseudo BE-algebras. We provide conditions for a pseudo subalgebra to be a pseudo filter. We consider the relationship between the set of pseudo atoms and pseudo subalgebras. We find conditions for a pseudo filter to be atomic.

2. Preliminaries

Definition 2.1 ([10]). A *BE-algebra*, denoted by $X := (X; *, 1)$, is defined to be a set X together with a binary operation " $*$ " and the unit element "1" satisfying the conditions:
(BE1) $(\forall a \in X)\ (a * a = 1)$,
(BE2) $(\forall a \in X)\ (a * 1 = 1)$,
(BE3) $(\forall a \in X)\ (1 * a = a)$,
(BE4) $(\forall a, b, c \in X)\ (a * (b * c) = b * (a * c))$.

Proposition 2.2 ([10]). *Every BE-algebra* $X := (X; *, 1)$ *satisfies the following conditions*

$$(\forall a, b \in X)\,(a * (b * a) = 1)\,. \tag{2.1}$$

$$(\forall a, b \in X)\,(a * ((a * b) * b) = 1)\,. \tag{2.2}$$

Proposition 2.3 ([10]). *A subset* F *of a BE-algebra* $X := (X; *, 1)$ *is called*

- *a subalgebra of* $X := (X; *, 1)$ *if it satisfies:*

$$(\forall a, b \in F)(a * b \in F), \tag{2.3}$$

- *a filter of* $X := (X; *, 1)$ *if it satisfies:*

$$1 \in F, \tag{2.4}$$

$$(\forall a, b \in X)(a * b \in F,\ a \in F\ \Rightarrow\ b \in F). \tag{2.5}$$

Definition 2.4 ([3]). An algebra $(X; *, \div, 1)$ of type (2,2,0) is called a *pseudo BE-algebra* if it satisfies:
(pBE1) $a * a = 1$ and $a \div a = 1$,
(pBE2) $a * 1 = 1$ and $a \div 1 = 1$,
(pBE3) $1 * a = a$ and $1 \div a = a$,
(pBE4) $a * (b \div c) = b \div (a * c)$,
(pBE5) $a * b = 1\ \Leftrightarrow\ a \div b = 1$,
for all $a, b, c \in X$.

In a pseudo BE-algebra $(X; *, \div, 1)$, a binary relation " $\leq$ " is defined as follows:

$$(\forall a, b \in X)(a \leq b\ \Leftrightarrow\ a * b = 1\ \Leftrightarrow\ a \div b = 1). \tag{2.6}$$

Proposition 2.5 ([3]). *Every pseudo BE-algebra* $(X; *, \div, 1)$ *satisfies:*

(i) $a * (b \div a) = 1$ *and* $a \div (b * a) = 1$,
(ii) $a * ((a \div b) * b) = 1$ *and* $a \div ((a * b) \div b) = 1$,

for all $x, y, z \in X$.

Definition 2.6 ([3]). A subset F of X is called

- a *pseudo subalgebra* of $(X; *, \div, 1)$ if it satisfies:

$$(\forall a, b \in X)(a, b \in F \;\Rightarrow\; a * b \in F,\; a \div b \in F). \tag{2.7}$$

- a *pseudo filter* of $(X; *, \div, 1)$ if it satisfies:

$$1 \in X, \tag{2.8}$$
$$(\forall a, b \in X)(a \in F,\; a * b \in F \;\Rightarrow\; b \in F). \tag{2.9}$$

Proposition 2.7 ([3]). *A subset* F *of* X *is a pseudo filter of* $(X; *, \div, 1)$ *if and only if it satisfies* (2.8) *and*

$$(\forall a, b \in X)(a \in F,\; a \div b \in F \;\Rightarrow\; b \in F). \tag{2.10}$$

Proposition 2.8 ([3]). *Every pseudo filter* F *of* $(X; *, \div, 1)$ *satisfies:*

$$(\forall a, b \in X)(a \leq b,\; a \in F \;\Rightarrow\; b \in F). \tag{2.11}$$

Proposition 2.9 ([3]). *Every pseudo filter* F *of* $(X; *, \div, 1)$ *satisfies:*

$$(\forall x, y \in F)(\forall z \in X)\begin{pmatrix} x \leq y * z \;\Rightarrow\; z \in F \\ x \leq y \div z \;\Rightarrow\; z \in F \end{pmatrix}. \tag{2.12}$$

3. Pseudo subalgebras and pseudo filters

In what follows, $(X; *, \div, 1)$ stands for pseudo BE-algebra, unless otherwise stated.

Proposition 3.1. *Every pseudo filter* F *of* $(X; *, \div, 1)$ *satisfies:*

$$(\forall x, y \in X)(y \in F \;\Rightarrow\; (y * x) \div x \in F,\; (y \div x) * x \in F). \tag{3.1}$$
$$(\forall x \in F)(\forall y \in X)\,(x * (x \div y) \in F \;\Rightarrow\; y \in F)\,. \tag{3.2}$$

Proof. Let $x \in X$ and $y \in F$. Then $y * ((y * x) \div x) = (y * x) \div (y * x) = 1 \in F$ and $y \div ((y \div x) * x) = (y \div x) * (y \div x) = 1 \in F$ by (pBE1) and (pBE4). It follows from (2.9) and (2.10) that $(y * x) \div x \in F$ and $(y \div x) * x \in F$. The assertion (3.2) is straightforward. □

The combination of (pBE4) and (3.2) induces the following corollary.

Corollary 3.2. *Every pseudo filter* F *of* $(X; *, \div, 1)$ *satisfies:*

$$(\forall x \in F)(\forall y \in X)\,(x \div (x * y) \in F \;\Rightarrow\; y \in F)\,. \tag{3.3}$$

Note that every pseudo filter is a pseudo subalgebra, but the converse may not be true (see [3]). We provide a condition for a pseudo subalgebra to be a pseudo filter.

Theorem 3.3. *If a pseudo subalgebra F of $(X; *, \div, 1)$ satisfies the condition* (3.2)*, then it is a pseudo filter of* $(X; *, \div, 1)$.

Proof. Let F be a pseudo subalgebra of $(X; *, \div, 1)$ that satisfies the condition (3.2). Clearly, $1 \in F$. Let $x \in F$ and $y \in X$ be such that $x \div y \in F$. Then $x * (x \div y) \in F$ since F is a pseudo subalgebra of $(X; *, \div, 1)$, and so $y \in F$ by (3.2). Hence F is a pseudo filter of $(X; *, \div, 1)$ by Proposition 2.7. □

We explore the conditions under which we can make a pseudo filter using a nonempty set.

Theorem 3.4. *Given a nonempty set F of X, the following are equivalent.*

(i) *F is a pseudo filter of* $(X; *, \div, 1)$.
(ii) *F satisfies* (2.11) *and*

$$(\forall a, b, x \in X)\left(a, b \in F \Rightarrow \begin{cases} (a * (b \div x)) * x \in F \\ (a \div (b * x)) \div x \in F \end{cases} \right). \tag{3.4}$$

Proof. Assume that F is a pseudo filter of $(X; *, \div, 1)$. Then F satisfies (2.11) (see Proposition 2.8). Let $a, b \in F$ and $x \in X$. Using (pBE1) and (pBE4) leads to

$$a * ((a * (b \div x)) * (b \div x)) = (a * (b \div x)) * (a * (b \div x)) = 1 \in F$$

and

$$a \div ((a \div (b * x)) \div (b * x)) = (a \div (b * x)) \div (a \div (b * x)) = 1 \in F.$$

It follows from (pBE4), (2.9) and (2.10) that

$$b \div ((a * (b \div x)) * x) = (a * (b \div x)) * (b \div x) \in F$$

and

$$b * ((a \div (b * x)) \div x) = (a \div (b * x)) \div (b * x) \in F.$$

Hence $(a * (b \div x)) * x \in F$ and $(a \div (b * x)) \div x \in F$ by (2.9) and (2.10).

Conversely, suppose that F satisfies (2.11) and (3.4). It is clear that $1 \in F$ by (pBE2) and (2.11). Let $x \in F$ and $y \in X$ be such that $x * y \in F$ or $x \div y \in F$. Then $y = 1 * y = ((x \div y) * (x \div y)) * y \in F$ or $y = 1 \div y = ((x * y) \div (x * y)) \div y \in F$ by (pBE1), (pBE3), and (3.4). Therefore F is a pseudo filter of $(X; *, \div, 1)$. □

Definition 3.5. A non-unit element b in $(X; *, \div, 1)$ is called a *pseudo atom* in $(X; *, \div, 1)$ if the following assertion is valid.

$$(\forall x \in X)\,(b \leq x \Rightarrow b = x \text{ or } x = 1)\,. \tag{3.5}$$

Example 3.6. (i) Let $X = \{1, 2, 3, 4\}$ and define the operations " $*$ " and " $\div$ " on X as follows:

$*$	1	2	3	4
1	1	2	3	4
2	1	1	1	1
3	1	2	1	4
4	1	3	1	1

$\div$	1	2	3	4
1	1	2	3	4
2	1	1	1	1
3	1	4	1	4
4	1	4	1	1

Then $(X;*,\div,1)$ is a pseudo BE-algebra (see [3]), and the element 3 is a pseudo atom in $(X;*,\div,1)$.

(ii) Let $X=\{0,1,2,3,4\}$ and define the operations " $*$ " and " $\div$ " on X as follows:

$*$	0	1	2	3	4
0	1	1	2	2	4
1	0	1	2	3	4
2	0	1	1	1	1
3	0	1	1	1	1
4	0	1	2	3	1

$\div$	0	1	2	3	4
0	1	1	2	2	4
1	0	1	2	3	4
2	0	1	1	1	1
3	0	1	1	1	1
4	1	1	1	1	1

It is routine to verify that $(X;*,\div,1)$ is a pseudo BE-algebra and the element 0 is the only pseudo atom in $(X;*,\div,1)$.

We explore conditions for a non-unit element to be a pseudo atom. First, we have the following question.

Question 1. *For every non-unit element b in $(X;*,\div,1)$, if $\{b,1\}$ is a pseudo subalgebra of $(X;*,\div,1)$, then is b a pseudo atom in $(X;*,\div,1)$?*

The example below gives a negative answer to Question 1.

Example 3.7. Let $X=\{1,2,3,4\}$ and define the operations " $*$ " and " $\div$ " on X as follows:

$*$	1	2	3	4
1	1	2	3	4
2	1	1	1	3
3	1	2	1	4
4	1	1	1	1

$\div$	1	2	3	4
1	1	2	3	4
2	1	1	1	2
3	1	2	1	2
4	1	1	1	1

Then $(X;*,\div,1)$ is a pseudo BE-algebra (see [3]). We can onserve that the set $F:=\{1,2\}$ is a pseudo subalgebra of $(X;*,\div,1)$. But 2 is not a pseudo atom because of $2\leq 3$ and $2\neq 3\neq 1$.

Theorem 3.8. *For every non-unit element b in $(X;*,\div,1)$, if $\{b,1\}$ is a pseudo filter of $(X;*,\div,1)$, then b is a pseudo atom in $(X;*,\div,1)$.*

Proof. Assume that $\{b,1\}$ is a pseudo filter of $(X;*,\div,1)$. Let $x\in X$ be such that $b\leq x$. Then $b*x=1\in\{b,1\}$ and $b\div x=1\in\{b,1\}$. It follows that $x\in\{b,1\}$. Hence $b=x$ or $x=1$. Therefore b is a pseudo atom in $(X;*,\div,1)$. □

The following example shows that the converse of Theorem 3.8 may not be true, that is, there exists a pseudo atom b in $(X;*,\div,1)$ for which $\{b,1\}$ is not a pseudo filter of $(X;*,\div,1)$.

Example 3.9. Let $X=\{0,1,2,3,4\}$ and define the operations " $*$ " and " $\div$ " on X as follows:

$*$	0	1	2	3	4
0	1	1	1	1	1
1	0	1	2	3	4
2	2	1	1	0	4
3	1	1	1	1	1
4	1	1	1	2	1

$\div$	0	1	2	3	4
0	1	1	1	1	4
1	0	1	2	3	4
2	2	1	1	0	4
3	1	1	1	1	4
4	1	1	1	1	1

It is routine to verify that $(X; *, \div, 1)$ is a pseudo BE-algebra. We can observe that 2 is a pseudo atom. But $\{1, 2\}$ is not a pseudo filter of $(X; *, \div, 1)$ since $2 \in \{1, 2\}$, $2 * 0 = 2 \in \{1, 2\}$ but $0 \notin \{1, 2\}$.

Definition 3.10. A pseudo BE-algebra $(X; *, \div, 1)$ is said to be *atomic* if every non-unit element of X is a pseudo atom in $(X; *, \div, 1)$.

Example 3.11. Consider a BE-algebra $X = \{1, a, b, c\}$ with the following Cayley table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

If we take $\div := *$, then $(X; *, \div, 1)$ is a pseudo BE-algebra. It is easily to check that a, b and c are non-units and they are pseudo atoms in $(X; *, \div, 1)$. Therefore $(X; *, \div, 1)$ is an atomic pseudo BE-algebra.

Theorem 3.12. *If* $(X; *, \div, 1)$ *is a pseudo BE-algebra, then the following assertions are equivalent.*

(i) $(X; *, \div, 1)$ *is atomic.*
(ii) $(X; *, \div, 1)$ *satisfies:*

$$(\forall x, y \in X)(x \neq y \ \Rightarrow \ x * y = y = x \div y). \tag{3.6}$$

(iii) $(X; *, \div, 1)$ *satisfies:*

$$(\forall x, y \in X)(x \neq y \ \Rightarrow \ (x * y) \div y = 1 = (x \div y) * y). \tag{3.7}$$

Proof. (i) $\Rightarrow$ (ii). Assume that $(X; *, \div, 1)$ is atomic. Let $x, y \in X$ be such that $x \neq y$. Note that $y \leq x \div y$ and $y \leq x * y$. Since y is a pseudo atom, we have $y = x \div y$ or $x \div y = 1$, and $y = x * y$ or $x * y = 1$. If $x \div y = 1$, then $x \leq y$ and so $x = y$ or $y = 1$. This is a contradiction. Similarly, $x * y = 1$ induces a contradiction. Hence $x * y = y = x \div y$.

(ii) $\Rightarrow$ (iii). Straightforward.

(iii) $\Rightarrow$ (i). Suppose that $(X; *, \div, 1)$ satisfies (3.7). Let x be an arbitrary non-unit element of X such that $x \leq y$ for all non-unit element $y \in X$. Then $x * y = 1$ and $x \div y = 1$. If $x \neq y$, then $1 = (x*y) \div y = 1 \div y = y$ and $1 = (x \div y) * y = 1 * y = y$ by (pBE3) and (3.7). This is a contradiction, and so $x = y$, which shows that x is a pseudo atom in $(X; *, \div, 1)$. Consequently, $(X; *, \div, 1)$ is atomic. □

Theorem 3.13. *If a pseudo BE-algebra* $(X; *, \div, 1)$ *is atomic, then every pseudo subalgebra is a pseudo filter.*

Proof. Let $(X; *, \div, 1)$ be an atomic pseudo BE-algebra. Let F be a pseudo subalgebra of $(X; *, \div, 1)$. Clearly, $1 \in F$. Let $x, y \in X$ be such that $x \in F$, $x * y \in F$ and $x \div y \in F$. It is clear that if $x = 1$ or $y = 1$, then $y \in F$. So we may assume that x and y are non-unit. Note that $y \leq x \div y$ and $y \leq x * y$. Since y is a pseudo atom, we have $y = x \div y \in F$ or $x \div y = 1$, and $y = x * y \in F$ or $x * y = 1$. If $x \div y = 1$ and $x * y = 1$, then $x \leq y$. Since x is a pseudo atom, it follows that $y = x \in F$ or $y = 1 \in F$. Therefore F is a pseudo filter of $(X; *, \div, 1)$. □

Given a subset F of X, we consider a set

$$\mathcal{A}(F) := \{y \in F \mid y \text{ is a pseudo atom in } (X; *, \div, 1)\}. \tag{3.8}$$

Note that $\mathcal{A}(X)$ is the set of all pseudo atoms in $(X; *, \div, 1)$. It is clear that $1 \notin \mathcal{A}(F)$ and $1 \notin \mathcal{A}(X)$.

Theorem 3.14. *If F is a pseudo subalgebra of $(X; *, \div, 1)$, then the set $\mathcal{A}(F) \cup \{1\}$ is also a pseudo subalgebra of $(X; *, \div, 1)$.*

Proof. Assume that F is a pseudo subalgebra of $(X; *, \div, 1)$. Let $x, y \in \mathcal{A}(F) \cup \{1\}$. It is clear that if $x = 1$, $y = 1$ or $x = y$, then $x * y$, $x \div y \in \mathcal{A}(F) \cup \{1\}$. Assume that $1 \neq x \neq y \neq 1$. Since $y \leq x \div y$ and $y \leq x * y$, we have $y = x \div y \in F$ or $x \div y = 1$, and $y = x * y \in F$ or $x * y = 1$. If $x \div y = 1$ or $x * y = 1$, then $x \leq y$. Since $x \in \mathcal{A}(F)$, it follows that $x = y$, a contradiction. Hence $x \div y \neq 1$ and $x * y \neq 1$, and so $x \div y = y \in \mathcal{A}(F) \cup \{1\}$ and $x * y = y \in \mathcal{A}(F) \cup \{1\}$. Therefore $\mathcal{A}(F) \cup \{1\}$ is a pseudo subalgebra of $(X; *, \div, 1)$. □

Corollary 3.15. *The set $\mathcal{A}(X) \cup \{1\}$ is a pseudo subalgebra of $(X; *, \div, 1)$.*

The following example shows that the converse of Theorem 3.14 may not be true.

Example 3.16. Consider the pseudo BE-algebra $(X; *, \div, 1)$ in Example 3.9. If we take $F := \{2, 3\}$, then $\mathcal{A}(X) = \{2\}$ and so $\mathcal{A}(F) \cup \{1\} = \{1, 2\}$ is a pseudo subalgebra of $(X; *, \div, 1)$. But F is not a pseudo subalgebra of $(X; *, \div, 1)$ since $2 * 3 = 0 \notin F$ or $3 \div 2 = 1 \notin F$.

Question 2. *If F is a pseudo filter of $(X; *, \div, 1)$, then is the set $\mathcal{A}(X) \cup \{1\}$ a pseudo filter of $(X; *, \div, 1)$?*

The following example provides a negative answer to Question 2.

Example 3.17. Let $X = \{0, 1, 2, 3, 4\}$ and define the operations " $*$ " and " $\div$ " on X as follows:

*	0	1	2	3	4
0	1	1	0	3	4
1	0	1	2	3	4
2	1	1	1	3	4
3	0	1	2	1	1
4	0	1	2	3	1

÷	0	1	2	3	4
0	1	1	0	3	4
1	0	1	2	3	4
2	1	1	1	3	4
3	0	1	2	1	4
4	1	1	0	3	1

It is routine to verify that $(X; *, \div, 1)$ is a pseudo BE-algebra. We can observe that $F = \{0, 1, 2\}$ is a pseudo filter of $(X; *, \div, 1)$. But $\mathcal{A}(F) \cup \{1\} = \{0, 1\}$, and it is not a pseudo filter of $(X; *, \div, 1)$ since $0 \in \mathcal{A}(F) \cup \{1\}$ and $0 * 2 = 0 \div 2 = 0 \in \mathcal{A}(F) \cup \{1\}$ but $2 \notin \mathcal{A}(F) \cup \{1\}$.

Definition 3.18. A pseudo filter F of $(X; *, \div, 1)$ is said to be *atomic* if $F = \mathcal{A}(F) \cup \{1\}$.

Example 3.19. Let $X = \{0, 1, 2, 3, 4\}$ and define the operations " $*$ " and " $\div$ " on X as follows:

*	0	1	2	3	4
0	1	1	2	3	4
1	0	1	2	3	4
2	0	1	1	3	4
3	0	1	1	1	4
4	0	1	1	3	1

÷	0	1	2	3	4
0	1	1	2	3	4
1	0	1	2	3	4
2	0	1	1	3	4
3	0	1	1	1	1
4	0	1	1	3	1

It is routine to verify that $(X; *, \div, 1)$ is a pseudo BE-algebra. Let $F = \{0, 1, 2\}$. Then F is a pseudo filter of $(X; *, \div, 1)$ and $F = \mathcal{A}(F) \cup \{1\}$. Hence F is an atomic pseudo filter of $(X; *, \div, 1)$.

Theorem 3.20. *In an atomic pseudo BE-algebra, every pseudo filter is atomic.*

Proof. Let F be a pseudo filter of an atomic pseudo BE-algebra $(X; *, \div, 1)$. It is clear that $\mathcal{A}(F) \cup \{1\} \subseteq F$. Let x be a non-unit element of F. Then x is a pseudo atom since $(X; *, \div, 1)$ is atomic. Hence $x \in \mathcal{A}(F)$, and so $F \subseteq \mathcal{A}(F) \cup \{1\}$. Therefore F is atomic. □

Theorem 3.21. *Let F be a pseudo filter of* $(X; *, \div, 1)$. *Then F is atomic if and only if it satisfies:*

$$(\forall x, y \in F)(x \neq y \ \Rightarrow \ x * y = y = x \div y). \tag{3.9}$$

Proof. Assume that F is an atomic pseudo filter of $(X; *, \div, 1)$. Let $x, y \in F$ be such that $x \neq y$. If $x = 1$, then $y \neq 1$ and so $1 * y = y = 1 \div y$ by (pBE3). If $y = 1$, then $x \neq 1$ and so $x * 1 = 1 = x \div 1$ by (pBE2). Suppose that $1 \neq x \neq y \neq 1$. Since $y \leq x \div y$ and $y \leq x * y$, we have $y = x \div y$ or $x \div y = 1$, and $y = x * y$ or $x * y = 1$. If $x \div y = 1$ or $x * y = 1$, then $x \leq y$. Since $x \in \mathcal{A}(F)$ and $y \neq 1$, it follows that $x = y$, a contradiction. Hence $x \div y \neq 1$ and $x * y \neq 1$, and thus $x * y = y = x \div y$.

Conversely, suppose that F satisfies the condition (3.9). It is clear that $\mathcal{A}(F) \cup \{1\} \subseteq F$. Let x be a non-unit element of F such that $x \leq y$ for all non-unit elements $y \in X$. Then $y \in F$. If $x \neq y$, then $1 = x * y = y = x \div y = 1$ by (3.9). This is a contradiction, and so $x = y$. Thus $x \in \mathcal{A}(F)$, and hence $F \subseteq \mathcal{A}(F) \cup \{1\}$. Therefore F is an atomic pseudo filter of $(X; *, \div, 1)$. □

4. Conclusions

As a generalization of BE-algebras, the pseudo BE-algebra was introduced by Borzooei et al. and has been intensively studied by several authors. In particular, the notions of pseudo subalgebras and pseudo filters were defined, and investigated some related properties. For the further study of pseudo subalgebras and pseudo filters in pseudo BE-algebras, we have introduced the concepts of pseudo atom, atomic pseudo BE-algebra, and atomic pseudo filter, and have investigated the related properties We have provided a condition for a pseudo subalgebra to be apseudo filter, and have explored the conditions under which we can make a pseudo filter using a nonempty set. We have established conditions for the non-unit element to be a pseudo atom, and have discussed the characterization of atomic pseudo BE-algebras. We have provided conditions for a pseudo subalgebra to be a pseudo filter, and have considered the relationship between the set of pseudo atoms and pseudo subalgebras. We have found conditions for a pseudo filter to be atomic. Future research will focus on the study of fuzzy and soft set theory about the ideas and results of this paper.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. S. S. Ahn, Y. H. Kim, J. M. Ko, Filters in commutative BE-algebras, *Commun. Korean Math. Soc.*, **27** (2012), 233–242. https://doi.org/10.4134/CKMS.2012.27.2.233

2. S. S. Ahn, K. S. So, On ideals and upper sets in BE-algebras, *Sci. Math. Jpn.*, **68** (2008), 279–285.
3. R. A. Borzooei, A. Borumand Saeid, A. Rezaei, A. Radfar, R. Ameri, On pseudo BE-algebras, *Discuss. Math., General Algebra Appl.*, **33** (2013), 95–108. https://doi.org/10.7151/dmgaa.1193
4. R. A. Borzooei, A. Borumand Saeid, A. Rezaei, A. Radfar, R. Ameri, Distributive pseudo BE-algebras, *Fasciculi Math.*, **54** (2015), 21–39. https://doi.org/10.1515/fascmath-2015-0002
5. Z. Ciloğlu, Y. Ceven, Commutative and bounded BE-algebras, *Algebra*, **2013** (2013), 1–6. http://dx.doi.org/10.1155/2013/473714
6. L. C. Ciungu, Commutative pseudo BE-algebras, *Iran. J. Fuzzy Syst.*, **13** (2016), 131–144.
7. W. A. Dudek, Y. B. Jun, Pseudo-BCI algebras, *East Asian Math. J.*, **24** (2008), 187–190.
8. K. Iséki, An algebra related with a propositional calculus, *Proc. Jpn Acad.*, **42** (1966), 26–29. https://doi.org/10.3792/pja/1195522171
9. Y. B. Jun, H. S. Kim, J. Neggers, On pseudo-BCI ideals of pseudo-BCI algebras, *Mat. Vesnik*, **58** (2006), 39–46.
10. H. S. Kim, Y. H. Kim, On BE-algebras, *Sci. Math. Jpn.*, **66** (2007), 113–116. https://doi.org/10.32219/isms.66.1_113
11. H. S. Kim, K. J. Lee, Extended upper sets in BE-algebras, *Bull. Malays. Math. Sci. Soc.*, **34** (2011), 511–520.
12. B. L. Meng, On filters in BE-algebras, *Sci. Math. Jpn.*, **71** (2010), 201–207.
13. A. Rezaei, A. Borumand Saeid, On fuzzy subalgebras of BE-algebras, *Afr. Mat.*, **22** (2011), 115–127. https://doi.org/10.1007/s13370-011-0011-4
14. A. Rezaei, A. Borumand Saeid, Some results in BE-algebras, *An. Univ. Oradea, Fasc. Mat.*, **19** (2012), 33–44.
15. A. Rezaei, A. Borumand Saeid, Commutative ideals in BE-algebras, *Kyungpook Math. J.*, **52** (2012), 483–494. https://doi.org/10.5666/KMJ.2012.52.4.483
16. A. Rezaei, A. Borumand Saeid, R. A. Borzooei, Relation between Hilbert algebras and BE-algebras, *Appl. Appl. Math.*, **8** (2013), 573–584.
17. A. Rezaei, A. Borumand Saeid, A. Radfar, R. A. Borzooei, Congruence relations on pseudo BE-algebras, *Ann. Univ. Craiova, Math. Comput. Sci. Ser.*, **41** (2014), 166–176.

AIMS Press