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*Research article*

# Complex-valued double controlled metric like spaces with applications to fixed point theorems and Fredholm type integral equations

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**Abstract:** In this paper we introduce the concept of complex-valued double controlled metric like spaces. These new results generalize and extend the corresponding results about complex-valued double controlled metric type spaces. We prove some complex-valued fixed point theorems in this new complex-valued metric like spaces and, as application, we give an existence and uniqueness of the solution of a Fredholm type integral equation result. Moreover, some examples are also presented in favor of our given results.

**Keywords:** fixed point; complex-valued double controlled metric like spaces; Fredholm type integral equations
**Mathematics Subject Classification:** 47H10, 54H25

## 1. Introduction and Preliminaries

Fixed point theory concept is a widely recognized as a subject with implications in different domains as mathematical sciences, engineering, computer sciences. This area interact with all the mathematics research branches, including geometry, algebra and topology. The start point of fixed point theory has been done by Banach [1] by introducing the notion of contraction mapping in a complete metric space, in order to find fixed point of the specified operators. This classical theorem of Banach [1], well-known as Banach contraction principle, has been studied and generalized by many researchers in diverse methods (see [5–23]). Moreover, fixed point techniques play a very important role in proving of the existence and uniqueness of the solution of different type of equations as: integral equations, differential equations, fractional differential equations, etc. In this direction we recall [24–32].

Classical definition of metric space was generalized by Harandi [3] by introducing the notion of metric like space. Azam et al. introduced in [2] the notion of complex valued metric space. Then, the notion was generalized by Hosseni and Karizaki [4], giving the notion complex valued metric like space. Bakhtin [5] and Czerwik [6] generalized the metric space by giving the idea of $b$-metric spaces. Aslam et al. [7] introduced later the notion of complex valued controlled metric type space.

Abdeljawad et al. presented the idea of double controlled metric type space (DCMLS) in [10], which was a generalization of Kamran et al. [8] and Mlaiki et al. [9] notions.

Further, let us recall some definitions and results useful in the introduction of our new concept.

Let $\mathbb{C}$ be the set of complex numbers and $w_1, w_2 \in \mathbb{C}$. Since we cannot compare in usual way two complex numbers, let us add to the complex set $\mathbb{C}$ the following partial order $\precsim$, known in related literature as lexicographic order

$$w_1 \precsim w_2 \text{ if and only if } Re(w_1) \leq Re(w_2) \text{ or } (Re(w_1) = Re(w_2) \text{ and } Im(w_1) \leq Im(w_2)).$$

Taking into account the previous definition, we have that $w_1 \precsim w_2$ if one of the next conditions is satisfied:

$(P_1)$ $Re(w_1) < Re(w_2)$ and $Im(w_1) < Im(w_2)$;
$(P_2)$ $Re(w_1) < Re(w_2)$ and $Im(w_1) = Im(w_2)$;
$(P_3)$ $Re(w_1) < Re(w_2)$ and $Im(w_1) > Im(w_2)$;
$(P_4)$ $Re(w_1) = Re(w_2)$ and $Im(w_1) < Im(w_2)$.

Let us recall the definition of complex valued extended $b$-metric given by N. Ullah et al. in [12].

**Definition 1.1.** *[12] Let $\mathbb{X}$ be a non empty set and let $\vartheta : \mathbb{X} \times \mathbb{X} \to [1, \infty)$ be a function. The function $h_{eb}$: $\mathbb{X} \times \mathbb{X} \to C$ is said to be complex valued extended b-metric if the following conditions are satisfied:*

$(CEB_1)$ $0 \precsim h_{eb}(p, q)$ *and* $h_{eb}(p, q) = 0$ *if and only if* $p = q$,

$(CEB_2)$ $h_{eb}(p, q) = h_{eb}(q, p)$,

$(CEB_3)$ $h_{eb}(p, r) \precsim \vartheta(p, r)[h_{eb}(p, q) + h_{eb}(q, r)]$,

*for all* $p$, $q$, $r \in \mathbb{X}$. *A pair* $(\mathbb{X}, h_{eb})$ *is called a complex valued extended b-metric space.*

Mlaiki et al. [9] generalized the notion of $b$-metric spaces as follows.

**Definition 1.2.** *[9] Given $\varrho$: $\mathbb{X} \times \mathbb{X} \to [1, \infty)$, where $\mathbb{X}$ is nonempty and let $h_c$: $\mathbb{X} \times \mathbb{X} \to [0, \infty)$. Suppose that*

$(CMT_1)$ $h_c(p, q) = 0$ *if and only if* $p = q$,

$(CMT_2)$ $h_c(p, q) = h_c(q, p)$,

$(CMT_3)$ $h_c(p, q) \leq \varrho(p, r)h_c(p, r) + \varrho(r, q)h_c(r, q)$,

*for all* $p$, $q$, $r \in \mathbb{X}$. *Then,* $h_c$ *is called a controlled metric type and the pair* $(\mathbb{X}, h_c)$ *is called a controlled metric type space.*

**Definition 1.3.** *[16] Be given two non-comparable functions $\varrho$, $\varsigma$: $\mathbb{X} \times \mathbb{X} \to [1, \infty)$, where $\mathbb{X}$ is nonempty. If $h_{dl}$: $\mathbb{X}^2 \to [0, \infty)$ satisfies*

$(DCML_1)$ $h_{dl}(p,q) = 0 \implies p = q$,

$(DCML_2)$ $h_{dl}(p,q) = h_{dl}(q,p)$,

$(DCML_3)$ $h_{dl}(p,q) \leq \varrho(p,r)h_{dl}(p,r) + \varsigma(r,q)h_{dl}(r,q)$,

*for all p, q, r ∈ $\mathbb{X}$, then, $h_{dl}$ is called a double controlled metric like by $\varrho$ and $\varsigma$, and $(\mathbb{X}, h_{dl})$ is called a double controlled metric like space (DCMLS).*

**Definition 1.4.** *[11] Be given two non-comparable functions $\varrho, \varsigma$: $\mathbb{X} \times \mathbb{X} \to [1,\infty)$, where $\mathbb{X}$ is nonempty. If $h_{cdt}$: $\mathbb{X}^2 \to [0,\infty)$ satisfies*

$(CDCMT_1)$ $h_{cdt}(p,q) = 0 \iff p = q$,

$(CDCMT_2)$ $h_{cdt}(p,q) = h_{cdt}(q,p)$,

$(CDCMT_3)$ $h_{cdt}(p,q) \precsim \varrho(p,r)h_{cdt}(p,r) + \varsigma(r,q)h_{cdt}(r,q)$,

*for all p, q, r ∈ $\mathbb{X}$, then, $h_{cdt}$ is called a complex valued double controlled metric type by $\varrho$ and $\varsigma$, and $(\mathbb{X}, h_{cdt})$ is called a complex valued double controlled metric type space(CDCMTS).*

Recently Panda et al. [11] presented idea of complex valued double controlled metric type space (CDCMTS). Inspired by Panda et al. [11], in this article we will present the concept of complex valued double controlled metric like space (CDCMLS). Then two fixed point theorems in CDCMLS are presented. One of them is the Banach contraction principle and the second one is the related to Reich type result. The theorems are validate with the help of some examples. Moreover, an application to prove the existence and the uniqueness of a solution of a Fredholm type integral equation is given.

## 2. Complex valued double controlled metric like spaces

This section is dedicated to the introduction of our generalization-complex valued double controlled metric like spaces (CDCMLS). Then let us present first the definition of such a type of space and then, an illustrative example of it.

**Definition 2.1.** *Be given two non-comparable functions $\varrho, \varsigma$: $\mathbb{X} \times \mathbb{X} \to [1,\infty)$, where $\mathbb{X}$ is nonempty. If $h_{cdl}$: $\mathbb{X}^2 \to [0,\infty)$ satisfies*

$(CDCML_1)$ $h_{cdl}(p,q) = 0 \implies p = q$,

$(CDCML_2)$ $h_{cdl}(p,q) = h_{cdl}(q,p)$,

$(CDCML_3)$ $h_{cdl}(p,r) \precsim \varrho(p,q)h_{cdl}(p,q) + \varsigma(q,r)h_{cdl}(q,r)$,

*for all p, q, r ∈ $\mathbb{X}$, then $h_{cdl}$ is called a complex valued double controlled metric like by $\varrho$ and $\varsigma$ and $(\mathbb{X}, h_{cdl})$ is called a complex valued double controlled metric like space(CDCMLS).*

**Remark 2.1.** *A complex valued double controlled metric type space is also a complex valued double controlled metric like space in general. The converse is not true in general. This conclude that, it is a more generalized version than the one of complex valued extended b- metric type space.*

**Example 2.1.** *Let $\mathbb{X} = \{1,2,3\}$. Consider the complex valued double controlled metric like $h = h_{cdl}$ defined by*

$$h(1,1) = h(2,2) = 0 \text{ and } h(3,3) = \tfrac{i}{2},$$
$$h(1,2) = h(2,1) = 2 + 4i,\ h(2,3) = h(3,2) = i,\ h(1,3) = h(3,1) = 1 - i.$$

*Taking $\varrho, \varsigma$: $\mathbb{X} \times \mathbb{X} \to [1, \infty)$ to be symmetric and defined by*

$$\varrho(1,1) = \varrho(2,2) = \varrho(3,3) = 1,\ \varrho(1,2) = \varrho(2,1) = \tfrac{6}{5},\ \varrho(2,3) = \varrho(3,2) = \tfrac{8}{5},\ \varrho(3,1) = \varrho(1,3) = \tfrac{151}{100},$$

*and*

$$\varsigma(1,1) = \varsigma(2,2) = \varsigma(3,3) = 1,\ \varsigma(1,2) = \varsigma(2,1) = \tfrac{6}{5},\ \varsigma(2,3) = \varsigma(3,2) = \tfrac{33}{20},\ \varsigma(3,1) = \varsigma(1,3) = \tfrac{8}{3}.$$

*The conditions $(CDCML_1)$ and $(CDCML_2)$ hold.*
*Next, we will verify $(CDCML_3)$.*
**Case 1.** *When $p = q = r = 1$,*

$$\begin{aligned} |h(p,r)| = |h(1,1)| = 0 \le 0 = 0 + 0 &= \varrho(1,1)|h(1,1)| + \varsigma(1,1)|h(1,1)| \\ &= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|. \end{aligned}$$

**Case 2.** *When $p = 2$, $q = r = 1$,*

$$\begin{aligned} |h(p,r)| = |h(2,1)| = \sqrt{20} \le \frac{6}{5}\sqrt{20} = \frac{6}{5}\sqrt{20} + 0 &= \varrho(2,1)|h(2,1)| + \varsigma(1,1)|h(1,1)| \\ &= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|. \end{aligned}$$

**Case 3.** *When $p = 3$, $q = r = 1$,*

$$\begin{aligned} |h(p,r)| = |h(3,1)| = \sqrt{2} \le \frac{151}{100}\sqrt{2} = \frac{151}{100}\sqrt{2} + 0 &= \varrho(3,1)|h(3,1)| + \varsigma(1,1)|h(1,1)| \\ &= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|. \end{aligned}$$

**Case 4.** *When $p = r = 1$, $q = 2$,*

$$\begin{aligned} |h(p,r)| = |h(1,1)| = 0 \le \frac{12}{5}\sqrt{20} = \frac{6}{5}\sqrt{20} + \frac{6}{5}\sqrt{20} &= \varrho(1,2)|h(1,2)| + \varsigma(2,1)|h(2,1)| \\ &= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|. \end{aligned}$$

**Case 5.** *When $p = r = 1$, $q = 3$,*

$$\begin{aligned} |h(p,r)| = |h(1,1)| = 0 \le \frac{1253}{300}\sqrt{2} = \frac{151}{100}\sqrt{2} + \frac{8}{3}\sqrt{2} &= \varrho(1,3)|h(1,3)| + \varsigma(3,1)|h(3,1)| \\ &= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|. \end{aligned}$$

**Case 6.** *When $p = q = 1$, $r = 2$,*

$$\begin{aligned} |h(p,r)| = |h(1,2)| = \sqrt{20} \le \frac{6}{5}\sqrt{20} = 0 + \frac{6}{5}\sqrt{20} &= \varrho(1,1)|h(1,1)| + \varsigma(1,2)|h(1,2)| \\ &= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|. \end{aligned}$$

**Case 7.** *When* $p = q = 1$*,* $r = 3$,

$$|h(p,r)| = |h(1,3)| = \sqrt{2} \leq \frac{8}{3}\sqrt{2} = 0 + \frac{8}{3}\sqrt{2} = \varrho(1,1)|h(1,1)| + \varsigma(1,3)|h(1,3)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 8.** *When* $p = q = 2$*,* $r = 1$,

$$|h(p,r)| = |h(2,1)| = \sqrt{20} \leq \frac{6}{5}\sqrt{20} = 0 + \frac{6}{5}\sqrt{20} = \varrho(2,2)|h(2,2)| + \varsigma(2,1)|h(2,1)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 9.** *When* $p = 2, q = 3$*,* $r = 1$,

$$|h(p,r)| = |h(2,1)| = \sqrt{20} \leq \frac{24 + 40\sqrt{2}}{15} = \frac{8}{5} + \frac{8}{3}\sqrt{2} = \varrho(2,3)|h(2,3)| + \varsigma(3,1)|h(3,1)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 10.** *When* $p = 3, q = 2$*,* $r = 1$,

$$|h(p,r)| = |h(3,1)| = \sqrt{2} \leq \frac{8 + 6\sqrt{20}}{5} = \frac{8}{5} + \frac{6}{5}\sqrt{20} = \varrho(3,2)|h(3,2)| + \varsigma(2,1)|h(2,1)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 11.** *When* $p = q = 3$*,* $r = 1$,

$$|h(p,r)| = |h(3,1)| = \sqrt{2} \leq \frac{3 + 16\sqrt{2}}{6} = \frac{1}{2} + \frac{8}{3}\sqrt{2} = \varrho(3,3)|h(3,3)| + \varsigma(3,1)|h(3,1)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 12.** *When* $p = 1$*,* $q = r = 2$,

$$|h(p,r)| = |h(1,2)| = \sqrt{20} \leq \frac{6}{5}\sqrt{20} = \frac{6}{5}\sqrt{20} + 0 = \varrho(1,2)|h(1,2)| + \varsigma(2,2)|h(2,2)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 13.** *When* $p = 1$*,* $q = 3$*,* $r = 2$,

$$|h(p,r)| = |h(1,2)| = \sqrt{20} \leq \frac{453\sqrt{2} + 800}{300} = \frac{151}{100}\sqrt{2} + \frac{8}{3} = \varrho(1,3)|h(1,3)| + \varsigma(3,2)|h(3,2)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 14.** *When* $p = r = 2$*,* $q = 1$,

$$|h(p,r)| = |h(2,2)| = 0 \leq \frac{12}{5}\sqrt{20} = \frac{6}{5}\sqrt{20} + \frac{6}{5}\sqrt{20} = \varrho(2,1)|h(2,1)| + \varsigma(1,2)|h(1,2)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 15.** *When $p = q = r = 2$,*

$$\begin{aligned}|h(p,r)| = |h(2,2)| = 0 \leq 0 = 0 + 0 = \varrho(2,2)|h(2,2)| + \varsigma(2,2)|h(2,2)|\\ = \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.\end{aligned}$$

**Case 16.** *When $p = r = 2$, $q = 3$,*

$$\begin{aligned}|h(p,r)| = |h(2,2)| = 0 \leq \frac{13}{4} = \frac{8}{5} + \frac{33}{20} = \varrho(2,3)|h(2,3)| + \varsigma(3,2)|h(3,2)|\\ = \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.\end{aligned}$$

**Case 17.** *When $p = 3$, $q = 1$, $r = 2$,*

$$\begin{aligned}|h(p,r)| = |h(3,2)| = 1 \leq \frac{151\sqrt{2} + 120\sqrt{20}}{100} = \frac{151}{100}\sqrt{2} + \frac{6}{5}\sqrt{20} = \varrho(3,1)|h(3,1)| + \varsigma(1,2)|h(1,2)|\\ = \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.\end{aligned}$$

**Case 18.** *When $p = 3$, $q = r = 2$,*

$$\begin{aligned}|h(p,r)| = |h(3,2)| = 1 \leq \frac{8}{5} = \frac{8}{5} + 0 = \varrho(3,2)|h(3,2)| + \varsigma(2,2)|h(2,2)|\\ = \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.\end{aligned}$$

**Case 19.** *When $p = q = 3$, $r = 2$,*

$$\begin{aligned}|h(p,r)| = |h(3,2)| = 1 \leq \frac{43}{20} = \frac{1}{2} + \frac{33}{20} = \varrho(3,3)|h(3,3)| + \varsigma(3,2)|h(3,2)|\\ = \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.\end{aligned}$$

**Case 20.** *When $p = 1$, $q = 2$, $r = 3$,*

$$\begin{aligned}|h(p,r)| = |h(1,3)| = \sqrt{2} \leq \frac{24\sqrt{20} + 33}{20} = \frac{6}{5}\sqrt{20} + \frac{33}{20} = \varrho(1,2)|h(1,2)| + \varsigma(2,3)|h(2,3)|\\ = \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.\end{aligned}$$

**Case 21.** *When $p = 1$, $q = r = 3$,*

$$\begin{aligned}|h(p,r)| = |h(1,3)| = \sqrt{2} \leq \frac{151\sqrt{2} + 50}{100} = \frac{151}{100}\sqrt{2} + \frac{1}{2} = \varrho(1,3)|h(1,3)| + \varsigma(3,3)|h(3,3)|\\ = \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.\end{aligned}$$

**Case 22.** *When $p = 2$, $q = 1$, $r = 3$,*

$$\begin{aligned}|h(p,r)| = |h(2,3)| = 1 \leq \frac{18\sqrt{20} + 40\sqrt{2}}{15} = \frac{6}{5}\sqrt{20} + \frac{8}{3}\sqrt{2} = \varrho(2,1)|h(2,1)| + \varsigma(1,3)|h(1,3)|\\ = \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.\end{aligned}$$

**Case 23.** *When* $p = q = 2$, $r = 3$,

$$|h(p,r)| = |h(2,3)| = 1 \precsim \frac{33}{20} = 0 + \frac{33}{20} = \varrho(2,2)|h(2,2)| + \varsigma(2,3)|h(2,3)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 24.** *When* $p = 2$, $q = r = 3$,

$$|h(p,r)| = |h(2,3)| = 1 \precsim \frac{21}{10} = \frac{8}{5} + \frac{1}{2} = \varrho(2,3)|h(2,3)| + \varsigma(3,3)|h(3,3)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 25.** *When* $p = r = 3$, $q = 1$,

$$|h(p,r)| = |h(3,3)| = \frac{1}{2} \precsim \frac{1253}{300}\sqrt{2} = \frac{151}{100}\sqrt{2} + \frac{8}{3}\sqrt{2} = \varrho(3,1)|h(3,1)| + \varsigma(1,3)|h(1,3)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 26.** *When* $p = r = 3$, $q = 2$,

$$|h(p,r)| = |h(3,3)| = \frac{1}{2} \precsim \frac{13}{4} = \frac{8}{5} + \frac{33}{20} = \varrho(3,2)|h(3,2)| + \varsigma(2,3)|h(2,3)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

**Case 27.** *When* $p = q = r = 3$,

$$|h(p,r)| = |h(3,3)| = \frac{1}{2} \precsim 1 = \frac{1}{2} + \frac{1}{2} = \varrho(3,3)|h(3,3)| + \varsigma(3,3)|h(3,3)|$$
$$= \varrho(p,q)|h(p,q)| + \varsigma(q,r)|h(q,r)|.$$

*Thus,* $h = h_{cdl}$ *is complex valued double controlled metric like space(CDCMLS).*

*But when* $p = 2$, $q = 3$, $r = 1$,

$$|h(p,r)| = |h(2,1)| = \sqrt{20} \not\precsim \frac{6}{5}(1+\sqrt{2}) = \frac{6}{5}[1+\sqrt{2}] = \vartheta(2,1)[|h(2,3)| + |h(3,1)|]$$
$$= \vartheta(p,q)[|h(p,r)| + |h(q,r)|].$$

*Thus,* $h = h_{cdl}$ *is not a complex valued extended b-metric type for the function* $\vartheta$.

Let us discuss in the following the continuity property in the complex valued double controlled metric like space (CDCMLS).

**Definition 2.2.** *Let* $(\mathbb{X}, h_{cdl})$ *be a complex valued double controlled metric like space (CDCMLS) by one or two functions.*

*(1) The sequence* $\{p_n\}$ *is convergent to some* $p$ *in* $\mathbb{X}$, *if for each positive* $\varepsilon$, *there is some integer* $Z_\varepsilon$ *such that* $h_{cdl}(p_n, p) \prec \varepsilon$ *for each* $n \geq Z_\varepsilon$. *It is written as* $\lim_{n\to\infty} p_n = p$.

*(2) The sequence* $\{p_n\}$ *is said Cauchy, if for every* $\varepsilon > 0$, $h_{cdl}(p_n, p_m) \prec \varepsilon$ *for all* $m, n \geq Z_\varepsilon$, *where* $Z_\varepsilon$ *is some integer.*

*(3)* $(\mathbb{X}, h_{cdl})$ *is said complete if every Cauchy sequence is convergent.*

**Definition 2.3.** *Let* $(\mathbb{X}, h_{cdl})$ *be a complex valued double controlled metric like space (CDCMLS) by either one function or two functions—for* $p \in \mathbb{X}$ *and* $l > 0$.

*(i) We define* $\mathcal{B}(p, l)$ *as*

$$\mathcal{B}(p, l) = \{y \in \mathbb{X}, h_{cdl}(p, y) \prec l\}.$$

*(ii) The self-map* $\Upsilon$ *on* $\mathbb{X}$ *is said to be continuous at p in* $\mathbb{X}$ *if for all* $\delta > 0$, *there exists* $l > 0$ *such that*

$$\Upsilon(\mathcal{B}(p, l)) \subseteq \mathcal{B}(\Upsilon p, \delta).$$

Note that if $\Upsilon$ is continuous at $p$ in $(\mathbb{X}, h_{cdl})$, then $p_n \to p$ implies that $\Upsilon p_n \to \Upsilon p$ when n tends to $\infty$.

One can prove the following lemmas for the specific case of CDCMLS, in a similar way as in [14].

**Lemma 2.1.** *Let* $(\mathbb{X}, h_{cdl})$ *be a CDCMLS and assume a sequence* $\{d_n\}$ *in* $\mathbb{X}$. *Then* $\{d_n\}$ *is Cauchy sequence* $\iff |h_{cdl}(d_m, d_n)| \to 0$ *as* $m, n \to \infty$, *where* $m, n \in N$.

**Lemma 2.2.** *Suppose* $(\mathbb{X}, h_{cdl})$ *be a CDCMLS and* $\{d_n\}$ *be sequence in* $\mathbb{X}$. *Then* $\{d_n\}$ *converges to* $d \iff |h_{cdl}(d_n, d)| \to 0$ *as* $n \to \infty$.

**Lemma 2.3.** *Let* $(\mathbb{X}, h_{cdl})$ *be a CDCMLS. Then a sequence* $\{d_n\}$ *in* $\mathbb{X}$ *is Cauchy sequence, such that* $d_m \neq d_n$, *whenever* $m \neq n$. *Then* $\{d_n\}$ *converges to at most one point.*

**Lemma 2.4.** *For a given complex valued controlled space* $(\mathbb{X}, h_{cdl})$, *the complex valued double controlled (c.v.dc) metric like function* $h_{cdl}$*:* $\mathbb{X} \times \mathbb{X} \to \mathbb{C}$ *is continuous, with respect to the partial order "* $\precsim$ *".*

**Lemma 2.5.** *Consider* $(\mathbb{X}, h_{cdl})$ *be a CDCMLS. Limit of every convergent sequence in* $\mathbb{X}$ *is unique, if the functional* $h_{cdl}$*:* $\mathbb{X} \times \mathbb{X} \to \mathbb{X}$ *is continuous.*

The next scheme it is necessary to draw the relations between the recently generalizations of complex valued metric space.

| | | | | | | |
|---|---|---|---|---|---|---|
| *b*- metr. sp. | → | extended *b*-metr. sp. | → | contr. metr. sp. | → | double contr. metr. sp. |
| ↓ | | ↓ | | ↓ | | ↓ |
| c.v. *b*- metr. sp. | → | c.v.extend *b*-metr. sp. | → | c.v. contr. metr. sp. | → | c.v. double contr. metr. sp. |

## 3. Fixed point theorems in CDCMLS

In our first theorem of this section we prove the Banach contraction type theorem in CDCMLS.

**Theorem 3.1.** *Let* $(\mathbb{X}, h_{cdl})$ *be a CDCMLS by the functions* $\varrho, \varsigma$*:* $\mathbb{X} \times \mathbb{X} \to [1, \infty)$. *Suppose that* $\Upsilon$*:* $\mathbb{X} \to \mathbb{X}$ *satisfies*

$$h(\Upsilon p, \Upsilon q) \precsim l h_{cdl}(p, q), \tag{3.1}$$

*for all $p, q \in \mathbb{X}$, where $l \in (0,1)$. For $p_0 \in \mathbb{X}$, choose $p_n = \Upsilon^n p_0$. Assume that*

$$\sup_{m\geq 1}\lim_{\iota\to\infty}\frac{\varrho(p_{\iota+1}, p_{\iota+2})}{\varrho(p_\iota, p_{\iota+1})}\varsigma(p_{\iota+1}, p_m) < \frac{1}{l}. \tag{3.2}$$

*In addition, for each $p \in \mathbb{X}$, suppose that*

$$\lim_{n\to\infty}\varrho(p, p_n) \text{ and } \lim_{n\to\infty}\varsigma(p_n, p) \text{ exist and are finite.} \tag{3.3}$$

*Then, $\Upsilon$ has a unique fixed point.*

*Proof.* Consider the sequence $\{p_n = \Upsilon^n p_0\}$ in $\mathbb{X}$ that satisfies the hypothesis of the theorem. By using (3.1), we get

$$h_{cdl}(p_n, p_{n+1}) \precsim l^n h_{cdl}(p_0, p_1), \text{ for all } n \geq 0. \tag{3.4}$$

Let $n, m$ be integers such that $n < m$. We have

$$\begin{aligned}
h_{cdl}(p_n, p_m) &\precsim \varrho(p_n, p_{n+1})h_{cdl}(p_n, p_{n+1}) + \varsigma(p_{n+1}, p_m)h_{cdl}(p_{n+1}, p_m)\\
&\precsim \varrho(p_n, p_{n+1})h_{cdl}(p_n, p_{n+1}) + \varsigma(p_{n+1}, p_m)\varrho(p_{n+1}, p_{n+2})h_{cdl}(p_{n+1}, p_{n+2})\\
&+ \varsigma(p_{n+1}, p_m)\varsigma(p_{n+2}, p_m)h_{cdl}(p_{n+2}, p_m)\\
&\precsim \varrho(p_n, p_{n+1})h_{cdl}(p_n, p_{n+1}) + \varsigma(p_{n+1}, p_m)\varrho(p_{n+1}, p_{n+2})h_{cdl}(p_{n+1}, p_{n+2})\\
&+ \varsigma(p_{n+1}, p_m)\varsigma(p_{n+2}, p_m)\varrho(p_{n+2}, p_{n+3})h_{cdl}(p_{n+2}, p_{n+3})\\
&+ \varsigma(p_{n+1}, p_m)\varsigma(p_{n+2}, p_m)\varsigma(p_{n+3}, p_m)h_{cdl}(p_{n+3}, p_m)\\
&\precsim \cdots\\
&\precsim \varrho(p_n, p_{n+1})h_{cdl}(p_n, p_{n+1}) + \sum_{\iota=n+1}^{m-2}\Big(\prod_{j=n+1}^{\iota}\varsigma(p_j, p_m)\Big)\varrho(p_\iota, p_{\iota+1})h_{cdl}(p_\iota, p_{\iota+1})\\
&+ \prod_{k=n+1}^{m-1}\varsigma(p_k, p_m)h_{cdl}(p_{m-1}, p_m)\\
&\precsim \varrho(p_n, p_{n+1})l^n h_{cdl}(p_0, p_1) + \sum_{\iota=n+1}^{m-2}\Big(\prod_{j=n+1}^{\iota}\varsigma(p_j, p_m)\Big)\varrho(p_\iota, p_{\iota+1})l^\iota h_{cdl}(p_0, p_1)\\
&+ \prod_{\iota=n+1}^{m-1}\varsigma(p_\iota, p_m)l^{m-1}h_{cdl}(p_0, p_1)\\
&\precsim \varrho(p_n, p_{n+1})l^n h_{cdl}(p_0, p_1) + \sum_{\iota=n+1}^{m-2}\Big(\prod_{j=n+1}^{\iota}\varsigma(p_j, p_m)\Big)\varrho(p_\iota, p_{\iota+1})l^\iota h_{cdl}(p_0, p_1)\\
&+ \Big(\prod_{\iota=n+1}^{m-1}\varsigma(p_\iota, p_m)\Big)l^{m-1}\varrho(p_{m-1}, p_m)h_{cdl}(p_0, p_1)\\
&= \varrho(p_n, p_{n+1})l^n h_{cdl}(p_0, p_1) + \sum_{\iota=n+1}^{m-1}\Big(\prod_{j=n+1}^{\iota}\varsigma(p_j, p_m)\Big)\varrho(p_\iota, p_{\iota+1})l^\iota h_{cdl}(p_0, p_1)\\
&\precsim \varrho(p_n, p_{n+1})l^n h_{cdl}(p_0, p_1) + \sum_{\iota=n+1}^{m-1}\Big(\prod_{j=0}^{\iota}\varsigma(p_j, p_m)\Big)\varrho(p_\iota, p_{\iota+1})l^i h_{cdl}(p_0, p_1).
\end{aligned}$$

We used $\varrho(p,q) \geq 1$. Let

$$R_g = \sum_{\iota=0}^{g}(\prod_{j=0}^{\iota} \varsigma(p_j, p_m))\varrho(p_\iota, p_{\iota+1})l^\iota.$$

Hence, we have

$$h_{cdl}(p_n, p_m) \precsim h_{cdl}(p_0, p_1)[l^n \varrho(p_n, p_{n+1}) + (R_{m-1}, R_n)]. \tag{3.5}$$

The ratio test together with (3.2) imply that the limit of the real number sequence $\{R_n\}$ exits. Then $\{R_n\}$ is a Cauchy sequence.

Indeed, the ration test is applied to the term $v_\iota = (\prod_{j=0}^{\iota} \varsigma(p_j, p_m))\varrho(p_\iota, p_{\iota+1})$. Letting $n, m$ tend to $\infty$ in (3.5) yields

$$\lim_{n,m\to\infty} h_{cdl}(p_n, p_m) = 0,$$

so the sequence $\{p_n\}$ is Cauchy. Since $(\mathbb{X}, h_{cdl})$ is a complete double controlled metric type space, there exists some $\kappa \in \mathbb{X}$ such that

$$\lim_{n\to\infty} h_{cdl}(p_n, \kappa) = 0.$$

We claim that $\Upsilon\kappa = \kappa$. By (DCML3), we have

$$h(\kappa, p_{n+1}) \precsim \varrho(\kappa, p_n)h_{cdl}(\kappa, p_n) + \varsigma(p_n, p_{n+1})h_{cdl}(p_n, p_{n+1}). \tag{3.6}$$

Using (3.3) and (3.6), we get that

$$\lim_{n\to\infty} h(\kappa, p_{n+1}) = 0. \tag{3.7}$$

By (3.1), we have

$$\begin{aligned} h(\kappa, \Upsilon\kappa) &\precsim \varrho(\kappa, p_{n+1})h_{cdl}(\kappa, p_{n+1}) + \varsigma(p_{n+1}, \Upsilon\kappa)h_{cdl}(p_{n+1}, \Upsilon\kappa) \\ &\precsim \varrho(p, p_{n+1})h_{cdl}(\kappa, p_{n+1}) + l\varsigma(p_{n+1}, \Upsilon\kappa)h_{cdl}(p_n, \kappa). \end{aligned}$$

Using (3.3) and (3.7), we get at the limit $h_{cdl}(\kappa, \Upsilon\kappa) = 0$, that is, $\Upsilon\kappa = \kappa$. Let $\varpi$ in $\mathbb{X}$ be such that $\Upsilon\eta = \varpi$ and $\kappa \neq \varpi$. We have

$$0 \prec h_{cdl}(\kappa, \varpi) = h_{cdl}(\Upsilon\kappa, \Upsilon\kappa) \preceq l h_{cdl}(\kappa, \varpi).$$

Contradiction. Then $\kappa = \varpi$. Hence, $\kappa$ is the unique fixed point of $\Upsilon$. □

**Remark 3.1.** *The assumption (3.3) of the Theorem 3.1 above given can be replaced by the assumptions that the mapping $\Upsilon$ and the complex valued double controlled metric $h$ are continuous. Indeed, when $p_n \to \kappa$, then $\Upsilon p_n \to \Upsilon\kappa$ and hence we have*

$$\lim_{n\to\infty} h_{cdl}(\Upsilon p_n, \Upsilon\kappa) = 0 = \lim_{n\to\infty} h_{cdl}(\Upsilon p_{n+1}, \Upsilon\kappa) = p(\kappa, \Upsilon\kappa),$$

*and hence $\Upsilon\kappa = \kappa$.*

The Theorem 3.1 is illustrated by the following examples.

**Example 3.1.** *We endow $\mathbb{X} = \{1, 2, 3\}$ by the following CDCMLS $h = h_{cdl}$*

$$h(1,1) = h(2,2) = 0 \text{ and } h(3,3) = \frac{i}{2},$$
$$h(1,2) = h(2,1) = 2 + 4i\,,\ h(2,3) = h(3,2) = i\,,\ h(1,3) = h(3,1) = 1 - i.$$

*We consider* $\varrho, \varsigma$*:* $\mathbb{X} \times \mathbb{X} \to [1, \infty)$ *to be symmetric and defined by*

$$\varrho(1,1) = \varrho(2,2) = \varrho(3,3) = 1,\ \varrho(1,2) = \varrho(2,1) = \frac{6}{5},\ \varrho(2,3) = \varrho(3,2) = \frac{8}{5},\ \varrho(3,1) = \varrho(1,3) = \frac{151}{100}.$$

*and*

$$\varsigma(1,1) = \varsigma(2,2) = \varsigma(3,3) = 1,\ \varsigma(1,2) = \varsigma(2,1) = \frac{6}{5},\ \varsigma(2,3) = \varsigma(3,2) = \frac{33}{20},\ \varsigma(3,1) = \varsigma(1,3) = \frac{8}{3}.$$

*Let us define the self mapping* $\Upsilon$ *on* $\mathbb{X}$ *as follows:*

$$\Upsilon 1 = \Upsilon 2 = \Upsilon 3 = 2.$$

*Next, we will verify the condition 1:*

**Case 1.** *When* $p = q = 1$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(1), \Upsilon(1))| = |h(2,2)| = 0 \precsim l|h(1,1)|.$$

**Case 2.** *When* $p = 1$*,* $q = 2$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(1), \Upsilon(2))| = |h(2,2)| = 0 \precsim l|h(1,2)|.$$

**Case 3.** *When* $p = 1$*,* $q = 3$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(1), \Upsilon(3))| = |h(2,2)| = 0 \precsim l|h(1,3)|.$$

**Case 4.** *When* $p = 2$*,* $q = 1$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(2), \Upsilon(1))| = |h(2,2)| = 0 \precsim l|h(2,1)|.$$

**Case 5.** *When* $p = q = 2$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(2), \Upsilon(2))| = |h(2,2)| = 0 \precsim l|h(2,2)|.$$

**Case 6.** *When* $p = 2$*,* $q = 3$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(2), \Upsilon(3))| = |h(2,2)| = 0 \precsim l|h(2,3)|.$$

**Case 7.** *When* $p = 3$*,* $q = 1$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(3), \Upsilon(1))| = |h(2,2)| = 0 \precsim l|h(3,1)|.$$

**Case 8.** *When* $p = 3$*,* $q = 2$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(3), \Upsilon(2))| = |h(2,2)| = 0 \precsim l|h(3,2)|.$$

**Case 9.** *When* $p = q = 3$,

$$|h(\Upsilon p, \Upsilon q)| = |h(\Upsilon(3), \Upsilon(3))| = |h(2,2)| = 0 \precsim l|h(3,3)|.$$

*For all* $k \in (0, 1)$*, it is clear that the above conditions are satisfied, these conditions are also satisfied for* $\Upsilon 1 = \Upsilon 2 = \Upsilon 3 = 1$*. For any* $p_0 \in \mathbb{X}$ *condition (2) holds along with conditions of Theorem 3.1. Therefore, there exists a unique fixed point at 1.*

**Definition 3.1.** *Given $p_0 \in \mathbb{X}$, the orbit $\mathbb{O}(u_0)$ of $p_0$ is defined as $\mathbb{O}(u_0) = \{p_0, \Upsilon p_0, \Upsilon^2 u_0, \cdots\}$, where $\Upsilon$ is a self-map on the set $\mathbb{X}$. The operator $\Gamma : \mathbb{X} \to \mathbb{R}$ is called $\Upsilon$-orbitally lower semi-continuous at $\varpi \in \mathbb{X}$ if when $\{p_n\}$ in $\mathbb{O}(p_0)$ such that $\lim_{n\to\infty} h_{cdl}(p_n, \varpi) = 0$, we get that $\Gamma(\varpi) \preceq \lim_{n\to\infty} \inf \Gamma(p_n)$.*

Following same steps as in [34] and using Definition 3.1, we have the following corollary, generalizing the Theorem 3.1 of [16].

**Corollary 3.1.** *Let $\Upsilon$ be a self-map on $(\mathbb{X}, h_{cdl})$ a complete complex valued double controlled metric lke space by two mappings $\varrho, \varsigma$. Given $p_0 \in \mathbb{X}$, let $l \in (0, 1)$ be such that*

$$h_{cdl}(\Upsilon z, \Upsilon^2 z) \precsim l h_{cdl}(z, \Upsilon z), \text{ for each } z \in \mathbb{O}(p_0).$$

*Take $p_n = \Upsilon^n p_0$ and suppose that*

$$\sup_{m\geq 1} \lim_{\iota\to\infty} \frac{\varrho(p_{\iota+1}, p_{\iota+2})}{\varrho(p_\iota, p_{\iota+1})} \varsigma(p_{\iota+1}, p_m) < \frac{1}{l}.$$

*Then, $\lim_{n\to\infty} h_{cdl}(p_n, \kappa) = 0$. We also we have that $\Upsilon\kappa = \kappa$ if and only if the operator $x \mapsto h_{cdl}(x, \Upsilon x)$ is $\Upsilon$-orbitally lower semi-continuous at $p$.*

Our next fixed point result involve a Reich type inequality, as follows.

**Theorem 3.2.** *Let $(\mathbb{X}, h_{cdl})$ be a CDCMLS by the functions $\varrho, \varsigma$: $\mathbb{X} \times \mathbb{X} \to [1, \infty)$ and $\Upsilon$ be a self mapping satisfying Reich condition. That is, $\Upsilon$ satisfies*

$$h_{cdl}(\Upsilon p, \Upsilon q) \precsim \alpha h_{cdl}(p, q) + \beta(p, \Upsilon p) + \gamma(q, \Upsilon q), \tag{3.8}$$

*for $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and $\gamma = \frac{\alpha+\beta}{1-\gamma} < 1$, for all $p, q \in \mathbb{X}$.*

*For $p_0 \in \mathbb{X}$ we choose $p_n = \Upsilon^n p_0$. Assume that*

$$\sup_{m\geq 1} \lim_{\iota\to\infty} \frac{\varrho(p_{\iota+1}, p_{\iota+2})}{\varrho(p_\iota, p_{\iota+1})} \varsigma(p_{\iota+1}, p_m) < \frac{1}{l}, \tag{3.9}$$

$$\lim_{n\to\infty} \varrho(p, p_n) < \infty \text{ exist and finite and } \lim_{n\to\infty} \varsigma(p_n, p) < \frac{1}{\gamma}. \tag{3.10}$$

*Then, $\Upsilon$ has a unique fixed point.*

*Proof.* Let $p_0 \in \mathbb{X}$. Consider the sequence $\{p_n\}$ with $p_{n+1} = \Upsilon p_n$ for all $n \in \mathbb{N}$. It is clear that if there exist $n_0$ for which $p_{n_0+1} = p_{n_0}$ then $\Upsilon p_{n_0} = p_{n_0}$. Then the proof is finished.

Thus, we suppose that $p_{n_o+1} \neq p_n$ for every $n \in \mathbb{N}$. Therefore, we may assume that $p_{n+1} = p_n$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned} h_{cdl}(p_n, p_{n+1}) = h_{cdl}(\Upsilon p_{n-1}, \Upsilon p_n) &\precsim \alpha h_{cdl}(p_{n-1}, p_n) + \beta h_{cdl}(p_{n-1}, \Upsilon p_{n-1}) + \gamma h_{cdl}(p_n, \Upsilon p_n) \\ &= \alpha h_{cdl}(p_{n-1}, p_n) + \beta h_{cdl}(p_{n-1}, \Upsilon p_n) + \gamma h_{cdl}(\Upsilon p_n, p_{n+1}). \end{aligned} \tag{3.11}$$

Therefore, we get

$$h_{cdl}(p_n, p_{n+1}) \precsim (\frac{\alpha + \beta}{1 - \gamma}) h_{cdl}(p_{n-1}, p_n) = l h_{cdl}(p_{n-1}, p_n). \tag{3.12}$$

Thus, we obtain

$$h_{cdl}(p_n, p_{n+1}) \precsim l h_{cdl}(p_{n-1}, p_n) \precsim l^2 h_{cdl}((p_{n-2}, p_{n-1} \precsim ... \precsim l^n h_{cdl}(p_0, p_1). \tag{3.13}$$

For all $n, m \in \mathbb{N}$ with $n < m$ we get

$$\begin{aligned}
h_{cdl}(p_n, p_m) &\precsim \varrho(p_n, p_{n+1}) h_{cdl}(p_n, p_{n+1}) + \varsigma(p_{n+1}, p_m) h_{cdl}(p_{n+1}, p_m) \\
&\precsim \varrho(p_n, p_{n+1}) h_{cdl}(p_n, p_{n+1}) + \varsigma(p_{n+1}, p_m) \varrho(p_{n+1}, p_{n+2}) h_{cdl}(p_{n+1}, p_{n+2}) \\
&+ \varsigma(p_{n+1}, p_m) \varsigma(p_{n+2}, p_m) h_{cdl}(p_{n+2}, p_m) \\
&\precsim \varrho(p_n, p_{n+1}) h_{cdl}(p_n, p_{n+1}) + \varsigma(p_{n+1}, p_m) \varrho(p_{n+1}, p_{n+2}) h_{cdl}(p_{n+1}, p_{n+2}) \\
&+ \varsigma(p_{n+1}, p_m) \varsigma(p_{n+2}, p_m) \varrho(p_{n+2}, p_{n+3}) h_{cdl}(p_{n+2}, p_{n+3}) \\
&+ \varsigma(p_{n+1}, p_m) \varsigma(p_{n+2}, p_m) \varsigma(p_{n+3}, p_m) h_{cdl}(p_{n+3}, p_m) \\
&\precsim \cdots \\
&\precsim \varrho(p_n, p_{n+1}) h_{cdl}(p_n, p_{n+1}) + \sum_{\iota=n+1}^{m-2} \Big( \prod_{j=n+1}^{\iota} \varsigma(p_j, p_m) \Big) \varrho(p_\iota, p_{\iota+1}) h_{cdl}(p_\iota, p_{\iota+1}) \\
&+ \prod_{k=n+1}^{m-1} \varsigma(p_k, p_m) h_{cdl}(p_{m-1}, p_m) \\
&\precsim \varrho(p_n, p_{n+1}) l^n h_{cdl}(p_0, p_1) + \sum_{i=n+1}^{m-2} \Big( \prod_{j=n+1}^{\iota} \varsigma(p_j, p_m) \Big) \varrho(p_\iota, p_{\iota+1}) l^\iota h_{cdl}(p_0, p_1) \\
&+ \prod_{\iota=n+1}^{m-1} \varsigma(p_\iota, p_m) l^{m-1} h_{cdl}(p_0, p_1) \\
&\precsim \varrho(p_n, p_{n+1}) l^n h_{cdl}(p_0, p_1) + \sum_{\iota=n+1}^{m-2} \Big( \prod_{j=n+1}^{\iota} \varsigma(p_j, p_m) \Big) \varrho(p_\iota, p_{\iota+1}) l^\iota h_{cdl}(p_0, p_1) \\
&+ \Big( \prod_{\iota=n+1}^{m-1} \varsigma(p_\iota, p_m) \Big) l^{m-1} \varrho(p_{m-1}, p_m) h_{cdl}(p_0, p_1) \\
&= \varrho(p_n, p_{n+1}) l^n h_{cdl}(p_0, p_1) + \sum_{\iota=n+1}^{m-1} \Big( \prod_{j=n+1}^{\iota} \varsigma(p_j, p_m) \Big) \varrho(p_\iota, p_{\iota+1}) l^\iota h_{cdl}(p_0, p_1) \\
&\precsim \varrho(p_n, p_{n+1}) l^n h_{cdl}(p_0, p_1) + \sum_{\iota=n+1}^{m-1} \Big( \prod_{j=0}^{\iota} \varsigma(p_j, p_m) \Big) \varrho(p_\iota, p_{\iota+1}) l^\iota h_{cdl}(p_0, p_1).
\end{aligned}$$

$$R_n = \sum_{i=0}^{n} \Big( \prod_{j=0}^{\iota} \varsigma(p_j, p_m) \Big) \varrho(p_\iota, p_{\iota+1}) l^\iota h_{cdl}(p_1, p_0) \cdot$$

Then, applying the ratio test, we have

$$g_n = \Big( \prod_{j=0}^{\iota} \varsigma(p_j, p_m) \Big) \varrho(p_\iota, p_{\iota+1}) l^\iota h_{cdl}(p_1, p_0).$$

Then, we have

$$\frac{g_{n+1}}{g_n} = l\varrho(p_{\iota+1}, p_m)\frac{\varsigma(p_{\iota+1}, p_{\iota+2})}{(\varsigma(p_\iota, p_{\iota+1})}.$$

Therefore under condition (3.9), the series $\sum_n g_n$ converges. Therefore, $\lim_{n\to\infty} R_n$ exist. So the real number sequence $\{R_n\}$ is Cauchy.

Thus we obtained the inequality $h_{cdl}(p_n, p_m) \precsim h_{cdl}(p_1, p_0)[l^n\varsigma(p_n, p_{n+1}) + (R_{m-1} - R_n]$.

Letting $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} h_{cdl}(p_n, p_m) = 0,$$

so the sequence $\{p_n\}$ is Cauchy. Since $(\mathbb{X}, h_{cdl})$ is a complete CDCMLS, then there exists some $p_0^* \in \mathbb{X}$ such that

$$\lim_{n\to\infty} h_{cdl}(p_n, p_0^*) = 0.$$

Which means $p_n \to p_0^*$ and $n \to \infty$.

Now, our claim is to show that $\Upsilon p_0^* = p_0^*$.

$$\begin{aligned}
h_{cdl}(p_0^*, \Upsilon p_0^*) &\precsim \varrho(p_0^*, p_{n+1})h_{cdl}(p_0^*, p_{n+1}) + \varsigma(p_{n+1}, \Upsilon p_0^*)h_{cdl}(p_n, p_{n+1})h_{cdl}(p_{n+1}, \Upsilon p_0^*)\\
&= \varrho(p_0^*, p_{n+1})h_{cdl}(p_0^*, p_{n+1}) + \varsigma(p_{n+1}, \Upsilon p_0^*)h_{cdl}(p_n, p_{n+1})h_{cdl}(p_{n+1}, \Upsilon p_0^*)\\
&\precsim \varrho(p_0^*, p_{n+1})h_{cdl}(p_0^*, p_{n+1}) + \varsigma(p_{n+1}, \Upsilon p_0^*)[\alpha h_{cdl}(p_n, \Upsilon p_0^*) + \beta h_{cdl}(p_n, \Upsilon p_n) + \gamma h_{cdl}(p_0^*, \Upsilon p_0^*)]\\
&= \varsigma(p_0^*, p_{n+1})h(p_0^*, p_{n+1}) + \varrho(p_{n+1}, \Upsilon p_0^*)[\alpha h_{cdl}(p_n, \Upsilon p_0^*) + \beta h_{cdl}(p_n, p_{n+1}) + \gamma h_{cdl}(p_0^*, \Upsilon p_0^*)].
\end{aligned}$$

Using this facts in (3.10) and letting the limit as $n \to \infty$ we obtain

$$h_{cdl}(p_0^*, \Upsilon p_0^*) \precsim \varrho(p_{n+1}, \Upsilon p_0^*)[\gamma \lim_{n\to\infty} h_{cdl}(p_0^*, \Upsilon(p_0, \Upsilon p_0^*)].$$

Suppose that $\Upsilon p_0^* \neq p_0^*$. Since $\lim_{n\to\infty} \varrho(p_{n+1}, \Upsilon p_n) < \frac{1}{l}$ we have

$$0 \prec h_{cdl}(p_0^*, \Upsilon p_0^*) \precsim \varrho(p_{n+1}, \Upsilon p_0^*)[\gamma h_{cdl}(p_0^*, \Upsilon p_0^*)] \precsim \varrho(p_0^*, \Upsilon p_0^*).$$

Contradiction. Which means $p_0^* = \Upsilon p_0^*$.

Finally, assume that $\Upsilon$ has two fixed points, say $p$ and $q$.

Then

$$h_{cdl}(p, q) = h_{cdl}(\Upsilon p, \Upsilon q) \precsim \alpha h_{cdl}(p, q) + \beta h_{cdl}(p, \Upsilon p) + \gamma h_{cdl}(q, \Upsilon q)$$

and so

$$h_{cdl}(p, q)(1 - \alpha) \precsim 0.$$

Since$\alpha \neq 1$. We get $h_{cdl}(p, q) = 0$ which implies $p = q$. This completes the proof. □

**Example 3.2.** *Let* $\mathbb{E} = \{1, 2, 3\}$. *Define* $h = h_{cdl}$: $\mathbb{E} \times \mathbb{E} \to \mathbb{C}$ *by*

$$h(1, 1) = h(2, 2) = 0 \textit{ and } h(3, 3) = \frac{i}{2},$$
$$h(1, 2) = h(2, 1) = 2 + i,\ h(2, 3) = h(3, 2) = i,$$
$$h(1, 3) = h(3, 1) = 1 - i.$$

*Define* $\varrho, \varsigma$: $\mathbb{E} \times \mathbb{E} \to [1, \infty)$ *by*

$$\varrho(1,1) = \varrho(2,2) = \varrho(3,3) = 1,$$
$$\varrho(1,2) = \varrho(2,1) = 1, \varrho(2,3) = \varrho(3,2) = \tfrac{8}{7}, \varrho(3,1) = \varrho(1,3) = \tfrac{3}{2}$$

*and*

$$\varsigma(1,1) = \varsigma(2,2) = \varsigma(3,3) = 1,$$
$$\varsigma(1,2) = \varsigma(2,1) = \tfrac{7}{6}, \varsigma(2,3) = \varsigma(3,2) = \tfrac{9}{2}, \varsigma(3,1) = \varsigma(1,3) = 1.$$

*Let* $\Upsilon(1) = 2$ *,*$\Upsilon(2) = 2$ *,*$\Upsilon(3) = 2$*,*

*Proof.* $h(\Upsilon p, \Upsilon q) \precsim \alpha h(p,q) + \beta h(p, \Upsilon p) + \gamma h(q, \Upsilon q)$.

**Case 1.** When $p = 1$, $q = 2$,

$$\begin{aligned}|h(\Upsilon p, \Upsilon q)| = |h(2,2)| = 0 \precsim \frac{7\sqrt{5}}{12} = \frac{1}{3}\sqrt{5} + \frac{1}{4}\sqrt{5} = \alpha|h(1,2)| + \beta|h(1,2)| + \gamma|h(2,2)|\\ = \alpha|h(p,q)| + \beta|h(p,\Upsilon p)| + \gamma|h(q,\Upsilon q)|.\end{aligned}$$

**Case 2.** When $p = 1$, $q = 1$,

$$\begin{aligned}|h(\Upsilon p, \Upsilon q)| = |h(2,2)| = 0 \precsim \frac{13\sqrt{5}}{36} = \frac{1}{3}\sqrt{5} + \frac{1}{9}\sqrt{5} = \alpha|h(1,1)| + \beta|h(1,2)| + \gamma|h(1,2)|\\ = \alpha|h(p,q)| + \beta|h(p,\Upsilon p)| + \gamma|h(q,\Upsilon q)|.\end{aligned}$$

**Case 3.** When $p = 2$, $q = 2$,

$$\begin{aligned}|h(\Upsilon p, \Upsilon q)| = |h(2,2)| = 0 \precsim 0 = 0 + 0 + 0 = \alpha|h(2,2)| + \beta|h(2,2)| + \gamma|h(2,2)|\\ = \alpha|h(p,q)| + \beta|h(p,\Upsilon p)| + \gamma|h(q,\Upsilon q)|.\end{aligned}$$

**Case 4.** When $p = 3$, $q = 3$,

$$\begin{aligned}|h(\Upsilon p, \Upsilon q)| = |h(2,2)| = 0 \precsim \frac{12 + 13\sqrt{2}}{36\sqrt{2}} = \frac{1}{3\sqrt{2}} + \frac{1}{4} + \frac{1}{9} = \alpha|h(3,3)| + \beta|h(3,2)| + \gamma|h(3,2)|\\ = \alpha|h(p,q)| + \beta|h(p,\Upsilon p)| + \gamma|h(q,\Upsilon q)|.\end{aligned}$$

**Case 5.** When $p = 2$, $q = 1$,

$$\begin{aligned}|h(\Upsilon p, \Upsilon q)| = |h(2,2)| = 0 \precsim \frac{4\sqrt{5}}{9} = \frac{1}{3}\sqrt{5} + 0 + \frac{1}{9}\sqrt{5} = \alpha|h(2,1)| + \beta|h(2,2)| + \gamma|h(1,2)|\\ = \alpha|h(p,q)| + \beta|h(p,\Upsilon p)| + \gamma|h(q,\Upsilon q)|.\end{aligned}$$

**Case 6.** When $p = 3$, $q = 1$,

$$\begin{aligned}|h(\Upsilon p, \Upsilon q)| = |h(2,2)| = 0 \precsim \frac{12\sqrt{2} + 9 + 4\sqrt{5}}{36} = \frac{1}{3}\sqrt{2} + \frac{1}{4} + \frac{1}{9}\sqrt{5} = \alpha|h(3,1)| + \beta|h(3,2)| + \gamma|h(1,2)|\\ = \alpha|h(p,q)| + \beta|h(p,\Upsilon p)| + \gamma|h(q,\Upsilon q)|.\end{aligned}$$

**Case 7.** When $p = 1$, $q = 3$,

$$|h(\Upsilon p, \Upsilon q)| = |h(2,2)| = 0 \precsim \frac{12\sqrt{2} + 9\sqrt{5} + 4}{36} = \frac{1}{3}\sqrt{2} + \frac{1}{4}\sqrt{5} + \frac{1}{9} = \alpha|h(1,3)| + \beta|h(1,2)| + \gamma|h(3,2)|$$
$$= \alpha|h(p,q)| + \beta|h(p,\Upsilon p)| + \gamma|h(q,\Upsilon q)|.$$

**Case 8.** When $p = 2$, $q = 3$,

$$|h(\Upsilon p, \Upsilon q)| = |h(2,2)| = 0 \precsim \frac{4}{9} = \frac{1}{3} + \frac{1}{4}(0) + \frac{1}{9} = \alpha|h(2,3)| + \beta|h(2,2)| + \gamma|h(3,2)|$$
$$= \alpha|h(p,q)| + \beta|h(p,\Upsilon p)| + \gamma|h(q,\Upsilon q)|.$$

For all $\alpha, \beta, \gamma \in (0,1)$with $\alpha + \beta + \gamma < 1$, it is clear that the above conditions are satisfied, these conditions are also satisfied for $\Upsilon 1 = \Upsilon 2 = \Upsilon 3 = 2$. For any $p_0 \in \mathbb{E}$ condition (3.9) holds along with conditions of theorem 3.2. Therefore, there exists a unique fixed point at 2. □

## 4. Existence and uniqueness of the solution of a Fredholm type integral equation

During this section we suppose the following Fredholm integral equation

$$p(u) = f(u) + \int_a^b B(u,v,p(v))dv, \quad u,v \in [a,b], \; p(u) \in \mathbb{X}, \tag{4.1}$$

where $B(u,v,p(v))$: $[a,b] \times [a,b] \times \mathbb{C} \to \mathbb{C}$ and $f(u)$: $[a,b] \to \mathbb{C}$ be two bounded and continuous functions.

To prove the existence of solution for integral Eq (4.1) we use Theorem 3.1. Then we give the following result.

**Theorem 4.1.** *Let $\mathbb{X} = C([a,b],\mathbb{C})$ is the set of all continuous and complex valued functions which are defined on $[a,b]$. Also let $\Upsilon$: $\mathbb{X} \to \mathbb{X}$ be an operator defined as:*

$$p(u) = f(u) + \int_a^b B(u,v,p(v))dv, \quad u,v \in [a,b]. \tag{4.2}$$

*Suppose the following conditions hold:*

(*i*) *The functions $B(u,v,p(v))$: $[a,b]\times[a,b]\times\mathbb{C} \to \mathbb{C}$ and $f(u)$: $[a,b] \to \mathbb{C}$ it's a continuous function.*

(*ii*) $\mid B(u,v,p(v)) - B(u,v,q(v)) \mid \precsim \frac{1}{\tau\sqrt{b-a}} \mid p(u) - q(u) \mid$, *for all $p,q \in \mathbb{X}$ and $\omega \in (1, \frac{1}{\lambda}]$ with $\lambda \in (0,1)$.*

*Then the Eq (4.1) has a unique solution.*

*Proof.* Let $\mathbb{X} = C([a,b],\mathbb{C})$ and $h_{cdl}$: $\mathbb{X} \times \mathbb{X} \to \mathbb{C}$ such that,

$$h_{cdl}(p,q) = \|p - q\|_\infty = \mid p(u) - q(u) \mid^2 e^{i\cos^{-1}\tau},$$

where $\mid x \mid = \sqrt{\alpha^2 + \beta^2}$, with $\alpha, \beta \in \mathbb{R}$, $\tau > 0$ and $i = \sqrt{-1} \in \mathbb{C}$.

Let $\vartheta_u, \varrho_u$: $\mathbb{X} \times \mathbb{X} \to [1, \infty)$ be defined as

$$\varrho_u(p,q) = \begin{cases} 1, & \text{if } p,q \in [0,1], \\ \max\{p(u),q(u)\}, & \text{otherwise.} \end{cases}$$

$$\varsigma_u(p,q) = \begin{cases} 1, & \text{if } p,q \in (0,1], \\ \frac{1+\max\{p(u),q(u)\}}{\min\{p(u),q(u)\}}, & \text{otherwise.} \end{cases}$$

We observe that $(\mathbb{X}, h_{cdl})$ is a complete CDCMLS. Then the problem (4.1) can be translated to find a fixed point of the operator $\Upsilon$.

Then we have the next inequality

$$\begin{aligned} |\Upsilon p(u) - \Upsilon q(u)|^2 &\precsim \left| \int_a^b B(u,v,p(v))dv - \int_a^b B(u,v,q(v))dv \right|^2 \\ &\precsim \int_a^b |B(u,v,p(v)) - B(u,v,q(v))|^2 \, dv \\ &\precsim \frac{1}{\tau^2(b-a)} \int_a^b \mid p(v) - q(v) \mid^2 \, dv \\ &= \frac{e^{-i\cos^{-1}\tau}}{\tau^2(b-a)} \int_a^b \mid p(v) - q(v) \mid^2 \, e^{i\cos^{-1}\tau} dv \\ &= \frac{e^{-i\cos^{-1}\tau}}{\tau^2(b-a)} \|p-q\|_\infty \left( \int_a^b dv \right). \end{aligned}$$

Following the calculus we obtain

$$\mid \Upsilon p(u) - \Upsilon q(u) \mid^2 e^{i\cos^{-1}\tau} = \|\Upsilon p - \Upsilon q\|_\infty \precsim \frac{1}{\tau^2} \mid p(u) - q(u) \mid^2 e^{i\cos^{-1}\tau} = \frac{1}{\tau^2} \|p-q\|_\infty.$$

Using the hypothesis (*ii*) we have

$$h_{cdl}(\Upsilon p, \Upsilon q) = \|\Upsilon p - \Upsilon q\|_\infty \precsim \frac{1}{\tau^2} \|p-q\|_\infty = \frac{1}{\tau^2} h_{cdl}(p,q).$$

It is easy to check that, for both cases of the expressions of $\varrho_u(p,q)$ and $\varsigma_u(p,q)$, the conditions (3.2) and (3.3) are true.

Then, for $0 < \delta = \frac{1}{\tau^2} < 1$, all the hypothesis of Theorem 3.1 holds.

In these conditions we get that Eq (4.1) has a unique solution. □

## 5. Conclusions

Considering the results from [15, 33] we have introduced the concept of complex valued double controlled metric like spaces (CDCMLS). Some fixed point results and supporting examples in this setting, the related Banach contraction principle and a Reich type fixed point result are presented. Fredholm integral equations are powerful tools on mathematics in order to model phenomena of real world. Then, the last section of the present work is dedicated to apply our main result in order to prove the existence and uniqueness of a solution of a Fredholm type integral equation.

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**Conflicts of interest**

The authors declare no conflicts of interest.

**References**

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, *Fundam. Math.*, **3** (1922), 133–181. http://doi.org/10.4064/FM-3-1-133-181
2. A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, *Numer. Funct. Anal. Optim.*, **32** (2011), 243–253. http://doi.org/10.1080/01630563.2011.533046
3. A. A. Harandi, Metric-like spaces, partial metric spaces and fxed points, *Fixed Point Theory Appl.*, **2012** (2012), 204. http://doi.org/10.1186/1687-1812-2012-204
4. A. Hosseini, M. M. Karizaki, On the complex valued metric-like spaces, *ArXiv*, 2022. http://doi.org/10.48550/ARXIV.2209.06551
5. I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Func. Anal.*, **30** (1989), 26–37.
6. S. Czerwik, Contraction mappings in *b*-metric spaces, *Acta Math. Inf. Univ. Ostra.*, **1** (1993), 5–11.
7. M. S. Aslam, M. S. R. Chowdhury, L. Guran, M. A. Alqudah, T. Abdeljawad, Fixed point theory in complex valued controlled metric spaces with an application, *AIMS Math.*, **7** (2022), 11879–11904. http://doi.org/10.3934/math.2022663
8. T. Kamran, M. Samreen, Q. Ul. Ain, A generalization of *b*-metric space and some fixed point theorems, *Mathematics*, **5** (2017), 1–7. http://doi.org/10.3390/math5020019
9. N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled metric type spaces and the related contraction principle, *Mathematics*, **6** (2018), 194. http://doi.org/10.3390/math6100194
10. T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double controlled metric type spaces and some fixed point results, *Mathematics*, **6** (2018), 320. https://doi.org/10.3390/math6120320
11. S. K. Panda, T. Abdeljawad, C. Ravichandran, A complex valued approach to the solutions of Riemann-Liouville integral, Atangana-Baleanu integral operator and non-linear Telegraph equation via fixed point method, *Chaos Solitons Fract.*, **130** (2020), 109439. https://doi.org/10.1016/j.chaos.2019.109439
12. N. Ullah, M. S. Shagari, A. Azam, Fixed point theoremsin complex valued extended *b*-metric spaces, *Moroccan J. Pure Appl. Anal.*, **5** (2019), 140–163. https://doi.org/10.2478/mjpaa-2019-0011
13. T. Abdeljawad, K. Abodayeh, N. Mlaiki, On fixed point generalizations to partial *b*-metric spaces, *J. Comput. Anal. Appl.*, **19** (2015), 883–891.

14. K. P. Rao, P. Swamy, J. Prasad, A common fixed point theorem in complex valued $b$-metric spaces, *Bull. Math. Stat. Res.*, **1** (2013).

15. N. Mlaiki, Double controlled metric-like spaces, *J. Inequal. Appl.*, **2020** (2020), 189. https://doi.org/10.1186/s13660-020-02456-z

16. T. L. Hicks, B. E. Rhodes, A Banach type fixed point theorem, *Math. Jpn.*, **24** (1979), 327–330.

17. J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, *Proc. Am. Math. Soc.*, **62** (1977), 344–348. https://doi.org/10.2307/2041041

18. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71–76. https://doi.org/10.4064/FM-74-3-181-187

19. H. Afshari, M. Atapour, H. Aydi, Generalized $\varrho - \psi - Geraghty$ multivalued mappings on $b$-metric spaces endowed with a graph, *J. Appl. Eng. Math.*, **7** (2017), 248–260.

20. M. S. Aslam, M. F. Bota, M. S. R. Chowdhury, L. Guran, N. Saleem, Common fixed points technique for existence of a solution of Urysohn type integral equations system in complex valued $b$-metric spaces, *Mathematics*, **9** (2021), 400. https://doi.org/10.3390/math9040400

21. N. Alharbi, H. Aydi, A. Felhi, C. Ozel, S. Sahmim, $\varrho$-contractive mappings on rectangular $b$-metric spaces and an application to integral equations, *J. Math. Anal.*, **9** (2018), 47–60.

22. H. Aydi, E. Karapinar, M. F. Bota, S. Mitrović, A fixed point theorem for set-valued quasi-contractions in $b$-metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 88. https://doi.org/10.1186/1687-1812-2012-88

23. H. Aydi, M. F. Bota, E. Karapinar, S. Moradi, A common fixed point for weak $\phi$-contractions on $b$-metric spaces, *Fixed Point Theory*, **13** (2012), 337–346.

24. O. A. Arqub, Numerical solutions for the Robin time-fractional partial differential equations of heat and fluid flows based on the reproducing kernel algorithm, *Int. J. Numer. Method Heat Fluid Flow*, **28** (2018), 828–856. https://doi.org/10.1108/HFF-07-2016-0278

25. O. A. Arqub, Numerical simulation of time-fractional partial differential equations arising in fluid flows via reproducing Kernel method, *Int. J. Numer. Meth. Heat Fluid Flow*, **11** (2020), 4711–4733. https://doi.org/10.1108/HFF-10-2017-0394

26. O. A. Arqub, Approximate solutions of DASs with nonclassical boundary conditions using novel reproducing kernel algorithm, *Fundam. Inf.*, **146** (2016), 231–254. https://doi.org/10.3233/FI-2016-1384

27. M. M. A. Khater, S. K. Elagan, M. A. El-Shorbagy, S. H. Alfalqi, J. F. Alzaidi, N. A. Alshehri, Folded novel accurate analytical and semi-analytical solutions of a generalized Calogero–Bogoyavlenskii–Schiff equation, *Commun. Theor. Phys.*, **73** (2021), 095003. https://dx.doi.org/10.1088/1572-9494/ac049f

28. M. M. A. Khater, D. Lu, Analytical versus numerical solutions of the nonlinear fractional time–space telegraph equation, *Mod. Phys. Let. B*, **35** (2021), 2150324. https://doi.org/10.1142/S0217984921503243

29. M. M. A. Khater, Abundant breather and semi-analytical investigation: on high-frequency waves' dynamics in the relaxation medium, *Mod. Phys. Let. B*, **35** (2021), 2150372. https://doi.org/10.1142/S0217984921503723

30. S. Sivasankaran, M. M. Arjunan, V. Vijayakumar, Existence of global solutions for second order impulsive abstract partial differential equations, *Nonlinear Anal.*, **74** (2011), 6747–6757. https://doi.org/10.1016/j.na.2011.06.054

31. V. Vijayakumar, Approximate controllability for a class of second-order stochastic evolution inclusions of Clarke's Subdifferential type, *Results Math.*, **73** (2018), 42. https://doi.org/10.1007/s00025-018-0807-8

32. A. Singh, A, Shukla, V. Vijayakumar, R. Udhayakumar, Asymptotic stability of fractional order (1,2] stochastic delay differential equations in Banach spaces, *Chaos Solitons Fract.*, **150** (2021), 111095. https://doi.org/10.1016/j.chaos.2021.111095

33. T. Aysegul, On double controlled metric-like spaces and related fixed point theorems, *Adv. Theory Nonlinear Anal. Appl.*, **5** (2021), 167–172. https://doi.org/10.31197/atnaa.869586

34. T. Kamran, M. Samreen, Q. U. Ain, A generalization of *b*-metric space and some fixed point theorems, *Mathematics*, **5** (2017), 19. https://doi.org/10.3390/math5020019


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