Mathematics

Research article

Robustness analysis of random hyper-networks based on the internal structure of hyper-edges

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Abstract: Random hyper-network is an important hyper-network structure. Studying the structure and properties of random hyper-networks, which helps researchers to understand the influence of the hyper-network structure on its properties. Currently, studies related to the influence of the internal structure of the hyper-edge on robustness have not been carried out for research on the robustness of hyper-networks. In this paper, we construct three k-uniform random hyper-networks with different structures inside hyper-edges. The nodes inside hyper-edges are connected in the ways randomly connected, preferentially connected and completely connected. Meanwhile, we propose a capacity-load model that can describe the relationship between the internal structure and the robustness of the hyper-edge, based on the idea of capacity-load model. The robustness of the three hyper-networks was obtained by simulation experiments. The results show the variation of the internal structure of hyper-edge has a large influence on the robustness of the k-uniform random hyper-network. In addition, the larger number of ordinary edges m_k inside the hyper-edges and the size of the hyper-network k, the more robust the k-uniform random hyper-network is.

1. Introduction

In recent years, hypergraph-based hyper-networks are superior in representing multivariate complex relationships, thus it is widely used to model many real complex systems [1–9]. Meanwhile, research on the robustness of hyper-networks has also made great progress. Ma studied the cascading failures of a hyper-network. They found that hyper-networks under attack exhibit similar properties to

ordinary networks. And when faced with the same external perturbations, the hyper-network is more robust [10]. Ma proposed a hyper-network cascade failure model based on the structure of the hyper-network. They obtained similar conclusions through the simulation analysis of this model [11]. Chen proposed a capacity-load cascading failure model based on hyper-edge diffusion. The model is applied to random hyper-networks and small-world hyper-networks. The comparison reveals that random hyper-networks have stronger robustness than small-world hyper-networks [12]. Luo found hyper-networks are more suitable than ordinary networks to represent City Bus Network. The node represents bus stops and the hyper-edge represents routes. The authors also found the bus hyper-network is robust to random attacks and vulnerable to deliberate attacks [13]. Wang constructed a hyper-network model for High-speed Rail Logistics in China. They identified the important nodes and routes by analyzing the robustness of the High-speed Rail Logistics hyper-network [14]. Zhang analyzed the differences robustness under different methods of brain function hyper-network construction [15]. Pearcy proposed a bacterial metabolic hyper-network capable of describing general metabolic processes. The concept of percolation process is extended to bacterial metabolic hyper-networks. And the robustness of the bacterial metabolic hyper-network was synthesized [16]. However, in the existing studies on the robustness of hyper-networks, most researchers have ignored the influence of the internal structure of the hyper-edges on the robustness of the hyper-networks. Yet, when the internal substructure of a complex system changes, it can also cause changes in the system performance. For instance, in the QQ group hyper-network [6]. The nodes represent users, and the hyper-edges represent different QQ groups. If the users in the group do not know each other, it will slow down the spread of information. Conversely, it will enhance the spread of information. Similarly, changes in the internal structure of the hyper-edge also have an impact on the robustness of the hyper-network. Researchers can gain more comprehensive understanding of factors, influencing the robustness of hyper-networks. It can also provide a reference for proposing a strategy to improve the robustness of the hyper-networks. But, there are fewer studies on how the internal structure of the hyper-edge affects the robustness of the hyper-network.

Although most complex systems in reality are not always random. But, random networks still hold an important network structure of interest to researchers [17–24]. Studying the properties of random networks can provide a reference for the research of other network properties. This enables a clear understanding of impact different network structures on system performance. In previous studies, scholars compared structural properties of some networks with random networks [23, 24], an objective and accurate evaluation of the performance of the network structure is made. We carried out a study on the influence of the internal structure of the hyper-edges on the robustness of k-uniform random hyper-networks.

In this paper, we construct three random hyper-networks with nodes connected in different ways inside the hyper-edges. We propose a new load assignment method based on the capacity-load model [25]. This model can better reflect the relationship between the hyper-edges and the nodes of the hyper-network. And we obtained the influence of different structures inside the hyper-edge on the robustness of the random hyper-network by simulation analysis.

2. Hypergraph-based hyper-network model

Berge [26, 27] first gave the concept of hypergraph, definition is as follows: If the binary relation $H = (V, E)$ satisfies the condition: $\emptyset = e_i \in P(V), i \in \{1, 2, \cdots, m\}; \cup_{i=1}^{m} = V$.

Where $V = \{v_1, v_2, \cdots, v_n\}, E = \{e_1, e_2, \cdots, e_m\}$, and $P(V)$ denotes the power set of the set V; then H is a hypergraph. The elements in the set V are the nodes of the hypergraph. The hyper-degree of node v_i is denoted by $d_H(v_i)$ [28] is defined as the number of hyper-edges containing node v_i. The node degree of node v_i is similar to that of the ordinary network. It is still defined as the number of ordinary edges associated with node v_i, denoted as the $d(v_i)$ [28]. The ordinary degree of a node v_i within a separate hyper-edge e_i is denoted by $k_{e_i}(v_i)$. Figure 1 is a hypergraph containing 10 nodes and 5 hyper-edges. Take the example of node v_3. The hyper-degree $d_H(v_3)$ of v_3 is 3. The node degree $d(v_3)$ is 5. The ordinary degree $k_{e_2}(v_3)$ inside the hyper-edge e_2 is 2.
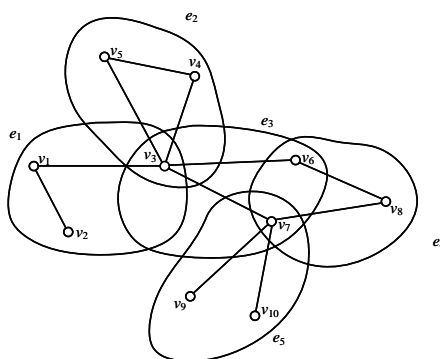
Figure 1. A hypergraph containing 10 nodes and 5 hyper-edges.

2.1. k-uniform random hyper-network model

A hypergraph is said to be k-uniform hypergraph if the number of nodes in each hyper-edge is k. A hyper-network with k-uniform hypergraph as topology is a k-uniform hyper-network. In this paper, we study the influence of the internal structure of the hyper-edges on the robustness of k-uniform random hyper-networks. The procedure for the construction of the k-uniform random hyper-network is given below, as described in Algorithm 1.

Algorithm 1 Modeling the k-uniform random hyper-network.

Input: Uniform variable, k; Initial number of isolated hyper-edges, M_0; Number of nodes, $N = M_0 \times k$;
  Hyper-edge connection probability, $p \in [0, 1]$;
Output: ***hyper-network-matrix***
  1: **do** Time step $T = 0$, ***hyper-network-matrix***;
  2: **for** $T \leq \binom{M_0}{2}$;
  3:   **do** Choose two e_1, e_2 randomly among M_0 isolated hyper-edges;
  4:   **do** Generate a random numbers, $s \in (0, 1)$;
  5:     **if** $s \leq p$;
  6:       **do** Randomly select k nodes from e_1, e_2 (select at least 1 node in each hyper-edge);
  7:       **do** Connecting them into a new hyper-edge;

8:     **end if**
9:   **do** index ← the number of k nodes;
10:   **for** i in index;
11:     **do** ***hyper-network-matrix*** $(M_0 + T, i) = 1$;
12:     **do** $T = T + 1$;
13:   **end for**
14: **end for**

2.2. Random hyper-network models based on different internal structures of hyper-edges

To better discover the influence of the internal structure of the hyper-edges in random hyper-networks on the robustness, we did further research. Based on the k-uniform random hyper-network, we construct three random hyper-network models with different connection methods of nodes inside the hyper-edges. Shown in Table 1.

Table 1. Three random hyper-network models.

Hyper-network type	Nodes connection method inside the hyper-edge		
	Randomly connected ER-R hyper-network	Preferentially connected ER-P hyper-network	Completely connected ER-C hyper-network

The construction process of the three hyper-network models is described in Algorithm 2.

Algorithm 2 Modeling three random hyper-networks

Input: Uniform variable, k; Total hyper-edges, M; Total nodes, N; Node reconnection edges probability, $p \in [0, 1]$;

Output: ***inner-hyper-network-matrix***

1: **do** Disconnect the original connections between nodes inside the hyper-edges;
2: **switch** (selecting the hyper-edge internal connection method);
3:   **case1** Hyper-edge internal nodes are randomly connected;
4:   **do** Make k nodes inside the hyper-edge e_i become isolated nodes; ordinary edge $m_k = 0$; ***inner-hyper-network-matrix***
5:   **for** $m_k \le p \times k(k-1)/2$;
6:     **do** Generate a random number r;
7:     **do** Randomly select a pair of nodes v_i, v_j inside the same hyper-edge;
8:     **if** $r < p$
9:       **if** ***inner-hyper-network-matrix*** $(v_i, v_j)! = 1$
10:         **do** ***inner-hyper-network-matrix*** $(v_i, v_j) = 1$
11:         **do** ***inner-hyper-network-matrix*** $(v_j, v_i) = 1$
12:       **end if**
13:     **end if**
14:   **end for**
15:   **end**
16:   **case2** Hyper-edge internal nodes are preferentially connected;

17:      **do** Make k nodes inside the hyper-edge e_i become isolated nodes; time step $t = 0$; ***inner-hyper-network-matrix***

18:      **do** Select 3 nodes to be completely connected. Make them the initial connected branches inside the hyper-edges C_0;

19:      **do** Γ_t (The set of nodes contained in the connected branch C_t at time t)

20:      **for** $\Gamma_t \leq k$

21:          **do** $t = t + 1$

22:          **do** Randomly select a node v_j outside C_t at time $t + 1$, $v_j (v_j \in V_{e_i}, v_j \notin C_t)$

23:          **do** $C_{t+1} = C_t + v_j$

24:          **do** $\Pi_i = \dfrac{k_{e_i}(v_i)}{\sum_{j=1}^{|\Gamma_t|} k_{e_i}(v_j)}, v_i, v_j \in C_t$

25:          **do** v_j selects a node v_i inside C_t with a preferential connection probability of $\Pi (v_i \in V_{e_i}, v_i \in C_t)$

26:              **if** ***inner-hyper-network-matrix*** $(v_i, v_j)! = 1$

27:                  **do** ***inner-hyper-network-matrix*** $(v_i, v_j) = 1$

28:                  **do** ***inner-hyper-network-matrix*** $(v_j, v_i) = 1$

29:              **end if**

30:      **end for**

31:      **end**

32:      **case3** Hyper-edge internal nodes are completely connected;

33:      **do** Make k nodes inside the hyper-edge e_i become isolated nodes; ordinary edge $m_k = 0$; ***inner-hyper-network-matrix***

34:      **do** Each node is connected to $k - 1$ nodes with edges

35:      **if** ***inner-hyper-network-matrix*** $(v_i, v_j)! = 1$

36:                  **do** ***inner-hyper-network-matrix*** $(v_i, v_j) = 1$

37:                  **do** ***inner-hyper-network-matrix*** $(v_j, v_i) = 1$

38:                  **do** ***diag-inner-hyper-network-matrix*** $= 0$;

39:      **end if**

40:      **end**

41: **end**

3. Hyper-network cascade failure model

Previous researchers have proposed some hyper-network cascade failure models. But, there is the problem of considering too single metrics. As an example, Chen proposed a hyper-network cascading failure model based on hyper-edge diffusion [12]. The authors converted the hyper-networks into line graphs for research. They default hyper-edge internal nodes are completely connected. When a hyper-edge is attacked and fails. Failure hyper-edge's load is spread to its neighboring unfailed hyper-edges. But it does not consider the case where the load is spread through the nodes inside the hyper-edges. To better characterize the influence of the internal structure of the hyper-edge on the robustness of the hyper-network. We propose a new load distribution method based on the capacity-load model.

In a hyper-network with N nodes. The initial load of node v_i is related to the hyper-degree $d_H(v_i)$

and the node degree $d(v_i)$ of this node. The initial load $L_{v_i}(0)$ is defined as:

$$L_{v_i}(0) = \alpha(d_H(v_i) + d(v_i))^\beta, i = 1, 2, \cdots, N, \alpha \geq 1, \beta \geq 1. \tag{1}$$

To control the initial load of node v_i, let α is a load parameter and β is an adjustable parameter. From (1), it shows that the initial load of the node is proportional to its hyper-degree and node degree.

Let node v_i and node vj connected by ordinary edges inside a certain hyper-edge. At time t, node v_i failed. Firstly, the load L_{v_i} of the failed node v_i will be equally divided by the hyper-degree $d_H(v_i)$ of v_i. And assigns the equalized load to all the unfailed hyper-edges associated with the failed node. Then the load received by all hyper-edges associated with the failed node v_i is $L_{v_i}/d_H(v_i)$. Then, the load received by the hyper-edge e_i is equalized again according to the v_i's node degree $k_{e_i}(v_i)$ in the hyper-edge e_i. The node v_j receives the load as shown in (2).

$$\Delta L_{v_i \to v_j}(t) = L_{v_i}(t) \times \frac{1}{d_H^\theta(v_i)} \times \frac{1}{k_{e_i}^\theta(v_i)}. \tag{2}$$

The θ is the perturbation parameter, and $0 \leq \theta \leq 1$. From (2), a node v_j inside the hyper-edge e_i at time t receives a load $\Delta L_{v_i \to v_j}(t)$ that is related to three factors: The hyper-degree of the failed node v_i at time t; The ordinary node degree of node v_i inside the hyper-edge e_i; and the initial load of node v_i. If node v_j has not failed at time t, its load is the sum of the load of node v_j at time $t - 1$ and the load redistributed from the failed node v_i. That is:

$$L_{v_j}(t) = L_{v_j}(t - 1) + \Delta L_{v_i \to v_j}(t). \tag{3}$$

The capacity is the largest amount of load can be handled by the node or hyper-edge. It is proportional to the initial load of the node. Let the capacity of node v_j be C_{v_j}, which is expressed by (4):

$$C_{v_j} = (1 + T)L_{v_j}, \ T \geq 0. \tag{4}$$

T is the capacity parameter. When the value of T is larger, the capacity of the node is larger. With a larger node capacity, the node is more resilient to failures. But the cost of resilience increases. The critical threshold T_C is the smallest value of the capacity parameter to avoid a global crash of the hyper-network. When $T > T_C$, there is no global crash in the whole hyper-network. When $T < T_C$, a global crash occurs in the whole hyper-network. So, the critical threshold T_C of T is an important measure of the robustness of the hyper-network. Obviously, the smaller T_C the more robust hyper-network is.

If node v_j fails after obtaining extra load, it will meet (5):

$$L_{v_j}(t - 1) + \Delta L_{v_i \to v_j}(t) > C_{v_j}. \tag{5}$$

If (5) is true, then node v_j will be overloaded and fail. When the load of v_j is redistributed, it may cause other nodes to fail. Combining with (2), (5) can be expressed as:

$$L_{v_j}(t - 1) + L_{v_i}(t) \times \frac{1}{d_H^\theta(v_i)} \times \frac{1}{k_{e_i}^\theta(v_i)} > C_{v_j}. \tag{6}$$

To measure the robustness of the hyper-network. We initially attack the node v_i and defeat it, and then redistribute its load. For other nodes after load redistribution, if (6) is fulfilled, the node fails.

When all nodes inside a hyper-edge failed, the failure of this hyper-edge occurs. After the number of failed nodes in the hyper-network attains steady state or all nodes failed, we count the number of failed hyper-edges $F_M(0 \leq F_M \leq M)$ in the hyper-network. The hyper-edge failure ratio f_M is shown in (7):

$$f_M = \frac{F_M}{M}, 0 \leq f_M \leq 1. \tag{7}$$

M is the total number of hyper-edges in the hyper-network. From (7), it can be seen that the larger the f_M, the higher the number of failed hyper-edges in the hyper-network. It means the less robustness of the hyper-network.

4. Simulation analysis

To analyze the influence of the node connection method inside the hyper-edge on the robustness of the k-uniform random hyper-network. We designed a simulation experiment as in Algorithm 3. The cascading failure process of k-uniform random hyper-network is simulated under both deliberate and random attack strategies. And record the relevant experimental data. Table 2 shows the experimental parameters in the simulation experiments. Each datum is averaged over 50 independent realizations.

Algorithm 3 Simulation steps

 1: **do** Construct a k-uniform random hyper-network with k of 20, 40, 60
 2: **do** Construct ER-R, ER-P and ER-C hyper-networks
 3: **for** $i = 1 : N$
 4:     **do** Set node v_i initial capacity and load
 5: **end for**
 6: **switch** (Choose an attack strategy)
 7:     **case1**: Random attack strategy: Randomly select a node
 8:     **case2**: Deliberate attack strategy: Select a node with the largest sum of hyper-degree and node degree in the hyper-network
 9: **end**
10: **while** Hyper-network not in stable state
11:     **do** Attacking a node disables it
12:     **do** The load of the failed node is distributed to the non-failed nodes adjacent to it
13:     **if** All nodes in a hyper-edge are invalid
14:         **do** This hyper-edge fails and counts the number of failed hyper-edges
15:     **end if**
16: **end**
17: **do** Calculate and output the hyper-edge failure ratio

Table 2. Experimental simulation parameters and values.

Parameter Name	Parameter Meaning	Value range
N	Total number of hyper-network nodes	[80,1600]
M	Total number of hyper-edges	50
α	Load parameters	10
β	Adjustable parameters	1
θ	Perturbation parameters	1
L	Node Load	–
d	Node-degree	–
d_H	Hyper-degree	–
k	Uniform number	20,40,60
k_{e_i}	Node-degree of the node inside a hyper-edge e_i	–
C	Node Capacity	–
T	Capacity Parameters	[0,1]
F_M	Failure hyper-edge number	[0,M]
f_M	Hyper-edge failure ratio	[0,1]
T_C	Critical Threshold	–

4.1. Robustness analysis of three k-uniform random hyper-networks

In this section, we make three k-uniform random hypernetworks under attack at different k. Subsequently, we analyze the variation of hyper-edge failure ratio f_M with capacity parameter T for three k-uniform random hypernetworks. The results show that: Under deliberate and random attacks, hyper-edge failure ratio tends to decrease with increasing value of capacity parameter. The hyper-network attains a critical point of global collapse under certain capacity parameters. When $T \leq T_C$, all three types of k-uniform random hyper-networks are in a state of global collapse. When $T > T_C$, the failure size starts to decrease and eventually attains the state of global non-failure.

4.1.1. ER-R hyper-network robustness analysis

Figure 2 shows the variation of the hyper-edge failure ratio f_M with the capacity parameter T, when the ER-R hyper-network is attacked with different sizes k. From Figures 2(a) and 2(b) it can be seen that the critical threshold T_C of the ER-R hyper-network gradually reduces with the increase of the size k. Since the smaller the T_C, the more robust the hyper-network is, i. e., the robustness of the ER-R hyper-network increases with the size k of the hyper-network.
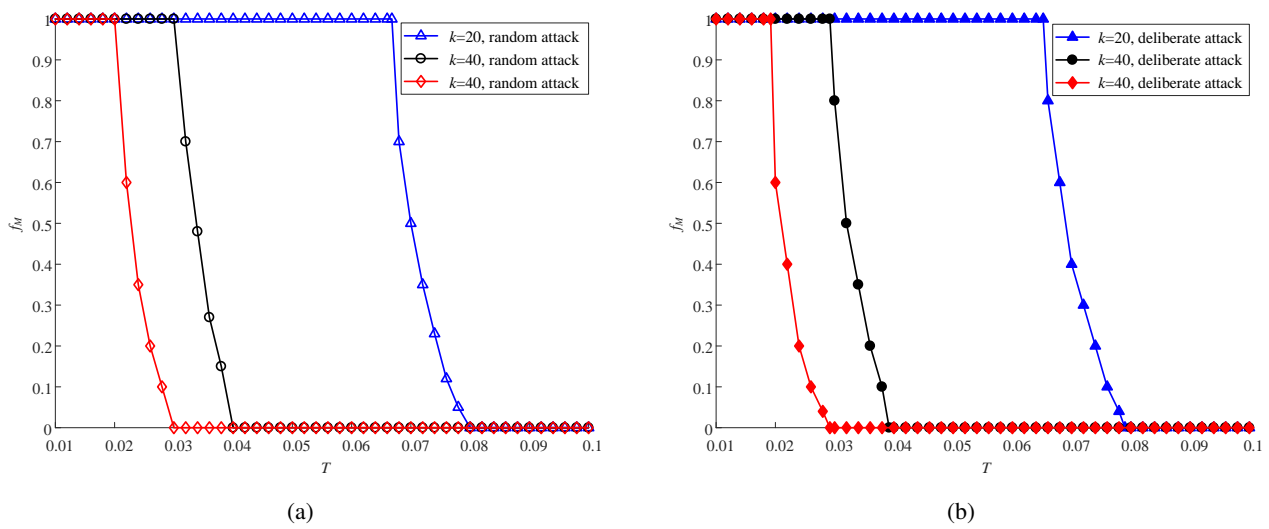
Figure 2. ER-R hyper-network hyper-edge failure ratio under different capacity parameters; (a) Random attack; (b) Deliberate attack.

The diagram cannot clearly distinguish the difference between the critical threshold T_C of the ER-R hyper-network when it is subjected to random attacks and deliberate attacks. So, we give the ER-R hyper-network specific critical thresholds when it is under attack, shown in Table 3. From the data in Table 3, it can be seen that ER-R hyper-network's T_C under deliberate attack is always smaller than T_C under random attack. Again, the smaller T_C is, the more robust the hyper-network is, which gives: The ER-R hyper-network is robust to deliberate attacks and vulnerable to random attacks.

**Table 3.** Critical threshold T_C of ER-R hyper-network with different size k ($k = 20, 40, 60$).

Hyper-network type	Attack strategy	Critical threshold T_C		
		$k = 20$	$k = 40$	$k = 60$
ER-R hyper-network	Random attack	0.0668	0.03	0.02
	Deliberate attack	0.0652	0.0292	0.0192

It can be seen that the ER-R hyper-network exhibits the same robustness as the random network when subjected to external attacks. That is, ER-R hyper-network is robust to deliberate attacks and vulnerable to random attacks. Next, we analyze the reasons for this robustness from the perspective of the ER-R hyper-network structure.

The connection between each hyper-edge of the k-uniform random hyper-network is a random connection. The distribution of the hyper-degree of the k-uniform random hyper-network at $k = 60$ is shown in Figure 3. When the connection between the nodes inside the hyper-edge is randomly connected, the node degree distribution and the hyper-degree distribution inside each hyper-edge are Poisson distributions. As shown in Figures 3 and 4. This connection enhances the random property of k-uniform random hyper-networks. Thus, k-uniform random hyper-networks exhibit properties consistent with random networks when attacked. That is, it is robust to deliberate attacks and vulnerable to random attacks.
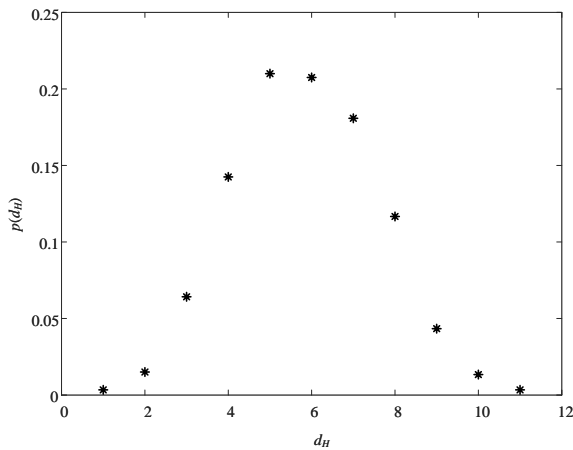
Figure 3. k-uniform random hyper-network hyper-degree distribution ($k = 60$).
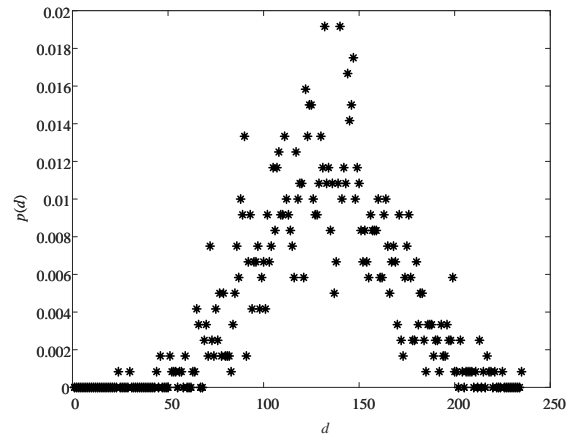
Figure 4. ER- hyper-network node degree distribution ($k = 60$).

4.1.2. ER-P hyper-network robustness analysis

Figure 5 shows the variation of the hyper-edge failure ratio f_M with the capacity parameter T, when the ER-P hyper-network is attacked with different sizes k. From Figures 5(a) and 5(b), it can be seen that the robustness of the ER-P hyper-network increases with the increase of the hyper-network size k.
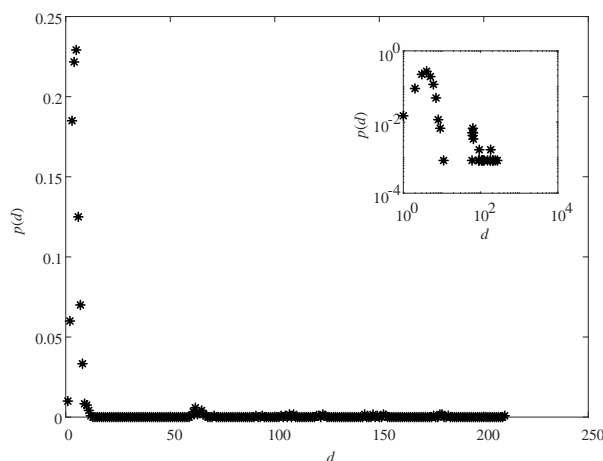

(a)

(b)
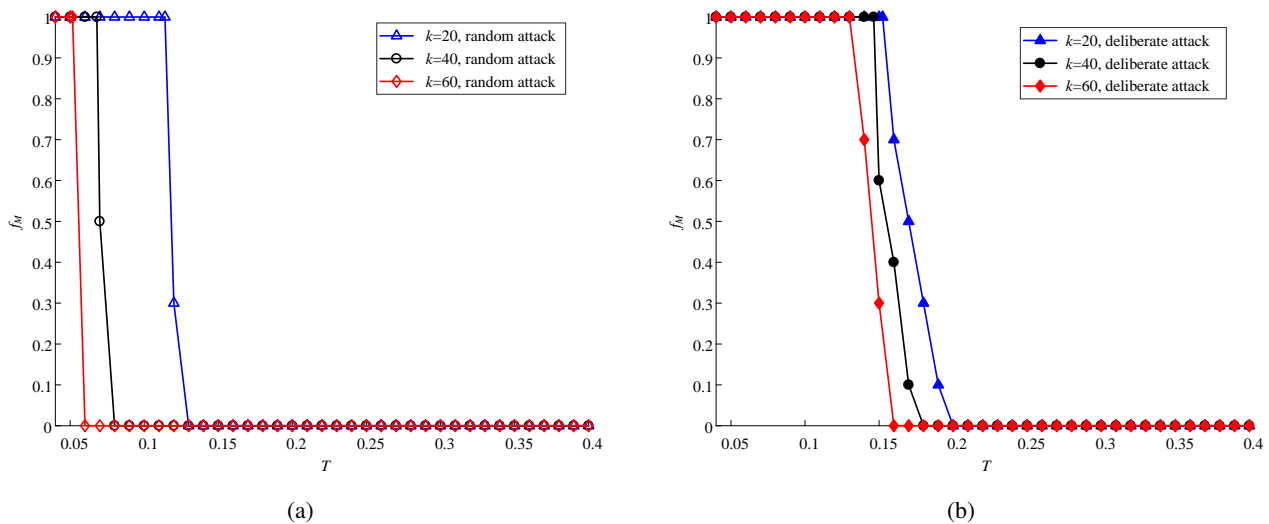
Figure 5. ER-P hyper-network hyper-edge failure ratio under different capacity parameters; (a) Random attack; (b) Deliberate attack.

Next, we give the ER-P hyper-network specific critical thresholds when it is under attack, shown in Table 4. From the data in Table 4, it can be seen that ER-P hyper-network's T_C under deliberate attack is always bigger than T_C under random attack. That is, the ER-P hyper-network is robust to random attacks and vulnerable to deliberate attacks.

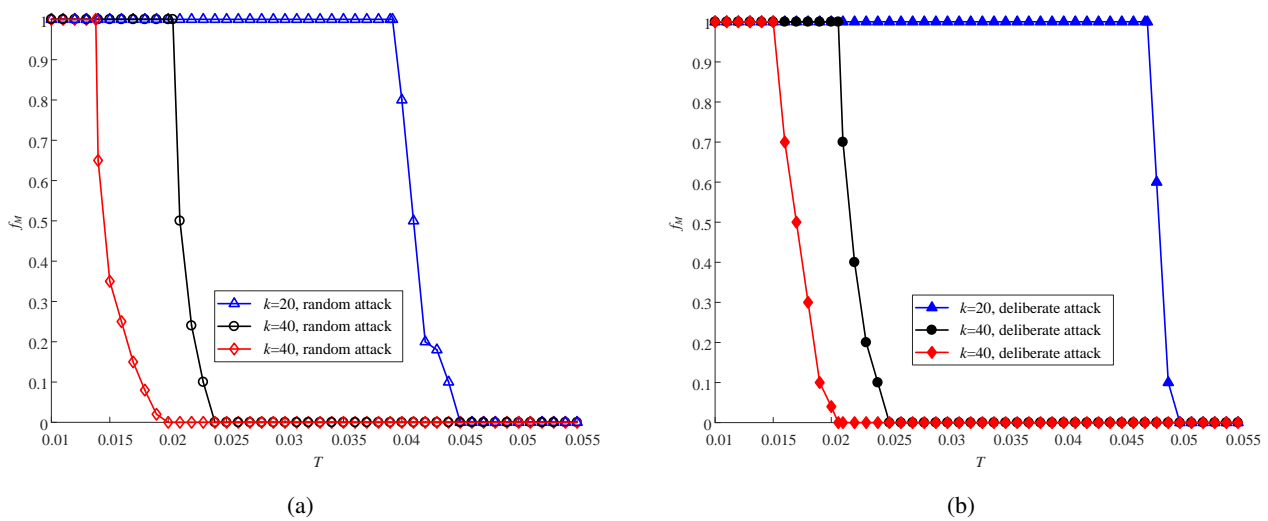**Table 4.** Critical threshold T_C of ER-P hyper-network with different size k ($k = 20, 40, 60$).

Hyper-network type	Attack strategy	Critical threshold T_C		
		$k = 20$	$k = 40$	$k = 60$
ER-P hyper-network	Random attack	0.114	0.068	0.0516
	Deliberate attack	0.1524	0.1464	0.13

It can be seen that the ER-P hyper-network exhibits the contrary robustness as the random network when subjected to external attacks. That is, ER-P hyper-network is robust to random attacks and vulnerable to deliberate attacks. Why does this kinetic phenomenon occur? Next, we analyze the reasons for this robustness from the perspective of the ER-P hyper-network structure. When the connection between nodes inside the hyper-edges is preferentially connected, the node degree distribution inside each hyper-edge is a power-law distribution. Since the hyper-degree distribution is still Poisson, it makes the entire node degree distribution of the ER-P hyper-network like a power-law distribution. As shown in Figure 6. This hyper-edge internal structure reduces the randomness of the k-uniform random hyper-network, so that ER-P hyper-network exhibits the consistent properties of a scale-free network when it is under attack. That is, it is robust to random attacks and vulnerable to deliberate attacks.

Figure 6. ER-P hyper-network node degree distribution ($k = 60$).

4.1.3. ER-C hyper-network robustness analysis

Figure 7 shows the variation of the hyper-edge failure ratio f_M with the capacity parameter T, when the ER-C hyper-network is attacked with different sizes k. From Figures 7(a) and 7(b), it can be seen that the robustness of the ER-C hyper-network increases with the increase of the hyper-network size k.

Figure 7. ER-C hyper-network hyper-edge failure ratio under different capacity parameters; (a) Random attack; (b) Deliberate attack.

Again, we give the ER-C hyper-network specific critical thresholds when it is under attack, shown in Table 5. From the data in Table 5, it can be seen that ER-C hyper-network's T_C under deliberate attack is always bigger than T_C under random attack. That is, the ER-C hyper-network is robust to random attacks and vulnerable to deliberate attacks.

**Table 5.** Critical threshold T_C of ER-C hyper-network with different size k ($k = 20, 40, 60$).

Hyper-network type	Attack strategy	Critical threshold T_C		
		$k = 20$	$k = 40$	$k = 60$
ER-C hyper-network	Random attack	0.0392	0.0204	0.0138
	Deliberate attack	0.0472	0.0206	0.015

The ER-C hyper-network, like the ER-P hyper-network, both exhibit the opposite robustness to the random network when subjected to attacks. That is, the ER-C hyper-network is robust to random attacks and vulnerable to deliberate attacks. Here we analyze the reasons in detail.

When the nodes inside the hyper-edge are completely connected to each other, the inside of each hyper-edge is a regular network. Due to the load assignment method defined in this paper, there will be some nodes with large hyper-degree in the load assignment with overlapping load assignment. Because the random hyper-network is a uniform hyper-network, the difference in hyper-degree from node to node is small. Attacking the node whose hyper-degree is large, the neighboring nodes of the failed node will receive more load. When a neighboring node receives the load assigned by the failed node, it is more likely to crash. So, the ER-C hyper-network is robust to random attacks and vulnerable to deliberate attacks. The difference between the critical threshold T_C of random and deliberate attacks is particularly small influenced by the hyper-network structure.

In addition, Chen studied random hyper-network that also used full connections within the hyper-edge [12]. However, it uses the method of converting the hyper-network into a line graph

network to study. In other words, the effect of the internal structure of the hyperedges on the robustness of the hypernetwork is ignored. Moreover, Chen proposed a capacity-load model with equal load distribution that does not consider the effect brought by the structure. i.e., the effect of the hyper-edge inside and outside together on the hyper-network robustness. We reproduced the simulation of Chen and compared it with the method in this paper, as shown in Figure 8. To ensure the validity of the comparison, the two methods are compared at the same hyper-network size $k = 20$. When the k-uniform random hyper-network under random attack, in Chen's method the critical threshold $T_C = 0.075$ and in this papers' method the critical threshold $T_C = 0.039$. When the k-uniform random hyper-network under deliberate attack, in Chen's method the critical threshold $T_C = 0.07$ and in this papers' method the critical threshold $T_C = 0.047$. Thus, it can be seen that the robustness of this paper's method is better than Chen's method under attack.
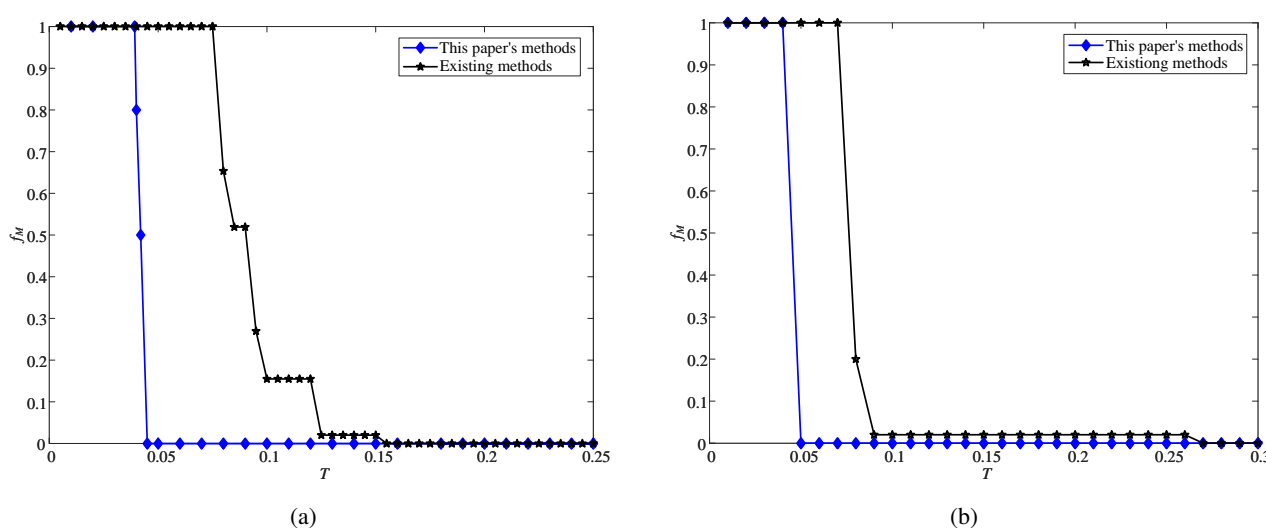
Figure 8. Robustness comparison of k-uniform random hyper-networks under two capacity-load models; (a) Random attack; (b) Deliberate attack.

This shows that, it can be seen that the robustness of the three k-uniform random hyper-networks is closely related to the connection method of nodes inside the hyper-edges. When the nodes inside the hyper-edges are randomly connected, the k-uniform random hyper-network exhibits robustness consistent with that of a random network. When the nodes inside the hyper-edges are preferentially connected and completely connected, the k-uniform random hyper-network exhibits robustness in contrast to the random network.

Analyzing again the critical threshold T_C for the three k-uniform random hyper-networks under attack reveals that: For the same hyper-network size k, the ER-C hyper-network has the best robustness, followed by the ER-R hyper-network, and the ER-P hyper-network has the worst robustness. Analysis of its causes: The robustness of k-uniform random hyper-networks under attack is related to the number of ordinary edges inside the hyper-edges. The size of the ordinary edges with different connection methods of nodes inside the hyper-edge is shown in Figure 9.
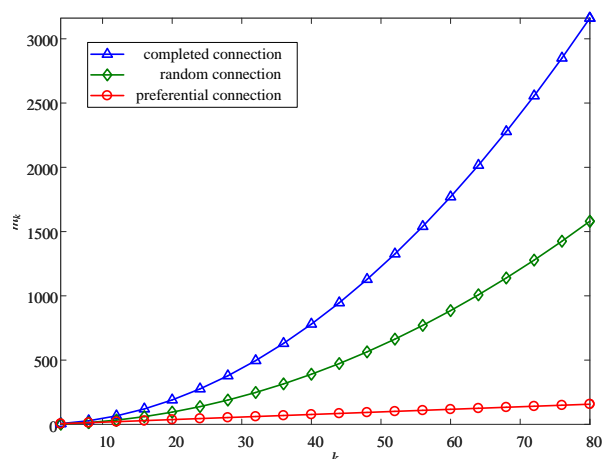
Figure 9. The variation of ordinary edge size m_k inside the hyper-edge with hyper-network size k for the three connection methods.

5. Conclusions

The innovations of this paper are mainly 2 points. First point is: Studying the robustness of k-uniform random hyper-networks from perspective of the internal structure of hyper-edges. Second point is: Based on the capacity-load model, we propose a capacity-load model that is more suitable for describing the load distribution inside and outside the hyper-edge. Applying this model to k-uniform random hyper-networks, its robustness under attack is obtained.

Based on the innovations of this paper, we have obtained conclusions as follows. When the connection between nodes inside a hyper-edge is same as the connection between hyper-edges, the k-uniform random hyper-network is robust to deliberate attacks and vulnerable to random attacks. When the connection between nodes inside a hyper-edge is not consistent with the connection between hyper-edges, the k-uniform random hyper-network is robust to random attacks and vulnerable to deliberate attacks. The ER-C hyper-network has the best robustness, followed by the ER-R hyper-network, and the ER-P hyper-network has the worst robustness. In other words, the larger number m_k of ordinary edges inside the hyper-edge, the better robustness of the k-uniform random hyper-network. The larger size k of the k-uniform random hyper-network, the more robust it is. To summarize, the internal structure of hyper-edges has an important impact on the robustness of hyper-networks.

Acknowledgments

Conflict of interest

All authors have no conflicts of interest to declare.

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