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Research article

Common fixed point results in $\mathcal{F}$-metric spaces with application to nonlinear neutral differential equation

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Abstract: The aim of this article is to obtain common fixed point results for generalized contractions involving control functions of two variables in the context of $\mathcal{F}$-metric spaces. We also furnish an example to show the originality of our main result. Some results in the context of $\mathcal{F}$-metric space equipped with a directed graph G are also established. As an application, we discuss the existence of solution to nonlinear neutral differential equation.

Keywords: fixed point; $\mathcal{F}$-metric space; generalized contractions; control functions; differential equation
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1. Introduction

The theory of fixed points is considered to be the most delightful and energetic field of investigations in the development of mathematical analysis. In this scope, the notion of metric space [1] is one of pillars of not only mathematics but also physical sciences. Due to its noteworthy and remarkable contribution in different fields, it has been extended, improved and generalized in various ways.

In recent years, many interesting generalizations (or extensions) of the metric space concept appeared. The famous extensions of the concept of metric spaces have been done by Bakhtin [2] which was formally defined by Czerwik [3] in 1993. Czerwik [3] gave the idea of b-metric space which broaden the notion of metric space by improving the triangle equality metric axiom by putting a constant $s \geq 1$ multiplied to the right-hand side, is one of the enormous applied extensions for metric spaces. Khamsi et al. [4] reintroduced this notion under the name metric-type and proved some fixed point results in this newly introduced space. In [5], Branciari gave the notion of rectangular metric space and generalized the classical metric space by replacing the triangle inequality with more general inequality that is called rectangular inequality. This inequality involves distance of four points. In 2018, Jleli et al. [6] gave a compulsive extension of a metric space, b-metric space and rectangular metric space which is known as $\mathcal{F}$-metric space. Later on, Al-Mazrooei et al. [7] utilized $\mathcal{F}$-metric space and investigated some fixed point theorems for rational contraction which involves non-negative constants. For more details, we refer the researchers [8–20]

In this research article, we improve rational contraction of Al-Mazrooei et al. [7] by adding one more rational expression in it and replacing non-negative constants with control functions of two variables. We prove some common fixed point results which are generalizations of fixed point results in the context of $\mathcal{F}$-metric spaces. As outcomes of our main results, we derive common fixed point theorems for rational contractions involving control functions of one variable. In this way, we derive the leading results of Jleli et al. [6] and Ahmad et al. [7]. We also establish some results in $\mathcal{F}$-metric space equipped with a directed graph G. As an application, we investigate the solution to nonlinear neutral differential equation.

2. Preliminaries

Let us recall some related material to be used to establish our main results. Recall that Czerwik [3] gave the notion of b-metric space as follows:

Definition 1. *(see [3]) Let $\Theta \neq \emptyset$ and $s \geq 1$ be a constant. A function $\varsigma : \Theta \times \Theta \to [0, \infty)$ is called a b-metric if the following assertions hold:*

$(b1)$ $\varsigma(\varrho, \hbar) \geq 0$ and $\varsigma(\varrho, \hbar) = 0$ if and only if $\varrho = \hbar$;
$(b2)$ $\varsigma(\varrho, \hbar) = \varsigma(\hbar, \varrho)$;
$(b3)$ $\varsigma(\varrho, \varphi) \leq s[\varsigma(\varrho, \hbar) + \varsigma(\hbar, \varphi)]$;
for all $\varrho, \hbar, \varphi \in \Theta$.

The pair (Θ, ς) is then said to be a b-metric space.
Jleli et al. [6] gave a fascinating extension of metric space and b-metric space as follows. Let $\mathcal{F}$ be a set of functions $f : (0, +\infty) \to \mathbb{R}$ satisfying

$(\mathcal{F}_1)$ f is non-decreasing,
$(\mathcal{F}_2)$ for each $\{\varrho_J\} \subseteq \mathbb{R}^+$, $\lim_{J\to\infty} \varrho_J = 0$ if and only if $\lim_{J\to\infty} f(\varrho_J) = -\infty$.

Definition 2. *(see [6]) Let $\Theta \neq \emptyset$ and $\varsigma : \Theta\times\Theta \to [0, +\infty)$. Assume that there exists $(f, \mathfrak{h}) \in \mathcal{F}\times[0, +\infty)$ such that*

(D_1) $(\varrho, \hbar) \in \Theta \times \Theta$, $\varsigma(\varrho, \hbar) = 0$ if and only if $\varrho = \hbar$,
(D_2) $\varsigma(\varrho, \hbar) = \varsigma(\hbar, \varrho)$, for all $(\varrho, \hbar) \in \Theta \times \Theta$,
(D_3) for every $(\varrho, \hbar) \in \Theta \times \Theta$, for every $N \in \mathbb{N}$, $N \geq 2$, and for every $(\varrho_i)_{i=1}^N \subset \Theta$, with

$$(\varrho_1, \varrho_N) = (\varrho, \hbar),$$

we have

$$\varsigma(\varrho, \hbar) > 0 \quad \text{implies} \quad f(\varsigma(\varrho, \hbar)) \leq f(\sum_{i=1}^{N-1} \varsigma(\varrho_i, \varrho_{i+1})) + \mathfrak{h}.$$

Then (Θ, ς) is called a $\mathcal{F}$-metric space.

Example 1. *Let* $\Theta = \mathbb{R}$ *and* $\varsigma : \Theta \times \Theta \to [0, +\infty)$ *be defined by*

$$\varsigma(\varrho, \hbar) = \begin{cases} (\varrho - \hbar)^2 & \text{if } (\varrho, \hbar) \in [0,2] \times [0,2] \\ |\varrho - \hbar| & \text{if } (\varrho, \hbar) \notin [0,2] \times [0,2] \end{cases}$$

with $f(t) = \ln(t)$ *and* $\mathfrak{h} = \ln(2)$*, then* (Θ, ς) *is a* $\mathcal{F}$*-metric space.*

Definition 3. *(see [6]) Let* (Θ, ς) *be* $\mathcal{F}$*-metric space,*

(i) a sequence $\{\varrho_\jmath\}$ in Θ is said to be $\mathcal{F}$-convergent to $\varrho \in \Theta$ if $\{\varrho_\jmath\}$ is convergent to ϱ with respect to the $\mathcal{F}$-metric ς;

(ii) a sequence $\{\varrho_\jmath\}$ is $\mathcal{F}$-Cauchy, if

$$\lim_{\jmath, m \to \infty} \varsigma(\varrho_\jmath, \varrho_m) = 0;$$

(iii) if every $\mathcal{F}$-Cauchy sequence in Θ is $\mathcal{F}$-convergent to a point of Θ, then (Θ, ς) is said to be a $\mathcal{F}$-complete.

Theorem 1. *(see [6]) Let* (Θ, ς) *be a* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\mathfrak{R} : \Theta \to \Theta$*. Assume that there exists* $\alpha \in [0, 1)$ *such that*

$$\varsigma(\mathfrak{R}(\varrho), \mathfrak{R}(\hbar)) \leq \alpha \varsigma(\varrho, \hbar)$$

for all $\varrho, \hbar \in \Theta$*, then* $\mathfrak{R}$ *has a unique fixed point* $\varrho^* \in \Theta$*. Moreover, for any* $\varrho_0 \in \Theta$*, the sequence* $\{\varrho_\jmath\} \subset \Theta$ *defined by*

$$\varrho_{\jmath+1} = \mathfrak{R}(\varrho_\jmath), \quad \jmath \in \mathbb{N},$$

is $\mathcal{F}$*-convergent to* ϱ^**.*

Subsequently, Hussain et al. [14] defined α-ψ-contraction in the background of $\mathcal{F}$-metric spaces and generalized the main result of Jleli et al. [6]. Later on, Ahmad et al. [7] defined a rational contraction in $\mathcal{F}$-metric space and proved the following result as generalization of main theorem of Jleli et al. [6].

Theorem 2. *(see [7]) Let* (Θ, ς) *be a* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\mathfrak{R} : \Theta \to \Theta$*. Assume that there exists* $\alpha, \beta \in [0, 1)$ *such that*

$$\varsigma(\mathfrak{R}(\varrho), \mathfrak{R}(\hbar)) \leq \alpha \varsigma(\varrho, \hbar) + \beta \frac{\varsigma(\varrho, \mathfrak{R}\varrho)\varsigma(\hbar, \mathfrak{R}\hbar)}{1 + \varsigma(\varrho, \hbar)}$$

for all $\varrho, \hbar \in \Theta$*, then* $\mathfrak{R}$ *has a unique fixed point.*

3. Main result

We start this section with the following proposition which is helpful in proving our main result.

Proposition 1. *Let* (Θ, ς) *be a* $\mathcal{F}$*-metric space and* $\mathfrak{R}_1, \mathfrak{R}_2 : (\Theta, \varsigma) \to (\Theta, \varsigma)$*. Let* $\varrho_0 \in \Theta$*. Define the sequence* $\{\varrho_\jmath\}$ *by*

$$\varrho_{2\jmath+1} = \mathfrak{R}_1 \varrho_{2\jmath} \ \text{ and } \ \varrho_{2\jmath+2} = \mathfrak{R}_2 \varrho_{2\jmath+1} \tag{3.1}$$

for all $\jmath = 0, 1, 2, ...$

Assume that there exist $\alpha:\Theta\times\Theta\to[0,1)$ satisfying

$$\alpha(\Re_2\Re_1\varrho,\hbar)\le\alpha(\varrho,\hbar)\ \text{and}\ \alpha(\varrho,\Re_1\Re_2\hbar)\le\alpha(\varrho,\hbar)$$

for all $\varrho,\hbar\in\Theta$. Then

$$\alpha\left(\varrho_{2\jmath},\hbar\right)\le\alpha(\varrho_0,\hbar)\ \text{and}\ \alpha\left(\varrho,\varrho_{2\jmath+1}\right)\le\alpha(\varrho,\varrho_1)$$

for all $\varrho,\hbar\in\Theta$ and $\jmath=0,1,2,....$

Proof. Let $\varrho,\hbar\in\Theta$ and $\jmath=0,1,2,...$ Then we have

$$\begin{aligned}\alpha\left(\varrho_{2\jmath},\hbar\right) &= \alpha\left(\Re_2\Re_1\varrho_{2\jmath-2},\hbar\right)\le\alpha\left(\varrho_{2\jmath-2},\hbar\right)\\ &= \alpha\left(\Re_2\Re_1\varrho_{2\jmath-4},\hbar\right)\le\alpha\left(\varrho_{2\jmath-4},\hbar\right)\\ &\le \cdots\le\alpha(\varrho_0,\hbar).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\alpha\left(\varrho,\varrho_{2\jmath+1}\right) &= \alpha\left(\varrho,\Re_1\Re_2\varrho_{2\jmath-1}\right)\le\alpha\left(\varrho,\varrho_{2\jmath-1}\right)\\ &= \alpha\left(\varrho,\Re_1\Re_2\varrho_{2\jmath-3}\right)\le\alpha\left(\varrho,\varrho_{2\jmath-3}\right)\\ &\le \cdots\le\alpha(\varrho,\varrho_1).\end{aligned}$$

□

Hence, the proof is completed.

Lemma 1. *Let (Θ,ς) be a $\mathcal{F}$-metric space and $\alpha,\beta:\Theta\times\Theta\to[0,1)$. If $\Re_1,\Re_2:\Theta\to\Theta$ satisfy*

$$\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)\le\alpha(\varrho,\Re_1\varrho)\varsigma(\varrho,\Re_1\varrho)+\beta(\varrho,\Re_1\varrho)\frac{\varsigma(\varrho,\Re_1\varrho)\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)}{1+\varsigma(\varrho,\Re_1\varrho)}$$

and

$$\varsigma(\Re_1\Re_2\hbar,\Re_2\hbar)\le\alpha(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\hbar)+\beta(\Re_2\hbar,\hbar)\frac{\varsigma(\Re_2\hbar,\Re_1\Re_2\hbar)\varsigma(\hbar,\Re_2\hbar)}{1+\varsigma(\Re_2\hbar,\hbar)}$$

for all $\varrho,\hbar\in\Theta$, then

$$\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)\le\alpha(\varrho,\Re_1\varrho)\varsigma(\varrho,\Re_1\varrho)+\beta(\varrho,\Re_1\varrho)\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)$$

and

$$\varsigma(\Re_1\Re_2\hbar,\Re_2\hbar)\le\alpha(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\hbar)+\beta(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\Re_1\Re_2\hbar).$$

Proof. Using the hypothesis, we have

$$\begin{aligned}\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho) &\le \alpha(\varrho,\Re_1\varrho)\varsigma(\varrho,\Re_1\varrho)+\beta(\varrho,\Re_1\varrho)\frac{\varsigma(\varrho,\Re_1\varrho)\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)}{1+\varsigma(\varrho,\Re_1\varrho)}\\ &\le \alpha(\varrho,\Re_1\varrho)\varsigma(\varrho,\Re_1\varrho)+\beta(\varrho,\Re_1\varrho)\frac{\varsigma(\varrho,\Re_1\varrho)}{1+\varsigma(\varrho,\Re_1\varrho)}\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)\\ &\le \alpha(\varrho,\Re_1\varrho)\varsigma(\varrho,\Re_1\varrho)+\beta(\varrho,\Re_1\varrho)\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\varsigma(\Re_1\Re_2\hbar,\Re_2\hbar) &\leq \alpha(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\hbar)+\beta(\Re_2\hbar,\hbar)\frac{\varsigma(\Re_2\hbar,\Re_1\Re_2\hbar)\varsigma(\hbar,\Re_2\hbar)}{1+\varsigma(\Re_2\hbar,\hbar)}\\ &\leq \alpha(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\hbar)+\beta(\Re_2\hbar,\hbar)\frac{\varsigma(\hbar,\Re_2\hbar)}{1+\varsigma(\Re_2\hbar,\hbar)}\varsigma(\Re_2\hbar,\Re_1\Re_2\hbar)\\ &\leq \alpha(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\hbar)+\beta(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\Re_1\Re_2\hbar).\end{aligned}$$

□

Theorem 3. *Let (Θ,ς) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\Re_1,\Re_2:\Theta\to\Theta$. If there exist mappings $\alpha,\beta,\gamma:\Theta\times\Theta\to[0,1)$ such that*

(a) $\alpha(\Re_2\Re_1\varrho,\hbar)\leq\alpha(\varrho,\hbar)$ and $\alpha(\varrho,\Re_1\Re_2\hbar)\leq\alpha(\varrho,\hbar)$
$\beta(\Re_2\Re_1\varrho,\hbar)\leq\beta(\varrho,\hbar)$ and $\beta(\varrho,\Re_1\Re_2\hbar)\leq\beta(\varrho,\hbar)$
$\gamma(\Re_2\Re_1\varrho,\hbar)\leq\gamma(\varrho,\hbar)$ and $\gamma(\varrho,\Re_1\Re_2\hbar)\leq\gamma(\varrho,\hbar)$,
(b) $\alpha(\varrho,\hbar)+\beta(\varrho,\hbar)+\gamma(\varrho,\hbar)<1$,
(c)

$$\varsigma(\Re_1\varrho,\Re_2\hbar)\leq\alpha(\varrho,\hbar)\varsigma(\varrho,\hbar)+\beta(\varrho,\hbar)\frac{\varsigma(\varrho,\Re_1\varrho)\varsigma(\hbar,\Re_2\hbar)}{1+\varsigma(\varrho,\hbar)}+\gamma(\varrho,\hbar)\frac{\varsigma(\hbar,\Re_1\varrho)\varsigma(\varrho,\Re_2\hbar)}{1+\varsigma(\varrho,\hbar)}, \quad (3.2)$$

for all $\varrho,\hbar\in\Theta$, then $\Re_1$ and $\Re_2$ have a unique common fixed point.

Proof. Let $\varrho,\hbar\in\Theta$. From (3.2), we have

$$\begin{aligned}\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho) &\leq \alpha(\varrho,\Re_1\varrho)\varsigma(\varrho,\Re_1\varrho)+\beta(\varrho,\Re_1\varrho)\frac{\varsigma(\varrho,\Re_1\varrho)\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)}{1+\varsigma(\varrho,\Re_1\varrho)}\\ &= \alpha(\varrho,\Re_1\varrho)\varsigma(\varrho,\Re_1\varrho)+\beta(\varrho,\Re_1\varrho)\frac{\varsigma(\varrho,\Re_1\varrho)\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)}{1+\varsigma(\varrho,\Re_1\varrho)}.\end{aligned}$$

By Lemma 1, we get

$$\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho)\leq\alpha(\varrho,\Re_1\varrho)\varsigma(\varrho,\Re_1\varrho)+\beta(\varrho,\Re_1\varrho)\varsigma(\Re_1\varrho,\Re_2\Re_1\varrho). \quad (3.3)$$

Similarly, we have

$$\begin{aligned}\varsigma(\Re_1\Re_2\hbar,\Re_2\hbar) &\leq \alpha(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\hbar)+\beta(\Re_2\hbar,\hbar)\frac{\varsigma(\Re_2\hbar,\Re_1\Re_2\hbar)\varsigma(\hbar,\Re_2\hbar)}{1+\varsigma(\Re_2\hbar,\hbar)}\\ &\quad+\gamma(\varrho,\hbar)\frac{\varsigma(\hbar,\Re_1\Re_2\hbar)\varsigma(\Re_2\hbar,\Re_2\hbar)}{1+\varsigma(\Re_2\hbar,\hbar)}\\ &= \alpha(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\hbar)+\beta(\Re_2\hbar,\hbar)\frac{\varsigma(\Re_2\hbar,\Re_1\Re_2\hbar)\varsigma(\hbar,\Re_2\hbar)}{1+\varsigma(\Re_2\hbar,\hbar)}.\end{aligned}$$

By Lemma 1, we get

$$\varsigma(\Re_1\Re_2\hbar,\Re_2\hbar)\leq\alpha(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\hbar)+\beta(\Re_2\hbar,\hbar)\varsigma(\Re_2\hbar,\Re_1\Re_2\hbar). \quad (3.4)$$

Let $\varrho_0 \in \Theta$ and the sequence $\{\varrho_\jmath\}$ be defined by (3.1). From Proposition 1, (3.3), (3.4) and for all $\jmath = 0, 1, 2, ...$

$$\begin{aligned}
\varsigma\left(\varrho_{2\jmath+1}, \varrho_{2\jmath}\right) &= \varsigma\left(\mathfrak{R}_1\mathfrak{R}_2\varrho_{2\jmath-1}, \mathfrak{R}_2\varrho_{2\jmath-1}\right) \leq \alpha\left(\mathfrak{R}_2\varrho_{2\jmath-1}, \varrho_{2\jmath-1}\right)\varsigma\left(\mathfrak{R}_2\varrho_{2\jmath-1}, \varrho_{2\jmath-1}\right) \\
&\quad +\beta\left(\mathfrak{R}_2\varrho_{2\jmath-1}, \varrho_{2\jmath-1}\right)\varsigma\left(\mathfrak{R}_2\varrho_{2\jmath-1}, \mathfrak{R}_1\mathfrak{R}_2\varrho_{2\jmath-1}\right) \\
&= \alpha\left(\varrho_{2\jmath}, \varrho_{2\jmath-1}\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath-1}\right) + \beta\left(\varrho_{2\jmath}, \varrho_{2\jmath-1}\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right) \\
&\leq \alpha\left(\varrho_0, \varrho_{2\jmath-1}\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath-1}\right) + \beta\left(\varrho_0, \varrho_{2\jmath-1}\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right) \\
&\leq \alpha\left(\varrho_0, \varrho_1\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath-1}\right) + \beta\left(\varrho_0, \varrho_1\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right)
\end{aligned}$$

which implies that

$$\varsigma\left(\varrho_{2\jmath+1}, \varrho_{2\jmath}\right) \leq \frac{\alpha\left(\varrho_0, \varrho_1\right)}{1 - \beta\left(\varrho_0, \varrho_1\right)}\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath-1}\right). \tag{3.5}$$

Similarly, we have

$$\begin{aligned}
\varsigma\left(\varrho_{2\jmath+2}, \varrho_{2\jmath+1}\right) &= \varsigma\left(\mathfrak{R}_2\mathfrak{R}_1\varrho_{2\jmath}, \mathfrak{R}_1\varrho_{2\jmath}\right) \leq \alpha\left(\varrho_{2\jmath}, \mathfrak{R}_1\varrho_{2\jmath}\right)\varsigma\left(\varrho_{2\jmath}, \mathfrak{R}_1\varrho_{2\jmath}\right) \\
&\quad +\beta\left(\varrho_{2\jmath}, \mathfrak{R}_1\varrho_{2\jmath}\right)\varsigma\left(\mathfrak{R}_1\varrho_{2\jmath}, \mathfrak{R}_2\mathfrak{R}_1\varrho_{2\jmath}\right) \\
&= \alpha\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right) + \beta\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right)\varsigma\left(\varrho_{2\jmath+1}, \varrho_{2\jmath+2}\right) \\
&\leq \alpha\left(\varrho_0, \varrho_{2\jmath+1}\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right) + \beta\left(\varrho_0, \varrho_{2\jmath+1}\right)\varsigma\left(\varrho_{2\jmath+1}, \varrho_{2\jmath+2}\right) \\
&\leq \alpha\left(\varrho_0, \varrho_1\right)\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right) + \beta\left(\varrho_0, \varrho_1\right)\varsigma\left(\varrho_{2\jmath+1}, \varrho_{2\jmath+2}\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\varsigma\left(\varrho_{2\jmath+2}, \varrho_{2\jmath+1}\right) &\leq \frac{\alpha\left(\varrho_0, \varrho_1\right)}{1 - \beta\left(\varrho_0, \varrho_1\right)}\varsigma\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right) \\
&= \frac{\alpha\left(\varrho_0, \varrho_1\right)}{1 - \beta\left(\varrho_0, \varrho_1\right)}\varsigma\left(\varrho_{2\jmath+1}, \varrho_{2\jmath}\right).
\end{aligned} \tag{3.6}$$

Let $\lambda = \frac{\alpha(\varrho_0,\varrho_1)}{1-\beta(\varrho_0,\varrho_1)} < 1$. Then from (3.5) and (3.6), we have

$$\varsigma\left(\varrho_{\jmath+1}, \varrho_\jmath\right) \leq \lambda\varsigma\left(\varrho_\jmath, \varrho_{\jmath-1}\right)$$

for all $\jmath \in \mathbb{N}$. Inductively, we can construct a sequence $\{\varrho_\jmath\}$ in Θ such that

$$\begin{aligned}
\varsigma\left(\varrho_{\jmath+1}, \varrho_\jmath\right) &\leq \lambda\varsigma\left(\varrho_\jmath, \varrho_{\jmath-1}\right) \\
\varsigma\left(\varrho_{\jmath+1}, \varrho_\jmath\right) &\leq \lambda^2\varsigma\left(\varrho_{\jmath-1}, \varrho_{\jmath-2}\right) \\
&\quad . \\
&\quad . \\
&\quad . \\
\varsigma\left(\varrho_{\jmath+1}, \varrho_\jmath\right) &\leq \lambda^\jmath\varsigma\left(\varrho_1, \varrho_0\right) = \lambda^\jmath\varsigma\left(\varrho_0, \varrho_1\right)
\end{aligned}$$

for all $\jmath \in \mathbb{N}$. Let $(f, \mathfrak{h}) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\delta) - \mathfrak{h}. \tag{3.7}$$

Hence, by (3.7), $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, we have

$$f(\sum_{i=\jmath}^{m-1}\varsigma(\varrho_i,\varrho_{i+1}))\leq f(\sum_{i=\jmath}^{m-1}\lambda^{\jmath}(\varsigma(\varrho_0,\varrho_1)))\leq f(\sum_{\jmath\geq \jmath(\epsilon)}\lambda^{\jmath}\varsigma(\varrho_0,\varrho_1))<f(\epsilon)-h \tag{3.8}$$

for $m>\jmath\geq \jmath(\epsilon)$. Using (D_3) and (3.8), we obtain $\varsigma(\varrho_\jmath,\varrho_m)>0$, $m>\jmath\geq \jmath(\epsilon)$ implies

$$f(\varsigma(\varrho_\jmath,\varrho_m))\leq f(\sum_{i=\jmath}^{m-1}\varsigma(\varrho_i,\varrho_{i+1}))+\mathfrak{h}<f(\epsilon),$$

which yields by $(\mathcal{F}_1)$ that $\varsigma(\varrho_\jmath,\varrho_m)<\epsilon$, $m>\jmath\geq \jmath(\epsilon)$. It shows that $\{\varrho_\jmath\}$ is $\mathcal{F}$-Cauchy. As (Θ,ς) is $\mathcal{F}$-complete, so there exists $\varrho^*\in\Theta$ such that $\{\varrho_\jmath\}$ is $\mathcal{F}$-convergent to ϱ^*, i.e.,

$$\lim_{\jmath\to\infty}\varsigma(\varrho_\jmath,\varrho^*)=0. \tag{3.9}$$

Now, we show that ϱ^* is fixed point of $\mathfrak{R}_1$. We contrary suppose that $\varsigma(\varrho^*,\mathfrak{R}_1\varrho^*)>0$. Then from (3.2), $(\mathcal{F}_1)$ and (D_3), we have

$$\begin{aligned}
f(\varsigma(\varrho^*,\mathfrak{R}_1\varrho^*)) &\leq f(\varsigma\left(\varrho^*,\mathfrak{R}_2\varrho_{2\jmath+1}\right)+\varsigma\left(\mathfrak{R}_2\varrho_{2\jmath+1},\mathfrak{R}_1\varrho^*\right))+\mathfrak{h}\\
&\leq f\left(\varsigma\left(\varrho^*,\mathfrak{R}_2\varrho_{2\jmath+1}\right)+\varsigma\left(\mathfrak{R}_1\varrho^*,\mathfrak{R}_2\varrho_{2\jmath+1}\right)\right)+\mathfrak{h}\\
&\leq f\left(\varsigma\left(\varrho^*,\varrho_{2\jmath+2}\right)+\left(\begin{array}{c}\alpha\left(\varrho^*,\varrho_{2\jmath+1}\right)\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)\\ +\beta\left(\varrho^*,\varrho_{2\jmath+1}\right)\frac{\varsigma\left(\varrho^*,\mathfrak{R}_1\varrho^*\right)\varsigma\left(\varrho_{2\jmath+1},\mathfrak{R}_2\varrho_{2\jmath+1}\right)}{1+\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)}\\ +\gamma\left(\varrho^*,\varrho_{2\jmath+1}\right)\frac{\varsigma\left(\varrho_{2\jmath+1},\mathfrak{R}_1\varrho^*\right)\varsigma\left(\varrho^*,\mathfrak{R}_2\varrho_{2\jmath+1}\right)}{1+\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)}\end{array}\right)\right)+\mathfrak{h}\\
&\leq f\left(\varsigma\left(\varrho^*,\varrho_{2\jmath+2}\right)+\left(\begin{array}{c}\alpha\left(\varrho^*,\varrho_{2\jmath+1}\right)\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)\\ +\beta\left(\varrho^*,\varrho_{2\jmath+1}\right)\frac{\varsigma\left(\varrho^*,\mathfrak{R}_1\varrho^*\right)\varsigma\left(\varrho_{2\jmath+1},\varrho_{2\jmath+2}\right)}{1+\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)}\\ +\gamma\left(\varrho^*,\varrho_{2\jmath+1}\right)\frac{\varsigma\left(\varrho_{2\jmath+1},\mathfrak{R}_1\varrho^*\right)\varsigma\left(\varrho^*,\varrho_{2\jmath+2}\right)}{1+\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)}\end{array}\right)\right)+\mathfrak{h}.
\end{aligned}$$

Taking the limit as $\jmath\to\infty$ and using $(\mathcal{F}_2)$ and (8), we have

$$\lim_{\jmath\to\infty}f(\varsigma(\varrho^*,\mathfrak{R}_1\varrho^*))\leq\lim_{\jmath\to\infty}f\left(\varsigma\left(\varrho^*,\varrho_{2\jmath+2}\right)+\left(\begin{array}{c}\alpha\left(\varrho^*,\varrho_{2\jmath+1}\right)\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)\\ +\beta\left(\varrho^*,\varrho_{2\jmath+1}\right)\frac{\varsigma\left(\varrho^*,\mathfrak{R}_1\varrho^*\right)\varsigma\left(\varrho_{2\jmath+1},\varrho_{2\jmath+2}\right)}{1+\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)}\\ +\gamma\left(\varrho^*,\varrho_{2\jmath+1}\right)\frac{\varsigma\left(\varrho_{2\jmath+1},\mathfrak{R}_1\varrho^*\right)\varsigma\left(\varrho^*,\varrho_{2\jmath+2}\right)}{1+\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)}\end{array}\right)\right)+\mathfrak{h}=-\infty,$$

which implies that $\varsigma(\varrho^*,\mathfrak{R}_1\varrho^*)=0$, a contradiction. Thus $\varrho^*=\mathfrak{R}_1\varrho^*$. Now we prove that ϱ^* is fixed point of $\mathfrak{R}_2$. Then from (3.2), $(\mathcal{F}_1)$ and (D_3), we have

$$\begin{aligned}
f(\varsigma(\varrho^*,\mathfrak{R}_2\varrho^*)) &\leq f(\varsigma\left(\varrho^*,\mathfrak{R}_1\varrho_{2\jmath}\right)+\varsigma\left(\mathfrak{R}_1\varrho_{2\jmath},\mathfrak{R}_2\varrho^*\right))+\mathfrak{h}\\
&\leq f\left(\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)+\left(\begin{array}{c}\alpha\left(\varrho_{2\jmath},\varrho^*\right)\varsigma\left(\varrho_{2\jmath},\varrho^*\right)\\ +\beta\left(\varrho_{2\jmath},\varrho^*\right)\frac{\varsigma\left(\varrho_{2\jmath},\mathfrak{R}_1\varrho_{2\jmath}\right)\varsigma\left(\varrho^*,\mathfrak{R}_2\varrho^*\right)}{1+\varsigma\left(\varrho_{2\jmath},\varrho^*\right)}\\ +\gamma\left(\varrho_{2\jmath},\varrho^*\right)\frac{\varsigma\left(\varrho^*,\mathfrak{R}_1\varrho_{2\jmath}\right)\varsigma\left(\varrho_{2\jmath},\mathfrak{R}_2\varrho^*\right)}{1+\varsigma\left(\varrho_{2\jmath},\varrho^*\right)}\end{array}\right)\right)+\mathfrak{h}
\end{aligned}$$

$$\leq \quad f\left(\varsigma\left(\varrho^*, \varrho_{2\jmath+1}\right) + \left(\begin{array}{c} \varsigma\left(\varrho^*, \varrho_{2\jmath+1}\right) + \alpha\left(\varrho_{2\jmath}, \varrho^*\right)\varsigma\left(\varrho_{2\jmath}, \varrho^*\right) \\ +\beta\left(\varrho_{2\jmath}, \varrho^*\right)\frac{\varsigma\left(\varrho_{2\jmath},\varrho_{2\jmath+1}\right)\varsigma\left(\varrho^*,\Re_2\varrho^*\right)}{1+\varsigma\left(\varrho_{2\jmath},\varrho^*\right)} \\ +\gamma\left(\varrho_{2\jmath}, \varrho^*\right)\frac{\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)\varsigma\left(\varrho_{2\jmath},\Re_2\varrho^*\right)}{1+\varsigma\left(\varrho_{2\jmath},\varrho^*\right)} \end{array}\right)\right) + \mathfrak{h}.$$

Taking the limit as $\jmath \to \infty$ and using $(\mathcal{F}_2)$ and (8), we have

$$\lim_{\jmath\to\infty} f(\varsigma(\varrho^*, \Re_2\varrho^*)) \leq \lim_{\jmath\to\infty} f\left(\varsigma\left(\varrho^*, \varrho_{2\jmath+1}\right) + \left(\begin{array}{c} \varsigma\left(\varrho^*, \varrho_{2\jmath+1}\right) + \alpha\left(\varrho_{2\jmath}, \varrho^*\right)\varsigma\left(\varrho_{2\jmath}, \varrho^*\right) \\ +\beta\left(\varrho_{2\jmath}, \varrho^*\right)\frac{\varsigma\left(\varrho_{2\jmath},\varrho_{2\jmath+1}\right)\varsigma\left(\varrho^*,\Re_2\varrho^*\right)}{1+\varsigma\left(\varrho_{2\jmath},\varrho^*\right)} \\ +\gamma\left(\varrho_{2\jmath}, \varrho^*\right)\frac{\varsigma\left(\varrho^*,\varrho_{2\jmath+1}\right)\varsigma\left(\varrho_{2\jmath},\Re_2\varrho^*\right)}{1+\varsigma\left(\varrho_{2\jmath},\varrho^*\right)} \end{array}\right)\right) + \mathfrak{h} = -\infty,$$

which implies that $\varsigma(\varrho^*, \Re_1\varrho^*) = 0$, a contradiction. Thus $\varrho^* = \Re_2\varrho^*$.Thus ϱ^* is a common fixed point of $\Re_1$ and $\Re_2$. Now we prove that ϱ^* is unique. We suppose that

$$\varrho^/ = \Re_1\varrho^/ = \Re_2\varrho^/$$

but $\varrho^* \neq \varrho^/$. Now from (3.2), we have

$$\begin{aligned} \varsigma\left(\varrho^*, \varrho^/\right) &= \varsigma\left(\Re_1\varrho^*, \Re_2\varrho^/\right) \\ &\leq \alpha\left(\varrho^*, \varrho^/\right)\varsigma\left(\varrho^*, \varrho^/\right) + \beta\left(\varrho^*, \varrho^/\right)\frac{\varsigma(\varrho^*, \Re\varrho^*)\varsigma\left(\varrho^/, \Re_2\varrho^/\right)}{1 + \varsigma(\varrho^*, \varrho^/)} \\ &\quad +\gamma\left(\varrho^*, \varrho^/\right)\frac{\varsigma\left(\varrho^/, \Re_1\varrho^*\right)\varsigma\left(\varrho^*, \Re_2\varrho^/\right)}{1 + \varsigma(\varrho^*, \varrho^/)} \\ &= \alpha\left(\varrho^*, \varrho^/\right)\varsigma\left(\varrho^*, \varrho^/\right) + \beta\left(\varrho^*, \varrho^/\right)\frac{\varsigma(\varrho^*, \varrho^*)\varsigma\left(\varrho^/, \varrho^/\right)}{1 + \varsigma(\varrho^*, \varrho^/)} \\ &\quad +\gamma\left(\varrho^*, \varrho^/\right)\frac{\varsigma\left(\varrho^/, \varrho^*\right)\varsigma\left(\varrho^*, \varrho^/\right)}{1 + \varsigma(\varrho^*, \varrho^/)}. \end{aligned}$$

This implies that, we have

$$\begin{aligned} \varsigma\left(\varrho^*, \varrho^/\right) &\leq \alpha\left(\varrho^*, \varrho^/\right)\varsigma\left(\varrho^*, \varrho^/\right) \\ &\quad +\gamma\left(\varrho^*, \varrho^/\right)\left\|\varsigma\left(\varrho^*, \varrho^/\right)\right\|\frac{\varsigma\left(\varrho^*, \varrho^/\right)}{1 + \varsigma(\varrho^*, \varrho^/)} \\ &\leq \alpha\left(\varrho^*, \varrho^/\right)\varsigma\left(\varrho^*, \varrho^/\right) + \gamma\left(\varrho^*, \varrho^/\right)\varsigma\left(\varrho^*, \varrho^/\right) \\ &= \left(\alpha\left(\varrho^*, \varrho^/\right) + \gamma\left(\varrho^*, \varrho^/\right)\right)\varsigma\left(\varrho^*, \varrho^/\right). \end{aligned}$$

As $\alpha\left(\varrho^*, \varrho^/\right) + \gamma\left(\varrho^*, \varrho^/\right) < 1$, we have

$$\varsigma\left(\varrho^*, \varrho^/\right) = 0.$$

Thus $\varrho^* = \varrho^/$. Hence, the proof is completed. □

Now, let us introduce the following example.

Example 2. *Let* $\Theta=\left\{S_\jmath = 2\jmath + 1 : \jmath \in \mathbb{N}\right\}$ *be endowed with the* $\mathcal{F}$*-metric*

$$\varsigma(\varrho,\hbar) = \begin{cases} 0, & if\ \varrho = \hbar, \\ e^{|\varrho-\hbar|}, & if\ \varrho \neq \hbar, \end{cases}$$

for all $\varrho,\hbar \in \Theta$ *and* $f(t) = \ln t$*. Then* (Θ,ς) *is an* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space. Define the mapping* $\mathfrak{R}_1,\mathfrak{R}_2 : \Theta \to \Theta$ *by*

$$\mathfrak{R}_1\left(S_\jmath\right) = \begin{cases} S_1, & if\ \jmath = 1, \\ S_2, & if\ \jmath = 2, \\ S_{\jmath-2}, & if\ \jmath \geq 3, \end{cases}$$

and

$$\mathfrak{R}_2\left(S_\jmath\right) = \begin{cases} S_1, & if\ \jmath = 1,2, \\ S_{\jmath-1}, & if\ \jmath \geq 3. \end{cases}$$

Suppose that $m \neq \jmath$*, then*

$$\begin{aligned} \varsigma\left(\mathfrak{R}_1\left(S_\jmath\right),\mathfrak{R}_2\left(S_m\right)\right) &= e^{\left|S_{\jmath-2}-S_{m-1}\right|} \\ &= e^{|2(\jmath-m)-2|} \\ &< e^{-1}\cdot e^{|2(\jmath-m)|} \\ &= \alpha\varsigma\left(S_\jmath,S_m\right) \\ &\leq \alpha\left(S_\jmath,S_m\right)\varsigma\left(S_\jmath,S_m\right)+\beta\left(S_\jmath,S_m\right)\frac{\varsigma\left(S_\jmath,\mathfrak{R}_1S_\jmath\right)\varsigma\left(S_m,\mathfrak{R}_2S_m\right)}{1+\varsigma\left(S_\jmath,S_m\right)} \\ &\quad +\gamma\left(S_\jmath,S_m\right)\frac{\varsigma\left(S_m,\mathfrak{R}_1S_\jmath\right)\varsigma\left(S_\jmath,\mathfrak{R}_2S_m\right)}{1+\varsigma\left(S_\jmath,S_m\right)}. \end{aligned}$$

Thus all the assertions of Theorem 3 are satisfied with $\alpha : \Theta\times\Theta\to[0,1)$ *defined by* $\alpha\left(S_\jmath,S_m\right) = e^{-1}$ *and any* $\beta,\gamma : \Theta\times\Theta\to[0,1)$*. Hence* S_1 *is a unique common fixed point of* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$*.*

Consequently, from Theorem 3, we have the following corollaries:

Corollary 1. *Let* (Θ,ς) *be a* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\mathfrak{R}_1,\mathfrak{R}_2 : \Theta \to \Theta$*. If there exist mappings* $\alpha,\beta : \Theta\times\Theta\to[0,1)$ *such that*

(a) $\alpha(\mathfrak{R}_2\mathfrak{R}_1\varrho,\hbar) \leq \alpha(\varrho,\hbar)$ and $\alpha(\varrho,\mathfrak{R}_1\mathfrak{R}_2\hbar) \leq \alpha(\varrho,\hbar)$,
$\beta(\mathfrak{R}_2\mathfrak{R}_1\varrho,\hbar) \leq \beta(\varrho,\hbar)$ and $\beta(\varrho,\mathfrak{R}_1\mathfrak{R}_2\hbar) \leq \beta(\varrho,\hbar)$;
(b) $\alpha(\varrho,\hbar)+\beta(\varrho,\hbar) < 1$;
(c)

$$\varsigma(\mathfrak{R}_1\varrho,\mathfrak{R}_2\hbar) \leq \alpha(\varrho,\hbar)\varsigma(\varrho,\hbar)+\beta(\varrho,\hbar)\frac{\varsigma(\varrho,\mathfrak{R}_1\varrho)\varsigma(\hbar,\mathfrak{R}_2\hbar)}{1+\varsigma(\varrho,\hbar)}$$

for all $\varrho,\hbar \in \Theta$, then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have a unique common fixed point.

Proof. Setting $\gamma : \Theta\times\Theta\to[0,1)$ by $\gamma(\varrho,\hbar) = 0$ in Theorem 3. □

Corollary 2. *Let* (Θ,ς) *be a* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\mathfrak{R}_1,\mathfrak{R}_2 : \Theta \to \Theta$*. If there exist mappings* $\alpha,\gamma : \Theta\times\Theta\to[0,1)$ *such that*

(a) $\alpha(\Re_2\Re_1\varrho,\hbar) \leq \alpha(\varrho,\hbar)$ and $\alpha(\varrho,\Re_1\Re_2\hbar) \leq \alpha(\varrho,\hbar)$,
$\gamma(\Re_2\Re_1\varrho,\hbar) \leq \gamma(\varrho,\hbar)$ and $\gamma(\varrho,\Re_1\Re_2\hbar) \leq \gamma(\varrho,\hbar)$;
(b) $\alpha(\varrho,\hbar) + \gamma(\varrho,\hbar) < 1$;
(c)

$$\varsigma(\Re_1\varrho,\Re_2\hbar) \leq \alpha(\varrho,\hbar)\varsigma(\varrho,\hbar) + \gamma(\varrho,\hbar)\frac{\varsigma(\hbar,\Re_1\varrho)\varsigma(\varrho,\Re_2\hbar)}{1+\varsigma(\varrho,\hbar)},$$

for all $\varrho,\hbar \in \Theta$, then $\Re_1$ and $\Re_2$ have a unique common fixed point.

Proof. Setting $\beta : \Theta\times\Theta\to[0,1)$ by $\beta(\varrho,\hbar) = 0$ in Theorem 3. □

Corollary 3. *Let* (Θ,ς) *be a* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\Re_1,\Re_2 : \Theta \to \Theta$. *If there exists a mapping* $\alpha : \Theta\times\Theta\to[0,1)$ *such that*

(a) $\alpha(\Re_2\Re_1\varrho,\hbar) \leq \alpha(\varrho,\hbar)$ and $\alpha(\varrho,\Re_1\Re_2\hbar) \leq \alpha(\varrho,\hbar)$;
(b)

$$\varsigma(\Re_1\varrho,\Re_2\hbar) \leq \alpha(\varrho,\hbar)\varsigma(\varrho,\hbar),$$

for all $\varrho,\hbar \in \Theta$, then $\Re_1$ and $\Re_2$ have a unique common fixed point.

Proof. Setting $\beta,\gamma : \Theta\times\Theta\to[0,1)$ by $\beta(\varrho,\hbar) = \gamma(\varrho,\hbar) = 0$ in Theorem 3. □

Corollary 4. *Let* (Θ,ς) *be a* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\Re : \Theta \to \Theta$. *If there exists mapping* $\alpha : \Theta\times\Theta\to[0,1)$ *such that*

(a) $\alpha(\Re\varrho,\hbar) \leq \alpha(\varrho,\hbar)$ and $\alpha(\varrho,\Re\hbar) \leq \alpha(\varrho,\hbar)$;
(b)

$$\varsigma(\Re\varrho,\Re\hbar) \leq \alpha(\varrho,\hbar)\varsigma(\varrho,\hbar),$$

for all $\varrho,\hbar \in \Theta$, then $\Re$ has a unique fixed point.

Corollary 5. *Let* (Θ,ς) *be a* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\Re : \Theta \to \Theta$. *If there exist mappings* $\alpha,\beta,\gamma : \Theta\times\Theta\to[0,1)$ *such that*

(a) $\alpha(\Re\varrho,\hbar) \leq \alpha(\varrho,\hbar)$ and $\alpha(\varrho,\Re\hbar) \leq \alpha(\varrho,\hbar)$,
$\beta(\Re\varrho,\hbar) \leq \beta(\varrho,\hbar)$ and $\beta(\varrho,\Re\hbar) \leq \beta(\varrho,\hbar)$,
$\gamma(\Re\varrho,\hbar) \leq \gamma(\varrho,\hbar)$ and $\gamma(\varrho,\Re\hbar) \leq \gamma(\varrho,\hbar)$;
(b) $\alpha(\varrho,\hbar) + \beta(\varrho,\hbar) + \gamma(\varrho,\hbar) < 1$;
(c)

$$\varsigma(\Re\varrho,\Re\hbar) \leq \alpha(\varrho,\hbar)\varsigma(\varrho,\hbar) + \beta(\varrho,\hbar)\frac{\varsigma(\varrho,\Re\varrho)\varsigma(\hbar,\Re\hbar)}{1+\varsigma(\varrho,\hbar)} + \gamma(\varrho,\hbar)\frac{\varsigma(\hbar,\Re\varrho)\varsigma(\varrho,\Re\hbar)}{1+\varsigma(\varrho,\hbar)}$$

for all $\varrho,\hbar \in \Theta$, then $\Re$ has a unique fixed point.

Proof. Setting $\Re_1 = \Re_2 = \Re$ in Theorem 3. □

Corollary 6. *Let* (Θ,ς) *be a* $\mathcal{F}$*-complete* $\mathcal{F}$*-metric space and* $\Re : \Theta \to \Theta$. *If there exist mappings* $\alpha,\beta,\gamma : \Theta\times\Theta\to[0,1)$ *such that*

(a) $\alpha(\Re\varrho,\hbar) \leq \alpha(\varrho,\hbar)$ and $\alpha(\varrho,\Re\hbar) \leq \alpha(\varrho,\hbar)$,
$\beta(\Re\varrho,\hbar) \leq \beta(\varrho,\hbar)$ and $\beta(\varrho,\Re\hbar) \leq \beta(\varrho,\hbar)$,
$\gamma(\Re\varrho,\hbar) \leq \gamma(\varrho,\hbar)$ and $\gamma(\varrho,\Re\hbar) \leq \gamma(\varrho,\hbar)$;
(b) $\alpha(\varrho,\hbar) + \beta(\varrho,\hbar) + \gamma(\varrho,\hbar) < 1$;
(c)

$$\varsigma(\Re^n\varrho,\Re^n\hbar) \leq \alpha(\varrho,\hbar)\varsigma(\varrho,\hbar) + \beta(\varrho,\hbar)\frac{\varsigma(\varrho,\Re^n\varrho)\varsigma(\hbar,\Re^n\hbar)}{1+\varsigma(\varrho,\hbar)} + \gamma(\varrho,\hbar)\frac{\varsigma(\hbar,\Re^n\varrho)\varsigma(\varrho,\Re^n\hbar)}{1+\varsigma(\varrho,\hbar)} \tag{3.10}$$

for all $\varrho,\hbar \in \Theta$, then $\Re$ has a unique fixed point.

Proof. From the Corollary (5), we have $\varrho \in \Theta$ such that $\Re^n\varrho = \varrho$. Now from

$$\begin{aligned}
\varsigma(\Re\varrho,\varrho) &= \varsigma(\Re\Re^n\varrho,\Re^n\varrho) \\
&= \varsigma(\Re^n\Re\varrho,\Re^n\varrho) \\
&\leq \alpha(\Re\varrho,\varrho)\varsigma(\Re\varrho,\varrho) + \beta(\Re\varrho,\varrho)\frac{\varsigma(\Re\varrho,\Re^n\Re\varrho)\varsigma(\varrho,\Re^n\varrho)}{1+\varsigma(\Re\varrho,\varrho)} \\
&\quad +\gamma(\Re\varrho,\varrho)\frac{\varsigma(\varrho,\Re^n\Re\varrho)\varsigma(\Re\varrho,\Re^n\varrho)}{1+\varsigma(\Re\varrho,\varrho)} \\
&\leq \alpha(\Re\varrho,\varrho)\varsigma(\Re\varrho,\varrho) + \beta(\Re\varrho,\varrho)\frac{\varsigma(\Re\varrho,\Re\varrho)\varsigma(\varrho,\varrho)}{1+\varsigma(\Re\varrho,\varrho)} + \gamma(\Re\varrho,\varrho)\frac{\varsigma(\varrho,\Re\varrho)\varsigma(\Re\varrho,\varrho)}{1+\varsigma(\Re\varrho,\varrho)} \\
&= \alpha(\Re\varrho,\varrho)\varsigma(\Re\varrho,\varrho) + \gamma(\Re\varrho,\varrho)\frac{\varsigma(\varrho,\Re\varrho)\varsigma(\Re\varrho,\varrho)}{1+\varsigma(\Re\varrho,\varrho)},
\end{aligned}$$

which implies that

$$\begin{aligned}
\varsigma(\Re\varrho,\varrho) &\leq \alpha(\Re\varrho,\varrho)\varsigma(\Re\varrho,\varrho) + \gamma(\Re\varrho,\varrho)\varsigma(\varrho,\Re\varrho) \\
&= (\alpha(\Re\varrho,\varrho) + \gamma(\Re\varrho,\varrho))\varsigma(\varrho,\Re\varrho),
\end{aligned}$$

which is possible only whenever $\varsigma(\Re\varrho,\varrho) = 0$. Thus $\Re\varrho = \varrho$. □

4. Deduced results

Corollary 7. *Let (Θ,ς) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\Re_1,\Re_2 : \Theta \to \Theta$. If there exist mappings $\alpha,\beta,\gamma : \Theta \to [0,1)$ such that*

(a) $\alpha(\Re_2\Re_1\varrho) \leq \alpha(\varrho)$,
$\beta(\Re_2\Re_1\varrho) \leq \beta(\varrho)$,
$\gamma(\Re_2\Re_1\varrho) \leq \gamma(\varrho)$;
(b) $\alpha(\varrho) + \beta(\varrho) + \gamma(\varrho) < 1$;
(c)

$$\varsigma(\Re_1\varrho,\Re_2\hbar) \leq \alpha(\varrho)\varsigma(\varrho,\hbar) + \beta(\varrho)\frac{\varsigma(\varrho,\Re_1\varrho)\varsigma(\hbar,\Re_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \gamma(\varrho)\frac{\varsigma(\hbar,\Re_1\varrho)\varsigma(\varrho,\Re_2\hbar)}{1+\varsigma(\varrho,\hbar)},$$

for all $\varrho,\hbar \in \Theta$, then $\Re_1$ and $\Re_2$ have a unique common fixed point.

Proof. Define $\alpha,\beta,\gamma : \Theta\times\Theta\to[0,1)$ by

$$\alpha(\varrho,\hbar) = \alpha(\varrho), \quad \beta(\varrho,\hbar) = \beta(\varrho) \quad \text{and} \quad \gamma(\varrho,\hbar) = \gamma(\varrho)$$

for all $\varrho,\hbar \in \Theta$. Then for all $\varrho,\hbar \in \Theta$, we have □

(a) $\alpha(\mathfrak{R}_2\mathfrak{R}_1\varrho,\hbar) = \alpha(\mathfrak{R}_2\mathfrak{R}_1\varrho) \leq \alpha(\varrho) = \alpha(\varrho,\hbar)$ and $\alpha(\varrho,\mathfrak{R}_1\mathfrak{R}_2\hbar) = \alpha(\varrho) = \alpha(\varrho,\hbar)$,
$\beta(\mathfrak{R}_2\mathfrak{R}_1\varrho,\hbar) = \beta(\mathfrak{R}_2\mathfrak{R}_1\varrho) \leq \beta(\varrho) = \beta(\varrho,\hbar)$ and $\beta(\varrho,\mathfrak{R}_1\mathfrak{R}_2\hbar) = \beta(\varrho) = \beta(\varrho,\hbar)$,
$\gamma(\mathfrak{R}_2\mathfrak{R}_1\varrho,\hbar) = \gamma(\mathfrak{R}_2\mathfrak{R}_1\varrho) \leq \gamma(\varrho) = \gamma(\varrho,\hbar)$ and $\gamma(\varrho,\mathfrak{R}_1\mathfrak{R}_2\hbar) = \gamma(\varrho) = \gamma(\varrho,\hbar)$;
(b) $\alpha(\varrho,\hbar) + \beta(\varrho,\hbar) + \gamma(\varrho,\hbar) = \alpha(\varrho) + \beta(\varrho) + \gamma(\varrho) < 1$;
(c)

$$\begin{aligned}
\varsigma(\mathfrak{R}_1\varrho,\mathfrak{R}_2\hbar) &\leq \alpha(\varrho)\varsigma(\varrho,\hbar) + \beta(\varrho)\frac{\varsigma(\varrho,\mathfrak{R}_1\varrho)\varsigma(\hbar,\mathfrak{R}_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \gamma(\varrho)\frac{\varsigma(\hbar,\mathfrak{R}_1\varrho)\varsigma(\varrho,\mathfrak{R}_2\hbar)}{1+\varsigma(\varrho,\hbar)} \\
&= \alpha(\varrho,\hbar)\varsigma(\varrho,\hbar) + \beta(\varrho,\hbar)\frac{\varsigma(\varrho,\mathfrak{R}_1\varrho)\varsigma(\hbar,\mathfrak{R}_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \gamma(\varrho,\hbar)\frac{\varsigma(\hbar,\mathfrak{R}_1\varrho)\varsigma(\varrho,\mathfrak{R}_2\hbar)}{1+\varsigma(\varrho,\hbar)};
\end{aligned}$$

(d) $\lambda = \frac{\alpha(\varrho_0,\varrho_1)}{1-\beta(\varrho_0,\varrho_1)} = \frac{\alpha(\varrho_0)}{1-\beta(\varrho_0)} < 1$.
By Theorem 3, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have a unique common fixed point.

Corollary 8. *Let $\mathfrak{R}_1,\mathfrak{R}_2 : \Theta \to \Theta$. If there exist $\alpha,\beta,\gamma \in [0,1)$ with $\alpha+\beta+\gamma < 1$ such that*

$$\varsigma(\mathfrak{R}_1\varrho,\mathfrak{R}_2\hbar) \leq \alpha\varsigma(\varrho,\hbar) + \beta\frac{\varsigma(\varrho,\mathfrak{R}_1\varrho)\varsigma(\hbar,\mathfrak{R}_2\hbar)}{1+\varsigma(\varrho,\hbar)} + \gamma\frac{\varsigma(\hbar,\mathfrak{R}_1\varrho)\varsigma(\varrho,\mathfrak{R}_2\hbar)}{1+\varsigma(\varrho,\hbar)},$$

for all $\varrho,\hbar \in \Theta$, then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have a unique common fixed point.

Proof. Taking $\alpha(\cdot) = \alpha$, $\beta(\cdot) = \beta$ and $\gamma(\cdot) = \gamma$ in Corollary 7. □

Corollary 9. *Let (Θ,ς) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{R}_1,\mathfrak{R}_2 : \Theta \to \Theta$. If there exist $\alpha,\beta \in [0,1)$ with $\alpha+\beta < 1$ such that*

$$\varsigma(\mathfrak{R}_1\varrho,\mathfrak{R}_2\hbar) \leq \alpha\varsigma(\varrho,\hbar) + \beta\frac{\varsigma(\varrho,\mathfrak{R}_1\varrho)\varsigma(\hbar,\mathfrak{R}_2\hbar)}{1+\varsigma(\varrho,\hbar)}$$

for all $\varrho,\hbar \in \Theta$, then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have a unique common fixed point.

Proof. Taking $\gamma = 0$ in Corollary 8. □

Remark 1. If we set $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}$ in the Corollary 9, the we get the main result of Al-Mazrooei et al. [7].

Corollary 10. *Let (Θ,ς) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space and $\mathfrak{R}_1,\mathfrak{R}_2 : \Theta \to \Theta$. If there exists $\alpha \in [0,1)$ such that*

$$\varsigma(\mathfrak{R}_1\varrho,\mathfrak{R}_2\hbar) \leq \alpha\varsigma(\varrho,\hbar)$$

for all $\varrho,\hbar \in \Theta$, then $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have a unique common fixed point.

Proof. Taking $\beta = 0$ in Corollary 9. □

Remark 2. If we set $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}$ in the Corollary 10, the we get the main result of Samet et al. [6].

5. Some results related to graphs

Let (Θ, ς) be an $\mathcal{F}$-metric space and G be a directed graph. Let us represent by G^{-1} the graph generated from G by changing the direction of $E(G)$. Hence,

$$E(G^{-1}) = \{(\varrho, \hbar) \in \Theta \times \Theta : \ (\hbar, \varrho) \in E(G)\}.$$

Definition 4. *An element $\varrho \in \Theta$ is said to be a common fixed point of the pair $(\Re_1, \Re_2)$, if $\Re_1(\varrho)$=$\Re_2(\varrho)$=ϱ. We denote by $CFix(\Re_1, \Re_2)$, the family of all common fixed points of the pair $(\Re_1, \Re_2)$, that is,*

$$CFix(\Re_1, \Re_2) = \{\varrho \in \Theta : \Re_1(\varrho) = \Re_2(\varrho) = \varrho\}.$$

Definition 5. *Let (Θ, ς) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a directed graph G and let $\Re_1, \Re_2 : \Theta \to \Theta$. Then the pair $(\Re_1, \Re_2)$ is called a G-orbital cyclic pair, if*

$$(\varrho, \Re_1\varrho) \in E(G) \Longrightarrow (\Re_1\varrho, \Re_2(\Re_1\varrho)) \in E(G),$$

$$(\varrho, \Re_2\varrho) \in E(G) \Longrightarrow (\Re_2\varrho, \Re_1(\Re_2\varrho)) \in E(G)$$

for any $\varrho \in \Theta$. Let us consider the following sets

$$\begin{aligned} \Theta^{\Re_1} &= \{\varrho \in \Theta : \ (\varrho, \Re_1\varrho) \in E(G)\}, \\ \Theta^{\Re_2} &= \{\varrho \in \Theta : \ (\varrho, \Re_2\varrho) \in E(G)\}. \end{aligned}$$

Remark 3. If $(\Re_1, \Re_2)$ is a G-orbital-cyclic pair, then $\Theta^{\Re_1} \neq \emptyset \Longleftrightarrow \Theta^{\Re_2} \neq \emptyset$.

Proof. Let $\varrho_0 \in \Theta^{\Re_1}$. Then $(\varrho_0, \Re_1\varrho_0) \in E(G) \Longrightarrow (\Re_1\varrho_0, \Re_2(\Re_1\varrho_0)) \in E(G)$. If we represent by $\varrho_1 = \Re_1\varrho_0$, then we get that $(\varrho_1, \Re_2(\varrho_1)) \in E(G)$, thus $\Theta^{\Re_2} \neq \emptyset$. □

Now let us prove the following main theorem.

Theorem 4. *Let (Θ, ς) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a directed graph G and $\Re_1, \Re_2 : \Theta \to \Theta$ is G-orbital cyclic pair. Assume that there exists $\alpha \in [0, 1)$ such that*

(i) $\Theta^{\Re_1} \neq \emptyset$,

(ii) $\forall\ \varrho \in \Theta^{\Re_1}$ and $\hbar \in \Theta^{\Re_2}$

$$\varsigma(\Re_1\varrho, \Re_2\hbar) \leq \alpha \max\left\{\varsigma(\varrho, \hbar), \frac{\varsigma(\varrho, \Re_1\varrho)\varsigma(\hbar, \Re_2\hbar)}{1 + \varsigma(\varrho, \hbar)}, \frac{\varsigma(\varrho, \Re_2\hbar)\varsigma(\hbar, \Re_1\varrho)}{1 + \varsigma(\varrho, \hbar)}\right\}, \tag{5.1}$$

(iii) $\Re_1$ and $\Re_2$ are continuous, or $\forall\ (\varrho_\jmath)_{\jmath \in \mathbb{N}} \subset \Theta$, with $\varrho_\jmath \to \varrho$ as $\jmath \to \infty$, and $(\varrho_\jmath, \varrho_{\jmath+1}) \in E(G)$ for $\jmath \in \mathbb{N}$, we have $\varrho \in \Theta^{\Re_1} \cap \Theta^{\Re_2}$. In these conditions CFix$(\Re_1, \Re_2) \neq \emptyset$,

(iv) if $(\varrho^*, \varrho^/) \in$CFix$(\Re_1, \Re_2)$ implies $\varrho^* \in \Theta^{\Re_1}$ and $\varrho^/ \in \Theta^{\Re_2}$, then the pair $(\Re_1, \Re_2)$ has a unique common fixed point.

Proof. Let $\varrho_0 \in \Theta^{\Re_1}$. Thus $(\varrho_0, \Re_1\varrho_0) \in E(G)$. As the pair $(\Re_1, \Re_2)$ is G-orbital cyclic, we get $(\Re_1\varrho_0, \Re_2\Re_1\varrho_0) \in E(G)$. Construct ϱ_1 by $\varrho_1 = \Re_1\varrho_0$, we have $(\varrho_1, \Re_2\varrho_1) \in E(G)$ and from here $(\Re_2\varrho_1, \Re_1\Re_2\varrho_1) \in E(G)$. Denoting by $\varrho_2 = \Re_2\varrho_1$, we have $(\varrho_2, \Re_1\varrho_2) \in E(G)$. Pursuing along these lines, we generate a sequence $(\varrho_\jmath)_{\jmath \in \mathbb{N}}$ with $\varrho_{2\jmath} = \Re_2\varrho_{2\jmath-1}$ and $\varrho_{2\jmath+1} = \Re_1\varrho_{2\jmath}$, such that $\left(\varrho_{2\jmath}, \varrho_{2\jmath+1}\right) \in$

$E(G)$. We assume that $\varrho_J \neq \varrho_{J+1}$. If, there exists $J_0 \in \mathbb{N}$, such that $\varrho_{J_0} = \varrho_{J_0+1}$, then in the view of the fact that $\Delta \subset E(G)$, $\left(\varrho_{J_0}, \varrho_{J_0+1}\right) \in E(G)$ and thus $\varrho^* = \varrho_{J_0}$ is a fixed point of $\mathfrak{R}_1$. Now to manifest that $\varrho^* \in$CFix$(\mathfrak{R}_1, \mathfrak{R}_2)$, we shall discuss these two cases for J_0. If J_0 is even, then $J_0 = 2J$. Then, $\varrho_{2J} = \varrho_{2J+1} = \mathfrak{R}_1\varrho_{2J}$ and thus, ϱ_{2J} is a fixed point of $\mathfrak{R}_1$. Assume that $\varrho_{2J} = \varrho_{2J+1} = \mathfrak{R}_1\varrho_{2J}$ but $\varsigma(\mathfrak{R}_1\varrho_{2J}, \mathfrak{R}_2\varrho_{2J+1}) > 0$, and let $\varrho = \varrho_{2J} \in \Theta^{\mathfrak{R}_1}$ and $\hbar = \varrho_{2J+1} \in \Theta^{\mathfrak{R}_2}$. So

$$\begin{aligned}
0 &< \varsigma(\varrho_{2J+1}, \varrho_{2J+2}) = \varsigma(\mathfrak{R}_1\varrho_{2J}, \mathfrak{R}_2\varrho_{2J+1}) \\
&\leq \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J+1}), \frac{\varsigma(\varrho_{2J}, \mathfrak{R}_1\varrho_{2J})\varsigma(\varrho_{2J+1}, \mathfrak{R}_2\varrho_{2J+1})}{1+\varsigma(\varrho_{2J}, \varrho_{2J+1})}, \frac{\varsigma(\varrho_{2J}, \mathfrak{R}_2\varrho_{2J+1})\varsigma(\varrho_{2J+1}, \mathfrak{R}_1\varrho_{2J})}{1+\varsigma(\varrho_{2J}, \varrho_{2J+1})}\right\} \\
&= \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J+1}), \frac{\varsigma(\varrho_{2J}, \varrho_{2J+1})\varsigma(\varrho_{2J+1}, \varrho_{2J+2})}{1+\varsigma(\varrho_{2J}, \varrho_{2J+1})}, \frac{\varsigma(\varrho_{2J}, \varrho_{2J+2})\varsigma(\varrho_{2J+1}, \varrho_{2J+1})}{1+\varsigma(\varrho_{2J}, \varrho_{2J+1})}\right\} \\
&\leq \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J+1}), \varsigma(\varrho_{2J+1}, \varrho_{2J+2})\right\} \\
&= \alpha\varsigma(\varrho_{2J+1}, \varrho_{2J+2}),
\end{aligned} \tag{5.2}$$

that is contradiction because $\alpha < 1$. Hence ϱ_{2J} is a fixed point of $\mathfrak{R}_2$ too. Likewise if J_0 is odd number, then $\exists\varrho^* \in \Theta$ such that $\mathfrak{R}_1\varrho^* \cap \mathfrak{R}_2\varrho^* = \varrho^*$. So we assume that $\varrho_J \neq \varrho_{J+1}$ for all $J \in \mathbb{N}$. Now we shall show that $(\varrho_J)_{J\in\mathbb{N}}$ is Cauchy sequence. We have these two possible cases to discuss:

Case 1. $\varrho = \varrho_{2J} \in \Theta^{\mathfrak{R}_1}$ and $\hbar = \varrho_{2J+1} \in \Theta^{\mathfrak{R}_2}$.

$$\begin{aligned}
0 &< \varsigma(\varrho_{2J+1}, \varrho_{2J+2}) = \varsigma(\mathfrak{R}_1\varrho_{2J}, \mathfrak{R}_2\varrho_{2J+1}) \\
&\leq \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J+1}), \frac{\varsigma(\varrho_{2J}, \mathfrak{R}_1\varrho_{2J})\varsigma(\varrho_{2J+1}, \mathfrak{R}_2\varrho_{2J+1})}{1+\varsigma(\varrho_{2J}, \varrho_{2J+1})}, \frac{\varsigma(\varrho_{2J}, \mathfrak{R}_2\varrho_{2J+1})\varsigma(\varrho_{2J+1}, \mathfrak{R}_1\varrho_{2J})}{1+\varsigma(\varrho_{2J}, \varrho_{2J+1})}\right\} \\
&= \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J+1}), \frac{\varsigma(\varrho_{2J}, \varrho_{2J+1})\varsigma(\varrho_{2J+1}, \varrho_{2J+2})}{1+\varsigma(\varrho_{2J}, \varrho_{2J+1})}, \frac{\varsigma(\varrho_{2J}, \varrho_{2J+2})\varsigma(\varrho_{2J+1}, \varrho_{2J+1})}{1+\varsigma(\varrho_{2J}, \varrho_{2J+1})}\right\} \\
&= \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J+1}), \varsigma(\varrho_{2J+1}, \varrho_{2J+2})\right\} \\
&\leq \alpha\left[\varsigma(\varrho_{2J}, \varrho_{2J+1}) + \varsigma(\varrho_{2J+1}, \varrho_{2J+2})\right]
\end{aligned}$$

that is

$$(1-\alpha)\varsigma(\varrho_{2J+1}, \varrho_{2J+2}) \leq \alpha\varsigma(\varrho_{2J}, \varrho_{2J+1})$$

which implies

$$\varsigma(\varrho_{2J+1}, \varrho_{2J+2}) \leq \frac{\alpha}{1-\alpha}\varsigma(\varrho_{2J}, \varrho_{2J+1}). \tag{5.3}$$

Case 2. $\varrho = \varrho_{2J} \in \Theta^{\mathfrak{R}_1}$ and $\hbar = \varrho_{2J-1} \in \Theta^{\mathfrak{R}_2}$.

$$\begin{aligned}
0 &< \varsigma(\varrho_{2J+1}, \varrho_{2J}) = \varsigma(\mathfrak{R}_1\varrho_{2J}, \mathfrak{R}_2\varrho_{2J-1}) \\
&\leq \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J-1}), \frac{\varsigma(\varrho_{2J}, \mathfrak{R}_1\varrho_{2J})\varsigma(\varrho_{2J-1}, \mathfrak{R}_2\varrho_{2J-1})}{1+\varsigma(\varrho_{2J}, \varrho_{2J-1})}, \frac{\varsigma(\varrho_{2J}, \mathfrak{R}_2\varrho_{2J-1})\varsigma(\varrho_{2J-1}, \mathfrak{R}_1\varrho_{2J})}{1+\varsigma(\varrho_{2J}, \varrho_{2J-1})}\right\} \\
&= \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J-1}), \frac{\varsigma(\varrho_{2J}, \varrho_{2J+1})\varsigma(\varrho_{2J-1}, \varrho_{2J})}{1+\varsigma(\varrho_{2J}, \varrho_{2J-1})}, \frac{\varsigma(\varrho_{2J}, \varrho_{2J})\varsigma(\varrho_{2J-1}, \varrho_{2J+1})}{1+\varsigma(\varrho_{2J}, \varrho_{2J-1})}\right\} \\
&\leq \alpha \max\left\{\varsigma(\varrho_{2J}, \varrho_{2J-1}), \varsigma(\varrho_{2J}, \varrho_{2J+1})\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \max\left\{\varsigma(\varrho_{2\jmath-1},\varrho_{2\jmath}),\varsigma(\varrho_{2\jmath},\varrho_{2\jmath+1})\right\}\\
&\leq \alpha\left[\varsigma(\varrho_{2\jmath-1},\varrho_{2\jmath})+\varsigma(\varrho_{2\jmath},\varrho_{2\jmath+1})\right]
\end{aligned}
$$

that is

$$(1-\alpha)\varsigma(\varrho_{2\jmath+1},\varrho_{2\jmath}) \leq \alpha\varsigma(\varrho_{2\jmath},\varrho_{2\jmath-1})$$

which implies

$$\varsigma(\varrho_{2\jmath},\varrho_{2\jmath+1}) \leq \frac{\alpha}{1-\alpha}\varsigma(\varrho_{2\jmath-1},\varrho_{2\jmath}). \tag{5.4}$$

Since $\tau = \frac{\alpha}{1-\alpha}$, so we have

$$\varsigma(\varrho_\jmath,\varrho_{\jmath+1}) \leq \tau\varsigma(\varrho_{\jmath-1},\varrho_\jmath). \tag{5.5}$$

Thus, we have

$$
\begin{aligned}
\varsigma(\varrho_\jmath,\varrho_{\jmath+1}) &\leq \tau\varsigma(\varrho_{\jmath-1},\varrho_\jmath)\\
&\leq \tau^2\varsigma(\varrho_{\jmath-2},\varrho_{\jmath-1})\\
&\leq\\
&\quad .\\
&\quad .\\
&\quad .\\
&\leq \tau^\jmath\varsigma(\varrho_0,\varrho_1).
\end{aligned}
\tag{5.6}
$$

Let $(f,\mathfrak{h}) \in \mathcal{F}\times[0,+\infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathcal{F}_2)$, $\exists\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\delta) - \mathfrak{h}. \tag{5.7}$$

Hence, by (5.6), $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, we have

$$f(\sum_{i=\jmath}^{m-1}\varsigma(\varrho_i,\varrho_{i+1})) \leq f(\sum_{i=\jmath}^{m-1}\lambda^\jmath(\varsigma(\varrho_0,\varrho_1))) \leq f(\sum_{\jmath\geq n(\epsilon)}\lambda^\jmath\varsigma(\varrho_0,\varrho_1)) < f(\epsilon) - h \tag{5.8}$$

for $m > \jmath \geq n(\epsilon)$. By (D_3) and (5.7), we get $\varsigma(\varrho_\jmath,\varrho_m) > 0$, $m > \jmath \geq n(\epsilon)$ implies

$$f(\varsigma(\varrho_\jmath,\varrho_m)) \leq f(\sum_{i=\jmath}^{m-1}\varsigma(\varrho_i,\varrho_{i+1})) + \mathfrak{h} < f(\epsilon)$$

which yields by $(\mathcal{F}_1)$ that $\varsigma(\varrho_\jmath,\varrho_m) < \epsilon$, $m > \jmath \geq n(\epsilon)$. It shows that $\{\varrho_\jmath\}$ is $\mathcal{F}$-Cauchy. As (Θ,ς) is $\mathcal{F}$-complete, so $\exists\varrho^* \in \Theta$ such that $\{\varrho_\jmath\}$ is $\mathcal{F}$-convergent to ϱ^*, i.e.,

$$\lim_{\jmath\to\infty}\varsigma(\varrho_\jmath,\varrho^*) = 0. \tag{5.9}$$

that is $\varrho_\jmath \to \varrho^*$ as $\jmath\to\infty$. It is obvious that

$$\lim_{\jmath\to\infty}\varrho_{2\jmath} = \lim_{\jmath\to\infty}\varrho_{2\jmath+1} = \varrho^*. \tag{5.10}$$

As $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are continuous, so we have

$$\begin{aligned}\varrho^* &= \lim_{\jmath\to\infty}\varrho_{2\jmath+1} = \lim_{\jmath\to\infty}\mathfrak{R}_1(\varrho_{2\jmath}) = \mathfrak{R}_1(\varrho^*)\\ \varrho^* &= \lim_{\jmath\to\infty}\varrho_{2\jmath+2} = \lim_{\jmath\to\infty}\mathfrak{R}_2(\varrho_{2\jmath+1}) = \mathfrak{R}_2(\varrho^*).\end{aligned} \tag{5.11}$$

Now letting $\varrho = \varrho^* \in \Theta^{\mathfrak{R}_1}$ and $\hbar = \varrho_{2\jmath+2} \in \Theta^{\mathfrak{R}_2}$, we have We contrary suppose that $\varsigma(\varrho^*, \mathfrak{R}_1\varrho^*) > 0$. Then from (3.2), $(\mathcal{F}_1)$ and (D_3), we have

$$\begin{aligned}
f(\varsigma(\mathfrak{R}_1\varrho^*, \varrho^*)) &\leq f(\varsigma\left(\varrho\mathfrak{R}_1\varrho^*, \mathfrak{R}_2\varrho_{2\jmath+1}\right) + \varsigma\left(\mathfrak{R}_2\varrho_{2\jmath+1}, \varrho^*\right)) + \mathfrak{h}\\
&\leq f\left(\varsigma\left(\varrho^*, \mathfrak{R}_2\varrho_{2\jmath+1}\right) + \varsigma\left(\varrho_{2\jmath+2}, \varrho^*\right)\right) + \mathfrak{h}\\
&\leq f\left(\alpha\max\left\{\begin{array}{c}\varsigma(\varrho^*, \varrho_{2\jmath+1}), \frac{\varsigma(\varrho^*,\mathfrak{R}_1\varrho^*)\varsigma(\varrho_{2\jmath+1},\mathfrak{R}_2(\varrho_{2\jmath+1}))}{1+\varsigma(\varrho^*,\varrho_{2\jmath+1})},\\ \frac{\varsigma(\varrho^*,\mathfrak{R}_2(\varrho_{2\jmath+1}))\varsigma(\varrho_{2\jmath+1},\mathfrak{R}_1\varrho^*)}{1+\varsigma(\varrho^*,\varrho_{2\jmath+1})}\end{array}\right\} + \varsigma\left(\varrho_{2\jmath+2}, \varrho^*\right)\right) + \mathfrak{h}\\
&= f\left(\alpha\max\left\{\begin{array}{c}\varsigma(\varrho^*, \varrho_{2\jmath+1}), \frac{\varsigma(\varrho^*,\mathfrak{R}_1\varrho^*)\varsigma(\varrho_{2\jmath+1},\varrho_{2\jmath+2})}{1+\varsigma(\varrho^*,\varrho_{2\jmath+1})},\\ \frac{\varsigma(\varrho^*,\varrho_{2\jmath+2})\varsigma(\varrho_{2\jmath+1},\mathfrak{R}_1\varrho^*)}{1+\varsigma(\varrho^*,\varrho_{2\jmath+1})}\end{array}\right\} + \varsigma\left(\varrho_{2\jmath+2}, \varrho^*\right)\right) + \mathfrak{h}.
\end{aligned}$$

Taking $\jmath\to\infty$ and using $(\mathcal{F}_2)$, (5.10) and (5.11), we get

$$\lim_{\jmath\to\infty} f(\varsigma(\mathfrak{R}_1\varrho^*, \varrho^*)) = -\infty,$$

which is a contradiction. Thus, we have $\varsigma(\varrho^*, \mathfrak{R}_1\varrho^*) = 0$. This yields that $\varrho^* = \mathfrak{R}_1\varrho^*$. Similarly, suppose that $\varrho = \varrho_{2\jmath+1} \in \Theta^{\mathfrak{R}_1}$ and $\hbar = \varrho^* \in \Theta^{\mathfrak{R}_2}$, we have

$$\begin{aligned}
f(\varsigma(\varrho^*, \mathfrak{R}_2\varrho^*)) &\leq f(\varsigma\left(\varrho^*, \mathfrak{R}_1(\varrho_{2\jmath})\right) + \varsigma\left(\mathfrak{R}_1(\varrho_{2\jmath}), \mathfrak{R}_2\varrho^*\right)) + \mathfrak{h}\\
&\leq f(\varsigma\left(\varrho^*, \mathfrak{R}_1(\varrho_{2\jmath})\right) + \varsigma\left(\mathfrak{R}_1(\varrho_{2\jmath}), \mathfrak{R}_2\varrho^*\right)) + \mathfrak{h}\\
&\leq f\left(\varsigma\left(\varrho^*, \mathfrak{R}_1(\varrho_{2\jmath})\right) + \alpha\max\left\{\begin{array}{c}\varsigma(\varrho_{2\jmath}, \varrho^*), \frac{\varsigma(\varrho_{2\jmath},\mathfrak{R}_1(\varrho_{2\jmath}))\varsigma(\varrho^*,\mathfrak{R}_2\varrho^*)}{1+\varsigma(\varrho_{2\jmath},\varrho^*)},\\ \frac{\varsigma(\varrho_{2\jmath},\mathfrak{R}_2\varrho^*)\varsigma(\varrho^*,\mathfrak{R}_1(\varrho_{2\jmath}))}{1+\varsigma(\varrho_{2\jmath},\varrho^*)}\end{array}\right\}\right) + \mathfrak{h}.
\end{aligned}$$

Taking the limit as $\jmath\to\infty$ and using $(\mathcal{F}_2)$, (5.10) and (5.11), we have

$$\lim_{\jmath\to\infty} f(\varsigma(\varrho^*, \mathfrak{R}_2\varrho^*)) = -\infty,$$

which is a contradiction. Thus, we have $\varsigma(\varrho^*, \mathfrak{R}_2\varrho^*) = 0$. This yields that $\varrho^* = \mathfrak{R}_2\varrho^*$. □

Corollary 11. *Let (Θ, ς) be a $\mathcal{F}$-complete $\mathcal{F}$-metric space equipped with a directed graph G and $\mathfrak{R} : \Theta \to \Theta$ is a G-orbital-cyclic. Suppose that there exists $\alpha \in [0, 1)$ such that*

(i) $\Theta^{\mathfrak{R}} \neq \emptyset$,

(ii) $\forall\ \varrho, \hbar \in \Theta^{\mathfrak{R}}$, we have

$$\varsigma(\mathfrak{R}\varrho, \mathfrak{R}\hbar) \leq \alpha\max\left\{\varsigma(\varrho, \hbar), \frac{\varsigma(\varrho, \mathfrak{R}\varrho)\varsigma(\hbar, \mathfrak{R}\hbar)}{1 + \varsigma(\varrho, \hbar)}, \frac{\varsigma(\varrho, \mathfrak{R}\hbar)\varsigma(\hbar, \mathfrak{R}\varrho)}{1 + \varsigma(\varrho, \hbar)}\right\},$$

(iii) $\mathfrak{R}$ is continuous, or $\forall\ (\varrho_\jmath)_{\jmath\in\mathbb{N}} \subset \Theta$, with $\varrho_\jmath \to \varrho$ as $\jmath \to \infty$, and $(\varrho_\jmath, \varrho_{\jmath+1}) \in E(G)$ for $\jmath \in \mathbb{N}$, we have $\varrho \in \Theta^{\mathfrak{R}}$.

Then $\mathfrak{R}$ has a unique fixed point.

6. Applications in differential equations

A representative stability result based on fixed point theory arguments follows a number of basic arguments adapted to the special structure of the equation under consideration. It leads to large number of results in the literature for different classes of equations, see [21, 22]. In the present section, we investigate the existence of solution to differential equation

$$\varrho^{/}(t) = -a(t)\varrho(t) + b(t)g(\varrho(t - r(t))) + c(t)\varrho^{/}(t - r(t)). \tag{6.1}$$

We state a lemma of Djoudi et al. [23] which will be used in proving of our theorem.

Lemma 2. *(see [23]) Assume that $r^{/}(t) \neq 1$ $\forall t \in \mathbb{R}$. Then $\varrho(t)$ is a solution of (6.1) if and only if*

$$\begin{aligned}\varrho(t) \;&=\; \left(\varrho(0) - \frac{c(0)}{1 - r^{/}(0)}\varrho(-r(0))\right)e^{-\int_0^t a(s)ds} + \frac{c(t)}{1 - r^{/}(t)}\varrho(t - r(t)) \\ &\quad - \int_0^t (h(\upsilon))\varrho(\upsilon - r(\upsilon))) - b(\upsilon)g\left(\varrho(\upsilon - r(\upsilon))\right))\, e^{-\int_\upsilon^t a(s)ds}d\upsilon, \end{aligned} \tag{6.2}$$

where

$$h(\upsilon) = \frac{r^{//}(\upsilon)c(\upsilon) + \left(c^{/}(\upsilon) + c(\upsilon)a(\upsilon)\right)(1 - r^{/}(\upsilon))}{(1 - r^{/}(\upsilon))^2}. \tag{6.3}$$

Now suppose that $\vartheta : (-\infty, 0] \to \mathbb{R}$ is a bounded and continuous function, then $\varrho(t) = \varrho(t, 0, \vartheta)$ is a solution of (6.1) if $\varrho(t) = \vartheta(t)$ for $t \leq 0$ and satisfies (6.1) for $t \geq 0$. Assume that $\mathfrak{C}$ is the collection of $\varrho : \mathbb{R} \to \mathbb{R}$ which are continuous. Define $\aleph_\vartheta$ by

$$\aleph_\vartheta = \{\varrho : \mathbb{R} \to \mathbb{R} \text{ such that } \vartheta(t) = \varrho(t) \text{ if } t \leq 0,\ \varrho(t) \to 0 \text{ as } t \to \infty,\ \varrho \in \mathfrak{C}\}.$$

Then $\aleph_\vartheta$ is a Banach space endowed with $\|\cdot\|$.

Lemma 3. *(see [14]) The space $(\aleph_\vartheta, \|\cdot\|)$ with the $\mathcal{F}$-metric d defined by*

$$d(\mathrm{t}, \mathrm{t}^*) = \|\mathrm{t} - \mathrm{t}^*\| = \sup_{\varrho \in I} |\mathrm{t}(\varrho) - \mathrm{t}^*(\varrho)|$$

for all $t, t^* \in \aleph_\vartheta$, is $\mathcal{F}$-metric space.

Theorem 5. *Let $\mathfrak{R} : \aleph_\vartheta \to \aleph_\vartheta$ be a mapping defined by*

$$\begin{aligned}(\mathfrak{R}\varrho)(\mathrm{t}) \;&=\; \left(\varrho(0) - \frac{c(0)}{1 - r^{/}(0)}\varrho(-r(0))\right)e^{-\int_0^{\mathrm{t}} a(s)ds} + \frac{c(\mathrm{t})}{1 - r^{/}(\mathrm{t})}\tau(\mathrm{t} - r(\mathrm{t})) \\ &\quad - \int_0^{\mathrm{t}} (h(\upsilon)\varrho(\upsilon - r(\upsilon)) - b(\upsilon)g\left(\varrho(\upsilon - r(\upsilon))\right))e^{-\int_\upsilon^{\mathrm{t}} a(s)ds}d\upsilon,\ t \geq 0 \end{aligned} \tag{6.4}$$

for all $\varrho \in \aleph_\vartheta$. Assume that there exists $\alpha : \aleph_\vartheta \times \aleph_\vartheta \to [0, 1)$ such that

$$\alpha\left(\varrho(t), \hbar(t)\right) = \left\{\left|\frac{c(\mathrm{t})}{1 - r^{/}(\mathrm{t})}\right| + \int_0^{\mathrm{t}} \left(|h(\upsilon)| + |b(\upsilon)|\right) e^{-\int_\upsilon^{\mathrm{t}} a(s)ds}\right\} < 1.$$

Then $\Re$ has a fixed point.

Proof. It follows from (6.3) that $\Re(\varrho), \Re(\hbar) \in \aleph_\vartheta$. Now from (6.4), we have

$$\begin{aligned}
\left|(\Re\varrho)(\mathrm{t}) - (\Re\hbar)(\mathrm{t})\right| \quad &\leq \quad \left|\frac{c(\mathrm{t})}{1 - r^{/}(\mathrm{t})}\right| \|\varrho - \hbar\| \\
&\quad + \int_0^{\mathrm{t}} |h(\upsilon)(\varrho(\upsilon - r(\upsilon))) - \hbar(\upsilon - r(\upsilon))| e^{-\int_\upsilon^{\mathrm{t}} a(s)ds} \\
&\quad + \int_0^{\mathrm{t}} |(b(\upsilon))g(\varrho(\upsilon - r(\upsilon))) - g(\hbar(\upsilon - r(\upsilon)))| e^{-\int_\upsilon^{\mathrm{t}} a(s)ds} \\
&\leq \quad \left\{\left|\frac{c(\mathrm{t})}{1 - r^{/}(\mathrm{t})}\right| + \int_0^{\mathrm{t}} (|h(\upsilon)| + |b(\upsilon)|)\, e^{-\int_\upsilon^{\mathrm{t}} a(s)ds}\right\} \|\varrho - \hbar\| \\
&\leq \quad \alpha\,(\varrho, \hbar)\, \|\varrho - \hbar\|\,.
\end{aligned}$$

Hence,

$$d(\Re\varrho, \Re\hbar) \leq \alpha\,(\varrho, \hbar)\, d(\varrho, \hbar).$$

Thus all the assumptions of Corollary 4 are satisfied and $\Re$ has a unique fixed point in $\aleph_\vartheta$ which solves (6.1). □

Open Problems

1) Can the notion of $\mathcal{F}$-metric space be extended to graphical $\mathcal{F}$-metric space?

2) Can the results proved in this article be extended to multivalued mappings and fuzzy set valued mappings?

3) Can differential inclusions can be solved as applications of fixed point results for multivalued mappings in the context of $\mathcal{F}$-metric space?

7. Conclusion

In this article, we have utilized the notion of $\mathcal{F}$-metric spaces and obtained common fixed point results for generalized rational contractions involving control functions of two variables. We have derived common fixed points and fixed points of single valued mappings for contractions involving control functions of one variable and constants. We also have established some common fixed point theorems in $\mathcal{F}$-metric spaces endowed with graph. We expect that the obtained theorems in this article will make new relations for those people who are employing in $\mathcal{F}$-metric spaces.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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