Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 4764–4785.
DOI: 10.3934/math.2023236
Received: 29 August 2022
Revised: 07 November 2022
Accepted: 16 November 2022
Published: 08 December 2022

Research article

Critical regularity of nonlinearities in semilinear effectively damped wave models

Abdelhamid Mohammed Djaouti[1,*] **and Michael Reissig**[2]

[1] Preparatory Year Deanship, King Faisal University, Hofuf 31982, Al-Ahsa, Saudi Arabia
[2] Faculty for Mathematics and Computer Science, Technical University Bergakademie, Prüferstr. 9, Freiberg 09596, Germany

* **Correspondence:** Email: adjaout@kfu.edu.sa.

Abstract: In this paper we consider the Cauchy problem for the semilinear effectively damped wave equation

$$u_{tt} - u_{xx} + b(t)u_t = |u|^3\mu(|u|), \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x).$$

Our goal is to propose sharp conditions on μ to obtain a threshold between global (in time) existence of small data Sobolev solutions (stability of the zero solution) and blow-up behaviour even of small data Sobolev solutions.

Keywords: damped wave equations; time dependent dissipation; global existence; blow-up; critical regularity
Mathematics Subject Classification: 35L52, 35L71

1. Introduction

Properties of solutions to the Cauchy problem for semilinear classical damped wave equations with power nonlinearity were treated in many papers. The model the authors have in mind is

$$u_{tt} - \Delta u + u_t = |u|^p, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \tag{1.1}$$

where $p > 1$.

The global (in time) existence of small data energy solutions was given in [14] for $p > 1 + \frac{2}{n}$ and by assuming suitable compactly supported small data from the energy space. In [7] the authors studied (1.1) under the assumption

$$(u_0, u_1) \in (H^1(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) \tag{1.2}$$

for the data, where additional regularity L^m, $m \in [1,2)$ was supposed. They obtained a new critical exponent $p_{crit} = 1 + \frac{2m}{n}$ from the global (in time) existence of small data Sobolev solutions side and the blow-up side as well. Here blow-up means the non-existence of global (in time) Sobolev solutions.

In [10], assuming that the right-hand side of (1.1) is given by $u|u|^{p-1}$, the authors proved for given compactly supported initial data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and for $p \leq p_{GN}(n) = \frac{n}{n-2}$ if $n \geq 3$ the local (in time) existence of energy solutions $u \in C([0,T), H^1(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n))$. In the same paper the global (in time) existence was proved for small data by using the technique of potential well and modified potential well. The authors proposed the critical exponent $p_{crit} = p_{crit}(n) = 1 + \frac{4}{n}$ which means that we have global (in time) existence of small data Sobolev solutions for some admissible $p > p_{crit}$, and local (in time) existence for $p > 1$ and large data as well.

In [5], the authors generalized the question for the critical exponent to the question for critical regularity of the right-hand side. The model of interest is

$$u_{tt} - \Delta u + u_t = |u|^{1+\frac{2}{n}}\mu(|u|), \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x),$$

where the function $\mu : \tau \in [0,\infty) \longrightarrow \mu(\tau) \in [0,\infty)$ is supposed to be a modulus of continuity. This means that μ is continuous, concave, strictly increasing and $\mu(0) = 0$.
In [5], a sharp condition on μ to get a threshold between global (in time) existence of small data solutions and blow-up behaviour was obtained.

Let us now include a time-dependent coefficient $b = b(t)$ in the dissipation term. A first step is to understand qualitative properties of solutions to the following Cauchy problem:

$$u_{tt} - \Delta u + b(t)u_t = 0, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x).$$

In [15, 16], a classification of dissipation terms $b(t)u_t$ in scattering to free waves producing, non-effective dissipation terms, effective dissipation terms and overdamping producing is proposed.

In [4], the corresponding semilinear model with power nonlinearity and effective dissipation, that is,

$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x)$$

is treated. The authors proved the global (in time) existence of small data energy solutions in the supercritical case $p > 1 + \frac{2m}{n}$ under assumption (1.2) for the data. Moreover, the Fujita type exponent $p_{crit} = 1 + \frac{2m}{n}$ was verified as critical exponent. Here the dissipation term $b(t)u_t$ is called effective if it satisfies the properties that are described in Section 2.1. The first goal of this paper is to deal with the semilinear effectively damped Cauchy problem in 1d with critical power nonlinearity and an additional modulus of continuity term which provides an additional regularity of the right-hand side. Namely, the Cauchy problem we have in mind is

$$u_{tt} - u_{xx} + b(t)u_t = |u|^3\mu(|u|), \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x). \tag{1.3}$$

Besides effectively damped semilinear models with scale-invariant dissipation are of special interest. Recently, several authors are interested in the model

$$u_{tt} - \Delta u + \frac{\nu}{1+t}u_t = |u|^p, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x). \tag{1.4}$$

In [1, 11, 12], the authors showed that the situation depends strongly on the value of ν. In other words, the transition of ν from 0 to ∞ describes the change from a hyperbolic to a parabolic like model from the point of decay estimates for solutions. Furthermore, they proved that the decay rate of solutions for large ν is the same that is obtained for solutions of the classical damped wave equation. A particular case of the Cauchy problem (1.4) with $\nu = 2$ was studied in [3].

The second goal of this paper is to study the semilinear Cauchy problem in 1d with scale-invariant dissipation, with power nonlinearity and an additional modulus of continuity term. The model of interest is

$$u_{tt} - u_{xx} + \frac{\nu}{1+t}u_t = |u|^3\mu(|u|), \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{1.5}$$

In the further considerations we assume that the modulus of continuity μ given in (1.3) and (1.5) satisfies the following two conditions:

$$\tau|\mu'(\tau)| \le C\mu(\tau) \qquad \text{for} \qquad \tau \in (0,\tau_0) \qquad \text{and} \qquad \int_0^{C_0} \frac{\mu(R)}{R}\,dR < \infty, \tag{1.6}$$

where C is a sufficiently large positive constant, τ_0 and C_0 are sufficiently small positive constants.

The paper is organized as follows: In Section 2 we present our main results for the global (in time) existence of small data Sobolev solutions. After introducing the philosophy of our approach in Section 3 the proofs of the results of Section 2 are given in Section 4. In Section 5 we turn to the question of blow-up for some cases which are treated in Sections 2 to 4. Finally, some concluding remarks from Section 6 complete the paper.

We introduce some notations used in this paper. We note that the letter C indicates a generic non-negative constant, which may change from line to line. The usual L^p norm for Lebesgue spaces is defined as follows:

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}^n} |f(x)|dx\Big)^{\frac{1}{p}}, \quad 1 \le p < \infty \quad \text{and} \quad \|f\|_{L^\infty} = ess\sup|f(x)|.$$

Moreover, $H^s(\mathbb{R}^n)$ denotes the Sobolev space based on $L^2(\mathbb{R}^n)$ with $s \ge 0$.

2. Global (in time) existence results of small data Sobolev solutions

2.1. *Effective dissipation*

Let us consider in 1d the Cauchy problem

$$u_{tt} - u_{xx} + b(t)u_t = |u|^3\mu(|u|), \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{2.1}$$

Here $b(t)u_t$ is called effective in the model (2.1) if $b = b(t)$ satisfies the following properties:

- b is a positive and monotonic function with $tb(t) \to \infty$ as $t \to \infty$,
- $((1+t)^2b(t))^{-1} \in L^1(0,\infty)$,
- $b \in C^3[0,\infty)$ and $|b^{(k)}(t)| \lesssim \frac{b(t)}{(1+t)^k}$ for $k = 1,2,3$,
- $\frac{1}{b} \notin L^1(0,\infty)$ and there exists a constant $a \in [0,1)$ such that $tb'(t) \le ab(t)$.

Typical examples are

$$b(t) = \frac{\upsilon}{(1+t)^r}, \quad b(t) = \frac{\upsilon}{(1+t)^r}(\log(e+t))^\gamma, \quad b(t) = \frac{\upsilon}{(1+t)^r(\log(e+t))^\gamma}$$

for some $\upsilon > 0$, $\gamma > 0$ and $r \in (-1, 1)$.

We denote by $B(t, 0)$ the primitive of $1/b(t)$ which vanishes at $t = 0$, that is,

$$B(t, 0) = \int_0^t \frac{1}{b(r)}\, dr.$$

We denote by $B(t, s)$ the primitive of $1/b(t)$ which vanishes at $t = s$, that is,

$$B(t, s) = \int_s^t \frac{1}{b(r)}\, dr = B(t, 0) - B(s, 0).$$

In [4], the authors proved that the primitive $B(t, s)$ satisfies the following properties:

$$B(t, s) \quad \approx \quad B(t, 0) \quad \text{if} \quad s \in \Big[0, \frac{t}{2}\Big], \tag{2.2}$$

$$B(t, 0) \quad \approx \quad B(s, 0) \quad \text{if} \quad s \in \Big[\frac{t}{2}, t\Big]. \tag{2.3}$$

Let us formulate the main results for the global (in time) existence of small data Sobolev solutions.

Theorem 2.1. *Let $(u_0, u_1) \in \mathcal{A} := (H^1(\mathbb{R}) \cap L^1(\mathbb{R})) \times (L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$. Assume that the modulus of continuity μ in (2.1) satisfies the condition (1.6). Then, the following statement holds for a sufficiently small $\varepsilon_0 > 0$: if*

$$\|(u_0, u_1)\|_{\mathcal{A}} \le \varepsilon \qquad \text{for} \qquad \varepsilon \le \varepsilon_0,$$

then there exists a unique globally (in time) Sobolev solution u to (2.1) belonging to the evolution space

$$C([0, \infty), H^1(\mathbb{R})).$$

Furthermore, the solution satisfies the following decay estimates:

$$\|u(t, \cdot)\|_{L^2} \lesssim (1 + B(t, 0))^{-\frac{1}{4}} \|(u_0, u_1)\|_{\mathcal{A}},$$

$$\|\partial_x u(t, \cdot)\|_{L^2} \lesssim (1 + B(t, 0))^{-\frac{3}{4}} \|(u_0, u_1)\|_{\mathcal{A}},$$

$$\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + B(t, 0))^{-\frac{1}{2}} \|(u_0, u_1)\|_{\mathcal{A}}.$$

Example 2.2. *The results of Theorem 2.1 can be used to treat the following Cauchy problems:*

$$u_{tt} - u_{xx} + b(t)u_t = |u|^{3+\alpha}, \;\; u(0, x) = u_0(x), u_t(0, x) = u_1(x), \;\; \alpha \in (0, 1],$$

$$u_{tt} - u_{xx} + b(t)u_t = |u|^3\Big(\log\frac{1}{|u|}\Big)^{-\alpha}, \;\; u(0, x) = u_0(x), u_t(0, x) = u_1(x), \;\; \alpha > 1.$$

2.2. Scale-invariant weak dissipation

Let us consider for $\nu > 1$ the following Cauchy problem in 1d:

$$u_{tt} - u_{xx} + \frac{\nu}{1 + t}u_t = |u|^3\mu(|u|), \;\; u(0, x) = u_0(x), \;\; u_t(0, x) = u_1(x). \tag{2.4}$$

2.2.1. The case $\nu > 3$

Theorem 2.3. *Let* $(u_0, u_1) \in \mathcal{A} = (H^1(\mathbb{R}) \cap L^1(\mathbb{R})) \times (L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$*. Assume that the modulus of continuity* μ *in (2.4) satisfies the condition (1.6). Then, the following statement holds for a sufficiently small* $\varepsilon_0 > 0$*: if*

$$\|(u_0, u_1)\|_{\mathcal{A}} \le \varepsilon \qquad \text{for} \qquad \varepsilon \le \varepsilon_0,$$

then there exists a unique globally (in time) Sobolev solution u *to (2.4) belonging to the evolution space*

$$C([0, \infty), H^1(\mathbb{R})).$$

Furthermore, the solution satisfies the decay estimates

$$\begin{aligned}
\|u(t, \cdot)\|_{L^2} &\lesssim (1 + t)^{-\frac{1}{2}} \|(u_0, u_1)\|_{\mathcal{A}},\\
\|\partial_x u(t, \cdot)\|_{L^2} &\lesssim (1 + t)^{-\frac{3}{2}} \|(u_0, u_1)\|_{\mathcal{A}},\\
\|u(t, \cdot)\|_{L^\infty} &\lesssim (1 + t)^{-1} \|(u_0, u_1)\|_{\mathcal{A}}.
\end{aligned}$$

2.2.2. The case $1 < \nu < 3$

Let us consider for $\nu \in (1, 3)$ the following Cauchy problem in 1d:

$$u_{tt} - u_{xx} + \frac{\nu}{1 + t} u_t = |u|^{3+\alpha(\nu)} \mu(|u|), \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \tag{2.5}$$

where $\alpha = \alpha(\nu)$ describes an additional exponent in the power nonlinearity.

Theorem 2.4. *Let* $(u_0, u_1) \in \mathcal{A} = (H^1(\mathbb{R}) \cap L^1(\mathbb{R})) \times (L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$*. Assume that the modulus of continuity* μ *in (2.5) satisfies the condition (1.6). Then, the following statement holds for a sufficiently small* $\varepsilon_0 > 0$*: if*

$$\|(u_0, u_1)\|_{\mathcal{A}} \le \varepsilon \qquad \text{for} \qquad \varepsilon \le \varepsilon_0,$$

then for $\alpha(\nu) = \frac{2(3-\nu)}{1+\nu}$ *there exists a unique globally (in time) Sobolev solution* u *to (2.5) belonging to the evolution space*

$$C([0, \infty), H^1(\mathbb{R})).$$

Furthermore, the solution satisfies the decay estimates

$$\begin{aligned}
\|u(t, \cdot)\|_{L^2} &\lesssim (1 + t)^{-\frac{1}{2}} \|(u_0, u_1)\|_{\mathcal{A}},\\
\|\partial_x u(t, \cdot)\|_{L^2} &\lesssim (1 + t)^{-\frac{\nu}{2}} \|(u_0, u_1)\|_{\mathcal{A}},\\
\|u(t, \cdot)\|_{L^\infty} &\lesssim (1 + t)^{-\frac{1+\nu}{4}} \|(u_0, u_1)\|_{\mathcal{A}}.
\end{aligned}$$

Example 2.5. *The following modulus of continuity can be used in the previous Theorems 2.3 and 2.4:*

- $\mu(\tau) = \tau^\alpha, \alpha \in (0, 1]$,
- $\mu(0) = 0$ *and for* $\tau > 0$ *it holds* $\mu(\tau) = \left(\log \frac{1}{\tau}\right)^{-\alpha}$, $\alpha > 1$,
- $\mu(0) = 0$ *and for* $\tau > 0$ *it holds* $\mu(\tau) = \left(\log \frac{1}{\tau}\right)^{-1} \left(\log \log \frac{1}{\tau}\right)^{-1} \ldots \left(\log^k \frac{1}{\tau}\right)^{-\alpha}$, $\alpha > 1$, $k \in \mathbb{N}$.

2.2.3. The case $\nu = 3$

Let us consider for $\nu = 3$ the following Cauchy problem in 1d:

$$u_{tt} - u_{xx} + \frac{3}{1+t}u_t = |u|^3\mu(|u|), \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{2.6}$$

Theorem 2.6. *Let $(u_0, u_1) \in \mathcal{A} = (H^1(\mathbb{R}) \cap L^1(\mathbb{R})) \times (L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$. Assume that the modulus of continuity μ in (2.6) satisfies instead of* (1.6) *the condition*

$$\tau|\mu'(\tau)| \leq C\mu(\tau) \qquad \textit{for} \qquad \tau \in (0, \tau_0) \qquad \textit{and} \qquad \int_0^{C_0} \frac{\mu(R)}{R}\Big(1 + \frac{4}{3}\log\frac{1}{R}\Big)^{\frac{1}{2}}\, dR < \infty. \tag{2.7}$$

Then, the following statement holds for a sufficiently small $\varepsilon_0 > 0$: if

$$\|(u_0, u_1)\|_{\mathcal{A}} \leq \varepsilon \qquad \textit{for} \qquad \varepsilon \leq \varepsilon_0,$$

then there exists a unique globally (in time) Sobolev solution u to (2.6) belonging to the evolution space

$$C([0,\infty), H^1(\mathbb{R})).$$

Furthermore, the solution satisfies the decay estimates

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{1}{2}}\|(u_0,u_1)\|_{\mathcal{A}}, \\ \|\partial_x u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{3}{2}}(1+\log(1+t))^{\frac{1}{2}}\|(u_0,u_1)\|_{\mathcal{A}}, \\ \|u(t,\cdot)\|_{L^\infty} &\lesssim (1+t)^{-1}(1+\log(1+t))^{\frac{1}{4}}\|(u_0,u_1)\|_{\mathcal{A}}. \end{aligned}$$

Example 2.7. *The following modulus of continuity can be used in the previous Theorem 2.6:*

- $\mu(\tau) = \tau^\alpha, \alpha \in (0, 1]$,
- $\mu(0) = 0$ *and for $\tau > 0$ it holds* $\mu(\tau) = \left(\log\frac{1}{\tau}\right)^{-\alpha}$, $\alpha > \frac{3}{2}$,
- $\mu(0) = 0$ *and for $\tau > 0$ it holds* $\mu(\tau) = \left(\log\frac{1}{\tau}\right)^{-\frac{3}{2}}\left(\log\log\frac{1}{\tau}\right)^{-1}\ldots\left(\log^k\frac{1}{\tau}\right)^{-\alpha}$, $\alpha > 1$, $k \in \mathbb{N}$.

3. Philosophy of our approach

Let us consider the Cauchy problem

$$u_{tt} - u_{xx} + a(t)u_t = f(|u|), \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x). \tag{3.1}$$

Here $a = a(t)$ can be either $b = b(t)$ of (2.1) or $\frac{\nu}{1+t}$ of (2.4).

Denote by $K_0 = K_0(t,0,x), K_1 = K_1(t,0,x)$ the fundamental solutions to the linear homogeneous Cauchy problem with the initial data $(u_0,u_1) = (\delta_0, 0)$ and $(u_0,u_1) = (0,\delta_0)$, respectively, where δ_0 is the Dirac distribution in $x = 0$ with respect to the spatial variables.

According to Duhamel's principle, solutions of (3.1) may be interpreted as solutions to the nonlinear integral equation

$$u(t,x) = K_0(t,0,x) \ast_{(x)} u_0(x) + K_1(t,0,x) \ast_{(x)} u_1(x) + \int_0^t K_1(t,s,x) \ast_{(x)} f(|u(s,x)|)\, ds, \tag{3.2}$$

where $K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x)$ is the Sobolev solution of the Cauchy problem

$$u_{tt} - u_{xx} + a(t)u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{3.3}$$

and $K_1(t,s,x) *_{(x)} g(s,x)$ is the Sobolev solution to the family of parameter-dependent Cauchy problems

$$u_{tt} - u_{xx} + a(t)u_t = 0, \quad u(s,x) = 0, \quad u_t(s,x) = g(s,x), \tag{3.4}$$

for $0 \le s \le t < \infty$. Here $*_{(x)}$ stands for the convolution with respect to the spatial variable. We want to underline again, that we understand a solution of (3.1) as a solution of the nonlinear integral equation (3.2).

In [4, 15, 16] the following results for the Cauchy problems (3.3) and (3.4) with $a(t) = b(t)$ are proved.

Proposition 3.1. *The Sobolev solutions to the Cauchy problem*

$$u_{tt} - u_{xx} + b(t)u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)$$

satisfy the following estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1 + B(t,0))^{-\frac{1}{4}} \|(u_0,u_1)\|_{\mathcal{A}},$$

$$\|\partial_x u(t,\cdot)\|_{L^2} \lesssim (1 + B(t,0))^{-\frac{1}{4}-\frac{1}{2}} \|(u_0,u_1)\|_{\mathcal{A}}.$$

Proposition 3.2. *The Sobolev solutions to the Cauchy problem*

$$u_{tt} - u_{xx} + b(t)u_t = 0, \quad u(s,x) = 0, \quad u_t(s,x) = g(s,x)$$

satisfy the following estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim b(s)^{-1}(1 + B(t,s))^{-\frac{1}{4}} \|g(s,\cdot)\|_{L^2 \cap L^1}, \tag{3.5}$$

$$\|\partial_x u(t,\cdot)\|_{L^2} \lesssim b(s)^{-1}(1 + B(t,s))^{-\frac{1}{4}-\frac{1}{2}} \|g(s,\cdot)\|_{L^2 \cap L^1}. \tag{3.6}$$

In [13] the following results for the Cauchy problem (3.3) and (3.4) with $a(t) = \frac{\nu}{1+t}$ are proved.

Proposition 3.3. *The Sobolev solutions to the Cauchy problem*

$$u_{tt} - u_{xx} + \frac{\nu}{1+t} u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{3.7}$$

satisfy the following estimates:

$$\begin{array}{lrlcl}
\textit{If} & \nu > 3: & \|\partial_x^k u(t,\cdot)\|_{L^2} & \lesssim & (1+t)^{-\frac{1}{2}-k} \|(u_0,u_1)\|_{\mathcal{A}}, \quad k = 0,1; \\
\textit{If} & \nu = 3: & \|u(t,\cdot)\|_{L^2} & \lesssim & (1+t)^{-\frac{1}{2}} \|(u_0,u_1)\|_{\mathcal{A}}, \\
 & & \|\partial_x u(t,\cdot)\|_{L^2} & \lesssim & (1+t)^{-\frac{3}{2}} \left(1 + \log(1+t)\right)^{\frac{1}{2}} \|(u_0,u_1)\|_{\mathcal{A}}; \\
\textit{If} & 1 < \nu < 3: & \|u(t,\cdot)\|_{L^2} & \lesssim & (1+t)^{-\frac{1}{2}} \|(u_0,u_1)\|_{\mathcal{A}}, \\
 & & \|\partial_x u(t,\cdot)\|_{L^2} & \lesssim & (1+t)^{-\frac{\nu}{2}} \|(u_0,u_1)\|_{\mathcal{A}}.
\end{array}$$

Using the Gagliardo-Nirenberg inequality from Proposition A.1 with $j = 0$, $m = 1$, $q = \infty$, $p = r = 2$, and taking account of the dimension $n = 1$ we may conclude the estimate

$$\|u\|_{L^\infty} \leq \|u\|_{L^2}^{\frac{1}{2}} \|u_x\|_{L^2}^{\frac{1}{2}}. \tag{3.8}$$

Here we use that due to a density argument this inequality is true for all functions $u \in H^1(\mathbb{R})$. So, we can get the following L^∞ estimates for the solutions to the Cauchy problem (3.7):

Remark 3.4. *The Sobolev solutions to the Cauchy problem (3.7) satisfy the following L^∞ estimates:*

$$\begin{array}{lll} \textit{If} \quad \nu > 3: & \|u(t,\cdot)\|_{L^\infty} \lesssim & (1+t)^{-1}\|(u_0,u_1)\|_{\mathcal{A}}; \\ \textit{If} \quad \nu = 3: & \|u(t,\cdot)\|_{L^\infty} \lesssim & (1+t)^{-1}\left(1+\log(1+t)\right)^{\frac{1}{4}} \|(u_0,u_1)\|_{\mathcal{A}}; \\ \textit{If} \quad 1 < \nu < 3: & \|u(t,\cdot)\|_{L^\infty} \lesssim & (1+t)^{-\frac{1+\nu}{4}}\|(u_0,u_1)\|_{\mathcal{A}}. \end{array}$$

Proposition 3.5. *The Sobolev solutions to the Cauchy problem*

$$u_{tt} - u_{xx} + \frac{\nu}{1+t}u_t = 0, \quad u(s,x) = 0, \quad u_t(s,x) = g(s,x)$$

satisfy the following estimates for $0 \leq s \leq t < \infty$*:*

$$\begin{array}{llll} \textit{If} \quad \nu > 3: & \|\partial_x^k v(t,\cdot)\|_{L^2} & \lesssim & (\|g(s,\cdot)\|_{L^1} + (1+s)^{\frac{1}{2}}\|g(s,\cdot)\|_{L^2})(1+s)(1+t)^{-\frac{1}{2}-k}, \quad k = 0,1; \\ \textit{If} \quad \nu = 3: & \|v(t,\cdot)\|_{L^2} & \lesssim & (1+t)^{-\frac{1}{2}}(1+s)\|g(s,\cdot)\|_{L^1} + (1+t)^{-\frac{1}{2}}(1+s)^{\frac{3}{2}}\|g(s,\cdot)\|_{L^2}, \\ & \|\partial_x v(t,\cdot)\|_{L^2} & \lesssim & \Big((1+t)^{-\frac{3}{2}}(1+s)\|g(s,\cdot)\|_{L^1} \\ & & & +(1+t)^{-\frac{3}{2}}(1+s)^{\frac{3}{2}}\|g(s,\cdot)\|_{L^2}\Big)\left(1+\log(1+t)\right)^{\frac{1}{2}}; \\ \textit{If} \quad 1 < \nu < 3: & \|v(t,\cdot)\|_{L^2} & \lesssim & (1+t)^{-\frac{1}{2}}(1+s)\|g(s,\cdot)\|_{L^1} + (1+t)^{-\frac{1}{2}}(1+s)^{\frac{3}{2}}\|g(s,\cdot)\|_{L^2}, \\ & \|\partial_x v(t,\cdot)\|_{L^2} & \lesssim & (1+t)^{-\frac{\nu}{2}}(1+s)^{-\frac{1}{2}+\frac{\nu}{2}}\|g(s,\cdot)\|_{L^1} + (1+t)^{-\frac{\nu}{2}}(1+s)^{\frac{\nu}{2}}\|g(s,\cdot)\|_{L^2}. \end{array} \tag{3.9}$$

Having all these estimates in hand we turn to (3.2). Our goal is to apply Banach's fixed point theorem to the fixed point equation $u = Nu$, where

$$Nu := K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x) + \int_0^t K_1(t,s,x) *_{(x)} f(|u(s,x)|)\, ds.$$

After introducing a family $\{X(t)\}_{t>0}$ of time-dependent solution spaces we will estimate $\|Nu\|_{X(t)}$ and $\|Nu - Nv\|_{X(t)}$ for all $u, v \in X(t)$. In this way we close the circle and obtain existence and uniqueness of Sobolev solutions as well.

4. Proofs of the main results

4.1. Proof of Theorem 2.1

Proof. Taking into consideration the decay estimates of Propositions 3.1 and 3.2 we introduce the following family $\{X(t)\}_{t>0}$ of time-dependent solution spaces: $X(t) = C([0,t], H^1(\mathbb{R}))$ with the norm

$$\|u\|_{X(t)} = \sup_{s\in[0,t]} \Big\{(1+B(s,0))^{\frac{1}{4}}\|u(s,\cdot)\|_{L^2} + (1+B(s,0))^{\frac{3}{4}}\|\partial_x u(s,\cdot)\|_{L^2} + (1+B(s,0))^{\frac{1}{2}}\|u(s,\cdot)\|_{L^\infty}\Big\}.$$

We introduce the operator N by

$$N : u \in X(t) \to Nu = Nu(t,x) := u^{ln}(t,x) + u^{nl}(t,x),$$

where

$$u^{ln}(t,x) := K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x)$$

is a Sobolev solution to the Cauchy problem

$$u^{ln}_{tt} - u^{ln}_{xx} + b(t)u^{ln}_t = 0, \;\; u^{ln}(0,x) = u_0(x), \;\; u^{ln}_t(0,x) = u_1(x),$$

and

$$u^{nl}(t,x) := \int_0^t K_1(t,s,x) *_{(x)} |u(s,x)|^3 \mu(|u(s,x)|)\, ds$$

is a Sobolev solution to the Cauchy problem

$$u^{nl}_{tt} - u^{nl}_{xx} + b(t)u^{nl}_t = |u|^3\mu(|u|), \;\; u^{nl}(0,x) = 0, \;\; u^{nl}_t(0,x) = 0.$$

Our aim is to prove the following inequalities:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}} + \|u\|^3_{X(t)}, \tag{4.1}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u - v\|_{X(t)}(\|u\|^2_{X(t)} + \|v\|^2_{X(t)}). \tag{4.2}$$

The statements of Proposition 3.1 lead together with the definition of $\|u\|_{X(t)}$ to the estimate

$$\left\|u^{ln}\right\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}.$$

So, it remains to prove

$$\left\|u^{nl}\right\|_{X(t)} \lesssim \|u\|^3_{X(t)}. \tag{4.3}$$

Let us estimate $\|u^{nl}(t,\cdot)\|_{L^2}$. From (3.5) we have

$$\left\|u^{nl}(t,\cdot)\right\|_{L^2} \lesssim \int_0^t b(s)^{-1}(1 + B(t,s))^{-\frac{1}{4}} \left\||u(s,\cdot)|^3\mu(|u(s,\cdot)|)\right\|_{L^2\cap L^1} ds.$$

It holds

$$\||u(s,\cdot)|^3\mu(|u(s,\cdot)|)\|_{L^2\cap L^1} \leq \mu(\|u(s,\cdot)\|_{L^\infty})\||u(s,\cdot)|^3\|_{L^2\cap L^1}.$$

To estimate the first term of the last right-hand side we use the definition of $\|\cdot\|_{X(t)}$ to get for $0 \leq s \leq t$ the estimate

$$\|u(s,\cdot)\|_{L^\infty} \lesssim (1 + B(s,0))^{-\frac{1}{2}}\|u\|_{X(t)}.$$

Let us assume $\|u\|_{X(t)} \leq \epsilon_0$ for all $t > 0$ and some sufficiently small $\epsilon_0 > 0$. Then

$$\|u(s,\cdot)\|_{L^\infty} \lesssim \epsilon_0(1 + B(s,0))^{-\frac{1}{2}}. \tag{4.4}$$

Using the Gagliardo-Nirenberg inequality from Proposition A.1 we get for $0 \leq s \leq t$ the estimates

$$\||u(s,\cdot)|^3\|_{L^1} \lesssim (1 + B(s,0))^{-1}\|u\|^3_{X(t)}, \tag{4.5}$$

$$\||u(s,\cdot)|^3\|_{L^2} \lesssim \big(1+B(s,0)\big)^{-1-\frac{1}{4}}\|u\|^3_{X(t)}. \tag{4.6}$$

Using (4.4)–(4.6) with the properties of $B = B(t,s)$ we may conclude

$$\begin{aligned}\big\|u^{nl}(t,\cdot)\big\|_{L^2} &\lesssim \int_0^t b(s)^{-1}(1+B(t,s))^{-\frac{1}{4}}\mu(\epsilon_0(1+B(s,0))^{-\frac{1}{2}})(1+B(s,0))^{-1}\|u\|^3_{X(t)}\,ds\\ &\lesssim \|u\|^3_{X(t)}(1+B(t,0))^{-\frac{1}{4}}\int_0^{\frac{t}{2}} b(s)^{-1}\mu(\epsilon_0(1+B(s,0))^{-\frac{1}{2}})(1+B(s,0))^{-1}\,ds\\ &\quad+\|u\|^3_{X(t)}\mu(\epsilon_0)(1+B(t,0))^{-1}\int_{\frac{t}{2}}^t b(s)^{-1}(1+B(t,s))^{-\frac{1}{4}}\,ds.\end{aligned}$$

For the first integral we obtain after the change of variables $R = \epsilon_0(1+B(s,0))^{-\frac{1}{2}}$ the relation

$$\int_0^{\frac{t}{2}} b(s)^{-1}\mu(\epsilon_0(1+B(s,0))^{-\frac{1}{2}})(1+B(s,0))^{-1}\,ds \lesssim \int_0^{\epsilon_0}\frac{\mu(R)}{R}dR < \infty.$$

For the second integral after putting the change of variables $r = B(t,s)$ we get

$$\int_{\frac{t}{2}}^t b(s)^{-1}(1+B(t,s))^{-\frac{1}{4}}\,ds = \int_0^{B(t,\frac{t}{2})}(1+r)^{-\frac{1}{4}}\,dr = \frac{4}{3}\Big(1+B\Big(t,\frac{t}{2}\Big)\Big)^{\frac{3}{4}} - \frac{4}{3} \lesssim (1+B(t,0))^{\frac{3}{4}}.$$

Then, we have

$$\mu(\epsilon_0)(1+B(t,0))^{-1}\int_{\frac{t}{2}}^t b(s)^{-1}(1+B(t,s))^{-\frac{1}{4}}\,ds \lesssim \mu(\epsilon_0)(1+B(t,0))^{-\frac{1}{4}} \lesssim (1+B(t,0))^{-\frac{1}{4}}.$$

All together implies

$$\big\|u^{nl}(t,\cdot)\big\|_{L^2} \lesssim (1+B(t,0))^{-\frac{1}{4}}\|u\|^3_{X(t)}. \tag{4.7}$$

In the same way we can prove

$$\big\|\partial_x u^{nl}(t,\cdot)\big\|_{L^2} \lesssim (1+B(t,0))^{-\frac{3}{4}}\|u\|^3_{X(t)}. \tag{4.8}$$

To estimate $\|u^{nl}(t,\cdot)\|_{L^\infty}$ we use (3.8) to get the desired estimate

$$\big\|u^{nl}(t,\cdot)\big\|_{L^\infty} \lesssim (1+B(t,0))^{-\frac{1}{2}}\|u\|^3_{X(t)}. \tag{4.9}$$

From (4.7)–(4.9) we get (4.3).

To prove (4.2) we assume that u and v belong to $X(t)$. Then

$$Nu - Nv = \int_0^t K_1(t,s,x) *_{(x)} \big(|u(s,x)|^3\mu(|u(s,x)|) - |v(s,x)|^3\mu(|v(s,x)|)\big)\,ds.$$

We control all norms appearing in $\|Nu-Nv\|_{X(t)}$. From (1.6), (3.5) and (3.6) together with Minkowski's integral inequality we get for $j = 0, 1$ the estimates

$$\left\|\partial_x^j(Nu-Nv)(t,\cdot)\right\|_{L^2} \lesssim \int_0^t b(s)^{-1}(1+B(t,s))^{-\frac14-\frac{j}{2}}\left\||u(s,\cdot)|^3\mu(|u(s,\cdot)|)-|v(s,\cdot)|^3\mu(|v(s,\cdot)|)\right\|_{L^2\cap L^1}ds$$
$$\lesssim \int_0^t b(s)^{-1}(1+B(t,s))^{-\frac14-\frac{j}{2}}\left\|\Big(\int_0^1 \mu(|u+\tau(v-u)|)|u+\tau(v-u)|^2\,d\tau\Big)|u-v|(s,\cdot)\right\|_{L^2\cap L^1}ds$$
$$\lesssim \int_0^t b(s)^{-1}(1+B(t,s))^{-\frac14-\frac{j}{2}}\left\|\left(|u|^2+|v|^2\right)(u-v)(s,\cdot)\right\|_{L^2\cap L^1}\int_0^1 \|\mu(|u+\tau(v-u)|)\|_{L^\infty}\,d\tau\,ds.$$

Similarly to verify (4.4)–(4.6) using Hölder's inequality and Gagliardo-Nirenberg inequality we obtain

$$\|\mu(|u(s,\cdot)+\tau(v-u)(s,\cdot)|)\|_{L^\infty} \lesssim \mu(\epsilon_0(1+B(s,0))^{-\frac12}) \text{ for all } \tau\in[0,1],$$
$$\left\|(|u|^2+|v|^2)(u-v)(s,\cdot)\right\|_{L^1} \lesssim (1+B(s,0))^{-1}\|u-v\|_{X(t)}(\|u\|^2_{X(t)}+\|v\|^2_{X(t)}),$$
$$\left\|(|u|^2+|v|^2)(u-v)(s,\cdot)\right\|_{L^2} \lesssim (1+B(s,0))^{-1-\frac14}\|u-v\|_{X(t)}(\|u\|^2_{X(t)}+\|v\|^2_{X(t)}).$$

Following the same steps to get (4.7)–(4.9) after using the last estimates one can complete the proof. □

4.2. *Proof of Theorem 2.3*

Proof. We follow the same steps as in the proof of the previous theorem with the same family of solution spaces $\{X(t)\}_{t>0}$. The norm in $X(t)$ is defined as follows:

$$\|u\|_{X(t)} = \sup_{s\in[0,t]}\left\{(1+s)^{\frac12}\|u(s,\cdot)\|_{L^2}+(1+s)^{\frac32}\|\partial_x u(s,\cdot)\|_{L^2}+(1+s)\|u(s,\cdot)\|_{L^\infty}\right\}.$$

Our goal is again to prove the following inequalities:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}+\|u\|^3_{X(t)}, \tag{4.10}$$

$$\|Nu-Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)}(\|u\|^2_{X(t)}+\|v\|^2_{X(t)}). \tag{4.11}$$

From the definition of the solution space $X(t)$ and the estimates of Proposition 3.3 one can get immediately

$$\left\|u^{ln}\right\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}}.$$

We complete the proof of (4.10) by showing

$$\left\|u^{nl}\right\|_{X(t)} \lesssim \|u\|^3_{X(t)}.$$

Using (3.9) for $k=0,1$ we get

$$\left\|\partial_x^k u^{nl}(t,\cdot)\right\|_{L^2} \lesssim \int_0^t (1+s)(1+t)^{-\frac12-k}\left\||u(s,\cdot)|^3\mu(|u(s,\cdot)|)\right\|_{L^1}ds \tag{4.12}$$

$$+\int_0^t (1+s)^{\frac32}(1+t)^{-\frac12-k}\left\||u(s,\cdot)|^3\mu(|u(s,\cdot)|)\right\|_{L^2}ds. \tag{4.13}$$

It holds,

$$\||u(s,\cdot)|^3\mu(|u(s,\cdot)|)\|_{L^r} \leq \mu(\|u(s,\cdot)\|_{L^\infty})\||u(s,\cdot)|^3\|_{L^r} \quad \text{for} \quad r = 1, 2. \tag{4.14}$$

Then we use in the first term of the last right-hand side

$$\|u(s,\cdot)\|_{L^\infty} \lesssim \epsilon_0(1+s)^{-1}. \tag{4.15}$$

Using the Gagliardo-Nirenberg inequality from Proposition A.1 together with the definition of $X(t)$ it follows

$$\||u(s,\cdot)|^3\|_{L^1} \lesssim (1+s)^{-2}\|u\|^3_{X(t)}, \tag{4.16}$$

$$\||u(s,\cdot)|^3\|_{L^2} \lesssim (1+s)^{-\frac{5}{2}}\|u\|^3_{X(t)}, \tag{4.17}$$

respectively, for $0 \leq s \leq t$. Replacing the last three estimates (4.15)–(4.17) in (4.14) and after that in (4.12) and (4.13) leads to

$$\begin{aligned}\left\|\partial_x^k u^{nl}(t,\cdot)\right\|_{L^2} &\lesssim \|u\|^3_{X(t)}\int_0^t (1+s)^{-1}(1+t)^{-\frac{1}{2}-k}\mu\big(\epsilon_0(1+s)^{-1}\big)\,ds \\ &\lesssim \|u\|^3_{X(t)}(1+t)^{-\frac{1}{2}-k}\int_0^t (1+s)^{-1}\mu\big(\epsilon_0(1+s)^{-1}\big)\,ds \lesssim \|u\|^3_{X(t)}(1+t)^{-\frac{1}{2}-k},\end{aligned}$$

where again the condition (1.6) is used. By inequality (3.8) we may conclude

$$\left\|u^{nl}(t,\cdot)\right\|_{L^\infty} \lesssim \|u\|^3_{X(t)}(1+t)^{-1}.$$

Summarizing the last estimates gives (4.10). To verify (4.11) we follow the same steps of the proof of (4.2) taking into consideration the definition of solution space $X(t)$. □

4.3. Proof of Theorem 2.4

Proof. In this case we use the same family $\{X(t)\}_{t>0}$ of spaces of Sobolev solutions as before. The norm $\|\cdot\|_{X(t)}$ is related to the estimates of Proposition 3.3 for $1 < \nu < 3$. Consequently, we introduce

$$\|u\|_{X(t)} = \sup_{s\in[0,t]}\Big\{(1+s)^{\frac{1}{2}}\|u(s,\cdot)\|_{L^2} + (1+s)^{\frac{\nu}{2}}\|\partial_x u(s,\cdot)\|_{L^2} + (1+s)^{\frac{1+\nu}{4}}\|u(s,\cdot)\|_{L^\infty}\Big\}.$$

Using (3.9) for u^{nl} we have

$$\begin{aligned}\left\|u^{nl}(t,\cdot)\right\|_{L^2} \lesssim &\int_0^t (1+s)(1+t)^{-\frac{1}{2}}\left\||u(s,\cdot)|^{3+\alpha(\nu)}\mu(|u(s,\cdot)|)\right\|_{L^1}\,ds \\ &+\int_0^t (1+s)^{\frac{3}{2}}(1+t)^{-\frac{1}{2}}\left\||u(s,\cdot)|^{3+\alpha(\nu)}\mu(|u(s,\cdot)|)\right\|_{L^2}\,ds.\end{aligned}$$

Similarly to (4.15)–(4.17) we obtain

$$\|u(s,\cdot)\|_{L^\infty} \lesssim \epsilon_0(1+s)^{-\frac{1+\nu}{4}},$$

$$\||u(s,\cdot)|^{3+\alpha(\nu)}\|_{L^1} \lesssim (1+s)^{-\frac{5+\nu}{4}-\alpha(\nu)\frac{1+\nu}{4}}\|u\|^{3+\alpha(\nu)}_{X(t)},$$

$$\||u(s,\cdot)|^{3+\alpha(\nu)}\|_{L^2} \lesssim (1+s)^{-1-\frac{\nu}{2}-\alpha(\nu)\frac{1+\nu}{4}}\|u\|^{3+\alpha(\nu)}_{X(t)}.$$

In the following we use the relation

$$\int_0^t (1+s)^A \mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds \lesssim \int_0^{\epsilon_0} R^{1-(A+\frac{5+\nu}{4})\frac{4}{1+\nu}} \frac{\mu(R)}{R}\,dR, \tag{4.18}$$

where $R = \epsilon_0(1+s)^{-\frac{1+\nu}{4}}$. Taking account of (3.9) and (4.18) we arrive at

$$\begin{aligned}
\big\|u^{nl}(t,\cdot)\big\|_{L^2} &\lesssim \int_0^t (1+s)(1+t)^{-\frac{1}{2}}(1+s)^{-\frac{5+\nu}{4}-\alpha(\nu)\frac{1+\nu}{4}}\mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds\,\|u\|_{X(t)}^{3+\alpha(\nu)}\\
&\quad+\int_0^t (1+s)^{\frac{3}{2}}(1+t)^{-\frac{1}{2}}(1+s)^{-1-\frac{\nu}{2}-\alpha(\nu)\frac{1+\nu}{4}}\mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds\,\|u\|_{X(t)}^{3+\alpha(\nu)}\\
&\lesssim (1+t)^{-\frac{1}{2}}\int_0^t (1+s)^{-\frac{1+\nu}{4}-\alpha(\nu)\frac{1+\nu}{4}}\mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds\,\|u\|_{X(t)}^{3+\alpha(\nu)}\\
&\quad+(1+t)^{-\frac{1}{2}}\int_0^t (1+s)^{\frac{1-\nu}{2}-\alpha(\nu)\frac{1+\nu}{4}}\mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds\,\|u\|_{X(t)}^{3+\alpha(\nu)}\\
&\lesssim (1+t)^{-\frac{1}{2}}\|u\|_{X(t)}^{3+\alpha(\nu)}\Big(\int_0^{\epsilon_0}\frac{\mu(R)}{R^{\frac{4}{1+\nu}-\alpha(\nu)}}\,dR+\int_0^{\epsilon_0}\frac{\mu(R)}{R^{\frac{7-\nu}{1+\nu}-\alpha(\nu)}}\,dR\Big)\\
&\lesssim (1+t)^{-\frac{1}{2}}\|u\|_{X(t)}^{3+\alpha(\nu)}\int_0^{\epsilon_0}\frac{\mu(R)}{R}\,dR \lesssim (1+t)^{-\frac{1}{2}}\|u\|_{X(t)}^{3+\alpha(\nu)},
\end{aligned}$$

where we use condition (1.6) and assume the following conditions:

$$\frac{4}{1+\nu}-\alpha(\nu)\le 1 \quad\text{and}\quad \frac{7-\nu}{1+\nu}-\alpha(\nu)\le 1. \tag{4.19}$$

Under conditions (1.6) and (4.19) we may conclude

$$\big\|u^{nl}(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\|u\|_{X(t)}^{3+\alpha(\nu)}. \tag{4.20}$$

For estimating $\partial_x u^{nl}$ we use the last estimates of (3.9) and (4.18) to get

$$\begin{aligned}
\big\|\partial_x u^{nl}(t,\cdot)\big\|_{L^2} &\lesssim \int_0^t (1+s)^{-\frac{1}{2}+\frac{\nu}{2}}(1+t)^{-\frac{\nu}{2}}(1+s)^{-\frac{5+\nu}{4}-\alpha(\nu)\frac{1+\nu}{4}}\mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds\,\|u\|_{X(t)}^{3+\alpha(\nu)}\\
&\quad+\int_0^t (1+s)^{\frac{\nu}{2}}(1+t)^{-\frac{\nu}{2}}(1+s)^{-1-\frac{\nu}{2}-\alpha(\nu)\frac{1+\nu}{4}}\mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds\,\|u\|_{X(t)}^{3+\alpha(\nu)}\\
&\lesssim (1+t)^{-\frac{\nu}{2}}\int_0^t (1+s)^{\frac{\nu-7}{4}-\alpha(\nu)\frac{1+\nu}{4}}\mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds\,\|u\|_{X(t)}^{3+\alpha(\nu)}\\
&\quad+(1+t)^{-\frac{\nu}{2}}\int_0^t (1+s)^{-1-\alpha(\nu)\frac{1+\nu}{4}}\mu\big(\epsilon_0(1+s)^{-\frac{1+\nu}{4}}\big)\,ds\,\|u\|_{X(t)}^{3+\alpha(\nu)}\\
&\lesssim (1+t)^{-\frac{\nu}{2}}\|u\|_{X(t)}^{3+\alpha(\nu)}\Big(\int_0^{\epsilon_0}\frac{\mu(R)}{R^{\frac{2\nu-2}{1+\nu}-\alpha(\nu)}}\,dR+\int_0^{\epsilon_0}\frac{\mu(R)}{R^{1-\alpha(\nu)}}\,dR\Big)\\
&\lesssim (1+t)^{-\frac{\nu}{2}}\|u\|_{X(t)}^{3+\alpha(\nu)}\int_0^{\epsilon_0}\frac{\mu(R)}{R}\,dR \lesssim (1+t)^{-\frac{\nu}{2}}\|u\|_{X(t)}^{3+\alpha(\nu)},
\end{aligned}$$

where we use condition (1.6) and assume the following conditions:

$$\alpha(\nu)\ge \frac{2\nu-2}{1+\nu}-1 \quad\text{and}\quad \alpha(\nu)\ge 0. \tag{4.21}$$

Under conditions (1.6) and (4.21) we may conclude

$$\left\|\partial_x u^{nl}(t,\cdot)\right\|_{L^2} \lesssim (1+t)^{-\frac{\nu}{2}} \|u\|_{X(t)}^{3+\alpha(\nu)}. \tag{4.22}$$

The minimal $\alpha(\nu)$ satisfying all the conditions (4.19) and (4.21) is $\alpha(\nu) = \frac{2(3-\nu)}{1+\nu}$ what we supposed in the theorem. By inequality (3.8) we may conclude

$$\left\|u^{nl}(t,\cdot)\right\|_{L^\infty} \lesssim (1+t)^{-\frac{1+\nu}{4}} \|u\|_{X(t)}^{3}. \tag{4.23}$$

From (4.20), (4.22) and (4.23) we obtain (4.10). To verify (4.11) we follow the same steps of the proof of (4.2) by taking into consideration the definition of solution spaces $X(t)$. □

4.4. Proof of Theorem 2.6

Proof. In this case we use the same family $\{X(t)\}_{t>0}$ of spaces of Sobolev solutions as before. The norm $\|\cdot\|_{X(t)}$ is related to the estimates of Proposition 3.3 for $\nu = 3$. Consequently, we introduce

$$\|u\|_{X(t)} = \sup_{s\in[0,t]} \Big\{ (1+s)^{\frac{1}{2}} \|u(s,\cdot)\|_{L^2} + (1+s)^{\frac{3}{2}} (1+\log(1+s))^{-\frac{1}{2}} \|\partial_x u(s,\cdot)\|_{L^2}$$
$$+ (1+s)(1+\log(1+s))^{-\frac{1}{4}} \|u(s,\cdot)\|_{L^\infty} \Big\}.$$

Similar to (4.15) and after using the Gagliardo-Nirenberg inequality from Proposition A.1 we obtain

$$\|u(s,\cdot)\|_{L^\infty} \lesssim \epsilon_0 (1+s)^{-1} (1+\log(1+s))^{\frac{1}{4}},$$
$$\||u(s,\cdot)|^3\|_{L^1} \lesssim (1+s)^{-2} (1+\log(1+s))^{\frac{1}{4}} \|u\|_{X(t)}^3,$$
$$\||u(s,\cdot)|^3\|_{L^2} \lesssim (1+s)^{-1-\frac{3}{2}} (1+\log(1+s))^{\frac{1}{2}} \|u\|_{X(t)}^3.$$

Using these estimates together with (3.9) we arrive at

$$\left\|u^{nl}(t,\cdot)\right\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}} \|u\|_{X(t)}^3 \int_0^t (1+s)^{-1} (1+\log(1+s))^{\frac{1}{4}} \mu\big(\epsilon_0 (1+s)^{-1} (1+\log(1+s))^{\frac{1}{4}}\big)\, ds$$
$$+ (1+t)^{-\frac{1}{2}} \|u\|_{X(t)}^3 \int_0^t (1+s)^{-1} (1+\log(1+s))^{\frac{1}{2}} \mu\big(\epsilon_0 (1+s)^{-1} (1+\log(1+s))^{\frac{1}{4}}\big)\, ds,$$

and

$$\left\|\partial_x u^{nl}(t,\cdot)\right\|_{L^2} \lesssim (1+t)^{-\frac{3}{2}} (1+\log(1+t))^{\frac{1}{2}} \|u\|_{X(t)}^3$$
$$\times \int_0^t (1+s)^{-1} (1+\log(1+s))^{\frac{1}{4}} \mu\big(\epsilon_0 (1+s)^{-1} (1+\log(1+s))^{\frac{1}{4}}\big)\, ds$$
$$+ (1+t)^{-\frac{3}{2}} (1+\log(1+t))^{\frac{1}{2}} \|u\|_{X(t)}^3$$
$$\times \int_0^t (1+s)^{-1} (1+\log(1+s))^{\frac{1}{2}} \mu\big(\epsilon_0 (1+s)^{-1} (1+\log(1+s))^{\frac{1}{4}}\big)\, ds.$$

Now we introduce the change of variables $R := \epsilon_0 (1+s)^{-1} (1+\log(1+s))^{\frac{1}{4}}$. Taking account of

$$ds \approx -\frac{\epsilon_0}{R^2} (1+\log(1+s))^{\frac{1}{4}}\, dR \quad \text{and} \quad \log(1+s) \le \frac{4}{3} \log\Big(\frac{\epsilon_0}{R}\Big) \le \frac{4}{3} \log\Big(\frac{1}{R}\Big),$$

we get

$$\int_0^t (1+s)^{-1}(1+\log(1+s))^{\frac{1}{4}}\mu(\epsilon_0(1+s)^{-1}(1+\log(1+s))^{\frac{1}{4}})\,ds$$
$$\lesssim \int_0^{C_0} \frac{\mu(R)}{R}\Big(1+\frac{4}{3}\log\frac{1}{R}\Big)^{\frac{1}{4}}\,dR < \infty,$$

due to condition (2.7). Similarly, we obtain

$$\int_0^t (1+s)^{-1}(1+\log(1+s))^{\frac{1}{2}}\mu(\epsilon_0(1+s)^{-1}(1+\log(1+s))^{\frac{1}{4}})\,ds \lesssim \int_0^{C_0} \frac{\mu(R)}{R}\Big(1+\frac{4}{3}\log\frac{1}{R}\Big)^{\frac{1}{2}}\,dR < \infty.$$

We can complete the proof in the same way as we did before in the proofs of the other theorems. □

5. A blow-up result

In this section we conclude the influence of the function μ on the non-existence of global (in time) small data Sobolev solutions or on the so-called blow-up of Sobolev solutions to the Cauchy problem (1.3). Then from Theorem 5.1 it follows that if the integral condition (1.6) is not satisfied, then, in general, the solution cannot exist globally (in time). This means the optimality of the integral condition in (1.6) for the Cauchy problem (1.3).

Theorem 5.1. *Let us consider the following Cauchy problem with effective dissipation $b(t)u_t$:*

$$u_{tt} - u_{xx} + b(t)u_t = |u|^3\mu(|u|), \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{5.1}$$

where $b = b(t)$ satisfies the assumptions from Section 2.1 and the further condition

$$\lim_{t\to\infty} b(t) = a_0 > 0. \tag{5.2}$$

Let $\mu = \mu(s)$, $s \in [0,\infty)$, be a modulus of continuity which satisfies

$$\int_0^{C_0} \frac{\mu(s)}{s}ds = \infty. \tag{5.3}$$

Here C_0 is a sufficiently small positive constant. The function $h : s \in \mathbb{R} \longmapsto h(s) := s^3\mu(s)$ is supposed to be convex on $\mathbb{R}$. The data $(u_0, u_1) \in C_0^\infty(\mathbb{R})$ are chosen such that

$$\int_{\mathbb{R}} (u_0(x) + b_0u_1(x))\,dx > 0, \tag{5.4}$$

where b_0 is defined in Lemma A.3. Then, we have no global (in time) existence of small data Sobolev solutions $u \in C([0,\infty), L^\infty(\mathbb{R}))$.

Example 5.2. *The following modulus of continuity satisfies the condition* (5.3) *of Theorem 5.1:*

- $\mu(0) = 0$ *and for* $\tau > 0$ *it holds* $\mu(\tau) = \Big(\log\frac{1}{\tau}\Big)^{-\alpha}$, $\alpha \in (0,1]$,
- $\mu(0) = 0$ *and for* $\tau > 0$ *it holds* $\mu(\tau) = \Big(\log\frac{1}{\tau}\Big)^{-1}\Big(\log\log\frac{1}{\tau}\Big)^{-1}\ldots\Big(\log^k\frac{1}{\tau}\Big)^{-\alpha}$, $\alpha \in (0,1]$, $k \in \mathbb{N}$.

Proof. We suppose that the solution $u \in C([0,\infty), L^\infty(\mathbb{R}))$ exists globally in time. Multiplying (5.1) by a positive function $g = g(t)$ which is defined in Lemma A.3 we obtain

$$(g(t)u(t,x))_{tt} - (g(t)u(t,x))_{xx} - (g'(t)u(t,x))_t + (-g'(t) + g(t)b(t))u_t(t,x) = g(t)|u(t,x)|^3\mu(|u(t,x)|).$$

From the definition of $g = g(t)$ we may conclude

$$(g(t)u(t,x))_{tt} - (g(t)u(t,x))_{xx} - (g'(t)u(t,x))_t + u_t(t,x) = g(t)|u(t,x)|^3\mu(|u(t,x)|).$$

For the further considerations we introduce the following functions:

$$\eta(s) = \begin{cases} 1 & \text{if} \quad s \in [0,\frac{1}{2}],\\ \text{decreasing} & \text{if} \quad s \in (\frac{1}{2},1),\\ 0 & \text{if} \quad s \geq 1, \end{cases} \qquad \eta^*(s) = \begin{cases} 0 & \text{if} \quad s \in [0,\frac{1}{2}],\\ \eta(s) & \text{if} \quad s \geq \frac{1}{2}. \end{cases}$$

We define for $(t,x) \in [0,\infty) \times \mathbb{R}$ the cut-off functions

$$\psi_{F(R)} = \psi_{F(R)}(t,x) = \eta\left(\frac{|x|^2 + t}{F(R)}\right)^3 \quad \text{and} \quad \psi^*_{F(R)} = \psi^*_{F(R)}(t,x) = \eta^*\left(\frac{|x|^2 + t}{F(R)}\right)^3, \tag{5.5}$$

where $F(R) = B^{-1}(R,0)$ and $B^{-1}(t,0)$ is the inverse function of $B(t,0)$. It follows that $F : R \in [0,\infty) \longrightarrow F(R) \in [0,\infty)$ is a strictly increasing function with $F(0) = 0$ and $\lim\limits_{R\longrightarrow\infty} F(R) = \infty$ thanks to $\frac{1}{b} \notin L^1(\mathbb{R}^+)$. Moreover, the support of ψ^* is contained in

$$Q^*_{F(R)} = Q_{F(R)} \backslash \Big\{(t,x) : |x|^2 + t \leq \frac{F(R)}{2}\Big\}.$$

After integrating by parts we arrive at

$$\begin{aligned}\int_{Q_{F(R)}} g(t)|u(t,x)|^3\mu(|u(t,x)|)\psi_{F(R)}(t,x)\,d(t,x) &= -\int_{B_{\sqrt{F(R)}}} (u_0(0,x) + b_0u_1(0,x))\psi_{F(R)}(0,x)\,dx\\ &+ \int_{Q_{F(R)}} \big(g(t)u(t,x)\partial_t^2\psi_{F(R)}(t,x) + (g'(t)-1)u(t,x)\partial_t\psi_{F(R)}(t,x) - g(t)u(t,x)\partial_x^2\psi_{F(R)}(t,x)\big)\,d(t,x).\end{aligned}$$

Due to the assumption $u \in C([0,\infty), L^\infty(\mathbb{R}))$ all integrals are well-defined. We define the functional

$$I_{F(R)} := \int_{Q_{F(R)}} g(t)|u(t,x)|^3\mu(|u(t,x)|)\psi_{F(R)}(t,x)\,d(t,x) = \int_{Q_{F(R)}} g(t)h(|u(t,x)|)\psi_{F(R)}(t,x)\,d(t,x).$$

Then, due to (5.4) it holds

$$I_{F(R)} \leq \int_{Q_{F(R)}} \big(g(t)u(t,x)\partial_t^2\psi_{F(R)}(t,x) + (g'(t)-1)u(t,x)\partial_t\psi_{F(R)}(t,x) - g(t)u(t,x)\partial_x^2\psi_{F(R)}(t,x)\big)\,d(t,x).$$

We have

$$\partial_t\psi_{F(R)} = \frac{3}{F(R)}\eta\left(\frac{|x|^2+t}{F(R)}\right)^2\eta'\left(\frac{|x|^2+t}{F(R)}\right),$$

$$\begin{aligned}
\partial_t^2\psi_{F(R)} &= \frac{6}{F(R)^2}\eta\left(\frac{|x|^2+t}{F(R)}\right)\eta'\left(\frac{|x|^2+t}{F(R)}\right)^2+\frac{3}{F(R)^2}\eta\left(\frac{|x|^2+t}{F(R)}\right)^2\eta''\left(\frac{|x|^2+t}{F(R)}\right),\\
\partial_x^2\psi_{F(R)} &= \frac{24x^2}{F(R)^2}\eta\left(\frac{|x|^2+t}{F(R)}\right)\eta'\left(\frac{|x|^2+t}{F(R)}\right)^2+\frac{12x^2}{F(R)^2}\eta\left(\frac{|x|^2+t}{F(R)}\right)^2\eta''\left(\frac{|x|^2+t}{F(R)}\right)\\
&\quad +\frac{6}{F(R)}\eta\left(\frac{|x|^2+t}{F(R)}\right)^2\eta'\left(\frac{|x|^2+t}{F(R)}\right).
\end{aligned}$$

From (5.5) together with the boundedness of η, η' and η'', there exists a constant $C>0$ such that

$$\left|\partial_t^2\psi_{F(R)}+\partial_t\psi_{F(R)}+\partial_x^2\psi_{F(R)}\right|\le\frac{C}{F(R)}\eta^*\left(\frac{|x|^2+t}{F(R)}\right)=\frac{C}{F(R)}(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}.$$

We have

$$I_{F(R)}\le\frac{C}{F(R)}\int_{Q_{F(R)}}(g(t)+C_0)|u(t,x)|(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}\,d(t,x),$$

where C_0 is the constant from Lemma A.3.

Due to (5.2), which implies that $g=g(t)$ is bounded to above, we can choose C_1 large enough such that $\frac{g(t)}{C_1}\le q_0<1$ for all $t\ge0$. Then let us choose $C>C_1>C_0$. Hence,

$$I_{F(R)}\le\frac{C}{F(R)}\int_{Q_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)|u(t,x)|(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}\,d(t,x). \tag{5.6}$$

Applying Lemma A.4 for $\alpha=\alpha(t):=g(t)>0$ since $b=b(t)$ is a positive function we get

$$h\left(\frac{\int_{Q^*_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)|u(t,x)|(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}\,d(t,x)}{\int_{Q^*_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)\,d(t,x)}\right)\le\frac{\int_{Q^*_{F(R)}}h\Big(\Big(\frac{g(t)}{C_1}+1\Big)|u(t,x)|(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}\Big)\,d(t,x)}{\int_{Q^*_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)\,d(t,x)}.$$

Moreover, we have

$$\int_{Q^*_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)|u(t,x)|(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}\,d(t,x)=\int_{Q_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)|u(t,x)|(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}\,d(t,x).$$

On the other hand we have

$$h\Big(|u(t,x)|(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}\Big)\le h\left(|u(t,x)|\right)(\psi^*_{F(R)}(t,x)).$$

All together with the monotonicity and continuity of μ and the existence of h^{-1} we can get the following estimate:

$$\int_{Q_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)|u(t,x)|(\psi^*_{F(R)}(t,x))^{\frac{1}{3}}\,d(t,x)\le CF(R)^{\frac{3}{2}}h^{-1}\left(\frac{\int_{Q_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)h\left(|u(t,x)|\right)(\psi^*_{F(R)}(t,x))\,d(t,x)}{CF(R)^{\frac{3}{2}}}\right), \tag{5.7}$$

where we use the following estimate:

$$\int_{Q^*_{F(R)}}\Big(\frac{g(t)}{C_1}+1\Big)\,d(t,x)\approx F(R)^{\frac{3}{2}}.$$

Let us now define the functions y and Y as follows:

$$y = y(r) = \int_{Q_{F(R)}} \Big(\frac{g(t)}{C_1} + 1\Big) h\,(|u(t,x)|)\,(\psi_r^*(t,x))\,d(t,x) \quad \text{and} \qquad Y = Y(F(R)) = \int_0^{F(R)} y(r) r^{-1}\,dr.$$

Then, we have

$$\begin{aligned} Y(F(R)) &= \int_0^{F(R)} \left(\int_{Q_{F(R)}} \Big(\frac{g(t)}{C_1} + 1\Big) h\,(|u(t,x)|)\,(\psi_r^*(t,x))\,d(t,x) \right) r^{-1}\,dr \\ &= \int_{Q_{F(R)}} \Big(\frac{g(t)}{C_1} + 1\Big) h(|u(t,x)|) \left(\int_0^{F(R)} (\psi_r^*(t,x)) r^{-1}\,dr \right) d(t,x) \\ &= \int_{Q_{F(R)}} \Big(\frac{g(t)}{C_1} + 1\Big) h(|u(t,x)|) \left(\int_0^{F(R)} \eta^* \left(\frac{|x|^2 + t}{r} \right)^3 r^{-1}\,dr \right) d(t,x). \end{aligned}$$

We apply the change of variables $s = \frac{|x|^2+t}{r}$ to obtain

$$\int_0^{F(R)} \eta^* \left(\frac{|x|^2 + t}{r} \right)^3 r^{-1}\,dr = \int_{\frac{|x|^2+t}{F(R)}}^{\infty} \eta^*\,(s)^3\, s^{-1}\,ds \le \eta \left(\frac{|x|^2 + t}{F(R)} \right)^3 \int_{\frac{1}{2}}^{1} s^{-1}\,ds = \log(2) \eta \left(\frac{|x|^2 + t}{F(R)} \right)^3.$$

Summarizing it follows

$$Y(F(R)) \lesssim \log(2) I_{F(R)}.$$

We notice that

$$\frac{d}{dF(R)} Y(F(R)) F(R) = y(F(R)).$$

Finally, from (5.6) and (5.7) we get

$$Y(F(R)) \lesssim C^2 \log(2) F(R)^{\frac{1}{2}} h^{-1} \left(\frac{d_{F(R)} Y(F(R))}{C F(R)^{\frac{1}{2}}} \right).$$

The last estimate implies

$$h \left(\frac{Y(F(R))}{C^2 \log(2) F(R)^{\frac{1}{2}}} \right) \lesssim \frac{d_{F(R)} Y(F(R))}{C F(R)^{\frac{1}{2}}}.$$

Then for $R \ge R_0$ we have

$$\left(\frac{Y(F(R))}{C^2 \log(2) F(R)^{\frac{1}{2}}} \right)^3 \mu \left(\frac{Y(F(R_0))}{C^2 \log(2) F(R)^{\frac{1}{2}}} \right) \lesssim \frac{d_{F(R)} Y(F(R))}{C F(R)^{\frac{1}{2}}}.$$

Consequently, we may conclude

$$\frac{1}{(C^2 \log(2))^3 F(R)} \mu \left(\frac{Y(F(R_0))}{C^2 \log(2) F(R)^{\frac{1}{2}}} \right) \lesssim \frac{d_{F(R)} Y(F(R))}{C Y(F(R))^3}.$$

After integration from $F(R_0)$ to $F(R)$ it follows

$$\int_{F(R_0)}^{F(R)} \frac{1}{(C^2 \log(2))^3 x} \mu \left(\frac{Y(F(R_0))}{C^2 \log(2) x^{\frac{1}{2}}} \right) dx \lesssim \int_{F(R_0)}^{F(R)} \frac{d_x Y(x)}{C Y(x)^3}\,dx.$$

Hence, there exist constants c_1 and c_2 such that after a change of variables we obtain

$$\int_{F(R_0)}^{F(R)} \frac{1}{x}\mu\left(c_1\frac{1}{\sqrt{x}}\right)dx = c_2\int_{F(R)^{-\frac{1}{2}}}^{F(R_0)^{-\frac{1}{2}}} \frac{1}{s}\mu(s)\,ds \lesssim \left[-\frac{1}{Y(s)^2}\right]_{F(R_0)}^{F(R)} \lesssim \frac{1}{Y\left(F(R_0)^2\right)^2} < \infty$$

uniformly for all $R \geq R_0$. Letting $R \longrightarrow \infty$ and taking account $\lim_{R\to\infty} F(R) = \infty$ the last chain of inequality contradicts to the condition (5.3). This completes the proof. □

6. Concluding remarks

Remark 6.1. *The results of this paper explain the critical regularity (not the critical exponent) for Sobolev solutions to the Cauchy problem*

$$u_{tt} - u_{xx} + b(t)u_t = |u|^3\mu(|u|),\;\; u(0,x) = u_0(x),\;\; u_t(0,x) = u_1(x).$$

Remark 6.2. *A reasonable application of Theorem 5.1 requires a local (in time) existence result. In the following we restrict ourselves for $p > 1$ to the effectively damped Cauchy problem*

$$u_{tt} - u_{xx} + b(t)u_t = |u|^p,\;\; u(0,x) = u_0(x),\;\; u_t(0,x) = u_1(x), \tag{6.1}$$

where the data u_0 and u_1 is supposed to belong to $C_0^\infty(\mathbb{R})$. Then due to [4] for $p \in (1,3]$ there exists a local (in time) energy solution $u \in C([0,T), H^1(\mathbb{R})) \cap C^1([0,T), L^2(\mathbb{R}))$ of the Cauchy problem (6.1). *Then it is clear that one can expect such a local (in time) existence result for the Cauchy problem (1.3), too because the right-hand side is more regular due to the presence of a modulus of continuity term.*

Remark 6.3. *A blow-up result for local (in time) Sobolev solutions to the effectively damped Cauchy problem*

$$u_{tt} - u_{xx} + b(t)u_t = |u|^3\mu(|u|),\;\; u(0,x) = u_0(x),\;\; u_t(0,x) = u_1(x),$$

where $\lim_{t\to\infty} b(t) = 0$, remains as an open problem.

Acknowledgments

The authors thank both referees for their valuable proposals to improve the readability of the paper.

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Project No. GRANT2124 (The number of the old project that is transferred to the new project is No. GRANT1509)].

Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. M. D'Abbicco, The threshold of effective damping for semilinear wave equations, *Math. Methods Appl. Sci.*, **38** (2015), 1032–1045. https://doi.org/10.1002/mma.3126
2. M. D'Abbicco, S. Lucente, A modified test function method for damped wave equations, *Adv. Nonlinear Stud.*, **13** (2013), 867–892. https://doi.org/10.1515/ans-2013-0407
3. M. D'Abbicco, S. Lucente, M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, *J. Differ. Equ.*, **259** (2015), 5040–5073. https://doi.org/10.1016/j.jde.2015.06.018
4. M. D'Abbicco, S. Lucente, M. Reissig, Semi-linear wave equations with effective damping, *Chin. Ann. Math. Ser. B*, **34** (2013), 345–380. https://doi.org/10.1007/s11401-013-0773-0
5. M. R. Ebert, G. Girardi, M. Reissig, Critical regularity of nonlinearities in semilinear classical damped wave equations, *Math. Ann.*, **378** (2020), 1311–1326. https://doi.org/10.1007/s00208-019-01921-5
6. A. Friedman, *Partial differential equations*, Corrected reprint of the original edition, Robert E. Krieger Publishing Co., New York, 1976.
7. R. Ikehata, M. Ohta, Critical exponents for semilinear dissipative wave equations in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **269** (2002), 87–97. https://doi.org/10.1016/S0022-247X(02)00021-5
8. J. Lin, K. Nishihara, J. Zhai, Critical exponent for the semilinear wave equation with time-dependent damping, *Discrete Contin. Dyn. Syst.*, **32** (2012), 4307–4320. https://doi.org/10.3934/dcds.2012.32.4307
9. A. Mohammed Djaouti, *Semilinear systems of weakly coupled damped waves*, Ph.D. Thesis, TU Bergakademie Freiberg, Freiberg, Germany, 2018.
10. M. Nakao, K. Ono, Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, *Math. Z.*, **214** (1993), 325–342. https://doi.org/10.1007/BF02572407
11. W. Nunes do Nascimento, A. Palmieri, M. Reissig, *Semi-linear wave models with power non-linearity and scale invariant time-dependent mass and dissipation*, *Math. Nachr.*, **290** (2017), 1779–1805. https://doi.org/10.1002/mana.201600069
12. A. Palmieri, *Global in time existence and blow-up results for a semilinear wave equation with scale-invariant damping and mass*, Ph.D. Thesis, TU Bergakademie Freiberg, Freiberg, Germany, 2018.
13. A. Palmieri, M. Reissig, Semi-linear wave models with power non-linearity and scale invariant time-dependent mass and dissipation, II, *Math. Nachr.*, **291** (2018), 1859–1892. https://doi.org/10.1002/mana.201700144
14. G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, *J. Differ. Equ.*, **174** (2001), 464–489. https://doi.org/10.1006/jdeq.2000.3933
15. J. Wirth, *Asymptotic properties of solutions to wave equations with time-dependent dissipation*, PhD Thesis, TU Bergakademie Freiberg, 2004.

16. J. Wirth, Wave equations with time-dependent dissipation II, Effective dissipation, *J. Differ. Equ.*, **232** (2007), 74–103. https://doi.org/10.1016/j.jde.2006.06.004

Appendix

Here we state some results which come into play in our proofs.

The next Proposition can be found in [6], Part 1, Theorem 9.3.

Proposition A.1. Let $j, m \in \mathbb{N}$ with $j < m$, and let $u \in C_0^m(\mathbb{R}^n)$, i.e. $u \in C^m(\mathbb{R}^n)$ with compact support. Let $\theta \in [\frac{j}{m}, 1]$, and let p, q, r in $[1, \infty]$ be such that

$$j - \frac{n}{q} = \Big(m - \frac{n}{r}\Big)\theta - \frac{n}{p}(1-\theta).$$

Then

$$\|D^j u\|_{L^q} \leq C_{n,m,j,p,r,\theta} \|D^m u\|_{L^r}^{\theta}\ \|u\|_{L^p}^{1-\theta}$$

provided that

$$\Big(m - \frac{n}{r}\Big) - j \notin \mathbb{N}, \ \text{ that is, } \ \frac{n}{r} > m - j \ \text{ or } \ \frac{n}{r} \notin \mathbb{N}.$$

If

$$\Big(m - \frac{n}{r}\Big) - j \in \mathbb{N},$$

then Gagliardo-Nirenberg inequality holds provided that $\theta \in [\frac{j}{m}, 1)$.

Proposition A.2. The operator N maps $X(t)$ into itself and has one and only one fixed point $u \in X(t)$ if the following inequalities hold:

$$\begin{aligned} \|Nu\|_{X(t)} &\leq C_0(t)\|(u_0, u_1)\|_{\mathcal{A}_{m,s}} + C_1(t)\|u\|_{X(t)}^p, \\ \|Nu - Nv\|_{X(t)} &\leq C_2(t)\|u - v\|_{X(t)}\big(\|u\|_{X(t)}^{p-1} + \|v\|_{X(t)}^{p-1}\big), \end{aligned}$$

where $C_1(t), C_2(t) \longrightarrow 0$ for $t \longrightarrow +0$ and $C_0(t), C_1(t), C_2(t) \leq C$ for all $t \in [0, \infty)$. For the proof see for example [9].

Lemma A.3. Let $g = g(t) \in C([0, \infty))$ be a solution of the following initial value problem for an ordinary differential equation:

$$-g'(t) + g(t)b(t) = 1, \ \ g(0) = \frac{1}{b_0} = \int_0^\infty e^{-\int_0^t b(\tau)\,d\tau}\,dt.$$

If $b = b(t)$ satisfies the assumptions of the effective case , then it holds $g(t) \approx \frac{1}{b(t)}$ and

$$|g'(t) - 1| \leq C_0 = C_0(b).$$

The proof of Lemma A.3 can be found in [2, 8].

In the following lemma we present the classic Jensen's inequality with respect to the weighted Lebesgue measure $\alpha(x)dx$.

Lemma A.4. Let Φ be a convex function on $\mathbb{R}$. Let $\alpha := \alpha(x)$ defined and non-negative almost everywhere on Ω, such that α is positive on a set of positive measure. Then, it holds

$$\Phi\left(\frac{\int_\Omega \alpha(x)u(x)dx}{\int_\Omega \alpha(x)dx}\right) \leq \frac{\int_\Omega \alpha(x)\Phi(u(x))dx}{\int_\Omega \alpha(x)dx}$$

provided that all integrals are meaningful and u is non-negative.