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Research article

Fixed point theorems in controlled $J-$metric spaces

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Abstract: In this article, we introduce a new extension to $J-$metric spaces, called C_J-metric spaces, where θ is the controlled function in the triangle inequality. We prove some fixed point results in this new type of metric space. In addition, we present some applications to systems of linear equations to illustrate our results.

Keywords: fixed point; C_J-metric space; $J-$metric spaces
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1. Introduction

The fixed point theory is a new, essential theory, and its application is utilized in many fields, including Mathematics, Economics, and many others. For example, the impact of the fixed point theory in the fractional differential equations appear clearly to all the observers, see [3, 4]. The fixed point theory and the proof of the uniqueness were introduced by Banach [2], which was encouraging to all subsequent researchers to start working on this theory; see [5, 8]. These days, the fixed point is an active area wildly generalizing Banach, see [6, 7, 9–18].

Generalization of the fixed point theory can be made in two ways, either a generalization of the Banach contraction to another linear or nonlinear contraction. The other way of extension is to generalize the metric spaces by either changing the triangle inequality, omitting the symmetry condition, or assuming that the self-distance is not necessarily zero.

Those generalizations are important due to the fact that more general spaces or contractions impact a greater number of applications that can be adapted to that results.

In this work, we generalize a $J-$metric spaces, which Souayah recently introduced [1], where it defined a metric space in three dimensions with a triangle inequality that includes a constant $b > 0$. We extend the notion of $J-$metric spaces to C_J metric spaces that include a control function θ in three dimensions instead of the constant b.

In the main result section, we prove the existence and the uniqueness of a fixed point for self-mappings on C_J-metric spaces, in Theorems 3.10 and 3.11, we consider self-mappings that satisfy linear contractions where in Theorem 3.12, we consider mappings that satisfy nonlinear contractions. Our finding generalizes many results in the literature. Moreover, in the last section, we present an application of our impact on the system of linear equations.

2. Preliminaries

We begin our preliminaries by recalling the definitions of $J-$metric spaces.

Definition 2.1. [1] Consider a nonempty set δ, and a function $J : \delta^3 \to [0, \infty)$. Let us define the set,

$$S(J, \delta, \phi) = \{\{\phi_n\} \subset \delta : \lim_{n\to\infty} J(\phi, \phi, \phi_n) = 0\}$$

for all $\phi \in \delta$.

Definition 2.2. [1] Let δ be a set with at least one element and, $J : \delta^3 \to [0, \infty)$ that satisfies the mentioned below conditions:

(i) $J(\alpha, \beta, \gamma) = 0$ implies $\alpha = \beta = \gamma$ for any $\alpha, \beta, \gamma \in \delta$.
(ii) There are some $b > 0$, where for each $(\alpha, \beta, \gamma) \in \delta^3$ and $\{\nu_n\} \in S(J, \delta, \nu)$

$$J(\alpha, \beta, \gamma) \leq b \limsup_{n\to\infty} \Big(J(\alpha, \alpha, \nu_n) + J(\beta, \beta, \nu_n) + J(\gamma, \gamma, \nu_n)\Big).$$

Then, (δ, J) is defined as a $J-$metric space. In addition, if $J(\alpha, \alpha, \beta) = J(\beta, \beta, \alpha)$ for each $\alpha, \beta \in \delta$, the pair (δ, J) is defined as a symmetric $J-$metric space.

3. Main result

In this part, we will define C_J-metric spaces and prove the existence and the uniqueness of the fixed point of self-mapping.

Definition 3.1. Let δ is a non empty set and a function $C_J : \delta^3 \to [0, \infty)$. Then the set is defined as follows

$$S(C_J, \delta, \alpha) = \{\{\alpha_n\} \subset \delta : \lim_{n\to\infty} C_J(\alpha, \alpha, \alpha_n) = 0\}$$

for each $\alpha \in \delta$

Definition 3.2. Let δ be a set with at least one element and $C_J : \delta^3 \to [0, \infty)$fulfill the following conditions:

(i) $C_J(\alpha, \beta, \gamma) = 0$ implies $\alpha = \beta = \gamma$ for all $\alpha, \beta, \gamma \in \delta$.
(ii) There exist a function $\theta : \delta^3 \to [0, \infty)$, where θ is a continuous function and

$$lim_{n\to\infty}\theta(\alpha, \alpha, \alpha_n)\}$$

is a finite and exist where,

$$C_J(\alpha, \beta, \gamma) \leq \theta(\alpha, \beta, \gamma) \limsup_{n\to\infty} \Big(C_J(\alpha, \alpha, \phi_n) + C_J(\beta, \beta, \phi_n) + C_J(\gamma, \gamma, \phi_n)\Big).$$

Then (δ, C_J) is defined as C_J-metric space. In addition, if

$$C_J(\alpha,\alpha,\beta) = C_J(\beta,\beta,\alpha)$$

for each $\alpha,\beta \in \delta$, then (δ, C_J) is defined as symmetric C_J-metric space.

Remark 3.3. Notice that, this symmetry hypothesis does not necessarily mean that

$$C_J(\alpha,\beta,\gamma) = C_J(\beta,\alpha,\gamma) = C_J(\gamma,\beta,\alpha) = \cdots .$$

We will start by presenting some properties in the topology of C_J-metric spaces.

Definition 3.4. (1) Let (δ, C_J) is a C_J-metric space. A sequence $\{\alpha_n\} \subset \delta$ is convergent to an element $\alpha \in \delta$ if $\lim_{n\to\infty} \alpha_n = \alpha$,for $\{\alpha_n\} \in S(C_J, \delta, \alpha)$.

(2) Let (δ, C_J) is a C_J-metric space.A sequence $\{\alpha_n\} \subset \delta$ is called Cauchy iff

$$\lim_{n,m\to\infty} C_J(\alpha_n, \alpha_n, \alpha_m) = 0.$$

(3) A C_J-metric space is called complete if each Cauchy sequence in δ is convergent.

(4) In a C_J-metric space (α, C_J), if ψ is a continuous map at $a_0 \in \delta$ then for each $\alpha_n \in S(C_J, \alpha, a_0)$ gives $\{\psi\alpha_n\} \in S(C_J, \alpha, \psi a_0)$.

Proposition 3.5. *In a C_J-metric space (δ, C_J), if $\{\alpha_n\}$ converges, then it is convergent to one exact element in δ.*

Proof. Let us start with $\{\alpha_n\}$ converges to α_1 and α_2. so by using the triangle inequality condition that is mentioned in the definition of the C_J metric space,

$$\begin{aligned} C_J(\alpha_1,\alpha_1,\alpha_2) &\le \theta(\alpha_1,\alpha_1,\alpha_2) \limsup_{n\to\infty} \Big(C_J(\alpha_1,\alpha_1,\alpha_n) + C_J(\alpha_1,\alpha_1,\alpha_n) + C_J(\alpha_2,\alpha_2,\alpha_n)\Big) \\ &= \theta(\alpha_1,\alpha_,\alpha_2) \limsup_{n\to\infty}(2C_J(\alpha_1,\alpha_1,\alpha_n) + C_J(\alpha_2,\alpha_2,\alpha_n)) = 0. \end{aligned}$$

Thus,

$$C_J(\alpha_1,\alpha_1,\alpha_2) = 0 \Rightarrow \alpha_1 = \alpha_2.$$

□

Definition 3.6. Let (δ, C_{J_1}) and (Γ, C_{J_1}) are two C_J-metric spaces and $\psi : \delta \to \Gamma$ is a map . Then ψ is said to be a continuous at $a_0 \in \delta$ if, for each $\varepsilon > 0$, there is $\xi > 0$ where, for each $\alpha \in \delta$, $C_{J_2}(\psi a_0, \psi a_0, \psi\alpha) < \varepsilon$ whenever $C_{J_1}(a_0, a_0, \alpha) < \xi$.

Example 3.7. Let $\delta = \mathbb{R}$ and, $C_J : \delta^3 \to [0,\infty)$ defined by

$$C_J(\alpha,\beta,\gamma) = |\alpha-\beta| + |\beta-\gamma|,$$

for all $\alpha,\beta,\gamma \in \mathbb{R}$.

Let $\alpha \in \mathbb{R}$, and the sequence α_n such that, $\alpha_n = \alpha + \frac{1}{n}$. It is not hard to observe that $\lim_{n\to\infty} C_J(\alpha, \alpha + \frac{1}{n}, \alpha + \frac{1}{n}) = 0$. For that reason and for each $\alpha \in \mathbb{R}$ there is a sequence $\alpha_n = \alpha + \frac{1}{n}$ where $S(C_J, \delta, \alpha) \neq \emptyset$.

Now, we present an example of C_J-metric space.

Example 3.8. Let $\delta = \mathbb{R}$, and $C_J(\alpha,\beta,\gamma) = |\alpha| + |\beta| + 2|\gamma|$, for all $\alpha,\beta,\gamma \in \delta$. And let $\theta : \delta^3 \to [0,\infty)$, where $\theta(\alpha,\beta,\gamma) = max(2, |\alpha|, |\beta|, |\gamma|)$. We have $C_J(\alpha,\beta,\gamma) = 0 \Rightarrow |\alpha| + |\beta| + 2|\gamma| = 0$, which gives $|\alpha| = |\beta| = |\gamma| = 0$.

Then, the hypotheses (2.2) are fulfilled, and the symmetry of C_J is satisfied too.

$$C_J(\alpha,\alpha,\gamma) = 2|\alpha| + 2|\gamma| = C_J(\gamma,\gamma,\alpha).$$

In the end, let's check the triangle inequality.

Let $\alpha,\beta,\gamma \in \delta$ and ϕ_n a convergent sequence in δ such that, $\lim_{n\to\infty} C_J(\phi,\phi,\phi_n) = 0$, we have

$$\begin{aligned}
C_J(\alpha,\beta,\gamma) &= |\alpha| + |\beta| + 2|\gamma| \\
&\leq 4|\alpha| + 4|\beta| + 4|\gamma| + 12|\phi_n| \\
&= 2(2|\alpha| + 2|\phi_n| + 2|\beta| + 2|\phi_n| + 2|\gamma| + 2|\phi_n|) \\
&= 2(C_J(\alpha,\beta,\phi_n) + C_J(\beta,\beta,\phi_n) + J(\gamma,\gamma,\phi_n)) \\
&\leq max(2, |\alpha|, |\beta|, |\gamma)\limsup_{n\to\infty}(C_J(\alpha,\beta,\phi_n) + C_J(\alpha,\beta,\phi_n) + C_J(\gamma,\gamma,\phi_n)) \\
&= \theta(\alpha,\beta,\gamma)\limsup_{n\to\infty}(C_J(\alpha,\beta,\phi_n) + C_J(\alpha,\beta,\phi_n) + C_J(\gamma,\gamma,\phi_n)).
\end{aligned}$$

So, this is an example of C_J metric space.

Next, we present an example of C_J-metric space that is not a $J-$metric spaces.

Example 3.9. Choose $\delta = \{1, 2, \cdots\}$. Take $C_J : \delta^3 \to [0,\infty)$ such that

$$C_J(\alpha,\beta,\gamma) = \begin{cases} 0, & \iff \alpha = \beta = \gamma \\ \frac{1}{\alpha+\beta}, & \text{if } \alpha,\beta \text{ are even and } \gamma = 2n+1 \\ \frac{1}{\gamma}, & \text{if } \alpha,\beta \text{ are odd and } \gamma = 2n \\ 1, & \text{otherwise.} \end{cases}$$

Consider $\theta : \delta^3 \to (0,\infty)$ as

$$\theta(\alpha,\beta,\gamma) = \begin{cases} \alpha+\beta, & \text{if } \alpha,\beta \text{ are even and } \gamma = 2n+1 \\ \gamma, & \text{if } \alpha,\beta \text{ are odd and } \gamma = 2n \\ 1, & \text{otherwise.} \end{cases}$$

Note that, it is not difficult to see that (δ, C_J) is a C_J-metric space.

However, (δ, C_J) is not a $J-$metric space.

Theorem 3.10. *Let (δ, C_J) is a C_J-complete symmetric metric space, and $g : \delta \to \delta$ is a continuous map satisfies*

$$C_J(g\alpha, g\beta, g\gamma) \leq P(C_J(\alpha,\beta,\gamma)) \quad \textit{for all } \alpha,\beta,\gamma \in \delta. \tag{3.1}$$

Where, $P : [0,+\infty) \to [0,+\infty)$ is a function and for all $t \in [0,+\infty)$,

$$t \succ x, P(t) \succ P(x).$$

And,

$$\lim_{n\to\infty} P^n(t) = 0 \ \textit{for each fixed} \ \ t > 0. \tag{3.2}$$

Then, g has a unique fixed point in δ.

Proof. Let's start with α_0 is an element in δ,and $\{\alpha_n\}_{n\geq 0} \subset \delta$ where,

$$\alpha_1 = g\alpha_0, \alpha_2 = g\alpha_1 ... \alpha_n = g^n\alpha_0, \quad n = 1, 2, \cdots . \tag{3.3}$$

First, we have to start proving $\{\alpha_n\}$ is a Cauchy in δ. Let $n, m \in \mathbb{N}$.

$$\begin{aligned} C_J(\alpha_n, \alpha_n, \alpha_m) = C_J(g\alpha_{n-1}, g\alpha_{n-1}, g\alpha_{m-1}) &\leq P(C_J(\alpha_{n-1}, \alpha_{n-1}, \alpha_{m-1})) \\ &= P(C_J(g\alpha_{n-2}, g\alpha_{n-2}, g\alpha_{m-2})) \\ &\leq \vdots \\ &\leq P^n(CJ(\alpha_0, \alpha_0, \alpha_{m-n})). \end{aligned}$$

After applying (3.1) n times and with assuming that $m = n + q$ for some constant $q \in \mathbb{N}$ to get

$$C_J(\alpha_n, \alpha_n, \alpha_m) \leq P^n(C_J(\alpha_0, \alpha_0, \alpha_q)). \tag{3.4}$$

By applying the limit in (3.4) as $n \longrightarrow \infty$ we get

$$\lim_{n\to\infty} C_J(\alpha_n, \alpha_n, \alpha_m) = 0. \tag{3.5}$$

Accordingly, $\{\alpha_n\}$ is a Cauchy sequence in δ and because of the completeness, there is $\alpha \in \delta$ such that $\alpha_k \to \alpha$ as $k \to \infty$.

Moreover, $\alpha = \lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \alpha_{k+1} = \lim_{k\to\infty} g\alpha_k = g\alpha$. Thus, g has α as a fixed point.

Assume that α_1 and α_2 are two fixed points of g.

$$\begin{aligned} C_J(\alpha_1, \alpha_1, \alpha_2) = C_J(g\alpha_1, g\alpha_1, g\alpha_2) &\leq P(C_J(\alpha_1, \alpha_1, \alpha_2)) \\ &\leq P^2(C_J(\alpha_1, \alpha_1, \alpha_2)) \\ &\vdots \\ &\leq P^n(CJ(\alpha_1, \alpha_1, \alpha_2)). \end{aligned}$$

By taking the limit for the above inequalities as $n \longrightarrow \infty$ we get $C_J(\alpha_1, \alpha_1, \alpha_2) = 0$ and $\alpha_1 = \alpha_2$.

Thus, g has a unique fixed point in δ as desired. □

Theorem 3.11. *Let (δ, C_J) is a C_J−complete symmetric metric space and $g : \delta \to \delta$ be a mapping that satisfies,*

$$C_J(g\alpha, g\beta, g\gamma) \leq \phi(\alpha,\beta,\gamma)C_J(\alpha,\beta,\gamma), \quad \forall\alpha,\beta,\gamma \in \delta, \tag{3.6}$$

where $\phi \in A$, and $\phi : \delta^3 \to (0, 1)$, such that

$$\phi(g(\alpha,\beta,\gamma)) \leq \phi(\alpha,\beta,\gamma) \ \textit{ and } \ \{g : \delta \to \delta\}$$

g is a given mapping. Then g has a unique fixed point in δ.

Proof. Let α_0 be an arbitrary element in δ. We establish the sequence $\{\alpha_n\}$as follows $\{\alpha_n = g^n\alpha_0\}$.

Let's start by proving that $\{\alpha_n\}$ is a Cauchy sequence. For all natural numbers n, m, we assume that that $n < m$ and assume that there is $q \in N$ where $m = n + q$. By applying (3.6) we get :

$$\begin{aligned} C_J(\alpha_n, \alpha_n, \alpha_m) =& C_J(g\alpha_{n-1}, g\alpha_{n-1}, g\alpha_{m-1}) \\ \leq& \phi(\alpha_{n-1}, \alpha_{n-1}, \alpha_{m-1}) C_J(\alpha_{n-1}, \alpha_{n-1}, \alpha_{m-1}) \\ &\vdots \\ \leq& \phi^n(\alpha_0, \alpha_0, \alpha_q) C_J(\alpha_0, \alpha_0, \alpha_q). \end{aligned}$$

By applying the limit as $n \to \infty$, and taking ϕ into consideration, we get $\lim_{n,m\to\infty} C_J(\alpha_n, \alpha_n, \alpha_m) = 0$, so$\{\alpha_n\}$ is a Cauchy sequence.

Then, by the completeness definition of δ, there is a $\alpha \in \delta$ such that

$$\alpha = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_{n-1}. \tag{3.7}$$

We will show that α is a fixed point of g. From (3.7), we conclude that $\alpha_n \in S(C_J, \delta, \alpha)$ and

$$\lim_{n\to\infty} C_J(\alpha, \alpha, \alpha_n) = 0 \tag{3.8}$$

and

$$\lim_{n\to\infty} C_J(\alpha, \alpha, \alpha_{n-1}) = 0. \tag{3.9}$$

By using the triangle inequality we get

$$\begin{aligned} C_J(g\alpha, g\alpha, \alpha) \leq& \theta(g\alpha, g\alpha, \alpha) \lim_{n\to\infty} \sup \left[2C_J(g\alpha, g\alpha, \alpha_n) + C_J(\alpha, \alpha, \alpha_n)\right] \\ =& 2\theta(g\alpha, g\alpha, \alpha) \lim_{n\to\infty} C_J(g\alpha, g\alpha, g\alpha_{n-1}) \\ \leq& 2\theta(g\alpha, g\alpha, \alpha) \lim_{n\to\infty} \phi(\alpha, \alpha, \alpha_n - 1) C_J(\alpha, \alpha, \alpha_{n-1}). \end{aligned} \tag{3.10}$$

By applying (3.9) in (3.10) we obtain that $C_J(g\alpha, g\alpha, \alpha) = 0$, that is $g\alpha = \alpha$. Therefore α is a fixed point of g.

To prove the uniqueness let, $\beta_1, \beta_2 \in \delta$ are two fixed points of g such that $\beta_1 \neq \beta_2$, $g\beta_1 = \beta_1$ and $g\beta_2 = \beta_2$.

$$\begin{aligned} C_J(\beta_1, \beta_1, \beta_2) =& C_J(G\beta_1, g\beta_1, g\beta_2) \\ \leq& \phi(\beta_1, \beta_1, \beta_2) C_J(\beta_1, \beta_1, \beta_2) \\ <& C_J(\beta_1, \beta_1, \beta_2). \end{aligned}$$

Where $\phi(\beta_1, \beta_1, \beta_2) < 1$, then $C_J(\beta_1, \beta_1, \beta_2) = 0$ which implies that $\beta_1 = \beta_2$. □

Theorem 3.12. *Let (δ, C_J) is a complete symmetric C_J-metric spaces, $g : \delta \to \delta$ is a continuous map where*

$$C_J(g\alpha, g\beta, g\gamma) \leq aC_J(\alpha, \beta, \gamma) + bC_J(\alpha, g\alpha, g\alpha) + cC_J(\beta, g\beta, g\beta) + dC_J(\gamma, g\gamma, g\gamma) \tag{3.11}$$

for each $\alpha, \beta, \gamma \in \delta$ *where*

$$0 < a + b < 1 - c - d, \tag{3.12}$$

$$0 < a < 1. \tag{3.13}$$

Then, there is a unique fixed point of g.

Proof. Let $\alpha_0 \in \delta$ be an arbitrary point of δ and $\{\alpha_n = g^n\alpha_0\}$ be a sequence in δ to get.

$$\begin{aligned} C_J(\alpha_n, \alpha_{n+1}, \alpha_{n+1}) = C_J(g\alpha_{n-1}, g\alpha_n, g\alpha_n) \\ \leq & aC_J(\alpha_{n-1}, \alpha_n, \alpha_n) + bC_J(\alpha_{n-1}, \alpha_n, \alpha_n) \\ & + cC_J(\alpha_n, \alpha_{n+1}, \alpha_{n+1}) + dC_J(\alpha_n, \alpha_{n+1}, \alpha_{n+1}) \\ \leq & (a+b)C_J(\alpha_{n-1}, \alpha_n, \alpha_n) + (c+d)C_J(\alpha_n, \alpha_{n+1}, \alpha_{n+1}). \end{aligned}$$

Then

$$C_J(\alpha_n, \alpha_{n+1}, \alpha_{n+1}) \leq \frac{a+b}{1-c-d} C_J(\alpha_{n-1}, \alpha_n, \alpha_n).$$

By taking $k = \dfrac{a+b}{1-c-d}$, then by using (3.12) we will have $0 < k < 1$.

$$C_J(\alpha_n, \alpha_{n+1}, \alpha_{n+1}) \leq k^n C_J(\alpha_0, \alpha_1, \alpha_1).$$

Which gives

$$\lim_{n\to\infty} C_J(\alpha_n, \alpha_{n+1}, \alpha_{n+1}) = 0. \tag{3.14}$$

We denote $C_{J_n} = C_J(\alpha_n, \alpha_{n+1}, \alpha_{n+1})$. For each $n, m \in N, n < m$,and suppose that there is a fixed $q \in N$ such that $m = n + q$. we have

$$\begin{aligned} C_J(\alpha_n, \alpha_n, \alpha_m) &= C_J(\alpha_n, \alpha_n, \alpha_{n+q}) = C_J(g\alpha_{n-1}, g\alpha_{n-1}, g\alpha_{n+q-1}) \\ &\leq aC_J(\alpha_{n-1}, \alpha_{n-1}, \alpha_{n+p-1}) + bC_J(\alpha_{n-1}, \alpha_n, \alpha_n) + cC_J(\alpha_{n-1}, \alpha_n, \alpha_n) \\ &+ dC_J(\alpha_{n+q-1}, \alpha_{n+q}, \alpha_{n+q}) \\ &= aC_J(\alpha_{n-1}, \alpha_{n-1}, \alpha_{n+q-1}) + (b+c)C_{J_{n-1}} + dC_{J_{n+q-1}} \\ &\leq a[aC_J(\alpha_{n-2}, \alpha_{n-2}, \alpha_{n+p-2}) + (c+d)C_{J_{n-2}} + \alpha J_{n+q-2}] + (b+c)C_{J_{n-1}} \\ &+ dC_{J_{n+q-1}} \\ &= a^2 C_J(\alpha_{n-2}, \alpha_{n-2}, \alpha_{n+q-2}) + a(c+d)C_{J_{n-2}} + a\alpha C_{J_{n+q-2}} + (b+c)C_{J_{n-1}} \\ &+ dC_{J_{n+q-1}} \\ &\vdots \\ &\leq a^n C_J(\alpha_0, \alpha_0, \alpha_q) + (b+c)\sum_{k=1}^{n} a^{k-1} C_{J_{(n-k)}} + d\sum_{k=1}^{n} a^{k-1} C_{J_{(n+q-k)}}. \end{aligned} \tag{3.15}$$

By applying the limit in (3.15) as $n \to \infty$ and using (3.13) and (3.14), we get

$$\lim_{n,m\to\infty} C_J(\alpha_n, \alpha_n, \alpha_m) = 0.$$

Then, $\{\alpha_n\}$ is a Cauchy sequence in δ. By the completeness definition, there is $\alpha \in \delta$ where $\alpha_n \to \alpha$ as $n \to \infty$ and

$$\lim_{n\to\infty} C_J(\alpha_n, \alpha_n, \alpha) = \lim_{n,m\to\infty} C_J(\alpha_n, \alpha_m, \alpha) = 0. \tag{3.16}$$

In addition, $u = \lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \alpha_{k+1} = \lim_{k\to\infty} g\alpha_k = g\alpha$. Therefore, g has u as a fixed point.

Let $\gamma_1, \gamma_2 \in \delta$ are two fixed point of g, $\gamma_1 \neq \gamma_2$,where, $g\gamma_1 = \gamma_1$, $g\gamma_2 = \gamma_2$.

$$\begin{aligned} C_J(\gamma_1, \gamma_1, \gamma_2) =& C_J(g\gamma_1, g\gamma_1, g\gamma_2) \\ \leq& aC_J(\gamma_1, \gamma_1, \gamma_2) + (b + c)C_J(\gamma_1, g\gamma_1, g\gamma_1) + dC_J(\gamma_2, g\gamma_2, g\gamma_2) \\ =& aC_J(\gamma_1, \gamma_1, \gamma_2) + (b + c)C_J(\gamma_1, \gamma_1, \gamma_1) + dC_J(\gamma_2, \gamma_2, \gamma_2). \end{aligned}$$

Then, $(1 - a)C_J(\gamma_1, \gamma_1, \gamma_2) \leq 0$. Using (3.13) so this gives $C_J(\gamma_1, \gamma_1, \gamma_2) = 0$ that is $\gamma_1 = \gamma_2$, which means that g has a unique a fixed point. □

4. Applications

Throughout this section, we represent an example of (3.10), where we have a linear system of equations in R, θ is a continuous function, and we prove that this system has a unique solution by applying the fixed point theory.

Let $\delta = \mathbb{R}^n$, and let the symmetric C_J-metric space (δ, C_J) introduced by

$$C_J(d, \xi, \nu) = \max_{1\leq i\leq n} |d_i - \xi_i| + |d_i - \nu_i|,$$

for all $d = (d_1, ..., d_n), \xi = (\xi_1, ..., \xi_n), \nu = (\nu_1, ..., \nu_n) \in \delta$. such that

$$\theta(d, \xi, \nu) = \max_{1\leq i\leq n}(2, |d_i|, |\xi_i|, |\nu_i|).$$

Theorem 4.1. *Consider the following system*

$$\begin{cases} \varpi_{11}d_1 + \varpi_{12}d_2 + \varpi_{13}d_3 + \varpi_{1n}d_n = r_1 \\ \varpi_{21}d_1 + \varpi_{22}d_2 + \varpi_{23}d_3 + \varpi_{2n}d_n = r_2 \\ \vdots \\ \varpi_{n1}d_1 + \varpi_{n2}d_2 + \varpi_{n3}d_3 + \varpi_{nn}d_n = r_n \end{cases},$$

if

$$\Upsilon = \max_{1\leq i\leq n}\Big(\sum_{j=1, j\neq i}^{n} |\varpi_{ij}| + |1 + \varpi_{ii}|\Big) < 1,$$

then the raised system of linear equations has a solution that is unique.

Proof. Let the map $\sigma : \delta \to \delta$ introduced as $\sigma d = (B + I_n)d - r$ where

$$B = \begin{pmatrix} \varpi_{11} & \varpi_{12} & \cdots & \varpi_{1n} \\ \varpi_{21} & \varpi_{22} & \cdots & \varpi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varpi_{n1} & \varpi_{n2} & \cdots & \varpi_{nn} \end{pmatrix},$$

$d = (d_1, d_2, \cdots, d_n); \xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and $\nu = (\nu_1, \nu_2, \cdots, \nu_n) \in \mathbb{R}^n$, I_n is the identity matrix for $n \times n$ matrices and $r = (r_1, r_2, \cdots, r_n) \in \mathbb{C}^n$. Let us show that $C_J(\sigma d, \sigma\xi, \sigma\nu) \leq \Upsilon C_J(d, \xi, \nu)$, $\forall d, \xi, \nu \in \mathbb{R}^n$.

We define

$$\tilde{B} = B + I_n = (\tilde{b}_{ij}), \quad i, j = 1, ..., n,$$

with $\tilde{b}_{ij} = \begin{cases} \varpi_{ij}, j \neq i \\ 1 + \varpi_{ii}, j = i \end{cases}$. Hence,

$$\max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{b}_{ij}| = \max_{1\leq i\leq n} \Big(\sum_{j=1, j\neq i}^{n} |\varpi_{ij}| + |1 + \varpi_{ii}| \Big) = \Upsilon < 1.$$

For all $i = 1, \ldots, n$, we have

$$(\sigma d)_i - (\sigma\xi)_i = \sum_{j=1}^{n} \tilde{b}_{ij}(d_j - \xi_j), \tag{4.1}$$

$$(\sigma d)_i - (\sigma\nu)_i = \sum_{j=1}^{n} \tilde{b}_{ij}(d_j - \nu_j). \tag{4.2}$$

Accordingly, using (4.1) and (4.2) we get

$$\begin{aligned} C_J(\sigma d, \sigma\xi, \sigma\nu) &= \max_{1\leq i\leq n} (|(\sigma d)_i - (\sigma\xi)_i| + |(\sigma d)_i - (\sigma\nu)_i|) \\ &\leq \max_{1\leq i\leq n} (\sum_{j=1}^{n} |\tilde{b}_{ij}||d_j - \xi_j| + \sum_{j=1}^{n} |\tilde{b}_{ij}||d_j - \nu_j|) \\ &\leq \max_{1\leq i\leq n} \sum_{j=1}^{n} |\tilde{b}_{ij}| \max_{1\leq k\leq n} (|d_k - \xi_k| + |d_k - \nu_k|) \\ &= \Upsilon C_J(d, \xi, \nu) = \Phi(C_J(d, \xi, \nu)), \end{aligned}$$

where, $\Phi(t) = \Upsilon t$, $\forall t \geq 0$. Notice that, all the conditions of Theorem 3.10 are fulfilled. Consequently, σ has a unique fixed point. Accordingly, the raised linear system has a unique solution.

□

5. Conclusions

We have introduced a new metric-type spaces, where we have proved fixed point theorems for self-mapping on such spaces. Our results generalize many well-known theorems in the field of fixed point theory. Also, we presented an application of our results to systems of linear equations. As a future work, our results can be used in fractional differential equations see [19–21], which include the most recent fractional definitions "Abu-Shady-Kaabar" fractional derivative. In closing, we would like to bring to the reader's attention the following open questions. Note that in Theorems 3.10 and 3.11, we assumed that the map is continuous. Can we replace the hypothesis of continuity with a weaker condition? How could C_J−metric space help in the "Abu-Shady-Kaabar" operators?

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Conflict of interest

The authors declare that they have no competing interests.

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