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Blow-up criteria for different fluid models in anisotropic Lorentz spaces

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Abstract: This paper establishes new blow-up criteria, in anisotropic Lorentz spaces, via one-directional derivatives of the velocity and magnetic fields for the Cauchy problem to the 3D magneto-micropolar model and via one-directional derivative of velocity for the Cauchy problem to the 3D nonlinear dissipative system.

Keywords: blow-up criterion; weak solutions; Navier–Stokes-Poisson–Nernst–Planck system; 3D magneto-micropolar model; anisotropic Lorentz spaces
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1. Introduction

In this study, we analyze the blow-up criteria for weak solutions in finite time for various fluid models. The first model consists of five equations governing the unsteady, viscous, incompressible magneto-micropolar flow:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U}\cdot\nabla\mathcal{U} - \Delta\mathcal{U} + \nabla(\Psi + \mathcal{V}^2) - \nabla\times\mathcal{W} - \mathcal{V}\cdot\nabla\mathcal{V} = 0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \frac{\partial \mathcal{W}}{\partial t} - \Delta\mathcal{W} + \mathcal{U}\cdot\nabla\mathcal{W} - \nabla\times\mathcal{U} + 2\mathcal{W} - \nabla div\,\mathcal{W} = 0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \frac{\partial \mathcal{V}}{\partial t} - \Delta\mathcal{V} + \mathcal{U}\cdot\nabla\mathcal{V} - \mathcal{V}\cdot\nabla\mathcal{U} = 0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \nabla\cdot\mathcal{U} = 0, \;\; \nabla\cdot\mathcal{V} = 0, \quad \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ (\mathcal{U},\mathcal{W},\mathcal{V})|_{t=0} = (\mathcal{U}_0,\mathcal{W}_0,\mathcal{V}_0), \quad \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

In the system (1.1), $\mathcal{U}(x,t)$ and $\mathcal{V}(x,t)$ are the velocity and magnetic fields. The micro-rotational velocity and hydrostatic pressure are given the notations $\mathcal{W}(x,t)$, $\Psi(x,t)$, while $\mathcal{U}_0$, $\mathcal{V}_0$ and $\mathcal{W}_0$ are the given initial velocity, magnetic field and micro-rotation velocity with $\nabla\cdot\mathcal{U}_0 = 0$ and $\nabla\cdot\mathcal{V}_0 = 0$ in the distributional sense. Galdi and Rionero [1] were the first who suggested the model (1.1).

Rojas-Medar and Boldrini [2] established the existence of global weak solutions to the system (1.1). Later on, the authors in [3] and [4], respectively, considered the problem of the existence of local and global strong solutions to the same system for small initial data. However, concerning the weak solutions to the system (1.1), there arises a question of the regularity of these solutions. In this regard, several publications discussing the regularity of weak solutions of system (1.1) have appeared in the literature, see for instance [5–12] and references therein. In this article, we also choose to discuss the blow-up criteria for the system (1.1) that guarantees the regularity of local smooth solutions for all time $[0, \infty)$. In view of the physical importance of system (1.1), it models the flow of microelements under the influence of a magnetic field. These micropolar fluids have a diluted suspension of tiny, stiff, cylindrical macromolecules that move independently and are affected by spin inertia. Such types of flows are significant in analysing animal and human blood, polymer fluids, liquid crystals, etc. Recently, enormous studies have been conducted on studying such fluids on different surfaces, including bounded and unbounded domains.

The second system we consider here for analysis is the Navier–Stokes-Poisson–Nernst–Planck system:

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{U} \cdot \nabla \mathcal{U} - \Delta \mathcal{U} + \nabla \Psi - \Delta \psi \nabla \psi = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ \nabla \cdot \mathcal{U} = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ \frac{\partial \theta}{\partial t} + \mathcal{U} \cdot \nabla \theta - \nabla \cdot (\nabla \theta + \theta \nabla \psi) = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ \frac{\partial \vartheta}{\partial t} + \mathcal{U} \cdot \nabla \vartheta - \nabla \cdot (\nabla \vartheta - \vartheta \nabla \psi) = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ \Delta \psi = \theta - \vartheta, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ (\mathcal{U}, \theta, \vartheta)|_{t=0} = (\mathcal{U}_0, \theta_0, \vartheta_0), \quad \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

In the system (1.2), $\mathcal{U}(x, t)$ and $\Psi(x, t)$ are the velocity and pressure, $\vartheta(x, t)$ and $\theta(x, t)$ are the densities of binary diffusive negative and positive charges, ψ is the electric potential, respectively. Rubinstein [13] proposed system (1.2), which can describe the drift, diffusion, and convection process for the charged ions in incompressible viscous fluids (see [14–17], and the references cited therein). The well-posedness problem of the system (1.2) has been tackled by Jerome [18] based on Kato's semigroup framework. The global existence of strong solutions for small initial data and the local existence of strong solutions for arbitary initial data has been established by Zhao et al. [19–21] in various function spaces. However, for arbitary initial data, the all time existence of local smooth solutions is one of the key open problem that we will investigate and present new blow-up conditions in anisotropic Lorentz space. Similar to system (1.1) the electro diffusion model covers various fluid models and could be considered as general formulation to Navier-Stokes, Micropolar, MHD, and Boussinesq systems. The momentum and mass conservation equations for the flow are $(1.2)_1$ and $(1.2)_2$, respectively, while the balance between diffusion and convective transport of charges by the flow and electric fields is modelled by $(1.2)_3$ and $(1.2)_4$, respectively, and the Poisson equation for the electrostatic potential is $(1.2)_5$. Keep in mind that the Lorentz force produced by the charges is represented in $(1.2)_1$. To learn more about the physical backdrop of this issue, we direct the reader to [22–25] and the references therein.

The regularity of weak solutions plays an important role in understanding the physical and mathematical significance of both models. Therefore, the question of regularity of weak solutions for all time $(0 \le t < \infty)$ is one of the outstanding open problems to be investigated.

In that regard, for the system (1.1), Yuan [26] presented the regularity critreia (1.3), (1.4), Lorenz et al. [27] presented conditions (1.5), (1.6) and Wang [28] established the regularity criteria (1.7)

$$\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \text{ where } \frac{3}{l}+\frac{2}{m}=1, \ 3<l\le\infty, \tag{1.3}$$

$$\nabla\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \text{ where } \frac{3}{l}+\frac{2}{m}=2, \ \frac{3}{2}<l\le\infty, \tag{1.4}$$

$$\nabla\mathcal{U}_3, \nabla_h\mathcal{V}, \nabla_h\mathcal{W} \in L^{\frac{32}{7}}(0,T;L^2(\mathbb{R}^3), \tag{1.5}$$

$$\partial_3\mathcal{U}_3, \partial_3\mathcal{V}, \partial_3\mathcal{W} \in L^{\infty}(0,T;L^2(\mathbb{R}^3)), \tag{1.6}$$

$$\partial_3\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)) \text{ where } \frac{3}{l}+\frac{2}{m}\le 1, \ 3<l\le\infty, \tag{1.7}$$

where $\nabla=(\partial_1,\partial_2,\partial_3)$ and $\nabla_h=(\partial_1,\partial_2)$.

For the system (1.2), Zhao and Bai [29] proved the regularity criteria (1.8),(1.9)

$$\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \text{ where } \frac{3}{l}+\frac{2}{m}\le 2, \ \frac{3}{2}<l\le\infty, \tag{1.8}$$

$$\nabla\mathcal{U} \in L^m(0,T;L^l(\mathbb{R}^3)), \text{ where } \frac{3}{l}+\frac{2}{m}\le 3, \ 1<l\le\infty. \tag{1.9}$$

Remark 1.1. The embedding relation $L^p \hookrightarrow L^{p,\infty}$ ensures that the anisotropic Lorentz space is larger than the anisotropic Lebesgue space and classical(simple) Lebesgue space. Furthermore, dropping ∞ and setting $l=m=n$ in the anisotropic Lorentz space we get anisotorpic Lebesgue space and simple Lebesgue space. This important observation is very useful because the results in anisotropic Lorentz sapces hold and improve numerous previous results in smaller spaces.

Remark 1.2. Throughout the paper the notation $\left\|\left\|\left\|(f,g)\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$ is expanded as $\left\|\left\|\left\|f\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \left\|\left\|\left\|g\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$.

As the blow-up of solution of the system (1.1) is controlled by four unknowns that is $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, Ψ. The important question regarding the regularity of weak solutions arises here. Can we propose a blow-up criteria for the system (1.1) only by controlling velocity and magnetic fields. In this paper, we give positive answer. Motivated by the above discussion, Remark 1.1 and conditions (1.5), (1.6) and (1.7), we present the following blow-up criteria in anisotropic Lorentz space for the system (1.1).

Theorem 1.1. Assume that $(\mathcal{U}_0,\mathcal{V}_0,\mathcal{W}_0) \in H^1(\mathbb{R}^3)$ with $\nabla\cdot\mathcal{U}_0=\nabla\cdot\mathcal{V}_0=0$ in the sense of distributions. The Leray-Hopf weak solution $(\mathcal{U},\mathcal{V},\mathcal{W})$ of the system (1.1) is smooth on the interval (0,T], if

$$\int_0^T \left\|\left\|\left\|(\partial_3\mathcal{U},\partial_3\mathcal{V})\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} < \infty, \tag{1.10}$$

where $2<l,m,n\le\infty$ and $1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})\ge 0$. Otherwise, if $T=T^\star<\infty$ is the maximal time for the existence of smooth solution then the solution blows up in finite time i.e.

$$\int_0^{T^\star} \left\| \left\| \left\| (\partial_3 \mathcal{U}, \partial_3 \mathcal{V}) \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} = \infty.$$

As the structure of the systems (1.1) and (1.2) suggests that the velocity plays more dominant role in the regularity of weak solutions than other unknowns. In view of these observations, we pose another problem. Can we prove a blow-up criterion that is only controlled by the one-directional derivative of velocity "$\partial_3 \mathcal{U}$"?. Thanks to the distributional methods, we give positive answer to this question and prove this criteria for the system (1.2). Because system (1.2) is also important for the theoretical and mathematical purposes having wide range of applications in electro-chemical and fluid-mechanical transport.

Theorem 1.2. Assume that $(\mathcal{U}_0, \mathcal{V}_0, \mathcal{W}_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot \mathcal{U}_0 = \nabla \cdot \mathcal{V}_0 = 0$ in the sense of distributions. The Leray-Hopf weak solution $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ to system (1.2) is regular on the interval (0,T], if

$$\int_0^{T} \left\| \left\| \left\| \partial_3 \mathcal{U} \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} dt < \infty, \tag{1.11}$$

where $2 < l, m, n \leq \infty$ and $1 - (\frac{1}{l} + \frac{1}{m} + \frac{1}{n}) \geq 0$. Otherwise, if $T = T^\star < \infty$ is the maximal time for the existence of smooth solution then the solution blows up to create finite time singularity that is

$$\int_0^{T^\star} \left\| \left\| \left\| \partial_3 \mathcal{U} \right\|_{L^{l,\infty}_{x_1}} \right\|_{L^{m,\infty}_{x_2}} \right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} = \infty.$$

Result (1.11) is refinement of the result (1.10).

Result (1.11) is also true for the system (1.1) and refines the result (1.10).

2. Preliminaries

Definition 2.1. [31] Let $l = (l_1, l_2, l_3)$ and $m = (m_1, m_2, m_3)$ with $0 < l_i \leq \infty$, $0 < m_i \leq \infty$. If $l_i = \infty$ then $m_i = \infty$ for every $i = 1, 2, 3$. An anisotropic Lorentz space $L^{l_1,m_1}(\mathbb{R}_{x_1}; L^{l_2,m_2}(\mathbb{R}_{x_2}; L^{l_3,m_3}((\mathbb{R}_{x_3}))))$ is the set of functions defined as

$$\left\| \left\| \left\| f \right\|_{L^{l_1,m_1}_{x_1}} \right\|_{L^{l_2,m_2}_{x_2}} \right\|_{L^{l_3,m_3}_{x_3}} := \Big(\int_0^\infty \Big(\int_0^\infty \Big(\int_0^\infty [t_1^{\frac{1}{l_1}} t_2^{\frac{1}{l_2}} t_3^{\frac{1}{l_3}} f^{*_1,*_2,*_3}(t_1, t_2, t_3)]^{m_1} \frac{dt_1}{t_1} \Big)^{\frac{m_2}{m_1}} \frac{dt_2}{t_2} \Big)^{\frac{m_3}{m_2}} \frac{dt_3}{t_3} \Big)^{\frac{1}{m_3}} < \infty.$$

Lemma 2.1. [30,31] (Holder's inequality for Lorentz spaces) If $1 \leq l_1, l_2, m_1, m_2 \leq \infty$, then for any $f \in L^{l_1,m_1}(\mathbb{R}^n)$, $g \in L^{l_2,m_2}(\mathbb{R}^n)$,

$$\|fg\|_{L^{l,m}(\mathbb{R}^n)} \leq C \|f\|_{L^{l_1,m_1}(\mathbb{R}^n)} \|g\|_{L^{l_2,m_2}(\mathbb{R}^n)},$$

where $\frac{1}{l} = \frac{1}{l_1} + \frac{1}{l_2}$ and $\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$.

Lemma 2.2. [30, 31] (Young's inequality for Lorentz spaces) Let $1 < l < \infty, 1 \leq m \leq \infty$ and $\frac{1}{l} + \frac{1}{l'} = 1$, $\frac{1}{m} + \frac{1}{m'} = 1$ with $1 < l < l'$ and $m' \leq m \leq \infty$. If $\frac{1}{l_2} + 1 = \frac{1}{l} + \frac{1}{l_1}$ and $\frac{1}{m_2} = \frac{1}{m} + \frac{1}{m_1}$ then the convolution operator

$$* : L^{l,m}(\mathbb{R}^n) \times L^{l_1,m_1}(\mathbb{R}^n) \mapsto L^{l_2,m_2}(\mathbb{R}^n)$$

is a bounded bilinear operator.

For any $s \geq 0$, we define homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ as

$$\dot{H}^s(\mathbb{R}^n) = \{f \in S' : \hat{f} \in L^1_{loc}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\beta|^{2s} |\widehat{f(\beta)}|^2 d\beta < \infty\},$$

where S' is the space of tempered distributions.

Lemma 2.3. [33] For $2 < l < \infty$, there exists a constant C=C(l) such that $f \in \dot{H}^{\frac{1}{l}}$, then $f \in L^{\frac{2l}{l-2},2}$ and

$$\|f\|_{\frac{2l}{l-2},2} \leq C\|f\|_{\dot{H}^{\frac{1}{l}}},$$

where $\dot{H}^{\frac{1}{l}}$ is the homogenous Sobolev space.

Lemma 2.4. [33] Let $2 \leq l, m, n \leq \infty$ and $1 - (\frac{1}{l} + \frac{1}{m} + \frac{1}{n}) \geq 0$, then $\exists\ C > 0$, such that $\forall$ $f \in C_0^\infty(\mathbb{R}^3)$

$$\left\| \left\| \|f\|_{L_{x_1}^{\frac{2l}{l-2},2}} \right\|_{L_{x_2,2}^{\frac{2m}{m-2},2}} \right\|_{L_{x_3}^{\frac{2n}{n-2},2}} \leq C\|\partial_1 f\|_{L^2}^{\frac{1}{l}} \|\partial_2 f\|_{L^2}^{\frac{1}{m}} \|\partial_3 f\|_{L^2}^{\frac{1}{n}} \|f\|_{L^2}^{1-(\frac{1}{m}+\frac{1}{n}+\frac{1}{l})}.$$

Lemma 2.5. [32] Let $1 \leq \alpha_1, \alpha_2, \alpha_3, \alpha_4 < \infty$, $\frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} > 1$ and $1 + \frac{3}{\alpha_1} = \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4}$. Suppose $\phi(x) = \phi(x_1, x_2, x_3)$ with $\partial_1\phi \in L^{\alpha_2}(\mathbb{R}^3)$, $\partial_2\phi \in L^{\alpha_3}(\mathbb{R}^3)$, $\partial_3\phi \in L^{\alpha_4}(\mathbb{R}^3)$ then $\exists$ a constant $C = C(\alpha_2, \alpha_3, \alpha_4)$ such that

$$\|\phi\|_{L^{\alpha_1}} \leq C\|\partial_1\phi\|_{L^{\alpha_2}}^{\frac{1}{3}} \|\partial_2\phi\|_{L^{\alpha_3}}^{\frac{1}{3}} \|\partial_3\phi\|_{L^{\alpha_4}}^{\frac{1}{3}}, \tag{2.1}$$

when $\alpha_2 = \alpha_3 = 2$ and $1 \leq \alpha_4 < \infty$, $\exists$ a $C = C(\alpha_4)$, such that

$$\|\phi\|_{L^{3\alpha_4}} \leq C\|\partial_1\phi\|_{L^2}^{\frac{1}{3}} \|\partial_2\phi\|_{L^2}^{\frac{1}{3}} \|\partial_3\phi\|_{L^2}^{\frac{1}{3}}, \tag{2.2}$$

holds for any ϕ with $\partial_1\phi \in L^2(\mathbb{R}^3)$, $\partial_2\phi \in L^2(\mathbb{R}^3)$, $\partial_3\phi \in L^{\alpha_4}(\mathbb{R}^3)$.

3. Proofs of Theorems 1.1 and 1.2

The proofs of Theorems 1.1 and 1.2 are based on distributional methods and setting up of a priori estimates under the blow-up conditions (1.10) and (1.11).

3.1. Proof of Theorem 1.1

In order to get the fundamental energy estimates of the system (1.1), taking inner product of $(1.1)_1$, $(1.1)_2$, $(1.1)_3$ over $\mathbb{R}^3$ with $\mathcal{U}$, $\mathcal{W}$, $\mathcal{V}$, respectively, then adding the resulting equations and integrating in time, we get

$$\|(\mathcal{U}, \mathcal{W}, \mathcal{V})\|_{L^2}^2 + 2\int_0^t (\|\nabla\mathcal{U}\|_{L^2}^2 + \|\nabla\mathcal{W}\|_{L^2}^2 + \|\nabla\mathcal{V}\|_{L^2}^2) d\tau + 2\int_0^t (\|\nabla \cdot \mathcal{W}\|_{L^2}^2 + \|\mathcal{W}\|_{L^2}^2) d\tau$$

$$\leq \|(\mathcal{U}_0, \mathcal{W}_0, \mathcal{V}_0)\|_{L^2}^2. \tag{3.1}$$

In order to find L^2-estimates for one-directional derivative of the velocity, take derivative of $(1.1)_1$ with respect to x_3, then multiply resulting equation with $\partial_3 u$ in $L^2(\mathbb{R}^3)$ inner product and integrating, we get the resulting equation as

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial_3\mathcal{U}|^2dx + \int_{\mathbb{R}^3}|\nabla\partial_3\mathcal{U}|^2dx - \int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\mathcal{V}\cdot\nabla\partial_3\mathcal{V}dx = -\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{U}dx$$
$$+\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{V}\cdot\nabla\mathcal{V}dx + \int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\nabla\times\partial_3\mathcal{W}dx. \tag{3.2}$$

Similarly, multiplying $(1.1)_2$ with $\partial_3\mathcal{W}$ and $(1.1)_3$ with $\partial_3\mathcal{V}$, integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial_3\mathcal{W}|^2dx + \int_{\mathbb{R}^3}|\nabla\partial_3\mathcal{W}|^2dx + \int_{\mathbb{R}^3}|\nabla\cdot\partial_3\mathcal{W}|^2dx + 2\int_{\mathbb{R}^3}|\partial_3\mathcal{W}|^2dx$$
$$= -\int_{\mathbb{R}^3}\partial_3\mathcal{W}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{W}dx + \int_{\mathbb{R}^3}\partial_3\mathcal{W}\cdot\nabla\times\partial_3\mathcal{U}dx. \tag{3.3}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\partial_3\mathcal{V}|^2dx + \int_{\mathbb{R}^3}|\nabla\partial_3\mathcal{V}|^2dx - \int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\mathcal{V}\cdot\nabla\partial_3\mathcal{U}dx = -\int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{V}dx$$
$$+\int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\partial_3\mathcal{V}\cdot\nabla\mathcal{U}dx. \tag{3.4}$$

Adding (3.2), (3.3) and (3.4), we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\partial_3\mathcal{U}\|_{L^2}^2 + \|\partial_3\mathcal{W}\|_{L^2}^2 + \|\partial_3\mathcal{V}\|_{L^2}^2) + (\|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + \|\nabla\partial_3\mathcal{W}\|_{L^2}^2 + \|\nabla\partial_3\mathcal{V}\|_{L^2}^2)$$
$$+\|div\ \partial_3\mathcal{W}\|_{L^2}^2 + 2\|\partial_3\mathcal{W}\|_{L^2}^2$$
$$= -\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{U}dx + \int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{V}\cdot\nabla\mathcal{V}dx - \int_{\mathbb{R}^3}\partial_3\mathcal{W}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{W}dx$$
$$+\int_{\mathbb{R}^3}\partial_3\mathcal{W}\cdot\nabla\times\partial_3\mathcal{U}dx - \int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{V}dx + \int_{\mathbb{R}^3}\partial_3\mathcal{V}\cdot\partial_3\mathcal{V}\cdot\nabla\mathcal{U}dx$$
$$= P_1 + P_2 + P_3 + P_4 + P_5 + P_6. \tag{3.5}$$

Now, we will find estimates for every term of (3.5), one by one, taking C as a generic constant.

$$|P_1| = \left|\int_{\mathbb{R}^3}\partial_3\mathcal{U}\cdot\partial_3\mathcal{U}\cdot\nabla\mathcal{U}dx\right|.$$

Using Holder's inequality and Lemma 2.4. we obtain

$$|P_1| \leq C\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L_{x_1}^{l,\infty}}\right\|_{L_{x_2}^{m,\infty}}\right\|_{L_{x_3}^{n,\infty}}\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L_{x_1}^{\frac{2l}{l-2},2}}\right\|_{L_{x_2}^{\frac{2m}{m-2},2}}\right\|_{L_{x_3}^{\frac{2n}{n-2},2}}\|\nabla\mathcal{U}\|_{L^2}$$

$$\leq C\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L_{x_1}^{l,\infty}}\right\|_{L_{x_2}^{m,\infty}}\right\|_{L_{x_3}^{n,\infty}}\|\partial_3\mathcal{U}\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\partial_1\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{l}}\|\partial_2\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{m}}\|\partial_3\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{n}}\|\nabla\mathcal{U}\|_{L^2}$$

$$\leq C\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\|\partial_3\mathcal{U}\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}\|\nabla\mathcal{U}\|_{L^2}.$$

Applying Young's inequality

$$\leq C\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\partial_3\mathcal{U}\|_{L^2}^{2.\frac{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\nabla\mathcal{U}\|_{L^2}^{\frac{2}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2.$$

Adjusting above inequality's exponents to apply again Young's inequality

$$\leq C\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\partial_3\mathcal{U}\|_{L^2}^2\right)^{\frac{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\left(\|\nabla\mathcal{U}\|_{L^2}^2\right)^{\frac{1}{2-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2$$

$$\leq C\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{U}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2.$$

Finally, we get an estimate for P_1 as

$$|P_1|\leq C(1+\|\partial_3\mathcal{U}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\mathcal{U}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2. \tag{3.6}$$

Similarly, we get bound for P_6 as

$$|P_6|\leq C(1+\|\partial_3\mathcal{V}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\mathcal{U}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2. \tag{3.7}$$

In case of P_4, using Holder's and Young's inequalities

$$|P_4|\leq\frac{1}{4}\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+C\|\partial_3\mathcal{W}\|_{L^2}^2. \tag{3.8}$$

Estimating P_2, P_3 and P_5

$$|P_2|\leq C\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{\frac{2l}{l-2},2}_{x_1}}\right\|_{L^{\frac{2m}{m-2},2}_{x_2}}\right\|_{L^{\frac{2n}{n-2},2}_{x_3}}\|\nabla\mathcal{V}\|_{L^2}$$

$$\leq C\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\|\partial_3\mathcal{U}\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\partial_1\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{l}}\|\partial_2\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{m}}\|\partial_3\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{n}}\|\nabla\mathcal{V}\|_{L^2}$$

$$\leq C\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}\|\partial_3\mathcal{U}\|_{L^2}^{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{l}+\frac{1}{m}+\frac{1}{n}}\|\nabla\mathcal{V}\|_{L^2}.$$

Following on the same steps as for (3.6)

$$\leq C\left(\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\|\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2.$$

$$|P_2| \leq C(1+\|\partial_3\mathcal{U}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{V}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \|\nabla\mathcal{V}\|_{L^2}^2\right) + \|\nabla\partial_3\mathcal{U}\|_{L^2}^2. \tag{3.9}$$

$$|P_3| \leq C(1+\|\partial_3\mathcal{W}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \|\nabla\mathcal{W}\|_{L^2}^2\right) + \|\nabla\partial_3\mathcal{W}\|_{L^2}^2. \tag{3.10}$$

$$|P_5| \leq C(1+\|\partial_3\mathcal{V}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}} + \|\nabla\mathcal{U}\|_{L^2}^2\right) + \|\nabla\partial_3\mathcal{V}\|_{L^2}^2. \tag{3.11}$$

Now we will find L^2-estimates for the gradient of velocity, magnetic field and micro-rotational velocity. In order to get required estimates, multiply $(1.1)_1$, $(1.1)_2$, $(1.1)_3$ with $-\Delta\mathcal{U}$, $-\Delta\mathcal{W}$, $-\Delta\mathcal{V}$, respectively, then integrating over $\mathbb{R}^3$, adding the resulting three equations, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{W}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2)+(\|\Delta\mathcal{U}\|_{L^2}^2+\|\Delta\mathcal{W}\|_{L^2}^2+\|\Delta\mathcal{V}\|_{L^2}^2)\\
&\quad+\|\nabla div\mathcal{W}\|_{L^2}^2+2\|\nabla\mathcal{W}\|_{L^2}^2\\
&\leq(\Delta\mathcal{U},\mathcal{U}\cdot\nabla\mathcal{U})-(\Delta\mathcal{U},\mathcal{V}\cdot\nabla\mathcal{V})+(\Delta\mathcal{V},\mathcal{V}\cdot\nabla\mathcal{V})-(\Delta\mathcal{V},\mathcal{V}\cdot\nabla\mathcal{U})\\
&\quad+(\Delta\mathcal{W},\mathcal{W}\cdot\nabla\mathcal{W})-2(\Delta\mathcal{W},\nabla\times\mathcal{U})\\
&=\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6.
\end{aligned} \tag{3.12}$$

The terms in (3.12) are bounded by Tang et al. [34] in inequality (33). For detailed prove see [34].

$$\Longrightarrow\ \|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{W}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2<\infty. \tag{3.13}$$

This implies the fact

$$(\mathcal{U},\mathcal{V},\mathcal{W})\in L^\infty(0,T,H^1(\mathbb{R}^3))\cap L^2(0,T,H^2(\mathbb{R}^3)).$$

Putting all estimates in (3.5), after simplifications, it yields

$$\begin{aligned}
&\frac{d}{dt}(\|\partial_3\mathcal{U}\|_{L^2}^2+\|\partial_3\mathcal{W}\|_{L^2}^2+\|\partial_3\mathcal{V}\|_{L^2}^2)+2(\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{W}\|_{L^2}^2+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2)\\
&\quad+2\|div\ \partial_3\mathcal{W}\|_{L^2}^2+2\|\partial_3\mathcal{W}\|_{L^2}^2\\
&\leq C\Big(1+\|\partial_3\mathcal{U}\|_{L^2}^2+\|\partial_3\mathcal{V}\|_{L^2}^2+\|\partial_3\mathcal{W}\|_{L^2}^2\Big)\left(\left\|\left\|\left\|(\partial_3\mathcal{U},\partial_3\mathcal{V})\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\right.\\
&\quad\left.+\|\nabla\mathcal{U}\|_{L^2}^2+\|\nabla\mathcal{V}\|_{L^2}^2+\|\nabla\mathcal{W}\|_{L^2}^2\right).
\end{aligned}$$

Invoking Gronwall's inequality with (3.13), we get

$$\sup_{0\leq t\leq T}(\|\partial_3\mathcal{U}\|_{L^2}^2+\|\partial_3\mathcal{W}\|_{L^2}^2+\|\partial_3\mathcal{V}\|_{L^2}^2)+2\int_0^t(\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{W}\|_{L^2}^2+\|\nabla\partial_3\mathcal{V}\|_{L^2}^2)d\tau$$

$$+2\int_0^t \|div\ \partial_3\mathcal{W}\|_{L^2}^2 d\tau + 2\int_0^t \|\partial_3\mathcal{W}\|_{L^2}^2 d\tau$$

$$\leq C\Big(1 + \|\partial_3\mathcal{U}_0\|_{L^2}^2 + \|\partial_3\mathcal{V}_0\|_{L^2}^2 + \|\partial_3\mathcal{W}_0\|_{L^2}^2\Big)\Big(\Big\|\Big\|\|(\partial_3\mathcal{U}, \partial_3\mathcal{V})\|_{L^{l,\infty}_{x_1}}\Big\|_{L^{m,\infty}_{x_2}}\Big\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}$$

$$+\|\nabla\mathcal{U}\|_{L^2}^2 + \|\nabla\mathcal{V}\|_{L^2}^2 + \|\nabla\mathcal{W}\|_{L^2}^2\Big)$$

$$\sup_{0\leq t\leq T}(\|\partial_3\mathcal{U}\|_{L^2}^2 + \|\partial_3\mathcal{W}\|_{L^2}^2 + \|\partial_3\mathcal{V}\|_{L^2}^2) + 2\int_0^t (\|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + \|\nabla\partial_3\mathcal{W}\|_{L^2}^2 + \|\nabla\partial_3\mathcal{V}\|_{L^2}^2)d\tau$$

$$+2\int_0^t \|div\ \partial_3\mathcal{W}\|_{L^2}^2 d\tau + 2\int_0^t \|\partial_3\mathcal{W}\|_{L^2}^2 d\tau \leq C.$$

Which completes the proof of Theorem 1.1 as desired.

3.2. *Proof of Theorem 1.2*

The proof of Theorem 1.2 will follow from setting up of a priori estimates for the blow-up conditions of the system (1.2).
As a first step we will find L^2-estimates for $\mathcal{U}$, θ, ϑ and $\nabla\psi$. Multiplying $(1.2)_3$ with θ and $(1.2)_4$ with ϑ, integrating over $\mathbb{R}^3$, using divergence free condition $(1.2)_2$ and $(1.2)_5$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\theta\|_{L^2}^2 + \|\vartheta\|_{L^2}^2) + (\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2) + \int_{\mathbb{R}^3}(\theta+\vartheta)(\theta-\vartheta)^2 dx = 0. \tag{3.14}$$

As masses of θ and ϑ are conserved, θ and ϑ are non-negative, we infer from (3.14) that for all $0 \leq t \leq T$

$$(\|\theta\|_{L^2}^2 + \|\vartheta\|_{L^2}^2) + 2\int_0^t (\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2)d\tau \leq \|\theta_0\|_{L^2}^2 + \|\vartheta_0\|_{L^2}^2. \tag{3.15}$$

Now, multiplying $(1.2)_1$ with $\mathcal{U}$, $(1.2)_3$, $(1.2)_4$ with ψ, integrating over $\mathbb{R}^3$, and using $(1.2)_5$, it gives

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{U}\|_{L^2}^2 + \|\nabla\mathcal{U}\|_{L^2}^2 - \int_{\mathbb{R}^3}(\theta-\vartheta)\mathcal{U}\cdot\nabla\psi dx = 0, \tag{3.16}$$

$$\int_{\mathbb{R}^3}\Big[\frac{\partial\theta}{\partial t}\psi + \nabla\cdot(\theta\nabla\psi)\psi - \Delta\theta\psi + (\mathcal{U}\cdot\nabla)\theta\psi\Big]dx = 0, \tag{3.17}$$

$$\int_{\mathbb{R}^3}\Big[\frac{\partial\vartheta}{\partial t}\psi + \nabla\cdot(\vartheta\nabla\psi)\psi - \Delta\vartheta\psi + (\mathcal{U}\cdot\nabla)\vartheta\psi\Big]dx = 0. \tag{3.18}$$

Subtracting (3.18) from (3.17), using integration by parts and $\Delta\psi = \theta - \vartheta$, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|_{L^2}^2 + \int_{\mathbb{R}^3}(\theta+\vartheta)|\nabla\psi|^2 dx + \int_{\mathbb{R}^3}|\Delta\psi|^2 dx + \int_{\mathbb{R}^3}(\theta-\vartheta)\mathcal{U}\cdot\nabla\psi dx = 0. \tag{3.19}$$

Adding (3.16) and (3.19), it follows that

$$\frac{1}{2}\frac{d}{dt}(\|\mathcal{U}\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2) + \|\nabla\mathcal{U}\|_{L^2}^2 + \|\Delta\psi\|_{L^2}^2 + \int_{\mathbb{R}^3}(\theta+\vartheta)|\nabla\psi|^2 dx = 0. \tag{3.20}$$

Because of the non-negativity of θ and ϑ, we obtained the final bound

$$\|\mathcal{U}\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + 2\int_0^t \|\nabla\mathcal{U}\|_{L^2}^2 + \|\Delta\psi\|_{L^2}^2 d\tau \leq C. \tag{3.21}$$

Now, we will find H^1-estimates for $\mathcal{U}$, θ and ϑ. For required bounds multiply $-\Delta\mathcal{U}$ with $(1.2)_1$, integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathcal{U}\|_{L^2}^2 + \|\Delta\mathcal{U}\|_{L^2}^2 &= \int_{\mathbb{R}^3} (\mathcal{U}\cdot\nabla)\mathcal{U}\cdot\Delta\mathcal{U}dx - \int_{\mathbb{R}^3} \Delta\psi\nabla\psi\cdot\Delta\mathcal{U}dx \\ &= Q_1 + Q_2. \end{aligned}\tag{3.22}$$

For Q_2, using Holder's and Young's inequalities, using $\Delta\psi = \theta - \vartheta$, interpolation inequality $\|\nabla f\|_{L^4} \leq \|f\|_{L^4}^{\frac{1}{8}}\|\Delta f\|_{L^4}^{\frac{7}{8}}$, and combining (3.15), (3.21), we obtain

$$\begin{aligned}|Q_2| &\leq \|\Delta\psi\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\mathcal{U}\|_{L^2} \\ &\leq \frac{1}{4}\|\Delta\mathcal{U}\|_{L^2}^2 + C\|\nabla\psi\|_{L^4}^2\|(\theta,\vartheta)\|_{L^4}^2 \\ &\leq \frac{1}{4}\|\Delta\mathcal{U}\|_{L^2}^2 + C\|\nabla\psi\|_{L^2}^2\|(\theta,\vartheta)\|_{L^2}^2 + C\|(\nabla\theta,\nabla\vartheta)\|_{L^2}^2\|(\theta,\vartheta)\|_{L^2}^2 \\ &\leq \frac{1}{4}\|\Delta\mathcal{U}\|_{L^2}^2 + C(\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2 + 1). \end{aligned}\tag{3.23}$$

For Q_1

$$\begin{aligned}|Q_1| &\leq \int_{\mathbb{R}^3} \nabla\mathcal{U}\nabla\mathcal{U}\nabla\mathcal{U}dx \\ &\leq C\|\nabla\mathcal{U}\|_{L^3}^3 \leq C\|\nabla\mathcal{U}\|_{L^3}^{\frac{3}{2}}\|\nabla\mathcal{U}\|_{L^6}^{\frac{3}{2}} \quad \text{(Interpolation inequality)} \\ &\leq C\|\nabla\mathcal{U}\|_{L^2}^{\frac{3}{2}}\|\nabla\partial_1\mathcal{U}\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_2\mathcal{U}\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{2}} \quad \text{(Lemma 2.5., for } \alpha_4 = 2) \\ &\leq \|\nabla\mathcal{U}\|_{L^2}^{\frac{3}{2}}\|\nabla^2\mathcal{U}\|_{L^2}\|\nabla\partial_3\mathcal{U}\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{1}{4}\|\Delta\mathcal{U}\|_{L^2}^2 + C\|\nabla\mathcal{U}\|_{L^2}^3\|\nabla\partial_3\mathcal{U}\|_{L^2}. \quad \text{(Young's inequality)} \end{aligned}\tag{3.24}$$

Putting (3.23) and (3.24) into (3.22), and employing Gronwall's inequality, it yields

$$\begin{aligned}\sup_{0\leq t\leq T}\|\nabla\mathcal{U}\|_{L^2}^2 + 2\int_0^t \|\Delta\mathcal{U}\|_{L^2}^2 d\tau &\leq (\|\nabla\mathcal{U}_0\|_{L^2}^2 + e)\exp(C\int_0^t (\|\nabla\partial_3\mathcal{U}\|_{L^2}^2 + \|\nabla\mathcal{U}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 \\ &\quad + \|\nabla\vartheta\|_{L^2}^2 + 1)d\tau). \end{aligned}\tag{3.25}$$

$$\Longrightarrow \mathcal{U} \in L^\infty(0,T,H^1(\mathbb{R}^3)) \cap L^2(0,T,H^2(\mathbb{R}^3)).$$

To get similar results for θ and ϑ. Multiply $-\Delta\theta$ with $(1.2)_3$ and $-\Delta\vartheta$ with $(1.2)_4$, we achieve

$$\sup_{0\leq t\leq T}(\|\nabla\theta\|_{L^2}^2 + \|\nabla\vartheta\|_{L^2}^2) + 2\int_0^t (\|\Delta\theta\|_{L^2}^2 + \|\Delta\vartheta\|_{L^2}^2)d\tau \leq C. \tag{3.26}$$

For our desired results, differentiate $(1.2)_1$ with respect to x_3, then multiply by $\partial_3\mathcal{U}$ and integrating by parts to get

$$\frac{1}{2}\frac{d}{dt}\|\partial_3\mathcal{U}\|_{L^2}^2+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2=-\int_{\mathbb{R}^3}\partial_3(\mathcal{U}\cdot\nabla\mathcal{U})\cdot\partial_3\mathcal{U}dx+\int_{\mathbb{R}^3}\partial_3(\Delta\psi\nabla\psi)\partial_3\mathcal{U}dx$$

$$=D_1+D_2. \tag{3.27}$$

Estimating D_2 as (3.23), we obtain

$$|D_2|\leq\int_{\mathbb{R}^3}\partial_3(\Delta\psi\nabla\psi)\partial_3\mathcal{U}dx$$

$$\leq\frac{1}{4}\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+C\|(\theta,\vartheta)\|_{L^2}^2\|(\nabla\theta,\nabla\vartheta)\|_{L^2}^2+C\|(\theta,\vartheta)\|_{L^2}^2\|\nabla\psi\|_{L^2}^2$$

$$\leq\frac{1}{4}\|\nabla\partial_3\mathcal{U}\|_{L^2}^2+C(\|\nabla\theta\|_{L^2}^2+\|\nabla\vartheta\|_{L^2}^2+1). \tag{3.28}$$

Similar to (3.6), D_1 is estimated as

$$|D_1|\leq C(1+\|\partial_3\mathcal{U}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|\nabla\mathcal{U}\|_{L^2}^2\right)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2. \tag{3.29}$$

putting (3.28), (3.29) into (3.27)

$$\frac{1}{2}\frac{d}{dt}(1+\|\partial_3\mathcal{U}\|_{L^2}^2)+\|\nabla\partial_3\mathcal{U}\|_{L^2}^2$$

$$\leq C(1+\|\partial_3\mathcal{U}\|_{L^2}^2)\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}+\|(\nabla\mathcal{U},\nabla\theta,\nabla\vartheta)\|_{L^2}^2+1\right).$$

Applying Gronwall's inequality together with (3.25) and (3.26) yields

$$(1+\|\partial_3\mathcal{U}\|_{L^2}^2)+2\int_0^T\|\nabla\partial_3\mathcal{U}\|_{L^2}^2d\tau\leq(1+\|\partial_3\mathcal{U}_0\|_{L^2}^2)\exp\left(C\int_0^T\left(\left\|\left\|\left\|\partial_3\mathcal{U}\right\|_{L^{l,\infty}_{x_1}}\right\|_{L^{m,\infty}_{x_2}}\right\|_{L^{n,\infty}_{x_3}}^{\frac{2}{1-(\frac{1}{l}+\frac{1}{m}+\frac{1}{n})}}\right.\right.$$

$$\left.\left.+\|(\nabla\mathcal{U},\nabla\theta,\nabla\vartheta)\|_{L^2}^2+1\right)d\tau.\right.$$

$$\sup_{0\leq t\leq T}(1+\|\partial_3\mathcal{U}\|_{L^2}^2)+2\int_0^T\|\nabla\partial_3\mathcal{U}\|_{L^2}^2d\tau\leq C. \tag{3.30}$$

The bound (3.30) ensures the smoothness of weak solutions of system (1.2) on the interval $(0,T]$. Hence proved.

4. Conclusions

This study investigates the regularity of magneto-micropolar system in terms of one-directional derivatives of velocity and magnetic fields that as a result generalize conditions (1.5), (1.6) and (1.7) in anisotorpic Lorentz space. For the dissipative system modeling electro-diffusion, we established an improved and new regularity condition in one-directional derivative of velocity, which is important as velocity plays more dominant role in controlling regularity than other unknowns of the system, in anisotropic Lorentz space. For future developments, it is interesting to establish the regularity criteria only in terms of velocity and its components in anisotropic Lebesgue and anisotropic Lorentz spaces for both systems.

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Conflict of interest

The authors declare no conflict of interest.

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