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Research article

Blow-up dynamic of solution to the semilinear Moore-Gibson-Thompson equation with memory terms

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Abstract: This article is mainly concerned with the formation of singularity for a solution to the Cauchy problem of the semilinear Moore-Gibson-Thompson equation with general initial values and different types of nonlinear memory terms $N_{\gamma,q}(u)$, $N_{\gamma,p}(u_t)$, $N_{\gamma,p,q}(u,\, u_t)$. The proof of the blow-up phenomenon for the solution in the whole space is based on the test function method ($\psi(x,t) = \varphi_R(x)D^{\alpha}_{t|T}(w(t))$). It is worth pointing out that the Moore-Gibson-Thompson equation with memory terms can be regarded as an approximation of the nonlinear Moore-Gibson-Thompson equation when $\gamma \to 1^-$. To the best of our knowledge, the results in Theorems 1.1–1.3 are new.

Keywords: Moore-Gibson-Thompson equation; general initial values; nonlinear memory terms; blow-up; test function method
Mathematics Subject Classification: 35L70, 58J45

1. Introduction

In this article, we are interested in exploring the Cauchy problem of the Moore-Gibson-Thompson equation (MGT) with memory terms

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta\Delta u_t = N(u,\, u_t), & x \in \mathbb{R}^n,\ t > 0, \\ (u,\, u_t,\, u_{tt})(x,0) = (u_0,\, u_1,\, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the nonlinear term $N(u,\, u_t)$ is illustrated in the forms of

$$N(u,\, u_t) = N_{\gamma,q}(u) = c_\gamma \int_0^t (t-s)^{-\gamma}|u(x,s)|^q ds,$$

$$N(u,\, u_t) = N_{\gamma,\, p}(u_t) = c_\gamma \int_0^t (t-s)^{-\gamma} |u_t(x,s)|^p ds,$$

and

$$\begin{aligned} N(u,\, u_t) &= N_{\gamma,\, p,\, q}(u,\, u_t) \\ &= c_\gamma \int_0^t (t-s)^{-\gamma} |u_t(x,s)|^p ds + c_\gamma \int_0^t (t-s)^{-\gamma} |u(x,s)|^q ds, \end{aligned}$$

with $c_\gamma = \frac{1}{\Gamma(1-\gamma)}$, $\gamma \in (0,1)$. $\Gamma(\cdot)$ represents the second kind of the Euler integral, namely, $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ for $s > 0$. The exponents of nonlinear terms satisfy $1 < p,\, q < \infty$. We assume $B_R(0) = \{x \big| \, |x| \le R\}$, where the constant R satisfies $R > 2$. The initial values $(u_0,\, u_1,\, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ are non-negative functions.

Recently, the research on the MGT equation which is a third order hyperbolic equation

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - b \Delta u_t = 0 \tag{1.2}$$

has caught a lot of attention. The MGT equation is one of the equations of nonlinear acoustics, and it describes acoustic wave propagation in gas and liquid. In the physical background of acoustic waves, a solution $u(x,t)$ to Eq (1.2) stands for scalar acoustic velocity. The coefficient τ represents thermal relaxation. The constant c denotes the speed of sound. The parameter $b = \beta c^2$ concerns the diffusivity of sound when $\tau \in (0,\beta]$. The behavior of the solution to Eq (1.2) is divided into the dissipative case when $\tau \in (0,\beta)$, and the conservative case when $\tau = \beta$. More precisely, there exists a transition from the case $\tau \in (0,\beta)$ with an energy being exponentially stable to the limit case $\tau = \beta$ with energy being conserved in the bounded domain. Employing the test function method, Chen and Ikehata [1] investigated the blow-up result of the solution to the Cauchy problem of the semilinear MGT equation in the dissipative case. Non-existence of the global solution to the semilinear MGT equation with the power nonlinear term $|u|^p$ in the conservative case is verified in [2]. Local (in time) existence of the solution is obtained by taking advantage of the fixed-point theorem method. Blow-up of the solution is derived by applying an iteration argument. Chen and Palmieri [3] discussed the blow-up phenomenon of the solution to the semilinear MGT equation with the derivative nonlinear term $|u_t|^p$ in the conservative case in n space dimensions. The lifespan estimate of the solution in the sub-critical and critical cases is established by exploiting the iteration method. Shi et al. [4] verified the global existence and blow-up of solutions to the viscous MGT equation. Ming et al. [5] presented the upper-bound lifespan estimate of the solution to the semilinear MGT equation with the nonlinear terms $|u|^p$, $|u_t|^p$ and $|u_t|^p + |u|^q$, respectively. The proof is based on the test function technique ($\phi(x,t) = \eta_T^{2p'}(t)\Phi(x,t)$, $\eta_M^{2p'}(t) b_q(x,t)$). Taking advantage of the test function method, Ming et al. [6] established the formation of singularities of solutions to the weakly coupled system of semilinear Moore-Gibson-Thompson equations with power nonlinearities $|v|^p$, $|u|^q$, derivative nonlinearities $|v_t|^p$, $|u_t|^q$, mixed nonlinearities $|v|^q$, $|u_t|^p$ and combined nonlinearities $|v_t|^{p_1} + |v|^{q_1}$, $|u_t|^{p_2} + |u|^{q_2}$, respectively. Upper bound lifespan estimates of solutions to the problem in the sub-critical and critical cases were obtained. We refer the readers to [7–17] for more details.

Let us state a historical overview concerning several results for the semilinear wave equation with memory terms

$$\begin{cases} u_{tt} - \Delta u = N(u,\, u_t), & x \in \mathbb{R}^n,\, t > 0, \\ u(x,0) = \varepsilon u_0(x), \quad u_t(x,0) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $N(u,\, u_t) = N_{\gamma,\,p}(u),\ N_{\gamma,\,p}(u_t)$. Chen and Palmieri [18] considered Problem (1.3) with a nonlinear memory term of the power type $N(u,\, u_t) = N_{\gamma,\,p}(u)$ in the sub-critical and critical cases. The blow-up dynamics of solutions were proved by making use of an iteration argument in the sub-critical case and the slicing approach in the critical case. Non-existence of the global solution to Problem (1.3) with a nonlinear memory term of the derivative type $N(u,\, u_t) = N_{\gamma,\,p}(u_t)$ was investigated in [19], where the iteration method and ODE (ordinary differential equation) blow-up approach were performed. It is worth noticing that Problem (1.3) with $N(u,\, u_t) = N_{\gamma,\,p}(u)$ possesses the critical exponent $p_c(n,\, \gamma)$, which is the maximal solution to the quadratic equation

$$-(n-1)p^2 + (n+3-2\gamma)p + 2 = 0.$$

Moreover, $p_c(n,\, \gamma)$ satisfies $\lim_{\gamma\to 1^-} p_c(n,\, \gamma) = p_c(n)$. We observe that $p_c(n)$ stands for the Strauss critical exponent of the classical wave equation

$$u_{tt} - \Delta u = |u|^p.$$

Here, the meaning of the Strauss critical exponent $p_c(n)$ represents the threshold between the global (in time) existence of the solution and blow-up dynamic of the solution with small initial values (see [20–25] and the references therein). The non-existence and existence of the global solution to the Cauchy problem of the classical wave equation with the derivative type nonlinearity $|u_t|^p$ are considered in [26, 27]. Applying the test function related to the hypergeometric function method ($\psi(x,t) = \Phi_{\beta,\,\lambda}(x,t)(\eta_R(t))^k$), Ikeda et al. [28] established the blow-up results and lifespan estimates of solutions to the classical wave equation with combined-type nonlinearities $|u_t|^p + |u|^q$ and a corresponding weakly coupled system.

Many scholars are committed to the Cauchy problem of the semilinear wave equation with nonlinear memory terms

$$\begin{cases} u_{tt} - \Delta u + h(u_t) = N_{\gamma,\,p}(u), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

(see detailed illustrations in [29–35]). Taking into account the test function method, Fino [33] deduced the blow-up result of the solution to Problem (1.4) with the weak damping term $h(u_t) = u_t$. Global existence of the solution to the problem when $1 \le n \le 3$ is demonstrated by using the weighted energy method. Chen and Ikehata [29] studied the formation of a singularity for the solution to Problem (1.4) with scaling invariant damping, i.e, $h(u_t) = \frac{\mu}{1+t}u_t$. The proof is based on the Kato lemma in the case $1 < p < p_S(n+\mu,\, \gamma)$ and the iteration method in the case $p = p_S(n+2,\, \gamma)$. Non-existence of the global solution to Problem (1.4) with the structural damping term $h(u_t) = 2(-\Delta)^{\frac{1}{2}}u_t$ is verified by taking advantage of the test function technique ($\phi(x,t) = D^{\alpha}_{t|T}(w(t))^{\beta}(\Psi_R(x))^{\ell}$) (see [30]). Problem (1.4) with $h(u_t) = \mu(-\Delta)^{\frac{\sigma}{2}}u_t\, (\mu > 0,\ 0 < \sigma < 2)$ is considered in [32]. The blow-up of the solution is obtained by applying the method of the test function. Dannawi et al. [31] showed the blow-up phenomenon of the solution to the Cauchy problem of the semilinear wave equation with space- and time-dependent potential, as well as nonlinear memory, where the test function method was employed.

Motivated by the works in [2, 3, 5, 29, 32], our main aim of this article is to verify the blow-up dynamic of the solution to Problem (1.1) with the different nonlinear memory terms $N_{\gamma,\,q}(u)$, $N_{\gamma,\,p}(u_t)$ and $N_{\gamma,\,p,\,q}(u,\, u_t)$, where the test function technique is performed ($\psi(x,t) = \varphi_R(x)D^{\alpha}_{t|T}(w(t))$). It is worth noting that Chen and Ikehata [29] presented the blow-up dynamic of the solution to the Cauchy problem for the semilinear wave equation with a scaling invariant damping term and memory term in the case $p = p_c(n+2, \gamma)$. The main tool performed in the proof is the iteration method together with a slicing procedure. Non-existence of the global solution to the semilinear wave equation with a structural damping term and memory term has been deduced by applying the test function technique ($\phi_R(x)D^{\alpha}_{t|T}(w(t))$) (see [32]). Taking advantage of the test function technique and iteration approach, Chen and Palmieri [2, 3] investigated the upper-bound lifespan estimate of the solution to the semilinear MGT equation with power nonlinearity ($|u|^p$) and derivative nonlinearity ($|u_t|^p$). We extend the problems studied in [2, 32] to Problem (1.1) with $N(u, u_t) = N_{\gamma,q}(u)$ by employing the test function technique (see Theorem 1.1). The blow-up phenomenon of the solution to Problem (1.1) with small initial values and nonlinear terms $|u|^p$, $|u_t|^p$, $|u_t|^p + |u|^q$ is discussed in [5]. From our observation, the non-existence of the global solution to Problem (1.1) with $N(u, u_t) = N_{\gamma,p}(u_t)$, $N_{\gamma,p,q}(u, u_t)$ is still unknown. Consequently, we apply the test function technique to supplement the blow-up result of the solution to Problem (1.1), which is different from the method in [5] (see Theorems 1.2 and 1.3). Moreover, our results in Theorems 1.1–1.3 exactly coincide with the blow-up results of solutions to Problem (1.1) with $N(u, u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$ when $\gamma \to 1^-$. Due to the similarity of structure in the equations, we observe that our results in Theorems 1.1–1.3 are the same as the blow-up of the solution to the classical wave equation with the nonlinear memory terms $N_{\gamma,q}(u)$, $N_{\gamma,p}(u_t)$ and $N_{\gamma,p,q}(u, u_t)$. To the best of our knowledge, the results in Theorems 1.1–1.3 are new.

Throughout this paper, we write $f \lesssim g$ when there exists a positive constant C such that $f \le Cg$.

The main results in this paper are presented as follows.

We consider Problem (1.1) with $N(u, u_t) = N_{\gamma,q}(u)$ with general initial values in $\mathbb{R}^n$.

Theorem 1.1. *Assume* $q \in (1, 1 + \frac{3-\gamma}{n+\gamma-2}]$ *and*

$$(u_0, u_1, u_2)(x) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n).$$

It holds that

$$\int_{\mathbb{R}^n} (\beta u_2(x) + u_1(x))dx > 0.$$

Then, there is no global weak solution to Problem (1.1) *with* $N(u, u_t) = N_{\gamma,q}(u)$.

We discuss Problem (1.1) with $N(u, u_t) = N_{\gamma,p}(u_t)$ with general initial values in $\mathbb{R}^n$.

Theorem 1.2. *Let* $p \in (1, 1 + \frac{2-\gamma}{n+\gamma-1}]$ *and*

$$(u_0, u_1, u_2)(x) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n).$$

It holds that

$$\int_{\mathbb{R}^n} (\beta u_2(x) + u_1(x))dx > 0.$$

Then, there is no global weak solution to Problem (1.1) *with* $N(u, u_t) = N_{\gamma,p}(u_t)$.

We study Problem (1.1) with $N(u, u_t) = N_{\gamma,p,q}(u, u_t)$ with general initial values in $\mathbb{R}^n$.

Theorem 1.3. *Let p and q satisfy*

$$\begin{cases} 1 < p < 1 + \dfrac{2-\gamma}{n+\gamma-1} \ \ \textit{and} \ \ q > 1, \\ p > 1 \ \ \textit{and} \ \ 1 < q < 1 + \dfrac{3-\gamma}{n+\gamma-2}. \end{cases}$$

Suppose that $(u_0,\, u_1,\, u_2)(x) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *and*

$$\int_{\mathbb{R}^n} (\beta u_2(x) + u_1(x))dx > 0.$$

Then, there is no global weak solution to Problem (1.1) *with*

$$N(u,\, u_t) = N_{\gamma,\,p,\,q}(u,\, u_t).$$

Remark 1.1. Applying the Poincare's inequality, we deduce

$$\int_{\mathbb{R}^n} |\nabla_x u|^q \psi dx \geq \frac{1}{(t+R)^q} \int_{\mathbb{R}^n} |u|^q \psi dx \geq C \int_{\mathbb{R}^n} |u|^q \psi dx.$$

This means that the blow-up result of the solution to Problem (1.1) with

$$N(u,\, u_t) = N_{\gamma,\,q}(\nabla_x u) = c_\gamma \int_0^t (t-s)^{-\gamma} |\nabla_x u(x,s)|^q ds$$

is the same as Theorem 1.1.

Remark 1.2. It is worth noting that our results in Theorems 1.1–1.3 exactly coincide with the blow-up dynamics of solutions to Problem (1.1) with $N(u,\, u_t) = |u|^p,\, |u_t|^p,\, |u_t|^p + |u|^q$ when $\gamma \to 1^-$. Due to the similarity of structure in the equations, our results in Theorems 1.1–1.3 are the same as the blow-up results of the solution to the classical wave equation with the nonlinear memory terms $N_{\gamma,\,q}(u)$, $N_{\gamma,\,p}(u_t)$ and $N_{\gamma,\,p,\,q}(u,\, u_t)$.

2. Proof of Theorem 1.1

2.1. Several definitions and related lemmas

Definition 2.1. [36] Let $f(t) \in L^1(0,T)$. The Riemann Liouville left- and right-sided fractional integrals of the order $\alpha \in (0,1)$ are defined by

$$I^\alpha_{0|t} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-(1-\alpha)} f(\tau) d\tau, \quad t > 0, \tag{2.1}$$
$$I^\alpha_{t|T} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (\tau-t)^{-(1-\alpha)} f(\tau) d\tau, \quad t < T.$$

Definition 2.2. [36] Set $f(t) \in AC[0,T]$. The Riemann Liouville left- and right-sided fractional derivatives of the order $\alpha \in (0,1)$ are defined by

$$D^\alpha_{0|t} f(t) = \frac{d}{dt} I^{1-\alpha}_{0|t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau, \quad t > 0,$$
$$D^\alpha_{t|T} f(t) = -\frac{d}{dt} I^{1-\alpha}_{t|T} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (\tau-t)^{-\alpha} f(\tau) d\tau, \quad t < T. \tag{2.2}$$

Definition 2.3. Let $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $u \in C([0,\infty), H^2(\mathbb{R}^n)) \cap C^1([0,\infty), H^1(\mathbb{R}^n)) \cap C^2([0,\infty), L^2(\mathbb{R}^n))$. $u \in L^q_{loc}([0,\infty) \times \mathbb{R}^n)$ when $N(u, u_t) = N_{\gamma,q}(u)$. $u_t \in L^p_{loc}([0,\infty) \times \mathbb{R}^n)$ when $N(u, u_t) = N_{\gamma,p}(u_t)$. $u \in L^q_{loc}([0,\infty) \times \mathbb{R}^n)$ and $u_t \in L^p_{loc}([0,\infty) \times \mathbb{R}^n)$ when $N(u, u_t) = N_{\gamma,p,q}(u, u_t)$. It holds that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} (-\beta u_{tt}\phi_t - u_t\phi_t - u\Delta\phi + \beta u\Delta\phi_t)(x,s)dxds \\
&-\beta \int_{\mathbb{R}^n} \phi(x,0)u_2(x)dx - \int_{\mathbb{R}^n} \phi(x,0)u_1(x)dx + \beta \int_{\mathbb{R}^n} \Delta\phi(x,0)u_0(x)dx \\
&= \int_0^\infty \int_{\mathbb{R}^n} N(u, u_t)\phi(x,s)dxds, \qquad (2.3)
\end{aligned}$$

where $\phi \in C_0^\infty([0,\infty) \times \mathbb{R}^n)$. Then, u is called a global (in time) weak solution to Problem (1.1).

Lemma 2.1. *[36] Let $T > 0$ and $\alpha \in (0,1)$. It holds that*

$$\int_0^T f(t)D^\alpha_{0|t}g(t)dt = \int_0^T g(t)D^\alpha_{t|T}f(t)dt \qquad (2.4)$$

for $f(t) \in I^\alpha_{t|T}(L^p(0,T))$ and $g(t) \in I^\alpha_{0|t}(L^q(0,T))$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$ with $1 < p, q < \infty$, where

$$I^\alpha_{0|t}(L^q(0,T)) = \{f(t) = I^\alpha_{0|t}h(t) \big| h(t) \in L^q(0,T)\},$$

$$I^\alpha_{t|T}(L^p(0,T)) = \{f(t) = I^\alpha_{t|T}h(t) \big| h(t) \in L^p(0,T)\}.$$

Lemma 2.2. *[37] Let $\alpha \in (0,1)$ and $t \in (0,T)$. Then*

$$D^\alpha_{0|t}I^\alpha_{0|t}f(t) = f(t)$$

for $f(t) \in L^r(0,T)$ with $1 \le r \le \infty$. Moreover, it holds that

$$(-1)^m D^m D^\alpha_{t|T}f(t) = D^{m+\alpha}_{t|T}f(t)$$

for $f(t) \in AC^{m+1}[0,T]$.

Lemma 2.3. *[32] Assume that $w(t) = (1 - \frac{t}{T})^\beta$ with $T > 0$ and $\beta \gg 1$. It holds that*

$$D^{m+\alpha}_{t|T}w(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-m-\alpha)}T^{-(m+\alpha)}(1-\frac{t}{T})^{\beta-\alpha-m},$$

where $m \ge 0$ and $0 < \alpha < 1$. For some positive constant C, it holds that

$$\int_0^T (w(t))^{-\frac{1}{p-1}}|D^{m+\alpha}_{t|T}w(t)|^{\frac{p}{p-1}}dt = CT^{1-(m+\alpha)\frac{p}{p-1}}.$$

Lemma 2.4. *[38] Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy*

$$\varphi(x) = \begin{cases} 1, & |x| \leq \dfrac{1}{2}, \\ 0, & |x| \geq 1. \end{cases}$$

Moreover, it holds that

$$\int_{\mathbb{R}^n} (\varphi_R(x))^{-\frac{1}{p-1}} |(-\Delta)^s \varphi_R(x)|^{\frac{p}{p-1}} dx \lesssim R^{-\frac{2sp}{p-1}+n},$$

where $\varphi_R(x) = \varphi(R^{-1}x)$, $0 < s \leq 1$.

2.2. Proof of Theorem 1.1

We set $\widetilde{\psi} = \varphi_R(x)w(t)$ and the test function $\psi(x,t) = D_{t|T}^\alpha(\widetilde{\psi}(x,t)) = \varphi_R(x)D_{t|T}^\alpha(w(t))$ with $\alpha = 1-\gamma$.

Replacing $\phi(x,s)$ in (2.3) with $N(u,\ u_t) = N_{\gamma,q}(u)$ by applying $\psi(x,s)$ and exploiting (2.1), (2.2) and Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
& J_R + T^{-\alpha}\int_{\mathbb{R}^n} (\beta u_2(x) + u_1(x))\varphi_R(x)dx \\
& -T^{-1-\alpha}\beta \int_{\mathbb{R}^n} (u_1(x) - T^{-1}u_0(x))\varphi_R(x)dx \\
& -T^{-\alpha}\beta \int_{\mathbb{R}^n} \Delta u_0(x)\varphi_R(x)dx - T^{-1-\alpha}\int_{\mathbb{R}^n} u_0(x)\varphi_R(x)dx \\
& = \int_0^T \int_{\mathbb{R}^n} u(x,s)(-\beta\psi_{ttt} + \psi_{tt} - \Delta\psi + \beta\Delta\psi_t)(x,s)dxds \\
& = J_1 + J_2 + J_3 + J_4,
\end{aligned} \tag{2.5}$$

where $J_R = \int_0^T \int_{\mathbb{R}^n} |u(x,s)|^q \widetilde{\psi}(x,s)dxds$.

Utilizing the change of variables $\widetilde{t} = \frac{t}{T}$, $\widetilde{x} = \frac{x}{R}$, Lemmas 2.2 and 2.3 and (2.5) gives rise to

$$\begin{aligned}
|J_1| & \lesssim \int_0^T \int_{\mathbb{R}^n} |u(x,s)|\varphi_R(x)|D_{t|T}^{3+\alpha}(w(s))|dxds \\
& \lesssim J_R^{\frac{1}{q}} (\int_0^T \int_{\mathbb{R}^n} \varphi_R(x)(w(s))^{-\frac{q'}{q}} |D_{t|T}^{3+\alpha}(w(s))|^{q'} dxds)^{\frac{1}{q'}} \\
& \lesssim J_R^{\frac{1}{q}} R^{\frac{n}{q'}} T^{\frac{1}{q'}-3-\alpha},
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
|J_2| & \lesssim \int_0^T \int_{\mathbb{R}^n} |u(x,s)|\varphi_R(x)|D_{t|T}^{2+\alpha}(w(s))|dxds \\
& \lesssim J_R^{\frac{1}{q}} R^{\frac{n}{q'}} T^{\frac{1}{q'}-2-\alpha}.
\end{aligned} \tag{2.7}$$

From Lemmas 2.3 and 2.4 with $s = 1$ and (2.5), we achieve

$$|J_3| \lesssim \int_0^T \int_{\mathbb{R}^n} |u(x,s)|\Delta\varphi_R(x)|D_{t|T}^\alpha(w(s))|dxds$$

$$
\begin{aligned}
&\lesssim \widetilde{J}_R^{\frac{1}{q}}\Big(\int_0^T (w(s))^{-\frac{q'}{q}}|D_{t|T}^{\alpha}(w(s))|^{q'}ds\Big)^{\frac{1}{q'}}\\
&\quad\times\Big(\int_{\{|x|\geq R\}}(\varphi_R(x))^{-\frac{q'}{q}}|\Delta\varphi_R(x)|^{q'}dx\Big)^{\frac{1}{q'}}\\
&\lesssim \widetilde{J}_R^{\frac{1}{q}}R^{\frac{n}{q'}-2}T^{\frac{1}{q'}-\alpha},
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
|J_4| &\lesssim \int_0^T\int_{\mathbb{R}^n}|u(x,s)|\Delta\varphi_R(x)|D_{t|T}^{1+\alpha}(w(s))|dxds\\
&\lesssim \widetilde{J}_R^{\frac{1}{q}}\Big(\int_0^T (w(s))^{-\frac{q'}{q}}|D_{t|T}^{1+\alpha}(w(s))|^{q'}ds\Big)^{\frac{1}{q'}}\\
&\quad\times\Big(\int_{\{|x|\geq R\}}(\varphi_R(x))^{-\frac{q'}{q}}|\Delta\varphi_R(x)|^{q'}dx\Big)^{\frac{1}{q'}}\\
&\lesssim \widetilde{J}_R^{\frac{1}{q}}R^{\frac{n}{q'}-2}T^{\frac{1}{q'}-1-\alpha},
\end{aligned}
\tag{2.9}
$$

where $\widetilde{J}_R = \int_0^T\int_{\{|x|\geq R\}}|u(x,s)|^q\widetilde{\psi}(x,s)dxds$.

Making use of (2.5)–(2.9) and $|\Delta\varphi_R(x)| \lesssim R^{-2}\varphi_R(x)$ and taking $R = T$ yields

$$
\begin{aligned}
&J_R + T^{-\alpha}\int_{\mathbb{R}^n}(\beta u_2(x) + u_1(x))\varphi_R(x)dx\\
&\lesssim J_R^{\frac{1}{q}}R^{\frac{n}{q'}}(T^{\frac{1}{q'}-3-\alpha} + T^{\frac{1}{q'}-2-\alpha}) + \widetilde{J}_R^{\frac{1}{q}}R^{\frac{n}{q'}-2}(T^{\frac{1}{q'}-\alpha} + T^{\frac{1}{q'}-1-\alpha})\\
&\quad+T^{-1-\alpha}\beta\int_{\mathbb{R}^n}(u_1(x) - T^{-1}u_0(x))\varphi_R(x)dx\\
&\quad+T^{-2-\alpha}\beta\int_{\mathbb{R}^n}u_0(x)\varphi_R(x)dx + T^{-1-\alpha}\int_{\mathbb{R}^n}u_0(x)\varphi_R(x)dx\\
&\lesssim J_R^{\frac{1}{q}}T^{\frac{n+1}{q'}-2-\alpha} + T^{-1-\alpha}\beta\int_{\mathbb{R}^n}(u_1(x) - T^{-1}u_0(x))\varphi_R(x)dx\\
&\quad+T^{-2-\alpha}\beta\int_{\mathbb{R}^n}u_0(x)\varphi_R(x)dx + T^{-1-\alpha}\int_{\mathbb{R}^n}u_0(x)\varphi_R(x)dx.
\end{aligned}
\tag{2.10}
$$

It is worth noting that $q \in (1, 1 + \frac{3-\gamma}{n+\gamma-2}]$ is equivalent to $-(2 + \alpha)q' + n + 1 \leq 0$. Therefore, our considerations are divided into two cases.

In the sub-critical case $-(2 + \alpha)q' + n + 1 < 0$, by sending $T \to \infty$ in (2.10), we arrive at a contradiction.

In the critical case $-(2 + \alpha)q' + n + 1 = 0$, according to (2.10), we acquire $J_R \leq C$ for a certain positive constant C as $T \to \infty$. Taking $R = TK^{-1}$ in (2.10), we conclude

$$
J_R \lesssim J_R^{\frac{1}{q}}(T^{-1}K^{-\frac{n}{q'}} + K^{-\frac{n}{q'}}) + \widetilde{J}_R^{\frac{1}{q}}(K^{-(\frac{n}{q'}-2)} + T^{-1}K^{-(\frac{n}{q'}-2)}).
\tag{2.11}
$$

It follows from $u \in L^q((0,\infty)\times\mathbb{R}^n)$ that

$$
\lim_{T\to\infty}\widetilde{J}_R = \lim_{T\to\infty}\int_0^T\int_{\{|x|\geq TK^{-1}\}}|u(x,s)|^q\widetilde{\psi}(x,s)dxds = 0.
$$

Consequently, we acquire the desired result by taking K big enough. This finishes the proof of Theorem 1.1. ■

3. Proof of Theorem 1.2

We assume

$$I_R = \int_0^T \int_{\mathbb{R}^n} |u_t(x,s)|^p \widetilde{\psi}(x,s)dxds,$$
$$\widetilde{I_R} = \int_0^T \int_{\{|x|\geq R\}} |u_t(x,s)|^p \widetilde{\psi}(x,s)dxds.$$

Let $\Psi(t) = \int_t^\infty w(\tau)d\tau$. Then, we have $\Psi'(t) = -w(t)$.

Replacing $\phi(x,s)$ in (2.3) with $N(u, u_t) = N_{\gamma,p}(u_t)$ by using $\psi = D^\alpha_{t|T}(\widetilde{\psi}(x,t)) = \varphi_R(x)D^\alpha_{t|T}(w(t))$ and applying (2.1) and (2.2), Lemmas 2.1 and 2.2 lead to

$$\begin{aligned}
&I_R + T^{-\alpha}\int_{\mathbb{R}^n}(\beta u_2(x) + u_1(x))\varphi_R(x)dx\\
&-\beta T^{-1-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)dx + \int_{\mathbb{R}^n} u_0(x)D^\alpha_{t|T}\Psi(0)\Delta\varphi_R(x)dx\\
&= \int_0^T\int_{\mathbb{R}^n} u_t(x,s)(\beta\psi_{tt} - \psi_t - D^\alpha_{t|T}\Psi(s)\Delta\varphi_R(x) - \beta\Delta\psi)(x,s)dxds\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.1}$$

Combining Lemmas 2.2 and 2.3 and (3.1), we acquire

$$\begin{aligned}
|I_1| &\lesssim \int_0^T\int_{\mathbb{R}^n} |u_t(x,s)|\varphi_R(x)|D^{2+\alpha}_{t|T}(w(s))|dxds\\
&\lesssim I_R^{\frac{1}{p}}\Big(\int_0^T\int_{\mathbb{R}^n}\varphi_R(x)(w(s))^{-\frac{p'}{p}}|D^{2+\alpha}_{t|T}(w(s))|^{p'}dxds\Big)^{\frac{1}{p'}}\\
&\lesssim I_R^{\frac{1}{p}} R^{\frac{n}{p'}} T^{\frac{1}{p'}-2-\alpha},
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
|I_2| &\lesssim \int_0^T\int_{\mathbb{R}^n} |u_t(x,s)|\varphi_R(x)|D^{1+\alpha}_{t|T}(w(s))|dxds\\
&\lesssim I_R^{\frac{1}{p}} R^{\frac{n}{p'}} T^{\frac{1}{p'}-1-\alpha}.
\end{aligned} \tag{3.3}$$

Employing Lemmas 2.3 and 2.4 with $s = 1$ and (3.1) gives rise to

$$\begin{aligned}
|I_3| &\lesssim \int_0^T\int_{\mathbb{R}^n} |u_t(x,s)||\Delta\varphi_R(x)||D^{\alpha}_{t|T}(\Psi(s))|dxds\\
&\lesssim \widetilde{I_R}^{\frac{1}{p}}\Big(\int_0^T (w(s))^{-\frac{p'}{p}}|D^{\alpha}_{t|T}(\Psi(s))|^{p'}ds\Big)^{\frac{1}{p'}}\\
&\quad\times\Big(\int_{\{|x|\geq R\}}(\varphi_R(x))^{-\frac{p'}{p}}|\Delta\varphi_R(x)|^{p'}dx\Big)^{\frac{1}{p'}}\\
&\lesssim \widetilde{I_R}^{\frac{1}{p}} R^{\frac{n}{p'}-2} T^{\frac{1}{p'}+1-\alpha},
\end{aligned} \tag{3.4}$$

where we have utilized the fact that $(w(s))^{-\frac{p'}{p}}(D^{\alpha}_{t|T}\Psi(s))^{p'} \leq T^{(1-\alpha)p'}$.

Analogously, we acquire

$$\begin{aligned}|I_4| &\lesssim \int_0^T \int_{\mathbb{R}^n} |u_t(x,s)|\Delta\varphi_R(x)|D^{\alpha}_{t|T}(w(s))|dxds \\ &\lesssim \widetilde{I}_R^{\frac{1}{p}}(\int_0^T (w(s))^{-\frac{p'}{p}}|D^{\alpha}_{t|T}(w(s))|^{p'}ds)^{\frac{1}{p'}} \\ &\quad\times(\int_{\{|x|\geq R\}} (\varphi_R(x))^{-\frac{p'}{p}}|\Delta\varphi_R(x)|^{p'}dx)^{\frac{1}{p'}} \\ &\lesssim \widetilde{I}_R^{\frac{1}{p}} R^{\frac{n}{p'}-2} T^{\frac{1}{p'}-\alpha}. \end{aligned} \tag{3.5}$$

Taking into account (3.1)–(3.5) and choosing $R = T$, we derive

$$\begin{aligned}& I_R + T^{-\alpha}\int_{\mathbb{R}^n}(\beta u_2(x) + u_1(x))\varphi_R(x)dx \\ &\lesssim I_R^{\frac{1}{p}} R^{\frac{n}{p'}}(T^{\frac{1}{p'}-2-\alpha} + T^{\frac{1}{p'}-1-\alpha}) + \widetilde{I}_R^{\frac{1}{p}} R^{\frac{n}{p'}-2}(T^{\frac{1}{p'}+1-\alpha} + T^{\frac{1}{p'}-\alpha}) \\ &\quad +\beta T^{-1-\alpha}\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)dx - \int_{\mathbb{R}^n} u_0(x) D^{\alpha}_{t|T}\Psi(0)\Delta\varphi_R(x)dx \\ &\lesssim I_R^{\frac{1}{p}} T^{\frac{n+1}{p'}-1-\alpha} + T^{-1-\alpha}\int_{\mathbb{R}^n}(\beta u_1(x) - u_0(x))\varphi_R(x)dx, \end{aligned} \tag{3.6}$$

where we have applied the following:

$$\int_{\mathbb{R}^n} u_0(x) D^{\alpha}_{t|T}\Psi(0)\Delta\varphi_R(x)dx \leq T^{-1-\alpha}\int_{\mathbb{R}^n} u_0(x)\varphi_R(x)dx.$$

It is worth noticing that $p \in (1, 1+\frac{2-\gamma}{n+\gamma-1}]$ is equivalent to $-(1+\alpha)p' + n + 1 \leq 0$. Thus, our discussions are divided into two cases.

In the sub-critical case $-(1+\alpha)p' + n + 1 < 0$, by letting $T \to \infty$ in (3.6), we obtain a contradiction.

In the critical case $-(1+\alpha)p' + n + 1 = 0$, employing (3.6) yields $I_R \leq C$ for a certain positive constant C as $T \to \infty$. Taking $R = TK^{-1}$ in (3.6), we arrive at

$$I_R \lesssim I_R^{\frac{1}{p}}(T^{-1}K^{-\frac{n}{p'}} + K^{-\frac{n}{p'}}) + \widetilde{I}_R^{\frac{1}{p}}(K^{-(\frac{n}{p'}-2)} + T^{-1}K^{-(\frac{n}{p'}-2)}).$$

As a result, we conclude that the solution blows up in finite time when $p \in (1, 1+\frac{2-\gamma}{n+\gamma-1}]$. This ends the proof of Theorem 1.2. ■

4. Proof of Theorem 1.3

Replacing J_R in (2.5) and I_R in (3.1) with $I_R + J_R$ when $N(u, u_t) = N_{\gamma,p,q}(u, u_t)$, we derive the same estimates of J_1–J_4 and I_1–I_4 as in Theorems 1.1 and 1.2. Therefore, we need to consider the following 16 cases in Table 1.

Table 1. Combination of the estimates I_1, J_1, I_2, J_2, I_3, J_3, I_4, J_4.

	Est. I_1, J_1	Est. I_2, J_2	Est. I_3, J_3	Est. I_4, J_4
Case 1	I_1	I_2	I_3	I_4
Case 2	J_1	I_2	I_3	I_4
Case 3	I_1	I_2	I_3	J_4
Case 4	J_1	I_2	I_3	J_4
Case 5	I_1	I_2	J_3	I_4
Case 6	J_1	I_2	J_3	I_4
Case 7	I_1	J_2	I_3	I_4
Case 8	J_1	J_2	I_3	I_4
Case 9	I_1	J_2	J_3	J_4
Case 10	J_1	J_2	J_3	J_4
Case 11	I_1	I_2	J_3	J_4
Case 12	J_1	I_2	J_3	J_4
Case 13	I_1	J_2	I_3	J_4
Case 14	J_1	J_2	I_3	J_4
Case 15	I_1	J_2	J_3	I_4
Case 16	J_1	J_2	J_3	I_4

To verify the blow-up dynamic of the solution, we recognize that it is sufficient to discuss Case 1 and Case 10 by direct calculation.

In Case 1, similar to the derivation in (3.6), we have

$$\begin{aligned}
& I_R + J_R + T^{-\alpha} \int_{\mathbb{R}^n} (\beta u_2(x) + u_1(x))\varphi_R(x)dx \\
\lesssim\ & I_R^{\frac{1}{p}} R^{\frac{n}{p'}} (T^{\frac{1}{p'}-2-\alpha} + T^{\frac{1}{p'}-1-\alpha}) + \widetilde{I_R^{\frac{1}{p}}} R^{\frac{n}{p'}-2} (T^{\frac{1}{p'}+1-\alpha} + T^{\frac{1}{p'}-\alpha}) \\
& + \beta T^{-1-\alpha} \int_{\mathbb{R}^n} u_1(x)\varphi_R(x)dx - \int_{\mathbb{R}^n} u_0(x) D_{t|T}^{\alpha} \Psi(0)\Delta\varphi_R(x)dx.
\end{aligned}$$

Therefore, we obtain that the solution blows up in finite time when $1 < p < 1 + \frac{2-\gamma}{n+\gamma-1}$ and $q > 1$.

In Case 10, similar to the derivation in (2.10), we achieve

$$\begin{aligned}
& I_R + J_R + T^{-\alpha} \int_{\mathbb{R}^n} (\beta u_2(x) + u_1(x))\varphi_R(x)dx \\
\lesssim\ & J_R^{\frac{1}{q}} R^{\frac{n}{q'}} (T^{\frac{1}{q'}-3-\alpha} + T^{\frac{1}{q'}-2-\alpha}) + \widetilde{J_R^{\frac{1}{q}}} R^{\frac{n}{q'}-2} (T^{\frac{1}{q'}-\alpha} + T^{\frac{1}{q'}-1-\alpha}) \\
& + T^{-1-\alpha}\beta \int_{\mathbb{R}^n} (u_1(x) - T^{-1}u_0(x))\varphi_R(x)dx \\
& + T^{-2-\alpha}\beta \int_{\mathbb{R}^n} u_0(x)\varphi_R(x)dx + T^{-1-\alpha} \int_{\mathbb{R}^n} u_0(x)\varphi_R(x)dx.
\end{aligned}$$

As a consequence, we conclude that the solution blows up in finite time when $p > 1$ and $1 < q \leq 1 + \frac{3-\gamma}{n+\gamma-2}$. The proof of Theorem 1.3 is completed. ∎

5. Conclusions

This paper is devoted to establishing the blow-up dynamic of the solution to the Cauchy problem of the semilinear Moore-Gibson-Thompson equation with general initial values and the nonlinear memory terms $N_{\gamma,q}(u)$, $N_{\gamma,p}(u_t)$ and $N_{\gamma,p,q}(u, u_t)$, respectively. We have presented the main results by utilizing the test function technique ($\psi(x,t) = \varphi_R(x) D^{\alpha}_{t|T}(w(t))$) (see Theorems 1.1–1.3). Our main new contribution is that the effects of nonlinear memory terms on the blow-up results of solutions have been obtained. The problems studied in [2, 32] were extended to Problem (1.1) with $N(u, u_t) = N_{\gamma,q}(u)$ (see Theorem 1.1). We have supplemented the formation of singularities for the solution to Problem (1.1) with $N(u, u_t) = N_{\gamma,p}(u_t)$ and $N_{\gamma,p,q}(u, u_t)$ by applying a method which is different from the approach used [5] (see Theorems 1.2 and 1.3). It is worth noting that our results in Theorems 1.1–1.3 exactly coincide with the blow-up results of the solution to Problem (1.1) with $N(u, u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$ when $\gamma \to 1^-$. To the best of our knowledge, the results in Theorems 1.1–1.3 are new.

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Conflict of interest

This work does not have any conflict of interest.

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