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Research article

Family of ruled surfaces generated by equiform Bishop spherical image in Minkowski 3-space

Emad Solouma[1,2,*] **and Mohamed Abdelkawy**[1,2]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Saudi Arabia

[2] Department of Mathematics and Information Science, Faculty of Science, Beni-Suef University, Egypt

* **Correspondence:** Email: emmahmoud@imamu.edu.sa, emadms74@gmail.com.

Abstract: The study of a family of equiform Bishop spherical image ruled surfaces created by some specific curves such as spherical image in Minkowski 3-space using equiform Bishop frame of that curve is presented in this paper. We also offer the necessary criteria for these surfaces to be equiform Bishop developable and equiform Bishop minimum in relation to equiform Bishop curvatures, as well as when the curve is enclosed in a plane. Finally, we provide an example, such as these surfaces.

Keywords: equiform Bishop frame; Minkowski 3-space; spherical image; ruled surfaces
Mathematics Subject Classification: 53B30, 53C40, 53C50

1. Introduction

One of several primary goals of classical differential geometry is to examine some surface's classes with unique properties in E^3, such as developable and minimal. There are numerous types of surfaces, including cyclic, revolution, helicoid, rotational, canal, and governed surfaces. This type of surface plays a vital role and has a wide range of applications Physics, Computer Aided Geometric Design, and the study of design challenges in spatial mechanisms, among other subjects [9, 12]. Many studies have been done on the features of these surfaces in Euclidean and Minkowski spaces, as well as certain characterizations [2, 3, 8, 10, 13, 14, 16, 18, 19].

In this paper, we look at a type of ruled surface known as equiform Bishop spherical image ruled surfaces using the equiform Bishop frame in E_1^3. The main results are presented in theorems that concert the necessary and sufficient conditions for those ruled surfaces to be equiform Bishop developable and equiform Bishop minimal.

2. Preliminaries

The Lorentzian product in Minkowski 3-dimentional space E_1^3 is define by

$$\mathcal{P} = d\varsigma_1^2 + d\varsigma_2^2 - d\varsigma_3^2,$$

where $(\varsigma_1, \varsigma_2, \varsigma_3)$ is a rectangular coordinate system of E_1^3. An arbitrary $r \in E_1^3$ vector is one of the following; spacelike if $\mathcal{P}(r,r) > 0$ or $r = 0$, timelike if $\mathcal{P}(r,r) < 0$ and null if $\mathcal{P}(r,r) = 0$ and $r \neq 0$. Similarly, a curve $\phi = \phi(s)$ can be spacelike, timelike or null if its $\phi'(s)$ is spacelike, timelike or null [15, 17].

Let $\varphi = \varphi(s)$ is a spacelike curve with a timelike binormal. Denoted $\{t, n, b\}$ be the moving Frenet frame of spacelike curve ϕ, then $\{t, n, b\}$ has the following properties [7, 15, 17]:

$$\begin{pmatrix} \dot{t}(s) \\ \dot{n}(s) \\ \dot{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}, \tag{2.1}$$

where $\left(\cdot = \dfrac{d}{ds}\right)$, $\mathcal{P}(t,t) = 1$, $\mathcal{P}(n,n) = 1$, $\mathcal{P}(b,b) = -1$ and $\mathcal{P}(t,n) = \mathcal{P}(t,b) = \mathcal{P}(n,b) = 0$.

The Bishop frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative [4, 5].

Let $\{t, n_1, n_2\}$ denote the Bishop frame of the spacelike curve $\phi(s)$ with timelike binormal. Then $\{t, n_1, n_2\}$ is expressed as [5, 11].

$$\begin{pmatrix} \dot{t}(s) \\ \dot{n}_1(s) \\ \dot{n}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & -\kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n_1(s) \\ n_2(s) \end{pmatrix}, \tag{2.2}$$

where $\mathcal{P}(t,t) = 1$, $\mathcal{P}(n_1,n_1) = 1$, $\mathcal{P}(n_2,n_2) = -1$ and $\mathcal{P}(t,n_1) = \mathcal{P}(t,n_2) = \mathcal{P}(n_1,n_2) = 0$. We call $\kappa_1(s)$ and $\kappa_2(s)$ as Bishop curvatures. The relation can be expressed as

$$\begin{aligned} t(s) &= t(s), \\ n(s) &= n_1 \cosh\vartheta(s) - n_2 \sinh\vartheta(s), \\ b(s) &= -n_1 \sinh\vartheta(s) + n_2 \cosh\vartheta(s), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \theta(s) &= arg\tanh\left(\frac{\kappa_2}{\kappa_1}\right), \ \kappa_1 \neq 0, \\ \tau(s) &= -\frac{d\vartheta(s)}{ds}, \\ \kappa(s) &= \sqrt{|\kappa_1^2(s) - \kappa_2^2(s)|}, \end{aligned} \tag{2.4}$$

and

$$\kappa_1(s) = \kappa(s)\cosh\vartheta(s), \qquad \kappa_2(s) = \kappa(s)\sinh\vartheta(s).$$

Let $\varphi : I \to E^3_1$ be a spacelike curve with a timelike binormal in E^3_1. The equiform parameter of φ by $\theta = \int \kappa_1 ds$. Then $\varrho = \dfrac{ds}{d\theta}$, where $\varrho = \dfrac{1}{\kappa_1}$ the radius of curvature of the curve φ. We recall $\{T, B_1, B_2,\}$ be the moving equiform Bishop frame where $T(\theta) = \varrho\, t(s)$, $B_1(\theta) = \varrho\, n_1(s)$ and $B_2(\theta) = \varrho\, n_2(s)$ and the equiform Bishop curvatures of the curve $\varphi = \varphi(\vartheta)$ are $k_1(\theta) = \dot{\varrho} = \dfrac{d\varrho}{ds}$ and $k_2(\theta) = \left(\dfrac{\kappa_2}{\kappa_1}\right)$. Then, the frame $\{T, B_1, B_2,\}$ of φ is given as [1,20–22]:

$$\begin{pmatrix} T'(\theta) \\ B_1'(\theta) \\ B_2'(\theta) \end{pmatrix} = \begin{pmatrix} k_1(\theta) & 1 & -k_2(\theta) \\ -1 & k_1(\theta) & 0 \\ -k_2(\theta) & 0 & k_1(\theta) \end{pmatrix} \begin{pmatrix} T(\theta) \\ B_1(\theta) \\ B_2(\theta) \end{pmatrix}, \tag{2.5}$$

where $\left(' = \dfrac{d}{d\theta}\right)$, $\mathcal{P}(T,T) = \varrho^2$, $\mathcal{P}(B_1,B_1) = \varrho^2$, $\mathcal{P}(B_2,B_2) = -\varrho^2$, and $\mathcal{P}(T,B_1) = \mathcal{P}(T,B_2) = \mathcal{P}(B_1,B_2) = 0$.

The pseudo-Riemannian sphere of unit radius and with center in the origin in the space E^3_1 is defined by

$$S^2_1 = \{x \in E^3_1 : \mathcal{P}(x,x) = 1\}.$$

A ruled surface Γ in E^3_1 can be representation as

$$\Theta(s,\upsilon) = \varphi(s) + \upsilon X(s), \tag{2.6}$$

where $\varphi(s)$ is Θ's base curve and $X(s)$ be the unit represents a space curve which representing the direction of straight line [6].

The Θ's normal vector field m defined by [17]:

$$m = \frac{\frac{\partial\Theta}{\partial s} \times \frac{\partial\Theta}{\partial\upsilon}}{\left\|\frac{\partial\Theta}{\partial s} \times \frac{\partial\Theta}{\partial\upsilon}\right\|}.$$

The Θ's components of the first and second fundamental forms are

$$e_{11} = \|\Theta_s\|^2,\ e_{12} = \langle\Theta_s, \Theta_\upsilon\rangle,\ e_{22} = \|\Theta_\upsilon\|^2,$$

$$L_{11} = \langle\Gamma_{ss}, m\rangle,\ L_{12} = \langle\Theta_{s\upsilon}, m\rangle,\ L_{22} = \langle\Theta_{\upsilon\upsilon}, m\rangle.$$

The Gaussian curvature K and the mean curvature H respectively are given by [17]:

$$\begin{aligned} K &= \frac{L_{11} L_{22} - L_{22}^2}{e_{11}e_{22} - e_{12}^2}, \\ H &= \frac{e_{11} L_{22} + e_{22} L_{11} - e_{12} L_{12}}{2(e_{11}e_{22} - e_{12}^2)}. \end{aligned} \tag{2.7}$$

A ruled surface is developable if and only if $K = 0$ and minimal if and only if $H = 0$ [17].

3. Some characterizations of equiform Bishop spherical image ruled surfaces

In this section, we introduce the equiform Bishop spherical image ruled surfaces in Mikowski 3-space E^3_1 via the equiform Bishop frame $\{T, B_1, B_2\}$. Also, we study some geometric properties that make these surfaces have $K = 0$ and $H = 0$.

3.1. T-equiform Bishop spherical image ruled surface

Definition 3.1. Let $\varphi = \varphi(\theta)$ be a regular spacelike curve in E_1^3 via equiform Bishop frame (2.5). The T-equiform Bishop spherical image ruled surface is defined as

$$\Phi(\theta, \upsilon) = T(\theta) + \upsilon\left[x_1 T(\theta) + x_2 B_1(\theta) + x_3 B_2(\theta)\right]: \quad x_1^2 + x_2^2 - x_3^2 = \varrho^2. \tag{3.1}$$

Consider T-equiform Bishop spherical image ruled surface (3.1), the natural frame is given by

$$\begin{aligned} \Phi_\theta &= \left[k_1(\upsilon x_1 + 1) + \upsilon(x_3 k_2 - x_2)\right] T(\theta) + \left[\upsilon(x_2 k_1 + x_1) + 1\right] B_1(\theta) \\ &\quad + \left[\upsilon(x_3 k_1 - x_1 k_1) - k_2\right] B_2(\theta), \\ \Phi_\upsilon &= x_1 T(\theta) + x_2 B_1(\theta) + x_3 B_2(\theta). \end{aligned} \tag{3.2}$$

With the above equation, we can derive the component parts of the first and second fundamental forms of Φ as follows:

$$\begin{aligned} E_\Phi &= \varrho^2\Big\{\left[k_1(\upsilon x_1 + 1) + \upsilon(x_3 k_2 - x_2)\right]^2 + \left[\upsilon(x_2 k_1 + x_1) + 1\right]^2 \\ &\quad - \left[\upsilon(x_3 k_1 - x_1 k_1) - k_2\right]^2\Big\}, \\ F_\Phi &= \varrho^2\left\{\upsilon\varrho^2 k_1 + 2\upsilon x_1 x_3 k_2 + x_1 k_1 + x_2\right\}, \\ G_\Phi &= \varrho^2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} e_\Phi &= \frac{\varrho(\mu_1\varepsilon_1 + \mu_2\varepsilon_2 - \mu_3\varepsilon_3)}{\sqrt{\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2}}, \\ f_\Phi &= \frac{\varrho}{\sqrt{\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_3^2}}\Big\{\varepsilon_1(x_1 k_1 + x_3 k_2 - x_2) + \varepsilon_2(x_2 k_1 + x_1) - \varepsilon_3(x_3 k_1 - x_1 k_2)\Big\}, \\ g_\Phi &= 0. \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \varepsilon_1 &= \upsilon x_3(x_1 + 1) + x_2 k_2(\upsilon x_1 + 1), \\ \varepsilon_2 &= x_3(\upsilon x_2 - k_1) - \upsilon k_2(x_1^2 + x_3^2) - x_1 k_2, \\ \varepsilon_3 &= \upsilon(x_1^2 + x_2^2) - x_2(\upsilon x_3 k_2 + k_1) + x_1, \end{aligned}$$

and

$$\begin{aligned} \mu_1 &= k_1\left[k_1(\upsilon x_1 + 1) + \upsilon(x_3 k_2 - x_2)\right] - k_2\left[\upsilon(x_3 k_1 - x_1 k_2) - k_2\right] - \upsilon(x_2 k_1 + 1)\upsilon(x_2 k_1 + x_1) \\ &\quad + \upsilon(x_1 k_1' + x_3 k_2') - 1, \\ \mu_2 &= k_1\left[\upsilon(x_2 k_1 + x_1) + 1\right] + k_1(\upsilon x_1 + 1) + \upsilon(x_3 k_2 - x_2) + \upsilon x_2 k_1', \\ \mu_3 &= -k_2\left[k_1(\upsilon x_1 + 1) + \upsilon(x_3 k_2 - x_2)\right] + \upsilon(x_2 k_1 + x_1) + \upsilon(x_3 k_1' - x_1 k_2') - k_2' + 1. \end{aligned}$$

Using the data described above, the equiform Bishop Gaussian curvature K_Φ and equiform Bishop mean curvature H_Φ are calculated as follows:

$$K_\Phi = \frac{-\varrho^2}{\Delta_1(\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2)}\Big\{\varepsilon_1(x_1k_1+x_3k_2-x_2)+\varepsilon_2(x_2k_1+x_1)-\varepsilon_3(x_3k_1-x_1k_2)\Big\}^2,$$

$$H_\Phi = \frac{\varrho^3}{2\Delta_1\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}}\Big\{[\upsilon\varrho^2k_1+2\upsilon x_1x_3k_2+x_1k_1+x_2][\varepsilon_1(x_1k_1+x_3k_2-x_2) \qquad (3.5)$$

$$+\varepsilon_2(x_2k_1+x_1)-\varepsilon_3(x_3k_1-x_1k_2)]+\mu_1\varepsilon_1+\mu_2\varepsilon_2-\mu_3\varepsilon_3\Big\},$$

where

$$\Delta_1 = \varrho^4\Big\{[k_1(\upsilon x_1+1)+\upsilon(x_3k_2-x_2)]^2+[\upsilon(x_2k_1+x_1)+1]^2-[\upsilon(x_3k_1-x_1k_1)-k_2]^2 - \Big[\upsilon\varrho^2k_1+2\upsilon x_1x_3k_2+x_1k_1+x_2\Big]^2\Big\}.$$

Theorem 3.1. *Let $\Phi=\Phi(\theta,\upsilon)$ is T-equiform Bishop spherical image ruled surface in E_1^3 given by (3.1). Then, Φ at the point $(\theta,0)$ is equiform Bishop flat if and only if*

$$x_2k_2(x_1k_1+x_3k_2-x_2)+(x_1-x_2k_1)(x_3k_1-x_1k_2)-(x_2k_1+x_1)(x_3k_1+x_1k_2)=0.$$

Theorem 3.2. *Let $\Phi=\Phi(\theta,\upsilon)$ is T-equiform Bishop spherical image ruled surface in E_1^3 given by (3.1). Then, Φ at the point $(\theta,0)$ is equiform Bishop minimal if and only if the equiform curvatures satisfy the following differential equation*

$$x_2k_2(k_1^2+k_2^2-1)-2k_1(x_3k_1+x_1k_2)-(k_2'+k_2-1)+(x_1k_1+x_2)[x_2k_2(x_1k_1+x_3k_2-x_2) - (x_3k_1+x_1k_2)(x_2k_1+x_1)+(x_2k_1+x_1)(x_3k_1-x_1k_2)]=0.$$

Case 3.1. *At $x_1=0$, the T-equiform Bishop spherical image ruled surface (3.1) has the following:*

$$K_\Phi = \frac{x_2^2k_2^2(x_3k_2-x_2)^2}{\varrho^2(\kappa^2+x_2+1)(x_3^2k_1^2-x_2^2\kappa^2)},$$

$$H_\Phi = \frac{x_2k_2(k_1^2+k_2^2-1)-x_2k_1(k_2'+k_2-1)-2x_3k_1^2+x_2[x_2x_3(k_1^2+k_2^2)-x_2^2k_2-x_3^2k_1^2]}{2\varrho(\kappa^2+x_2+1)\sqrt{x_3^2k_1^2-x_2^2\kappa^2}}. \qquad (3.6)$$

Corollary 3.1. *At the point $(\theta,0)$ the T-equiform Bishop spherical image ruled surface (3.1) with $x_1=0$ is:*

1). *Equiform Bishop flat surface if the equiform curve $\varphi(\theta)$ is contained in a plane.*

2). *Equiform Bishop flat surface if only if $k_2=\left(\frac{x_2}{x_3}\right)$.*

3). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$x_2k_2(k_1^2+k_2^2-1)-x_2k_1(k_2'+k_2-1)-2x_3k_1^2+x_2[x_2x_3(k_1^2+k_2^2)-x_2^2k_2-x_3^2k_1^2]=0.$$

Case 3.2. *At $x_2 = 0$, the T-equiform Bishop spherical image ruled surface (3.1) has the following:*

$$
\begin{aligned}
K_\Phi &= \frac{4x_1^4 k_2^2}{\varrho^2(\kappa^2 + x_2 + 1)\left[(x_3k_1 + x_1k_2)^2 - x_1^2\right]}, \\
H_\Phi &= \frac{x_1(k_2' + k_2 - 1) - 2k_1(x_3k_1 + x_1k_2) + x_1k_1(2x_1^2k_2^2 - x_3^2k_1^2 - x_1x_3k_1)}{2\varrho(\kappa^2 + x_2 + 1)\sqrt{(x_3k_1 + x_1k_2)^2 - x_1^2}}.
\end{aligned}
\tag{3.7}
$$

Corollary 3.2. *At the point $(\theta, 0)$ the T-equiform Bishop spherical image ruled surface (3.1) with $x_2 = 0$ is:*

1). *Equiform Bishop flat surface if the equiform curve $\varphi(\theta)$ is contained in a plane.*

2). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$x_1(k_2' + k_2 - 1) - 2k_1(x_3k_1 + x_1k_2) + x_1k_1(2x_1^2k_2^2 - x_3^2k_1^2 - x_1x_3k_1) = 0.$$

Case 3.3. *At $x_3 = 0$, the T-equiform Bishop spherical image ruled surface (3.1) has the following:*

$$
\begin{aligned}
K_\Phi &= \frac{k_2^2(x_1x_2k_1 + 2x_1^2 + x_2^2)^2}{\varrho^2(\kappa^2 + x_2 + 1)\left[(x_1^2 + x_2^2)k_2^2 - (x_1 - x_2k_1)^2\right]}, \\
H_\Phi &= \frac{1}{2\varrho(\kappa^2 + x_2 + 1)\sqrt{(x_1^2 + x_2^2)k_2^2 - (x_1 - x_2k_1)^2}}\Big\{(x_1 - x_2k_1)(k_2' + k_2 - 1) \\
&\quad - x_2k_2\left[(x_1^2 + 1)k_1^2 + k_2^2 - x_2^2 - 1\right] + x_1k_1(x_1k_1 + x_2)(2x_1k_2 - x_2k_1) - 2x_1k_1k_2\Big\}.
\end{aligned}
\tag{3.8}
$$

Corollary 3.3. *At the point $(\theta, 0)$, the T-equiform Bishop spherical image ruled surface (3.1) with $x_3 = 0$ is:*

1). *Equiform Bishop flat surface if the equiform curve $\varphi(\theta)$ is contained in a plane.*

2). *Equiform Bishop flat surface if only if $k_1 = -\dfrac{2x_1^2 + x_2^2}{x_1x_2}$.*

3). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$(x_1 - x_2k_1)(k_2' + k_2 - 1) - x_2k_2\left[(x_1^2 + 1)k_1^2 + k_2^2 - x_2^2 - 1\right] + x_1k_1(x_1k_1 + x_2)(2x_1k_2 - x_2k_1) - 2x_1k_1k_2 = 0.$$

Case 3.4. *At $x_1 = x_3 = 0,\ x_2 = \varrho$, the T-equiform Bishop spherical image ruled surface (3.1) has the following:*

$$
\begin{aligned}
K_\Phi &= \frac{-k_2^2}{\kappa^2(\kappa^2 + \varrho + 1)}, \\
H_\Phi &= \frac{k_2(k_1^2 + k_2^2 - \varrho^2 - 1) - k_1(k_2' + k_2 - 1)}{2\varrho(\kappa^2 + \varrho + 1)\sqrt{\left|k_2^2 - k_1^2\right|}}
\end{aligned}
\tag{3.9}
$$

Corollary 3.4. *At the point* $(\theta, 0)$ *the* T*-equiform Bishop spherical image ruled surface (3.1) with* $x_1 = x_3 = 0,\ x_2 = \varrho$ *is:*

1). *Equiform Bishop flat surface if the equiform curve* $\varphi(\theta)$ *is contained in a plane.*

2). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$k_2(k_1^2 + k_2^2 - \varrho^2 - 1) - k_1(k_2' + k_2 - 1) = 0.$$

Case 3.5. *At* $x_2 = x_3 = 0,\ x_1 = \varrho$*, the* T*-equiform Bishop spherical image ruled surface (3.1) has the following:*

$$\begin{aligned} K_\Phi &= \frac{4k_2^2}{(k_2^2 - 1)(\kappa^2 + \varrho k_1 + 1)}, \\ H_\Phi &= \frac{k_2' + k_2 - \varrho^2 k_2(k_2 - 1) - 2k_1k_2 - 1}{2\varrho(\kappa^2 + \varrho k_1 + 1)\sqrt{k_2^2 - 1}}. \end{aligned} \tag{3.10}$$

Corollary 3.5. *At the point* $(\theta, 0)$ *the* T*-equiform Bishop spherical image ruled surface (3.1) with* $x_2 = x_3 = 0,\ x_1 = \varrho$ *is:*

1). *Equiform Bishop flat surface if the equiform curve* $\varphi(\theta)$ *is contained in a plane.*

2). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$k_2' + k_2 - \varrho^2 k_2(k_2 - 1) - 2k_1k_2 - 1 = 0.$$

3.2. B_1-equiform Bishop spherical image ruled surface

Definition 3.2. Let $\varphi = \varphi(\theta)$ be a regular spacelike curve in E_1^3 via equiform Bishop frame (2.5). The B_1-equiform Bishop spherical image ruled surface is defined as

$$\Omega(\theta, \upsilon) = B_1(\theta) + \upsilon\left[x_1T(\theta) + x_2B_1(\theta) + x_3B_2(\theta)\right] : \quad x_1^2 + x_2^2 - x_3^2 = \varrho^2. \tag{3.11}$$

Now, the natural frame B_1-equiform Bishop spherical image ruled surface (3.11)is given by

$$\begin{aligned} \Omega_\theta &= \left[\upsilon(x_1k_1 - x_3k_2 - x_2) - 1\right]T(\theta) + \left[\upsilon(x_2k_1 + x_1) + k_1\right]B_1(\theta) \\ &\quad + \upsilon(x_3k_1 - x_1k_1)B_2(\theta), \\ \Omega_\upsilon &= x_1T(\theta) + x_2B_1(\theta) + x_3B_2(\theta). \end{aligned} \tag{3.12}$$

By the above equation, we can surmise the component parts of the first and second fundamental forms of Ω as shown in:

$$\begin{aligned} E_\Omega &= \varrho^2\Big\{\left[\upsilon(x_1k_1 - x_3k_2 - x_2) - 1\right]^2 + \left[\upsilon(x_2k_1 + x_1) + k_1\right]^2 - \upsilon^2(x_3k_1 - x_1k_1)^2\Big\}, \\ F_\Omega &= \varrho^2\left(\upsilon\varrho^2k_1 + x_2k_1 - x_1\right), \\ G_\Omega &= \varrho^2, \end{aligned} \tag{3.13}$$

$$
\begin{aligned}
e_{\Omega} &= \frac{\varrho(y_1\gamma_1 + y_2\gamma_2 - y_3\gamma_3)}{\sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}},\\
f_{\Omega} &= \frac{\varrho}{\sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}}\Big\{\gamma_1(x_1k_1 + x_3k_2 - x_2) + \gamma_2(x_2k_1 + x_1) - \gamma_3(x_3k_1 - x_1k_2)\Big\}, \qquad (3.14)\\
g_{\Omega} &= 0.
\end{aligned}
$$

Where

$$
\begin{aligned}
\gamma_1 &= \upsilon x_1(x_2k_2 + x_3) + x_3k_1,\\
\gamma_2 &= \upsilon k_2(x_3^2 - x_1^2) + x_3(\upsilon x_2 + 1),\\
\gamma_3 &= \upsilon(x_1^2 + x_2^2) + x_3(\upsilon x_3k_2 + 1) + x_1k_1,
\end{aligned}
$$

and

$$
\begin{aligned}
y_1 &= k_1\left[\upsilon(x_1k_1 - x_3k_2 - x_2) - 1\right] - \upsilon k_2(x_3k_1 - x_1k_2) - \upsilon(x_2k_1 + x_1) + \upsilon(x_1k_1' + x_3k_2') - k_1,\\
y_2 &= k_1\left[\upsilon(x_2k_1 + x_1) + k_1\right] + \upsilon(x_1k_1 - x_3k_2 - x_2) + \upsilon x_2k_1' + k_2' - 1,\\
y_3 &= -k_2\left[\upsilon(x_1k_1 - x_3k_2 - x_2) - 1\right] + \upsilon k_1(x_3k_1 - x_1k_2) + \upsilon(x_3k_1' - x_1k_2').
\end{aligned}
$$

Using the data described above, the equiform Bishop Gaussian curvature K_{Ω} and equiform Bishop mean curvature H_{Ω} are calculated as follows:

$$
\begin{aligned}
K_{\Omega} &= \frac{-\varrho^2}{\Delta_2(\gamma_1^2 + \gamma_2^2 - \gamma_3^2)}\Big\{\gamma_1(x_1k_1 + x_3k_2 - x_2) + \gamma_2(x_2k_1 + x_1) - \gamma_3(x_3k_1 - x_1k_2)\Big\}^2,\\
H_{\Omega} &= \frac{\varrho^3}{2\Delta_2\sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}}\Big\{[x_1 - \upsilon\varrho^2k_1 - x_2k_1][\gamma_1(x_1k_1 + x_3k_2 - x_2) \qquad (3.15)\\
&\qquad + \gamma_2(x_2k_1 + x_1) - \gamma_3(x_3k_1 - x_1k_2)] + y_1\gamma_1 + y_2\gamma_2 - y_3\gamma_3\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_2 =&\varrho^4\Big\{\left[\upsilon(x_1k_1 - x_3k_2 - x_2) - 1\right]^2 + \left[\upsilon(x_2k_1 + x_1) + k_1\right]^2 - \upsilon^2(x_3k_1 - x_1k_1)^2\\
&- \left(\upsilon\varrho^2k_1 + x_2k_1 - x_1\right)^2\Big\}.
\end{aligned}
$$

Theorem 3.3. *Let $\Omega = \Omega(\theta, \upsilon)$ is B_1-equiform Bishop spherical image ruled surface in E_1^3 given by (3.11). Then, Ω at the point $(\theta, 0)$ is equiform Bishop flat if and only if*

$$x_1k_1k_2(x_1 + x_3) + x_2k_1(x_3 - x_1) + x_3(x_1k_2 - x_3k_1) + x_1x_3 = 0.$$

Theorem 3.4. *Let $\Omega = \Omega(\theta, \upsilon)$ is B_1-equiform Bishop spherical image ruled surface in E_1^3 given by (3.11). Then, Ω at the point $(\theta, 0)$ is equiform Bishop minimal if and only if the equiform curvatures satisfy the following differential equation*

$$
\begin{aligned}
&x_3(k_1' + k_1^2 - 1) - k_2(x_1k_1 + x_3) - 2x_3k_1^2 + (x_2k_1 - x_1)[x_3k_1(x_1k_1 + x_3k_2 - x_2)\\
&+ x_3(x_2k_1 + x_1) - (x_1k_1 + x_3)(x_3k_1 - x_1x_2)] = 0.
\end{aligned}
$$

Case 3.6. *At $x_1 = 0$, the B_1-equiform Bishop spherical image ruled surface (3.11) has the following:*

$$\begin{aligned} K_\Omega &= \frac{(x_2 - x_3)^2}{\varrho^2[(x_2^2 - 1)k_1^2 - 1]}, \\ H_\Omega &= \frac{k_1^2 + k_2 - k_2' - x_2x_3k_1^2(k_2 - 1) + 1}{2\varrho k_1[(x_2^2 - 1)k_1^2 - 1]}. \end{aligned} \tag{3.16}$$

Corollary 3.6. *At the point $(\theta, 0)$ the B_1-equiform Bishop spherical image ruled surface (3.11) with $x_1 = 0$ is:*

1). *Equiform Bishop flat surface if and only if $x_2 = x_3$.*

2). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$k_1^2 + k_2 - k_2' - x_2x_3k_1^2(k_2 - 1) + 1 = 0.$$

Case 3.7. *At $x_2 = 0$, the B_1-equiform Bishop spherical image ruled surface (3.11) has the following:*

$$\begin{aligned} K_\Omega &= -\frac{[x_1k_1k_2(x_1 + x_3) - x_3(x_3k_1 - x_1)]^2}{\varrho^2(k_1^2 - x_1^2 + 1)[(x_3^2 - x_1^2)k_1^2 - 2x_1x_3k_1]}, \\ H_\Omega &= \frac{x_3(k_1' - k_1^2 - 1) - k_2(x_1k_1 + x_3) - x_1[x_3^2k_1(k_2 - 1) + x_1x_3]}{2\varrho(k_1^2 - x_1^2 + 1)\sqrt{(x_3^2 - x_1^2)k_1^2 - 2x_1x_3k_1}}. \end{aligned} \tag{3.17}$$

Corollary 3.7. *At the point $(\theta, 0)$ the B_1-equiform Bishop spherical image ruled surface (3.11) with $x_2 = 0$ is:*

1). *Equiform Bishop flat surface if and only if $x_1k_1k_2(x_1 + x_3) - x_3(x_3k_1 - x_1) = 0$.*

2). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$x_3(k_1' - k_1^2 - 1) - k_2(x_1k_1 + x_3) - x_1[x_3^2k_1(k_2 - 1) + x_1x_3] = 0.$$

Remark 3.1. *On the B_1-equiform Bishop spherical image ruled surface (3.11) at the point $(\theta, 0)$ with $x_3 = 0$, $x_1 = x_3 = 0$, $x_2 = \varrho$ and $x_2 = x_3 = 0$, $x_2 = \varrho$ the equiform Bishop mean curvature H_Ω is undefined.*

3.3. B_2-equiform Bishop spherical image ruled surface

Definition 3.3. Let $\varphi = \varphi(\theta)$ be a regular spacelike curve in E_1^3 via equiform Bishop frame (2.5). The B_2-equiform Bishop spherical image ruled surface is defined as

$$\Psi(\theta, \upsilon) = B_2(\theta) + \upsilon\,[x_1T(\theta) + x_2B_1(\theta) + x_3B_2(\theta)] : \quad x_1^2 + x_2^2 - x_3^2 = \varrho^2. \tag{3.18}$$

Now, the natural frame B_2-equiform Bishop spherical image ruled surface (3.18)is given by

$$\begin{aligned} \Psi_\theta &= [\upsilon(x_1k_1 - x_3k_2 - x_2) - k_2]\,T(\theta) + \upsilon(x_2k_1 + x_1)B_1(\theta) \\ &\quad + [\upsilon(x_3k_1 - x_1k_2) + k_1]\,B_2(\theta), \\ \Psi_\upsilon &= x_1T(\theta) + x_2B_1(\theta) + x_3B_2(\theta). \end{aligned} \tag{3.19}$$

By the above equation, we can surmise the component parts of the first and second fundamental forms of Ψ as shown in:

$$\begin{aligned} E_\Psi &= \varrho^2\Big\{ [\upsilon(x_1k_1 - x_3k_2 - x_2) - k_2]^2 + \upsilon^2(x_2k_1 + x_1)^2 - [\upsilon(x_3k_1 - x_1k_2) + k_1]^2 \Big\}, \\ F_\Psi &= \varrho^2 \Big[\upsilon\varrho^2 k_1 - (x_1k_2 + x_3k_1)\Big], \\ G_\Psi &= \varrho^2, \end{aligned} \tag{3.20}$$

$$\begin{aligned} e_\Psi &= \frac{\varrho(\lambda_1\nu_1 + \lambda_2\nu_2 - \lambda_3\nu_3)}{\sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_3^2}}, \\ f_\Psi &= \frac{\varrho}{\sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_3^2}}\Big\{\lambda_1(x_1k_1 + x_3k_2 - x_2) + \lambda_2(x_2k_1 + x_1) - \lambda_3(x_3k_1 - x_1k_2)\Big\}, \\ g_\Psi &= 0, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \lambda_1 &= \upsilon x_1x_3(k_2 + 1) - x_3k_1, \\ \lambda_2 &= \upsilon k_2(x_3^2 - x_1^2) + x_3(\upsilon x_2 + k_2) + x_1k_1, \\ \lambda_3 &= \upsilon(x_1^2 + x_2^2) + x_2k_2(\upsilon x_3 + 1), \end{aligned}$$

and

$$\begin{aligned} \nu_1 &= k_1 [\upsilon(x_1k_1 - x_3k_2 - x_2) - k_2] - k_2 [\upsilon(x_3k_1 - x_1k_2) + k_1] - \upsilon(x_2k_1 + x_1) \\ &\quad + \upsilon(x_1k_1' - x_3k_2') - k_2', \\ \nu_2 &= \upsilon k_1(x_2k_1 + x_1) + \upsilon(x_1k_1 - x_3k_2 - x_2) + \upsilon x_2k_1' - k_2, \\ \nu_3 &= k_1 [\upsilon(x_3k_1 - x_1k_2) + k_1] - k_2 [\upsilon(x_1k_1 - x_3k_2 - x_2) - k_2] + \upsilon(x_3k_1' - x_1k_2') + k_1'. \end{aligned}$$

Using the data described above, the equiform Bishop Gaussian curvature K_Ψ and equiform Bishop mean curvature H_Ψ are calculated as follows:

$$\begin{aligned} K_\Psi &= \frac{-\varrho^2}{\Delta_3(\lambda_1^2 + \lambda_2^2 - \lambda_3^2)}\Big\{\lambda_1(x_1k_1 + x_3k_2 - x_2) + \lambda_2(x_2k_1 + x_1) - \lambda_3(x_3k_1 - x_1k_2)\Big\}^2, \\ H_\Psi &= \frac{\varrho^3}{2\Delta_3\sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_3^2}}\Big\{\lambda_1\nu_1 + \lambda_2\nu_2 - \lambda_3\nu_3 - [\upsilon\varrho^2k_1 - (x_1k_2 + x_3k_1)][\lambda_1(x_1k_1 \\ &\quad + x_3k_2 - x_2) + \lambda_2(x_2k_1 + x_1) - \lambda_3(x_3k_1 - x_1k_2)]\Big\}, \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \Delta_3 =& \varrho^4\Big\{ [\upsilon(x_1k_1 - x_3k_2 - x_2) - k_2]^2 + \upsilon^2(x_2k_1 + x_1)^2 - [\upsilon(x_3k_1 - x_1k_2) + k_1]^2 \\ & - \Big[\upsilon\varrho^2k_1 - (x_1k_2 + x_3k_1)\Big]^2 \Big\}. \end{aligned}$$

Theorem 3.5. *Let $\Psi = \Psi(\theta, \upsilon)$ is B_2-equiform Bishop spherical image ruled surface in E_1^3 given by (3.18). Then, Ψ at the point $(\theta, 0)$ is equiform Bishop flat if and only if*

$$x_1x_2(k_1^2 + k_2^2) + x_3(x_1k_2 + x_2k_1 - x_3k_1k_2) + x_1k_1(x_1 - x_3k_1) = 0.$$

Theorem 3.6. *Let $\Psi = \Psi(\theta, \upsilon)$ is B_2-equiform Bishop spherical image ruled surface in E_1^3 given by (3.18). Then, Ψ at the point $(\theta, 0)$ is equiform Bishop minimal if and only if the equiform curvatures satisfy the following differential equation*

$$k_2(x_3k_2 + x_1k_1) + x_2k_2(k_1' + k_1^2 + k_2^2) - x_3k_1(k_2' + 2k_1k_2) + (x_1k_2 + x_3k_1)[x_3k_1(x_1k_1 + x_3k_2 - x_2) + x_2k_2(x_3k_1 - x_1k_2) - (x_2k_1 + x_1)(x_3k_2 + x_1k_1)] = 0.$$

Case 3.8. *At $x_1 = 0$, the B_2-equiform Bishop spherical image ruled surface (3.18) has the following:*

$$K_\Psi = \frac{x_3^2k_1^2(x_2 - x_3k_2)^2}{\varrho^2(\kappa^2 + x_3^2k_1^2)(2x_3^2k_1^2 - x_2^2k_2^2)},$$
$$H_\Psi = \frac{x_2k_2(k_1' + k_1^2 + k_2^2) + x_3k_1(k_2' + 2k_1k_2) + x_3^2k_1^2(x_3k_2 - x_2) + x_3k_2^2}{2\varrho(\kappa^2 + x_3^2k_1^2)\sqrt{2x_3^2k_1^2 - x_2^2k_2^2}}. \tag{3.23}$$

Corollary 3.8. *At the point $(\theta, 0)$ the B_1-equiform Bishop spherical image ruled surface (3.18) with $x_1 = 0$ is:*

1). *Equiform Bishop flat surface if the equiform curve $\varphi(\theta)$ is a straight line.*

2). *Equiform Bishop flat surface if and only if $k_2 = \left(\frac{x_2}{x_3}\right)$.*

3). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$x_2k_2(k_1' + k_1^2 + k_2^2) + x_3k_1(k_2' + 2k_1k_2) + x_3^2k_1^2(x_3k_2 - x_2) + x_3k_2^2 = 0.$$

Case 3.9. *At $x_2 = 0$, the B_2-equiform Bishop spherical image ruled surface (3.18) has the following:*

$$K_\Psi = \frac{(x_1 - x_3k_1)^2(x_1k_1 + x_3k_2)}{\varrho^2[\kappa^2 + (x_1k_2 + x_3k_1)^2][x_3^2k_1^2 + (x_1k_1 + x_3k_2)^2]},$$
$$H_\Psi = \frac{k_2(x_3k_2 + x_1k_1) - x_3k_1(k_2' + 2k_1k_2) + (x_1k_2 + x_3k_1)(x_1k_1 + x_3k_2)(x_3k_1 - x_1)}{2\varrho[\kappa^2 + (x_1k_2 + x_3k_1)^2]\sqrt{x_3^2k_1^2 + (x_1k_1 + x_3k_2)^2}}. \tag{3.24}$$

Corollary 3.9. *At the point $(\theta, 0)$ the B_2-equiform Bishop spherical image ruled surface (3.18) with $x_2 = 0$ is:*

1). *Equiform Bishop flat surface if and only if $k_1 = \left(\frac{x_1}{x_3}\right)$.*

2). *Equiform Bishop flat surface if and only if the equiform curve $\varphi(\theta)$ is a circular helix i.e., $\frac{k_2}{k_1} = -\left(\frac{x_1}{x_3}\right)$.*

3). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$k_2(x_3k_2 + x_1k_1) - x_3k_1(k_2' + 2k_1k_2) + (x_1k_2 + x_3k_1)(x_1k_1 + x_3k_2)(x_3k_1 - x_1).$$

Case 3.10. *At $x_3 = 0$, the B_2-equiform Bishop spherical image ruled surface (3.18) has the following:*

$$K_\Psi = \frac{x_1^2[x_2(k_1^2 + k_2^2) + x_1 k_1]^2}{\varrho^2(\kappa^2 + x_1^2 k_2^2)(x_1^2 k_1^2 - x_2^2 k_2^2)^2},$$
$$H_\Psi = \frac{x_2 k_2(k_1' + k_1^2 + k_2^2) + x_1 k_1 k_2[1 - x_1(2x_2 k_1 + x_1)]}{2\varrho(\kappa^2 + x_1^2 k_2^2)\sqrt{x_1^2 k_1^2 - x_2^2 k_2^2}}. \tag{3.25}$$

Corollary 3.10. *At the point $(\theta, 0)$, the B_2-equiform Bishop spherical image ruled surface (3.18) with $x_3 = 0$ is:*

1). *Equiform Bishop flat surface if only if $x_2(k_1^2 + k_2^2) + x_1 k_1 = 0$.*

2). *Equiform Bishop minimal surface if the equiform curvatures satisfy the following differential equation*

$$x_2 k_2(k_1' + k_1^2 + k_2^2) + x_1 k_1 k_2[1 - x_1(2x_2 k_1 + x_1) = 0.$$

Case 3.11. *At $x_2 = x_3 = 0,\ x_1 = \varrho$, the B_2-equiform Bishop spherical image ruled surface (3.18) has the following:*

$$K_\Psi = \frac{1}{\kappa^2 + \varrho^2 k_2^2},$$
$$H_\Psi = \frac{k_2(1 - \varrho^2)}{2\varrho(\kappa^2 + \varrho^2 k_2^2)}. \tag{3.26}$$

Corollary 3.11. *At the point $(\theta, 0)$ the B_2-equiform Bishop spherical image ruled surface (3.18) with $x_2 = x_3 = 0,\ x_1 = \varrho$ is:*

1). *Equiform Bishop minimal surface surface if the equiform curve $\varphi(\theta)$ is contained in a plane.*

2). *Equiform Bishop minimal surface if and only if $\varrho = \pm 1$.*

Remark 3.2. *On the B_2-equiform Bishop spherical image ruled surface (3.18) at the point $(\theta, 0)$ with $x_1 = x_3 = 0,\ x_2 = \varrho$ the equiform Bishop mean curvature H_Ψ is undefined.*

3.4. Example

We build computational example of equiform-Bishop spherical image ruled surfaces curves in E_1^3 using the moving equiform-Bishop frame $\{T, B_1, B_2\}$ of the spacelike equiform-Bishop curve $\psi(s) = (s \sin(\ln s), s\cos(\ln s), s)$ with timelike binormal vector (see Figure 1). Then it is simple to demonstrate that

$$t(s) = (\sin(\ln s) + \cos(\ln s), \cos(\ln s) - \sin(\ln s), 1),$$
$$n(s) = \frac{1}{\sqrt{2}}(\cos(\ln s) - \sin(\ln s), -\sin(\ln s) - \cos(\ln s), 0),$$
$$b(s) = \frac{1}{\sqrt{2}}\Big(\sin(\ln s) + \cos(\ln s), \cos(\ln s) - \sin(\ln s), 2\Big).$$

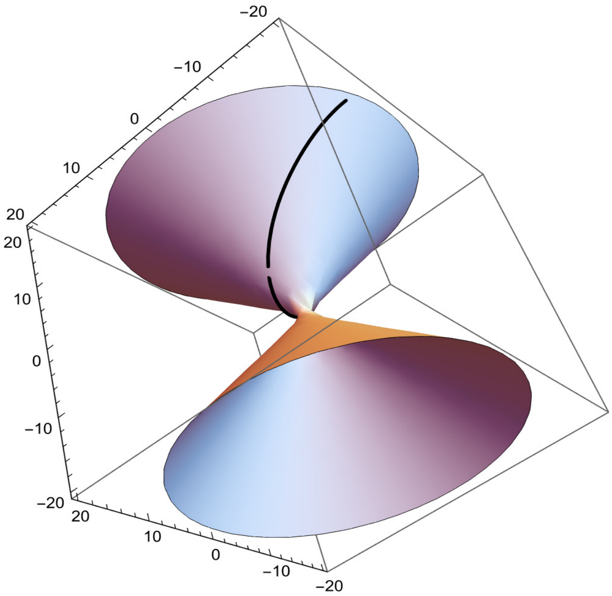

Figure 1. Spacelike curve $\psi = \psi(s)$ on S^2_1.

The curvature functions are $\kappa = \dfrac{\sqrt{2}}{s}$ and $\tau = \dfrac{1}{s}$. Also, $\vartheta(s) = \int_0^s \left(\frac{1}{s}\right) ds = \ln s$. From (2.4), we get $\kappa_1(s) = \left(\frac{\sqrt{2}}{s}\right)\cosh(\ln s)$, $\kappa_2(s) = \left(\frac{\sqrt{2}}{s}\right)\sinh(\ln s)$. Also from (2.3), we get

$$n_1(s) = \frac{1}{\sqrt{2}\,s}\Big(\cos(\ln s)[\cosh(\ln s) + \sinh(\ln s)] - \sin(\ln s)[\cosh(\ln s) - \sinh(\ln s)],$$
$$\cos(\ln s)[\cosh(\ln s) - \sinh(\ln s)] - \sin(\ln s)[\cosh(\ln s) + \sinh(\ln s)], 2\cosh(\ln s)\Big),$$

$$n_2(s) = \frac{1}{\sqrt{2}\,s}\Big(\cos(\ln s)[\cosh(\ln s) + \sinh(\ln s)] + \sin(\ln s)[\cosh(\ln s) - \sinh(\ln s)],$$
$$-\cos(\ln s)[\cosh(\ln s) - \sinh(\ln s)] - \sin(\ln s)[\cosh(\ln s) + \sinh(\ln s)], 2\sinh(\ln s)\Big).$$

Now, the equiform-Bishop parameter is $\theta = \int_0^s \kappa_1\, ds = \sqrt{2}\sinh(\ln s)$. Then we have $s = \left(\frac{\theta + \sqrt{\theta^2+2}}{\sqrt{2}}\right)$ and $\varrho = \left(\frac{\theta + \sqrt{\theta^2+2}}{\sqrt{2}\sqrt{\theta^2+2}}\right)$. Furthermore, the equiform-Bishop curvatures are given by

$$k_1(\theta) = \frac{1 - \theta\sqrt{\theta^2+2}}{\sqrt{2}(\theta^2+2)}, \quad k_2(\theta) = \frac{\theta^2 - \theta\sqrt{\theta^2+2}}{\theta^2 + \theta\sqrt{\theta^2+2} + 1}.$$

So the equiform-Bishop curve $\psi = \psi(\theta)$ is define as (see Figure 2)

$$\psi(\theta) = \left(\frac{\theta + \sqrt{\theta^2+2}}{\sqrt{2}}\right)\left\{\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right], \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right], 1\right\}.$$

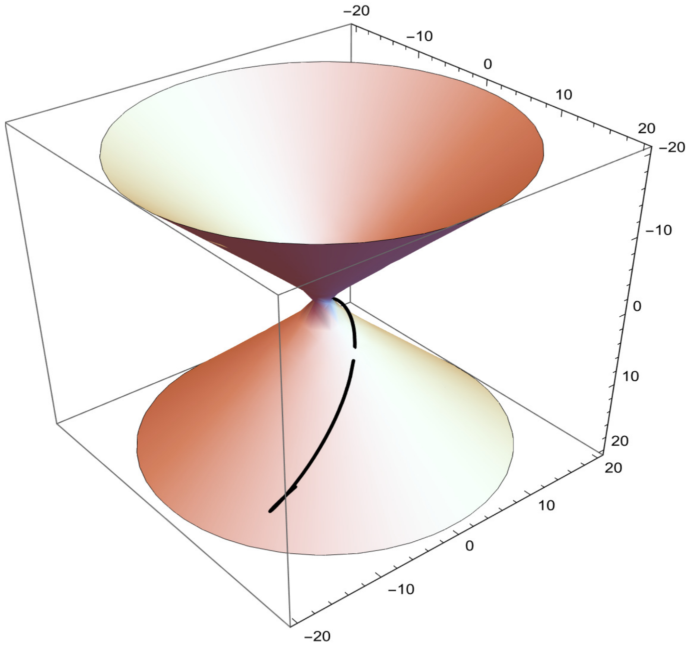

Figure 2. Equiform spacelike curve $\psi = \psi(\theta)$ on S_1^2.

Additionally, the equiform-Bishop frame are given by

$$T(\theta) = \left(\frac{\theta + \sqrt{\theta^2+2}}{\sqrt{2}\sqrt{\theta^2+2}}\right)\left\{\cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] + \sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right], \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] - \sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right], 1\right\},$$

$$B_1(\theta) = \left(\frac{\theta + \sqrt{\theta^2+2}}{2\sqrt{\theta^2+2}}\right)\begin{pmatrix} \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] + \frac{\theta}{\sqrt{2}}\right) \\ -\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] - \frac{\theta}{\sqrt{2}}\right), \\ -\cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] - \frac{\theta}{\sqrt{2}}\right) \\ -\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] + \frac{\theta}{\sqrt{2}}\right), \\ \sqrt{2}\,\theta \end{pmatrix},$$

$$B_2(\theta) = \left(\frac{\theta + \sqrt{\theta^2+2}}{2\sqrt{\theta^2+2}}\right)\begin{pmatrix} \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] + \frac{\theta}{\sqrt{2}}\right) \\ +\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] - \frac{\theta}{\sqrt{2}}\right), \\ \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] - \frac{\theta}{\sqrt{2}}\right) \\ -\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] + \frac{\theta}{\sqrt{2}}\right), \\ 2\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] \end{pmatrix}.$$

Thus, the T-equiform Bishop spherical image ruled surface, B_1-equiform Bishop spherical image ruled surface and B_2-equiform Bishop spherical image ruled surface are respectively given as (see

Figures 3–5)

$$\Phi(\theta,\upsilon)=\frac{\theta}{\sqrt{2}}\begin{pmatrix} \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3+\upsilon x_2)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]+\frac{\theta}{\sqrt{2}}\right)\right. \\ \left.+\sqrt{2}\,(\upsilon x_1+1)\right\}+\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3-\upsilon x_2)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\right.\right. \\ \left.\left.-\frac{\theta}{\sqrt{2}}\right)+\sqrt{2}\,(\upsilon x_1+1)\right\}, \\ \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3-\upsilon x_2)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]-\frac{\theta}{\sqrt{2}}\right)\right. \\ \left.+\sqrt{2}\,(\upsilon x_1+1)\right\}-\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3+\upsilon x_2)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\right.\right. \\ \left.\left.+\frac{\theta}{\sqrt{2}}\right)+\sqrt{2}\,(\upsilon x_1+1)\right\}, \\ \sqrt{2}\left\{\upsilon\left(x_1+x_2\theta+\sqrt{2}x_3\right)+\sqrt{2}\theta\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]+1\right\} \end{pmatrix},$$

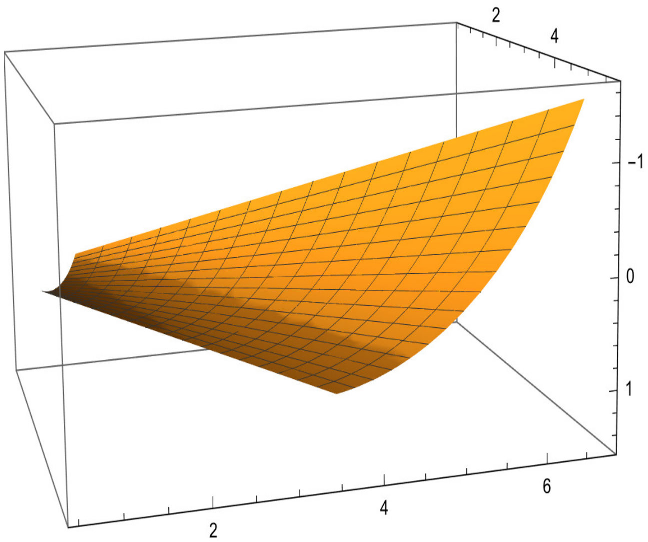

Figure 3. T-equiform Bishop spherical image ruled surface $\Phi(\theta,\upsilon)$.

$$\Omega(\theta,\upsilon)=\frac{\theta}{\sqrt{2}}\begin{pmatrix} \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_2+\upsilon x_3+1)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]+\frac{\theta}{\sqrt{2}}\right)\right. \\ \left.+\sqrt{2}\upsilon x_1\right\}+\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3-\upsilon x_2-1)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\right.\right. \\ \left.\left.-\frac{\theta}{\sqrt{2}}\right)+\sqrt{2}\upsilon x_1\right\}, \\ \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3-\upsilon x_2-1)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]-\frac{\theta}{\sqrt{2}}\right)\right. \\ \left.+\sqrt{2}\upsilon x_1\right\}-\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3+\upsilon x_2+1)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\right.\right. \\ \left.\left.+\frac{\theta}{\sqrt{2}}\right)+\sqrt{2}\upsilon x_1\right\}, \\ \sqrt{2}\left\{\upsilon\left(x_1+x_2\theta+\sqrt{2}x_3\right)+\sqrt{2}\theta\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]+\theta\right\} \end{pmatrix},$$

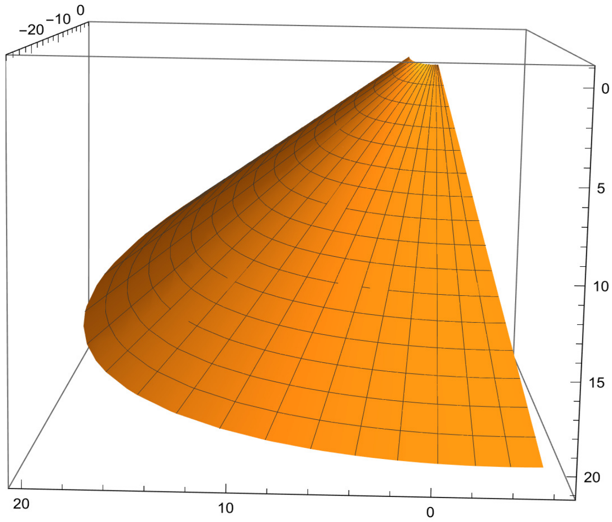

Figure 4. B_1-equiform Bishop spherical image ruled surface $\Omega(\theta, \upsilon)$.

$$\Psi(\theta,\upsilon)=\frac{\theta}{\sqrt{2}}\begin{pmatrix} \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_2+\upsilon x_3+1)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]+\frac{\theta}{\sqrt{2}}\right)\right. \\ \left.+\sqrt{2}\upsilon x_1\right\}+\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3-\upsilon x_2+1)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\right.\right. \\ \left.\left.-\frac{\theta}{\sqrt{2}}\right)+\sqrt{2}\upsilon x_1\right\}, \\ \cos\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3-\upsilon x_2+1)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]-\frac{\theta}{\sqrt{2}}\right)\right. \\ \left.+\sqrt{2}\upsilon x_1\right\}-\sin\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\left\{(\upsilon x_3+\upsilon x_2+1)\left(\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right]\right.\right. \\ \left.\left.+\frac{\theta}{\sqrt{2}}\right)+\sqrt{2}\upsilon x_1\right\}, \\ \sqrt{2}\upsilon(x_1+x_2\theta)+2(\upsilon x_3+1)\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2}}\right)\right] \end{pmatrix}.$$

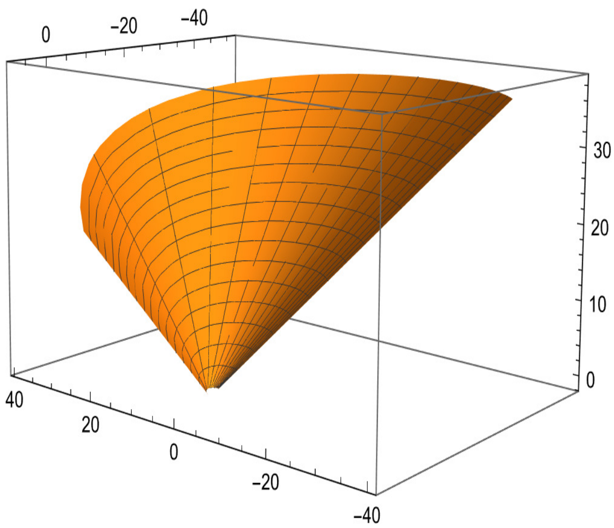

Figure 5. B_2-equiform Bishop spherical image ruled surface $\Psi(\theta, \upsilon)$.

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Conflicts of interest

The authors declare no competing interest.

References

1. M. Aydin, M. Ergut, The equiform differential geometry of curves in 4-dimensional galilean space G_4, *Stud. Univ. Babes-Bolyai Math.*, **58** (2013), 399–406.
2. I. Al-Dayel, E. Solouma, Characteristic properties of type-2 Smarandache ruled surfaces according to the type-2 Bishop frame in E^3, *Adv. Math. Phys.*, **2021** (2021), 8575443. https://doi.org/10.1155/2021/8575443
3. I. Al-Dayel, E. Solouma, M. Khan1, On geometry of focal surfaces due to B-Darboux and type-2 Bishop frames in Euclidean 3-space, *AIMS Mathematics*, **7** (2022), 13454–13468. https://doi: 10.3934/math.2022744
4. R. Bishop, There is more than one way to frame a curve, *The American Mathematical Monthly*, **82** (1975), 246–251. https://doi.org/10.2307/2319846
5. B. Bukcu, M. Karacan, Bishop frame of the spacelike curve with a spacelike principal normal in Minkowski 3-space, *Commun. Fac. Sci. Univ.*, **57** (2008), 13–22. https://doi.org/10.1501/Commua1_0000000185
6. J. Barbosa, A. Gervasio Colares, *Minimal surfaces in R^3*, Berlin: Springer Verlag, 1986. https://doi.org/10.1007/BFb0077105
7. M. Do Carmo, *Differential geometry of curves and surfaces*, 2Eds, Dover: Courier Dover Publications, 2016.
8. F. Dillen, W. Sodsiri, Ruled surfaces of Weingarten type in Minkowski 3-space, *J. Geom.*, **83** (2005), 10–21. https://doi.org/10.1007/s00022-005-0002-4
9. O. Gursoy, On the integral invariants of a closed ruled surface, *J. Geome.*, **39** (1990), 80–91. https://doi.org/10.1007/BF01222141
10. G. Hu, H. Cao, J. Wu, G. Wei, Construction of developable surfaces using generalized C-Bézier bases with shape parameters, *Comp. Appl. Math.*, **39** (2020), 157. https://doi.org/10.1007/s40314-020-01185-9
11. H. Kocayigit, M. Cetin, Spacelike curves of constant breadth according to Bishop frame in Minkowski 3-space, *Mathematical Sciences and Applications E-Notes*, **3** (2015), 86–93. https://doi.org/10.36753/mathenot.421222
12. O. Kose, Contribution to the theory of integral invariants of a closed ruled surface, *Mech. Mach. Theory*, **32** (1997), 261–277. https://doi.org/10.1016/S0094-114X(96)00034-1

13. Y. Kim, D. Yoon, Classification of ruled surfaces in Minkowski 3-space, *J. Geom. Phys.*, **49** (2004), 89–100. https://doi.org/10.1016/S0393-0440(03)00084-6
14. A. Kucuk, On the developable time-like trajectory ruled surfaces in Lorentz 3-space E_1^3, *Appl. Math. Comput.*, **157** (2004), 483–489. https://doi.org/10.1016/j.amc.2003.09.001
15. R. López, Differential geometry of curves and surfaces in Lorentz-Minkowski space, *Int. Electron. J. Geom.*, **7** (2014), 44–107. https://doi.org/10.36890/iejg.594497
16. W. Lam, Minimal surfaces from infinitesimal deformations of circle packings, *Adv. Math.*, **362** (2020), 106939. https://doi.org/10.1016/j.aim.2019.106939
17. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, New York: Academic press, 1983.
18. S. Ouarab, Smarandache ruled surfaces according to Frenet-Serret frame of a regular curve in E^3, *Abstr. Appl. Anal.*, **2021** (2021), 1–8. https://doi.org/10.1155/2021/5526536
19. E. Solouma, Generalized Smarandache curves of spacelike and equiform spacelike curves via timelike second binormal in R_1^4, *Appl. Appl. Math.*, **15** (2020), 1369–1380.
20. E. Solouma, W. Mahmoud, On spacelike equiform Bishop Smarandache curves on S_1^2, *J. Egypt. Math. Soc.*, **27** (2019), 7. https://doi.org/10.1186/s42787-019-0009-x
21. E. Solouma, Equiform spacelike Smarandache curves of anti-Eqiform Salkowski curve according to Equiform frame, *International Journal of Mathematical Analysis*, **15** (2021), 43–59. https://doi.org/10.12988/ijma.2021.912141
22. E. Solouma, I. Al-Dayel, Harmonic evolute surface of tubular surfaces via *B*-Darboux frame in Euclidean 3-space, *Adv. Math. Phys.*, **2021** (2021), 5269655. https://doi.org/10.1155/2021/5269655

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