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Research article

A modified proximal point algorithm in geodesic metric space

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Abstract: Proximal point algorithm is one of the most popular technique to find either zero of monotone operator or minimizer of a lower semi-continuous function. In this paper, we propose a new modified proximal point algorithm for solving minimization problems and common fixed point problems in CAT(0) spaces. We prove Δ and strong convergence of the proposed algorithm. Our results extend and improve the corresponding recent results in the literature.

Keywords: minimization problem; resolvent operator; CAT(0) space; proximal point algorithm; nonexpansive mappings
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Let (X, d) be a geodesic metric space and $f : X \to (-\infty, \infty]$ be a proper and convex function. One of the major problem in optimization is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{1.1}$$

We denote by

$$\arg\min_{y \in X} f(y),$$

the set of a minimizer of a convex function. One of the most effective way of solving problem (1.1) is the Proximal Point Algorithm (for short term, PPA). Its origin goes back to Martinet [1], Rockafellar [2], and Brézis and Lions [3]. Martinet studied the PPA for variational inequalities whereas Rockafellar showed the weak convergence of the sequence generated by the proximal point algorithm to a zero of the maximal monotone operator in Hilbert spaces. Güler's counterexample [4] showed that the sequence generated by the proximal point algorithm does not necessarily converge strongly even if the maximal monotone operator is the subdifferential of a convex, proper, and lower semicontinuous function. Kamimura and Takahashi [5] combined the PPA with Halpern's algorithm [6] so that the strong convergence is guaranteed. The proximal point algorithm can be used in numerous problems such as equilibrium problems, saddle point problems, convex minimization problems, and variational inequality problems.

Recently, many convergence results for the PPA for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds [7–10]. The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems on manifolds [11–14].

In 2014, Bačák [15] obtained few results using the proximal point algorithm in CAT(0) spaces. Also, he employed a splitting version of the PPA to find minimizer of a sum of convex functions, thereby extending the results of Bertsekas [16] into Hadamard spaces. Following this, many mathematicians have obtained numerous results involving the proximal point algorithm in the framework of CAT(0) spaces [17–21, 27, 28]. It is worth mentioning here that approximating the common fixed points has its own importance as it has a direct link with the minimization problems. Takahashi [22] and Izhar Uddin et al. [23] has applied common fixed point approximation to solve split feasibility and optimization problem. In 2020, Dung and Hieu [24] and Yambangwai et al. [25] studied approximating fixed points of three mappings and applied their results for image debluggring. Very recently, Yambangwai and Thianwan [26] applied approximating fixed points of three mappings into mage deblurring and signal recovering problems. They also showed that results involving three mappings are better than the results involving one or two mappings.

Fascinated by the ongoing research, in this paper, we propose a new modified proximal point algorithm for finding a common element of the set of fixed points of three single-valued nonexpansive mappings, the set of fixed points of three multi-valued nonexpansive mappings and the set of minimizers of convex and lower semi-continuous functions. We prove few convergence results for the proposed algorithm under some mild conditions.

2. Preliminaries

In this section, we present some fundamental concepts, definitions, and some results, which will be used in the next section.

A metric space (X, d) is said to be a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane (see more details in [29]). A complete CAT(0) space is then called a Hadamard space. Euclidean spaces, Hilbert spaces, the Hilbert ball [30], hyperbolic spaces [31], R-tress [32] and a complete, simply connected Riemannian manifold having non-positive sectional curvature are some examples of a CAT(0) space.

Definition 1. A subset D of a CAT(0) space X is said to be convex if D includes every geodesic segment joining ant two of its points, that is, for any $x, y \in D$, we have $[x, y] \subset D$, where $[x, y] := \{\alpha x \oplus (1-\alpha) y : 0 \leq \alpha \leq 1\}$ is the unique geodesic joining x and y.

Definition 2. A single-valued mapping $T : D \to D$ is said to be

(i) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$;
(ii) semi-compact if for any sequence $\{x_n\}$ in D such that

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0,$$

there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $x^* \in D$.

We denote the set of all fixed points of T is denoted by $F(T)$. Now, we state the following lemma to be used later on.

Lemma 1. *([33]) Let (X, d) be a CAT(0) space, then the following assertions hold:*

(i) For $x, y \in X$ and $t \in [0, 1]$, there exists a unique $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \ \ and \ \ d(y, z) = (1 - t)d(x, y).$$

(ii) For $x, y, z \in X$ and $t \in [0, 1]$, we have

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$$

and

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y).$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z of the above lemma.

Now, we collect some basic geometric properties which are instrumental throughout the discussions.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X. For $x \in X$ we write:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in X : r(x, x_n) = r(\{x_n\})\}.$$

It is well known that, in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point [34]. We now present the definition and some basic properties of the Δ-convergence which will be fruitful for our subsequent discussion.

Definition 3. ([35]) A sequence $\{x_n\}$ in a CAT(0) space X is said to be Δ-convergent to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Lemma 2. *([35]) Every bounded sequence in a complete CAT(0) space admits a Δ-convergent subsequence.*

Lemma 3. *([36]) If D is a closed convex subset of a complete CAT(0) space X and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 4. *([33]) Let D be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $T : D \to D$ be a nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in D such that $\Delta - \lim_n x_n = x$ and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, then x is a fixed point of T.*

Lemma 5. *([33]) If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 6. *([23, 37]) Let D be a nonempty closed and convex subset of a CAT(0) space X. Then, for any $\{x_i\}_{i=1}^n \in D$ and $\alpha_i \in (0, 1)$, $i = 1, 2, ..., n$ with $\sum_{i=1}^n \alpha_i = 1$, we have the following inequalities:*

$$d(\oplus_{i=1}^n \alpha_i x_i, z) \leq \sum_{i=1}^n \alpha_i d(x_i, z), \quad \forall \ z \in D \tag{2.1}$$

and

$$d^2(\oplus_{i=1}^n \alpha_i x_i, z) \leq \sum_{i=1}^n \alpha_i d^2(x_i, z) - \sum_{i,j=1, i\neq j}^n \alpha_i \alpha_j d^2(x_i, x_j), \quad \forall \ z \in D. \tag{2.2}$$

Convex and lower semi-continuous functions on CAT(0) spaces are our principal object of interest in this paper. Recall that a function $f : D \to (-\infty, \infty]$ defined on a convex subset D of a CAT(0) space is convex if, for any geodesic $\gamma : [a, b] \to D$, the function $fo\gamma$ is convex, i.e., $f(\alpha x \oplus (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ for all $x, y \in D$. For some important examples one can refer [38]. Now, a function f defined on D is said to be lower semi-continuous at $x \in D$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n)$$

for each sequence $\{x_n\}$ such that $x_n \to x$ as $n \to \infty$. A function f is said to be lower semi-continuous on D if it is lower semi-continuous at any point in D.

For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in CAT(0) space as follows:

$$J_\lambda(x) = \arg\min_{y\in D}[f(y) + \frac{1}{2\lambda} d^2(y, x)]$$

for all $x \in D$. The mapping J_λ is well defined for all $\lambda \geq 0$, see [4]. If f is a proper, convex and lower semi-continuous function, then the set $F(J_\lambda)$ of the fixed point of the resolvent J_λ associated with f coincides with the set $\arg\min_{y\in D} f(y)$ of minimizers of f; refer [38]. Also, for any $\lambda > 0$, the resolvent J_λ of f is nonexpansive, see [39].

Lemma 7. *([40]) Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function, then for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda} d^2(J_\lambda x, y) - \frac{1}{2\lambda} d^2(x, y) + \frac{1}{2\lambda} d^2(x, J_\lambda x) + f(J_\lambda x) \leq f(y).$$

Lemma 8. *([39,41]) Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Then the following identity holds:*

$$J_{\lambda}x = J_{\mu}(\frac{\lambda-\mu}{\lambda}J_{\lambda}x \oplus \frac{\mu}{\lambda}x)$$

for all $x \in X$ and $\lambda > \mu > 0$.

Let CB(D), CC(D) and KC(D) denote the families of nonempty closed bounded subsets, closed convex subsets and compact convex subsets of D, respectively. The Pompeiu-Hausdorff distance [42] on CB(D) is defined by

$$H(A, B) = max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\}$$

for $A, B \in CB(D)$, where $dist(x, D) = inf\{d(x, y) : y \in D\}$ is the distance from a point x to a subset D. An element $x \in D$ is said to be a fixed point of a multi-valued mapping $S : D \to CB(D)$ if $x \in Sx$. We denote the set of all fixed points of S by $F(S)$.

Definition 4. A multi-valued mapping $S : D \to CB(D)$ is said to be

(i) nonexpansive if $H(Sx, Sy) \leq d(x, y)$ for all $x, y \in D$;
(ii) hemi-compact if for any sequence $\{x_n\}$ in D with $\lim_{n\to\infty} dist(Sx_n, x_n) = 0$, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $x^* \in D$.

3. Main results

Theorem 1. *Let D be a nonempty closed and convex subset of a complete CAT(0) space X. Let $T_i : D \to D$, $i = 1, 2, 3$ be single-valued nonexpansive mappings, $S_i : D \to CB(D)$, $i = 1, 2, 3$ be multi-valued nonexpansive mappings and $g : D \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega = F(T_1)\cap F(T_2)\cap F(T_3)\cap F(S_1)\cap F(S_2)\cap F(S_3)\cap \arg\min_{y\in D} \neq \emptyset$ and $S_i q = \{q\}$, $i = 1, 2, 3$ for $q \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ is generated in the following manner:*

$$\begin{cases} w_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \\ z_n = \alpha_n x_n \oplus \beta_n w'_n \oplus \gamma_n w''_n, \\ y_n = \psi_n x_n \oplus \kappa_n w'''_n \oplus \phi_n T_1 x_n, \\ x_{n+1} = \delta_n x_n \oplus \eta_n T_2 x_n \oplus \xi_n T_3 y_n, \quad for \ \ all \ \ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\psi_n\}$, $\{\kappa_n\}$, $\{\phi_n\}$, $\{\delta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ are sequences in $(0, 1)$ such that

$$0 < a \leq \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\psi_n\}, \{\kappa_n\}, \{\phi_n\}, \{\delta_n\}, \{\eta_n\}, \{\xi_n\} \leq b < 1,$$

$$\alpha_n + \beta_n + \gamma_n = 1, \psi_n + \kappa_n + \phi_n = 1, \delta_n + \eta_n + \xi_n = 1,$$

for all $n \in \mathbb{N}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ. Then, the following statements hold:

(i) $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in \Omega$;

(ii) $\lim_{n\to\infty} d(x_n, w_n) = 0$;
(iii) $\lim_{n\to\infty} dist(x_n, S_i x_n) = 0, i = 1, 2, 3$;
(iv) $\lim_{n\to\infty} d(x_n, T_i x_n) = 0, i = 1, 2, 3$;
(v) $\lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0$.

Proof. Let $q \in \Omega$, then

$$q = T_1 q = T_2 q = T_3 q \in (S_1 q \cap S_2 q \cap S_3 q)$$

and

$$f(q) \le f(y), \quad \forall \; y \in D.$$

Therefore, we have

$$f(q) + \frac{1}{2\lambda_n} d^2(q, q) \le f(y) + \frac{1}{2\lambda_n} d^2(y, q),$$

for all $y \in D$ and hence $q = J_\lambda q$.

(i) Note that $w_n = J_{\lambda_n} x_n$ and J_{λ_n} is nonexpansive map for each $n \in \mathbb{N}$. So, we have

$$d(w_n, q) = d(J_{\lambda_n} x_n, J_{\lambda_n} q) \le d(x_n, q). \tag{3.2}$$

As $q \in S_i(q)$ for $i = 1, 2, 3$, using (3.2) and Lemma 6 we have

$$\begin{aligned} d(z_n, q) &= d(\alpha_n x_n \oplus \beta_n w_n' \oplus \gamma_n w_n'', q) \\ &\le \alpha_n d(x_n, q) + \beta_n d(w_n', q) + \gamma_n d(w_n'', q) \\ &\le \alpha_n d(x_n, q) + \beta_n d(S_1 x_n, S_1 q) + \gamma_n d(S_2 w_n, S_2 q) \\ &\le d(x_n, q) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(y_n, q) &= d(\psi_n x_n \oplus \kappa_n w_n''' \oplus \phi_n T_1 x_n, q) \\ &\le \psi_n d(x_n, q) + \kappa_n d(w_n''', q) + \phi_n d(T_1 x_n, q) \\ &\le \psi_n d(x_n, q) + \kappa_n d(S_3 z_n, q) + \phi_n d(T_1 x_n, q) \\ &\le d(x_n, q). \end{aligned} \tag{3.4}$$

Now, consider

$$\begin{aligned} d(x_{n+1}, q) &= d(\delta_n x_n \oplus \eta_n T_2 x_n \oplus \xi_n T_3 y_n, q) \\ &\le \delta_n d(x_n, q) + \eta_n d(T_2 x_n, q) + \xi_n d(T_3 y_n) \\ &\le d(x_n, q). \end{aligned} \tag{3.5}$$

This shows that $\lim_{n\to\infty} d(x_n, q)$ exists and so we assume that

$$\lim_{n\to\infty} d(x_n, q) = r \ge 0. \tag{3.6}$$

(ii) Next, we show that $\lim_{n\to\infty} d(x_n, w_n) = 0$. By Lemma 7 , we get

$$\frac{1}{2\lambda_n}\{d^2(w_n, q) - d^2(x_n, q) + d^2(x_n, w_n)\} \leq f(q) - f(w_n).$$

Since $f(p) \leq f(w_n)$ for each $n \in \mathbb{N}$, it follows that

$$d^2(x_n, w_n) \leq d^2(x_n, q) - d^2(w_n, q). \tag{3.7}$$

So, in order to show that $\lim_{n\to\infty} d(x_n, w_n) = 0$, it is sufficient to show that

$$\lim_{n\to\infty} d(w_n, q) = r.$$

From (3.3), we have

$$\limsup_{n\to\infty} d(z_n, q) \leq \limsup_{n\to\infty} d(x_n, q) = r. \tag{3.8}$$

Also, using (3.4), we get

$$\limsup_{n\to\infty} d(y_n, q) \leq \limsup_{n\to\infty} d(x_n, q) = r. \tag{3.9}$$

Using (3.5) along with the fact that $\delta_n + \eta_n + \xi_n = 1$ for all $n \geq 1$, we obtain

$$\begin{aligned} d(x_{n+1}, q) &\leq \delta_n d(x_n, q) + \eta_n d(T_2 x_n, q) + \xi_n d(T_3 y_n, q) \\ &\leq (1 - \xi_n) d(x_n, q) + \xi_n d(y_n, q), \end{aligned}$$

which is same as

$$\begin{aligned} d(x_n, q) &\leq \frac{1}{\xi_n}[d(x_n, q) - d(x_{n+1}, q)] + d(y_n, q) \\ &\leq \frac{1}{a}[d(x_n, q) - d(x_{n+1}, q)] + d(y_n, q), \end{aligned}$$

which gives

$$\liminf_{n\to\infty} d(x_n, q) \leq \liminf_{n\to\infty}\{\frac{1}{a}[d(x_n, q) - d(x_{n+1}, q)] + d(y_n, q)\}.$$

On using (3.6), we get

$$r \leq \liminf_{n\to\infty} d(y_n, q). \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\lim_{n\to\infty} d(y_n, q) = r. \tag{3.11}$$

Similarly, (3.4) yields

$$\begin{aligned} d(y_n, q) &\leq \psi_n d(x_n, q) + \kappa_n d(z_n, q) + \phi_n d(x_n, q) \\ &\leq d(x_n, q) - \kappa_n d(x_n, q) + \kappa_n d(z_n, q), \end{aligned}$$

which results into

$$d(x_n, q) \leq \frac{1}{\kappa_n}[d(x_n, q) - d(y_n, q)] + d(z_n, q) \leq \frac{1}{a}[d(x_n, q) - d(y_n, q)] + d(z_n, q),$$

which on using (3.6) and (3.11) gives

$$r \leq \liminf_{n\to\infty} d(z_n, q). \tag{3.12}$$

From (3.8) and (3.12), we get

$$\lim_{n\to\infty} d(z_n, q) = r. \tag{3.13}$$

Now, on using (3.3), we have

$$d(x_n, q) \leq \frac{1}{a}[d(x_n, q) - d(z_n, q)] + d(w_n, q),$$

which along with (3.6) and (3.13) gives

$$r \leq \liminf_{n\to\infty} d(w_n, q). \tag{3.14}$$

Also, (3.2) results into

$$\limsup_{n\to\infty} d(w_n, q) \leq \limsup_{n\to\infty} d(x_n, q) = r. \tag{3.15}$$

On using (3.14) and (3.15), we obtain

$$\lim_{n\to\infty} d(w_n, q) = r. \tag{3.16}$$

From (3.6), (3.7) and (3.16), we get

$$\lim_{n\to\infty} d(x_n, w_n) = 0. \tag{3.17}$$

(iii) Now, we prove $\lim_{n\to\infty} d(x_n, S_i x_n) = 0$ for $i = 1, 2, 3$.

Consider

$$\begin{aligned}
d^2(z_n, q) &= d^2(\alpha_n x_n \oplus \beta_n w'_n \oplus \gamma_n w''_n, q) \\
&\leq \alpha_n d^2(x_n, q) + \beta_n d^2(w'_n, q) + \gamma_n d^2(w''_n, q) \\
&\quad -\alpha_n\beta_n d^2(x_n, w'_n) - \alpha_n\gamma_n d^2(x_n, w''_n) - \beta_n\gamma_n d^2(w'_n, w''_n) \\
&\leq d^2(x_n, q) - \alpha_n\beta_n d^2(x_n, w'_n) - \alpha_n\gamma_n d^2(x_n, w''_n) - \beta_n\gamma_n d^2(w'_n, w''_n),
\end{aligned}$$

which is equivalent to

$$\alpha_n\beta_n d^2(x_n, w'_n) + \alpha_n\gamma_n d^2(x_n, w''_n) + \beta_n\gamma_n d^2(w'_n, w''_n) \leq d^2(x_n, q) - d^2(z_n, q).$$

On using (3.6) and (3.8), we obtain

$$\lim_{n\to\infty} d(x_n, w'_n) = 0, \tag{3.18}$$

$$\lim_{n\to\infty} d(x_n, w''_n) = 0, \tag{3.19}$$

and

$$\lim_{n\to\infty} d(w'_n, w''_n) = 0. \tag{3.20}$$

Now, triangle inequality gives

$$dist(x_n, S_1 x_n) \leq d(x_n, w'_n) + dist(w'_n, S_1 x_n),$$

which on using (3.18) results into

$$\lim_{n\to\infty} dist(x_n, S_1 x_n) = 0. \tag{3.21}$$

Again, consider

$$dist(x_n, S_2 x_n) \leq d(x_n, w''_n) + dist(w''_n, S_2 x_n) \leq d(x_n, w''_n) + d(w_n, x_n),$$

which on using (3.17) and (3.19) gives

$$\lim_{n\to\infty} dist(x_n, S_2 x_n) = 0. \tag{3.22}$$

Now, we have

$$\begin{aligned} d^2(y_n, q) &\leq \psi_n d^2(x_n, q) + \kappa_n d^2(w'''_n, q) + \phi_n d^2(T_1 x_n, q) \\ &\quad -\psi_n \kappa_n d^2(x_n, w'''_n) - \psi_n \phi_n d^2(x_n, T_1 x_n) - \kappa_n \phi_n d^2(w'''_n, T_1 x_n) \\ &\leq d^2(x_n, q) - \psi_n \kappa_n d^2(x_n, w'''_n) - \psi_n \phi_n d^2(x_n, T_1 x_n) - \kappa_n \phi_n d^2(w'''_n, T_1 x_n), \end{aligned}$$

which is equivalent to

$$\psi_n \kappa_n d^2(x_n, w'''_n) + \psi_n \phi_n d^2(x_n, T_1 x_n) + \kappa_n \phi_n d^2(w'''_n, T_1 x_n) \leq d^2(x_n, q) - d^2(y_n, q),$$

this on using (3.6) and (3.11) gives

$$\lim_{n\to\infty} d(x_n, w'''_n) = 0, \tag{3.23}$$

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = 0, \tag{3.24}$$

and

$$\lim_{n\to\infty} d(T_1 x_n, w'''_n) = 0. \tag{3.25}$$

On using (3.18) and (3.19), we have

$$\begin{aligned} d(z_n, x_n) &\leq \alpha_n d(x_n, x_n) + \beta_n d(w'_n, x_n) + \gamma_n d(w''_n, x_n) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned} \tag{3.26}$$

Thus, with the help of (3.23) and (3.26), we obtain

$$\begin{aligned} dist(x_n, S_3 x_n) &\leq d(x_n, w'''_n) + dist(w'''_n, S_3 x_n) \\ &\leq d(x_n, w'''_n) + d(z_n, x_n) \\ &\to 0 \quad as \quad n \to \infty. \end{aligned} \tag{3.27}$$

(*iv*) Next, we show that

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = \lim_{n\to\infty} d(x_n, T_3 x_n) = 0.$$

In (3.24), we have already proved that $\lim_{n\to\infty} d(x_n, T_1 x_n) = 0$.

So, consider

$$d^2(x_{n+1}, q) \leq d^2(x_n, q) - \delta_n \eta_n d^2(x_n, T_2 x_n) - \delta_n \xi_n d^2(x_n, T_3 y_n) - \eta_n \xi_n d^2(T_2 x_n, T_3 y_n),$$

which results into

$$\lim_{n\to\infty} d(x_n, T_2x_n) = 0, \tag{3.28}$$

$$\lim_{n\to\infty} d(x_n, T_3y_n) = 0, \tag{3.29}$$

and

$$\lim_{n\to\infty} d(T_2x_n, T_3y_n) = 0. \tag{3.30}$$

On using (3.23) and (3.24), we obtain

$$\begin{aligned} d(y_n, x_n) &\leq \psi_n d(x_n, x_n) + \kappa_n d(w_n''', x_n) + \phi_n d(T_1x_n, x_n) \\ &\rightarrow \ 0 \ \ as \ \ n \to \infty. \end{aligned} \tag{3.31}$$

Now, (3.28), (3.30) and (3.31) yields

$$\begin{aligned} d(x_n, T_3x_n) &\leq d(x_n, T_2x_n) + d(T_2x_n, T_3y_n) + d(T_3y_n, T_3x_n) \\ &\rightarrow \ 0 \ \ as \ \ n \to \infty. \end{aligned} \tag{3.32}$$

(v) Now, as $w_n = J_{\lambda_n}x_n$, from Lemma 8 we have

$$\begin{aligned} d(J_\lambda x_n, x_n) &\leq d(J_\lambda x_n, w_n) + d(w_n, x_n) \\ &= d(J_\lambda x_n, J_{\lambda_n}x_n) + d(w_n, x_n) \\ &= d(J_\lambda x_n, J_\lambda(\frac{\lambda_n - \lambda}{\lambda_n}J_{\lambda_n}x_n \oplus \frac{\lambda}{\lambda_n}x_n)) + d(w_n, x_n) \\ &\leq d(x_n, (1 - \frac{\lambda}{\lambda_n})J_{\lambda_n}x_n \oplus \frac{\lambda}{\lambda_n}x_n) + d(w_n, x_n) \\ &\leq (1 - \frac{\lambda}{\lambda_n})d(x_n, J_{\lambda_n}x_n) + \frac{\lambda}{\lambda_n}d(x_n, x_n) + d(w_n, x_n) \\ &= (1 - \frac{\lambda}{\lambda_n})d(x_n, w_n) + d(w_n, x_n) \\ &\rightarrow \ 0 \ \ as \ \ n \to \infty. \end{aligned}$$

□

We now present the Δ-convergence result in CAT(0) spaces.

Theorem 2. *Let D be a nonempty closed and convex subset of a complete CAT(0) space X. Let $T_i : D \to D$, $i = 1,2,3$ be single-valued nonexpansive mappings, $S_i : D \to KC(D)$, $i = 1,2,3$ be multi-valued nonexpansive mappings, and $f : D \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \cap F(S_1) \cap F(S_2) \cap F(S_3) \cap \arg\min_{y\in D} \neq \emptyset$ and $S_iq = \{q\}$, $i = 1,2,3$ for $q \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ is generated by (3.1), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\psi_n\}$, $\{\kappa_n\}$, $\{\phi_n\}$, $\{\delta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ are sequences in $(0,1)$ such that*

$$0 < a \leq \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\psi_n\}, \{\kappa_n\}, \{\phi_n\}, \{\delta_n\}, \{\eta_n\}, \{\xi_n\} \leq b < 1,$$

$$\alpha_n + \beta_n + \gamma_n = 1, \psi_n + \kappa_n + \phi_n = 1, \delta_n + \eta_n + \xi_n = 1,$$

for all $n \in \mathbb{N}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ. Then, the sequence $\{x_n\}$ Δ-converges to a point in Ω.

Proof. Let $W_\omega(\{x_n\}) = \cup A(\{u_n\})$, where union is taken over all subsequences $\{u_n\}$ over $\{x_n\}$. In order to show the Δ-convergence of $\{x_n\}$ to a point of Ω, firstly we will prove $W_\omega(\{x_n\}) \subset \Omega$ and thereafter argue that $W_\omega(\{x_n\})$ is a singleton set.

To show $W_\omega(\{x_n\}) \subset \Omega$, let $q \in W_\omega(\{x_n\})$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = q$. By Lemmas 2 and 3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v$ and $v \in D$. From Theorem 1, we have

$$\lim_{n\to\infty} d(v_n, T_i v_n) = 0, \quad i = 1,2,3$$

and

$$\lim_{n\to\infty} d(v_n, J_\lambda v_n) = 0.$$

Since T_i, $i = 1,2,3$ and J_λ are nonexpansive mappings, with the use of Lemma 4, we obtain

$$v = T_1 v = T_2 v = T_3 v = J_\lambda v.$$

So, we have

$$v \in F(T_1) \cap F(T_2) \cap F(T_3) \cap \arg\min_{y\in D} f(y). \tag{3.33}$$

Since S_i, $i = 1,2,3$ is compact valued, for each $n \in \mathbb{N}$, there exist $r_n^i \in S_i v_n$ and $p_n^i \in S_i v$, $i = 1,2,3$ such that

$$d(v_n, r_n^i) = dist(v_n, S_i v_n), \quad i = 1,2,3,$$

and

$$d(r_n^i, p_n^i) = dist(r_n^i, S_i v), \quad i = 1,2,3.$$

From Theorem 1, we get

$$\lim_{n\to\infty} d(v_n, r_n^i) = 0, \quad i = 1,2,3.$$

By using the compactness of $S_i v$, $i = 1,2,3$, there exists a subsequence $\{p_{n_j}^i\}$ of $\{p_n^i\}$ such that $\lim_{j\to\infty} p_{n_j}^i = p^i \in S_i v$, $i = 1,2,3$. With the help of Opial condition, we have

$$\begin{aligned}
\limsup_{j\to\infty} d(v_{n_j}, p^i) &\leq \limsup_{j\to\infty}(d(v_{n_j}, r_{n_j}^i) + d(r_{n_j}^i, p_{n_j}^i) + d(p_{n_j}^i, p^i)) \\
&\leq \limsup_{j\to\infty}(d(v_{n_j}, r_{n_j}^i) + dist(r_{n_j}^i, S_i v) + d(p_{n_j}^i, p^i)) \\
&\leq \limsup_{j\to\infty}(d(v_{n_j}, r_{n_j}^i) + H(S_i v_{n_j}, S_i v) + d(p_{n_j}^i, p^i)) \\
&\leq \limsup_{j\to\infty}(d(v_{n_j}, r_{n_j}^i) + d(v_{n_j}, v) + d(p_{n_j}^i, p^i)) \\
&= \limsup_{j\to\infty} d(v_{n_j}, v).
\end{aligned}$$

Since asymptotic center is unique, we get $v = p^i \in S_i v$, $i = 1,2,3$. By using (3.33), we obtain

$$v \in F(T_1) \cap F(T_2) \cap F(T_3) \cap F(S_1) \cap F(S_2) \cap F(S_3) \cap \arg\min_{y\in D} f(y) = \Omega.$$

From Theorem 1 and Lemma 5 , we get $q = v$, and $W_\omega(\{x_n\}) \subset \Omega$.

Now it is left to show that $W_\omega(\{x_n\})$ consists of single element only. For this, let $\{u_n\}$ be a subsequence of $\{x_n\}$. Again, by using Lemma 2, we can find a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v$. Let $A(\{u_n\}) = u$ and $A(\{x_n\}) = x$. It is enough to show that $v = x$. Since $v \in \Omega$, by Theorem 1, $\{d(x_n, v)\}$ is convergent. Again, by Lemma 5, we have $v = x$ which proves that $W_\omega(\{x_n\}) = \{x\}$. Hence the conclusion follows. □

The following results are strong convergence theorems for the proposed algorithm in CAT(0) spaces.

Theorem 3. *Under the hypothesis of Theorem 2, the sequence $\{x_n\}$ converges to an element of Ω if J_λ is semi-compact or T_1 is semi-compact or T_2 is semi-compact or T_3 is semi-compact or S_1 is hemi-compact or S_2 is hemi-compact or S_3 is hemi-compact.*

Proof. Without loss of generality, we assume that S_1 is hemi-compact. Therefore, there exist a subsequence $\{v_n\}$ of $\{x_n\}$ which is having a strong limit p in D. From Theorem 1, we get

$$\lim_{n\to\infty} d(T_i u_n, u_n) = 0, \quad i = 1, 2, 3,$$

$$\lim_{n\to\infty} d(J_\lambda u_n, u_n) = 0,$$

and

$$\lim_{n\to\infty} dist(S_i u_n, u_n) = 0, \quad i = 1, 2, 3.$$

From Lemma 4 , we obtain

$$p \in F(T_1) \cap F(T_2) \cap F(T_3) \cap \arg\min_{y\in D} f(y). \tag{3.34}$$

By using nonexpansiveness of S_1, we have

$$\begin{aligned} dist(p, S_1 p) &\leq d(p, u_n) + dist(u_n, S_1 u_n) + H(S_1 u_n, S_1 p) \\ &\leq 2d(p, u_n) + dist(u_n, S_1 u_n) \\ &\to 0 \ \ as \ \ n \ \to \ \infty. \end{aligned}$$

This results into $dist(p, S_1 p) = 0$, which is same as $p \in S_1 p$. Thus, $p \in F(S_1)$. Similarly, we can show that $p \in F(S_2)$ and $p \in F(S_3)$. Therefore, from (3.34), we get

$$p \in F(T_1) \cap F(T_2) \cap F(T_3) \cap F(S_1) \cap F(S_2) \cap F(S_3) \cap \arg\min_{y\in D} f(y) = \Omega.$$

By using double extract subsequence principle, we get that the sequence $\{x_n\}$ converges strongly to $p \in \Omega$. □

Since every multi-valued mapping $S : D \to CB(D)$ is hemi-compact if D is a compact subset of X. So, the following result can be obtained from Theorem 3 immediately.

Theorem 4. *Let D be a nonempty compact and convex subset of a complete CAT(0) space X. Let $T_i : D \to D$, $i = 1, 2, 3$ be single-valued nonexpansive mappings, $S_i : D \to KC(D)$, $i = 1, 2, 3$ be multi-valued nonexpansive mappings, and $f : D \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \cap F(S_1) \cap F(S_2) \cap F(S_3) \cap \arg\min_{y\in D} \neq \emptyset$*

and $S_i q = \{q\}$, $i = 1, 2, 3$ for $q \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ is generated by (3.1), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\psi_n\}$, $\{\kappa_n\}$, $\{\phi_n\}$, $\{\delta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ are sequences in $(0, 1)$ such that

$$0 < a \le \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\psi_n\}, \{\kappa_n\}, \{\phi_n\}, \{\delta_n\}, \{\eta_n\}, \{\xi_n\} \le b < 1,$$

$$\alpha_n + \beta_n + \gamma_n = 1, \psi_n + \kappa_n + \phi_n = 1, \delta_n + \eta_n + \xi_n = 1,$$

for all $n \in \mathbb{N}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in \mathbb{N}$ and some λ. Then, the sequence $\{x_n\}$ converges strongly to a point in Ω.

Remarks:

(i) Since any CAT(k) space is a CAT(k') space for $k' \ge k$ (refer [29]), all our results immediately apply to any CAT(k) space with $k \le 0$.

(ii) Every real Hilbert space H is a complete CAT(0) space, so we have the following convergence results which can be obtained from Theorems 2 and 3.

Corollary 1. *Let D be a nonempty closed and convex subset of a real Hilbert space X. Let $T_i : D \to D$, $i = 1, 2, 3$ be single-valued nonexpansive mappings, $S_i : D \to CB(D)$, $i = 1, 2, 3$ be multi-valued nonexpansive mappings and $g : D \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \cap F(S_1) \cap F(S_2) \cap F(S_3) \cap \arg\min_{y \in D} \neq \emptyset$ and $S_i q = \{q\}$, $i = 1, 2, 3$ for $q \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ is generated in the following manner:*

$$\begin{cases} w_n = \arg\min_{y \in X} [f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2], \\ z_n = \alpha_n x_n + \beta_n w_n' + \gamma_n w_n'', \\ y_n = \psi_n x_n + \kappa_n w_n''' + \phi_n T_1 x_n, \\ x_{n+1} = \delta_n x_n + \eta_n T_2 x_n + \xi_n T_3 y_n, \quad for \;\; all \;\; n \in \mathbb{N}, \end{cases} \tag{3.35}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\psi_n\}$, $\{\kappa_n\}$, $\{\phi_n\}$, $\{\delta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ are sequences in $(0, 1)$ such that

$$0 < a \le \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\psi_n\}, \{\kappa_n\}, \{\phi_n\}, \{\delta_n\}, \{\eta_n\}, \{\xi_n\} \le b < 1,$$

$$\alpha_n + \beta_n + \gamma_n = 1, \psi_n + \kappa_n + \phi_n = 1, \delta_n + \eta_n + \xi_n = 1,$$

for all $n \in \mathbb{N}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in \mathbb{N}$ and some λ. Then, the sequence $\{x_n\}$ Δ-converges to a point in Ω.

Corollary 2. *Let D be a nonempty closed and convex subset of a real Hilbert space X. Let $T_i : D \to D$, $i = 1, 2, 3$ be single-valued nonexpansive mappings, $S_i : D \to CB(D)$, $i = 1, 2, 3$ be multi-valued nonexpansive mappings, and $f : D \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Omega = F(T_1) \cap F(T_2) \cap F(T_3) \cap F(S_1) \cap F(S_2) \cap F(S_3) \cap \arg\min_{y \in D} \neq \emptyset$ and $S_i q = \{q\}$, $i = 1, 2, 3$ for $q \in \Omega$. For $x_1 \in D$, let the sequence $\{x_n\}$ is generated by (3.35), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\psi_n\}$, $\{\kappa_n\}$, $\{\phi_n\}$, $\{\delta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ are sequences in $(0, 1)$ such that*

$$0 < a \le \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\psi_n\}, \{\kappa_n\}, \{\phi_n\}, \{\delta_n\}, \{\eta_n\}, \{\xi_n\} \le b < 1,$$

$$\alpha_n + \beta_n + \gamma_n = 1, \psi_n + \kappa_n + \phi_n = 1, \delta_n + \eta_n + \xi_n = 1,$$

for all $n \in \mathbb{N}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in \mathbb{N}$ and some λ. Then, the sequence $\{x_n\}$ converges to an element of Ω if J_λ is semi-compact or T_1 is semi-compact or T_2 is semi-compact or T_3 is semi-compact or S_1 is hemi-compact or S_2 is hemi-compact or S_3 is hemi-compact.

4. Conclusions

In this article, we present a new proximal point algorithm for solving the constrained convex minimization problem as well as the fixed point problem of nonexpansive single-valued and multi-valued mappings in CAT(0) spaces. Theorems 2–4 are the main convergence results of the paper. We also driven some corollaries in the class of Hilbert spaces. Our results extend and improves the corresponding results of Cholamjiak [18], Suantai and Phuengrattana [43], Kumam et al. [44], Weng et al. [45] and Weng et al. [46].

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Conflict of interest

The authors declare that they have no conflicts of interests.

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