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Research article

Geometric properties of q-spiral-like with respect to $(\ell, \jmath)$-symmetric points

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Abstract: In this paper, the concepts of $(\ell, \jmath)$-symmetrical functions and the concept of q-calculus are combined to define a new subclasses defined in the open unit disk. In particular. We look into a convolution property, and we'll use the results to look into our task even more, we deduce the sufficient condition, coefficient estimates investigate related neighborhood results for the class $\mathcal{S}_q^{\ell,\jmath}(\lambda)$ and some interesting convolution results are also pointed out.

Keywords: spiral-like functions; q-calculus; $(\ell, \jmath)$-symmetrical functions (ρ, q)-neighborhood
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1. Introduction

Let $\mathcal{H}(\Sigma)$ denote the space of all analytic functions in the open unit disk $\Sigma = \{k \in \mathbb{C} : |k| < 1\}$ and let $\mathcal{H}$ denote the class of functions $\hbar \in \mathcal{H}(\Sigma)$ which has the form

$$\hbar(k) = k + \sum_{v=2}^{\infty} a_v k^v, \tag{1.1}$$

and suppose $\widetilde{\mathcal{S}}$ denote the subclass of $\mathcal{H}$ which are univalent in Σ. Then the convolution or Hadamard product of $\hbar$ of the form (1.1) and g of the form $g(k) = k + \sum_{v=2}^{\infty} b_v k^v$, is defined as:

$$(\hbar * g)(k) = k + \sum_{v=2}^{\infty} a_v b_v k^v.$$

In order to define new classes of q-spiral-like with respect to $(\ell, \jmath)$-symmetric points defined in Σ, we first recall the concept of quantum calculus (or q-calculus), the goal of quantum calculus, often known as q-calculus, is to find q-analogues without using limits, owing to the fact that it is widely used in a variety of scientific fields, in particular q-calculus has a great interest because of its applications in geometric function theory, so this method becomes a crucial component of the subject of our investigation. Jackson was the first participant who introduced fundamental ideas and developed the q-calculus theory [1–3]. In recent years, using quantum calculus approach to studying geometric properties several subclasses of analytic functions by many authors. For example, Naeem et al. [4] investigated subclesse of q-convex functions. Srivastava et al. [5] studied subclasses of q-starlike functions. Alsarari et al. [6] investigated the convolution conditions of q-Janowski symmetrical functions classes. Ovindaraj and Sivasubramanian in [7] found subclasses connected with q-conic domain. Khan et al. [8] used the symmetric q-derivative operator. Srivastava [9] published survey-cum-expository review paper which is useful for researchers. Many scholars introduced specific classes using q-calculus, which helped to advance the theory. See for further information on these contributions [10–12].

We provide some basic definitions and concept details of q-calculus which are used in our work and we will assume that q satisfies the condition $0 < q < 1$ throughout our work. Jackson [1] introduced q-derivative $\partial_q \hbar(k)$ as

$$\partial_q \hbar(k) = \begin{cases} \frac{\hbar(k)-\hbar(qk)}{k(1-q)}, & k \neq 0, \\ \hbar'(0), & k = 0. \end{cases} \tag{1.2}$$

Equivalently (1.2), may be written as

$$\partial_q \hbar(k) = 1 + \sum_{v=2}^{\infty} [v]_q a_v k^{v-1} \quad k \neq 0,$$

where

$$[v]_q = \frac{1-q^v}{1-q} = 1 + q + q^2 + ... + q^{v-1}. \tag{1.3}$$

For $\hbar$ a function defined in a subset of $\mathbb{C}$, provided $\hbar'(0)$ exists, then (1.2) yields

$$\lim_{q\to 1^-} (\partial_q \hbar(k)) = \lim_{q\to 1^-} \frac{\hbar(k) - \hbar(qk)}{k(1-q)} = \hbar'(k).$$

By using (1.2) it can easily be seen that for n and m any real (or complex) constants

$$\partial_q (n\hbar(k) \pm mg(k)) = n\partial_q \hbar(k) \pm m\partial_q g(k),$$

$$\partial_q (\hbar(k)g(k)) = \hbar(qk)\partial_q g(k) + \partial_q g\hbar(k)g(k) = \hbar(k)\partial_q g(k) + \partial_q \hbar(k)g(qk)$$

$$\partial_q \left(\frac{\hbar(k)}{g(k)}\right) = \frac{g(k)\partial_q \hbar(k) - \hbar(k)\partial_q g(k)}{g(qk)g(k)}.$$

As a right inverse Jackson [2] presented the q-integral of a function $\hbar$ as:

$$\int_0^k \hbar(z) d_q z = k(1-q) \sum_{v=0}^{\infty} q^v \hbar(kq^v),$$

provided that $\sum_{v=0}^{\infty} q^v \hbar(kq^v)$ is converges.

Recent work in the family of analytical functions has shown the use of the idea of $(\ell, \jmath)$-symmetrical functions to take a more general approach. This definition applies concepts of odd, even, and planar

Sakaguchi's functions to the $\jmath$-dimensional case. Liczberski and Polubinski [13] constructed the concept of $(\ell, \jmath)$-symmetrical functions for the positive integer $\jmath$ and $(\ell = 0, 1, 2, \ldots, \jmath - 1)$. A non-empty subset $\mathbf{Q}$ of the complex plane $\mathbb{C}$ will be called *$\jmath$-fold symmetric domain* if $\varepsilon\mathbf{Q} = \mathbf{Q}$, where $\varepsilon = e^{\frac{2\pi i}{\jmath}}$. function $\hbar : \mathbf{Q} \to \mathbb{C}$ is called $(\ell, \jmath)$-symmetrical if for each $k \in \mathbf{Q}$, $\hbar(\varepsilon k) = \varepsilon^{\ell}\hbar(k)$.

Theorem 1.1. *[13] For the $\jmath$-fold symmetric set Σ, then for every function $\hbar : \Sigma \mapsto \mathbb{C}$, can be written in the form,*

$$\hbar(k) = \sum_{\ell=0}^{\jmath-1} \hbar_{\ell,\jmath}(k), \quad \text{where } \hbar_{\ell,\jmath}(k) = \frac{1}{\jmath} \sum_{r=0}^{\jmath-1} \varepsilon^{-r\ell} \hbar\left(\varepsilon^{r} k\right), \ k \in \Sigma. \tag{1.4}$$

Remark 1.1. *Equivalently,* (1.4) *may be written as:*

$$h_{\ell,\jmath}(k) = \sum_{v=1}^{\infty} \delta_{v,\ell} a_v k^v, \quad \text{where } \delta_{v,\ell} = \frac{1}{\jmath} \sum_{r=0}^{\jmath-1} \varepsilon^{(v-\ell)r} = \begin{cases} 1, & v = I\jmath + \ell; \\ 0, & v \neq I\jmath + \ell; \end{cases} \tag{1.5}$$

for $I \in \mathbb{N}$.

Recently, many authors have conducted some studies about the concept of $(\ell, \jmath)$-symmetrical functions obtained interesting results for various classes see [14–17].

The function $\hbar$ is called λ-spirallike if $\Re\left\{e^{i\lambda}\frac{k\hbar'(k)}{\hbar(k)}\right\} > 0$, λ is real and $|\lambda| < \frac{\pi}{2}$. Furthermore, let $\mathcal{P}$ the Carathĕodory class of functions form $p(k) = 1 + \sum_{v=1}^{\infty} c_v k^v$ defined on Σ and satisfying $p(0) = 1$, $\mathrm{Re}\{p(k)\} > 0$, $k \in \Sigma$ and $p \in \mathcal{P} \Leftrightarrow p(k) = \dfrac{1 + s(k)}{1 - s(k)}$, where $s \in \Delta$ denote for the family of *Schwarz functions*, that is

$$\Delta := \{s \in \mathcal{H}, \ s(0) = 0, |s(k)| < 1, \ \ k \in \Sigma\}. \tag{1.6}$$

We amalgamate the notion of $(\ell, \jmath)$-symmetrical functions and q-derivative to originate new classes of q-spirallike functions with respect to $(\ell, \jmath)$-symmetric points $\widetilde{\mathcal{S}}_q^{\ell,\jmath}(\lambda)$.

Definition 1.1. *For arbitrary fixed numbers λ and q, $|\lambda| < \frac{\pi}{2}$, $0 < q < 1$, let $\widetilde{\mathcal{S}}_q^{\ell,\jmath}(\lambda)$ denote the family of functions $\hbar \in \mathcal{H}$ which satisfies*

$$\Re\left\{e^{i\lambda}\frac{k\partial_q \hbar(k)}{\hbar_{\ell,\jmath}(k)}\right\} \in \mathcal{P}, \quad \text{for all } k \in \Sigma, \tag{1.7}$$

where $\hbar_{\ell,\jmath}$ is defined in (1.4).

For special cases for the parameters q, λ, ℓ and $\jmath$ the class $\widetilde{\mathcal{S}}_q^{\ell,\jmath}(\lambda)$ yield several known subclasses of $\mathcal{H}$, namely $\widetilde{\mathcal{S}}_1^{1,\jmath}(0) := \widetilde{\mathcal{S}}_\jmath$ by defined by Sakaguchi [18], $\widetilde{\mathcal{S}}_q^{1,1}(1) = \widetilde{\mathcal{S}}_q$ which was first introduced by Ismail et al. [3], etc.

We denote by $\mathcal{T}_q^{\ell,\jmath}(\lambda)$ consisting all functions $\hbar$, satisfying

$$\hbar \in \mathcal{T}_q^{\ell,\jmath}(\lambda) \Leftrightarrow k\partial_q \hbar(k) \in \widetilde{\mathcal{S}}_q^{\ell,\jmath}(\lambda). \tag{1.8}$$

The following neighborhood principle was first proposed by Goodman [19] and generalized by Ruscheweyh [20].

Definition 1.2. *[19, 20] For $\rho \geq 0$ and any $\hbar \in \mathcal{H}$, ρ-neighborhood of function $\hbar$ defined as:*

$$\mathcal{N}_\rho(\hbar) = \left\{ g \in \mathcal{H} : \ g(k) = k + \sum_{v=2}^{\infty} b_v k^v, \quad \sum_{v=2}^{\infty} v|a_v - b_v| \leq \rho \right\}. \tag{1.9}$$

For the identity function $e(k) = k$, defined as:

$$\mathcal{N}_\rho(e) = \left\{ g \ \in \mathcal{H} : g(k) = k + \sum_{v=2}^{\infty} b_v k^v, \quad \sum_{v=2}^{\infty} v|b_v| \leq \rho \right\}. \tag{1.10}$$

For all $\eta \in \mathbb{C}$, with $|\eta| < \rho$, Ruscheweyh [20] proved

$$\frac{\hbar(k) + \eta k}{1 + \eta} \in \widetilde{\mathcal{S}}^* \Rightarrow \mathcal{N}_\rho(\hbar) \subset \widetilde{\mathcal{S}}^*.$$

Lemma 1.1. *[19] Let $P(k) = 1 + \sum_{v=1}^{\infty} p_v k^v$, $\quad (k \in \Sigma)$, with the condition $\Re\{p(k)\} > 0$, then*

$$|p_v| \leq 2, \quad (v \geq 1).$$

The goal of this research to investigate a convolution conditions and coefficient estimates for a function $\hbar$ to be in the classes $\widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$ and $\hbar \in \mathcal{T}_q^{\ell,J}(\lambda)$, which will be used as a assisting result for to discuss a sufficient prerequisites and associated neighborhood results.

2. Main results

Theorem 2.1. *A function $\hbar \in \mathcal{T}_q^{\ell,J}(\lambda)$ if and only if*

$$\frac{1}{k}\left[\hbar * \left(\frac{(k - qk^3)(1 - e^{i\phi})}{(1 - k)(1 - qk)(1 - q^2 k)} - \frac{(1 + e^{i(\phi - 2\lambda)})k}{(1 - \alpha_\ell k)(1 - \alpha_\ell qk)}\right)\right] \neq 0, \ \ |k| < 1,$$

where, $0 \leq \phi < 2\pi, 0 < q < 1$ and α_ℓ are defined by (2.3).

Proof. We have, $\hbar \in \mathcal{T}_q^{\ell,J}(\lambda)$ if and only if

$$\frac{e^{i\lambda \frac{\partial_q(k\partial_q \hbar(k))}{\partial_q \hbar_{\ell,J}(k)}} - i \sin\lambda}{\cos\lambda} \neq \frac{1 + e^{i\phi}}{1 - e^{i\phi}}, \ \ (|k| < 1),$$

which implies

$$\partial_q(k\partial_q \hbar(k))(1 - e^{i\phi}) - \partial_q \hbar_{\ell,J}(k)\{1 + e^{i(\phi - 2\lambda)}\} \neq 0. \tag{2.1}$$

Setting $\hbar(k) = k + \sum_{v=2}^{\infty} a_v k^v$, we have

$$\partial_q \hbar = 1 + \sum_{v=2}^{\infty} [v]_q a_v k^{v-1},$$

$$\partial_q \left(k\partial_q \hbar\right) = 1 + \sum_{v=2}^{\infty} [v]_q^2 a_v k^{v-1} = \partial_q \hbar * \frac{1}{(1 - k)(1 - qk)}.$$

$$\partial_q \hbar_{\ell,\jmath}(k) = \partial_q \hbar * \frac{1}{(1-\alpha_\ell k)} = \sum_{v=1}^{\infty} [v]_q \alpha_\ell^v a_v k^{v-1}, \tag{2.2}$$

where

$$\alpha_\ell^v = \delta_{v,\ell}^v \text{ and } \delta_{v,\ell} \text{ is given by (1.5).} \tag{2.3}$$

The left hand side of (2.1) is equivalent to

$$\partial_q \hbar * \left(\frac{1-e^{i\phi}}{(1-k)(1-qk)} - \frac{1+e^{i(\phi-2\lambda)}}{1-\alpha_\ell k} \right), \tag{2.4}$$

simplifying (2.4), we get

$$\frac{1}{k}\left[k\partial_q \hbar * \left(\frac{(1-e^{i\phi})k}{(1-k)(1-qk)} - \frac{(1+e^{i(\phi-2\lambda)})k}{1-\alpha_\ell k} \right) \right] \neq 0, \tag{2.5}$$

since $k\partial_q \hbar * g = \hbar * k\partial_q g$, then the above equation can be written as:

$$\frac{1}{k}\left[\hbar * \left(\frac{(k-qk^3)(1-e^{i\phi})}{(1-k)(1-qk)(1-q^2k)} - \frac{(1+e^{i(\phi-2\lambda)})k}{(1-\alpha_\ell k)(1-\alpha_\ell qk)} \right) \right] \neq 0.$$

□

Remark 2.1. *As $q \to 1^-$ and particular values of $\ell, \jmath$ and λ Theorem 2.1 yields to the results found in [21, 22].*

Theorem 2.2. *A function $\hbar \in \widetilde{\mathcal{S}}_q^{\ell,\jmath}(\lambda)$ if and only if*

$$\frac{1}{k}\left[\hbar * \left(\frac{(1-e^{i\phi})k}{(1-k)(1-qk)} - \frac{(1+e^{i(\phi-2\lambda)})k}{1-\alpha_\ell k} \right) \right] \neq 0, \quad |k| < 1,$$

where $0 < q < 1$, $0 \leq \phi < 2\pi$ and α_ℓ are defined by (2.3).

Proof. Since $\hbar \in \widetilde{\mathcal{S}}_q^{\ell,\jmath}(\lambda)$ if and only if $g(k) = \int_0^k \frac{\hbar(\zeta)}{\zeta} d_q\zeta \in \mathcal{T}_q^{\ell,\jmath}(\lambda)$, we have

$$\begin{aligned} &\frac{1}{k}\left[g * \left(\frac{(k-qk^3)(1-e^{i\phi})}{(1-k)(1-qk)(1-q^2k)} - \frac{(1+e^{i(\phi-2\lambda)})k}{(1-\alpha_\ell k)(1-\alpha_\ell qk)} \right) \right] \\ &= \frac{1}{k}\left[\hbar * \left(\frac{(1-e^{i\phi})k}{(1-k)(1-qk)} - \frac{(1+e^{i(\phi-2\lambda)})k}{1-\alpha_\ell k} \right) \right]. \end{aligned}$$

Thus the result follows from Theorem 2.2. □

Remark 2.2. *Note that from Theorem 2.2, we can easily get*

$$\hbar \in \widetilde{\mathcal{S}}_q^{\ell,\jmath}(\lambda) \Leftrightarrow \frac{(\hbar * g)(k)}{k} \neq 0, \quad g \in \mathcal{H}, k \in \Sigma, \tag{2.6}$$

where $g(k)$ has the form

$$g(k) = k - \sum_{v=2}^{\infty} t_v k^v, \qquad t_v = \frac{[v]_q - \delta_{v,\ell} - ([v]_q + \delta_{v,\ell} e^{-2i\lambda})e^{i\phi}}{(1+e^{-2i\lambda})e^{i\phi}}. \tag{2.7}$$

Theorem 2.3. *Let $\hbar(k) \in \mathcal{H}$, for $|\lambda| < \frac{\pi}{2}$ and $0 < q < 1$, if*

$$\sum_{v=2}^{\infty} \left\{ \frac{([v]_q - \delta_{v,\ell}) + \left|[v]_q + \delta_{v,\ell} e^{-2i\lambda}\right|}{|e^{-2i\lambda} + 1|} \right\} |a_v| \leq 1, \tag{2.8}$$

then $\hbar(k) \in \widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$.

Proof. Theorem 2.3 can be proved by demonstrating Remark 2.2 by showing $\frac{(\hbar * g)(k)}{k} \neq 0$. For $\hbar$ and g given by (1.1) and (2.7) respectfully.

$$\frac{(\hbar * g)(k)}{k} = 1 - \sum_{v=2}^{\infty} t_v a_v k^{v-1}, k \in \Sigma.$$

It is known from Remark 2.2 that $\hbar(k) \in \widetilde{\mathcal{S}}_q^{\ell,J}(\lambda) \Leftrightarrow \frac{(\hbar * g)(k)}{k} \neq 0$. Using (2.7) and (2.8), we get

$$\left|\frac{(\hbar * g)(k)}{k}\right| \geq 1 - \sum_{v=2}^{\infty} \frac{[v]_q - \delta_{v,\ell} + |[\ell]_q + \delta_{v,\ell} e^{-2i\lambda}|}{|e^{-2i\lambda} + 1|} |a_v||k|^{v-1} > 0, \ k \in \Sigma.$$

Thus, $\hbar(k) \in \widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$. □

Theorem 2.4. *If $\hbar(k) \in \widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$, then*

$$|a_v| \leq \prod_{r=1}^{v-1} \frac{\delta_{r,J} + [r]_q}{[r+1]_q - \delta_{r+1,J}}, \quad v \geq 2, \tag{2.9}$$

where $\delta_{m,J}$ is given by (1.5).

Proof. Let $\hbar(k) \in \widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$ from Definition 1.1, we have

$$p(k) = \left(e^{i\lambda} \frac{k\partial_q \hbar(k)}{\hbar_{\ell,J}(k)} \right) = 1 + \sum_{v=1}^{\infty} p_v z^v,$$

where $p(k)$ is Carath*é*odory function.

Since

$$e^{i\lambda} k\partial_q \hbar(k) = \hbar_{\ell,J}(k) p(k),$$

we have

$$e^{i\lambda} \sum_{v=2}^{\infty} ([v]_q - \delta_{v,J}) a_v k^v = \left(k + \sum_{v=2}^{\infty} a_v \delta_{v,J} k^v \right) \left(\sum_{v=1}^{\infty} p_v k^v \right), \tag{2.10}$$

where $\delta_{v,J}$ is given by (1.5), $\delta_{1,J} = 1$.

By equating coefficients of k^v in (2.10) both sides we have

$$a_v = \frac{e^{-i\lambda}}{([v]_q - \delta_{v,J})} \sum_{m=1}^{v-1} \delta_{v-m,J} a_{v-m} p_m, \quad a_1 = 1.$$

By Lemma1.1 and using the fact that $|e^{i\lambda}| = 1$, we get

$$|a_v| \leq \frac{2}{|[v]_q - \delta_{v,J}|} \sum_{m=1}^{v-1} \delta_{m,J}|a_m|, \quad a_1 = 1 = \delta_{1,J}. \tag{2.11}$$

It now suffices the prove that

$$\frac{2}{[v]_q - \delta_{v,J}} \sum_{r=1}^{v-1} \delta_{v,J}|a_J| \leq \prod_{r=1}^{v-1} \frac{\delta_{r,J} + [r]_q}{[r+1]_q - \delta_{r+1,J}}. \tag{2.12}$$

For this, we use the induction method. (2.12) is true for $v = 2$ and 3.

Let us suppose (2.12)holds for all $v \leq m$.

From (2.11), we have

$$|a_m| \leq \frac{2}{[m]_q - \delta_{m,J}} \sum_{r=1}^{m-1} \delta_{r,J}|a_r|, \quad a_1 = 1 = \delta_{1,J}.$$

From (2.9), we have

$$|a_m| \leq \prod_{r=1}^{m-1} \frac{\delta_{r,J} + [r]_q}{[r+1]_q - \delta_{r+1,J}}.$$

By the induction hypothesis , we have

$$\frac{2}{[m]_q - \delta_{m,J}} \sum_{r=1}^{m-1} \delta_{r,J}|a_r| \leq \prod_{r=1}^{m-1} \frac{\delta_{r,J} + [r]_q}{[r+1]_q - \delta_{r+1,J}}.$$

Multiplying both sides by

$$\frac{\delta_{m,J} + [m]_q}{[m+1]_q - \delta_{m+1,J}},$$

we have

$$\begin{aligned}
\prod_{r=1}^{m} \frac{\delta_{r,J} + [r]_q}{[r+1]_q - \delta_{r+1,J}} &\geq \frac{\delta_{m,J} + [m]_q}{[m+1]_q - \delta_{m+1,J}} \left[\frac{2}{[m]_q - \delta_{m,J}} \sum_{r=1}^{m-1} \delta_{r,J}|a_r| \right] \\
&= \frac{2}{[m+1]_q - \delta_{m+1,J}} \left\{ \frac{2\delta_{m,J}}{[m]_q - \delta_{m,J}} \sum_{r=1}^{m-1} \delta_{r,J}|a_r| + \sum_{r=1}^{m-1} \delta_{r,J}|a_r| \right\} \\
&\geq \frac{2}{[m+1]_q - \delta_{m+1,J}} \left\{ \delta_{m,J}|a_m| + \sum_{r=1}^{m-1} \delta_{r,j}|a_r| \right\} \\
&\geq \frac{2}{[m+1]_q - \delta_{m+1,J}} \sum_{r=1}^{m} \delta_{r,J}|a_r|.
\end{aligned}$$

Hence

$$\frac{2}{[m+1]_q - \delta_{m+1,J}} \sum_{r=1}^{m} \delta_{r,J}|a_r| \leq \prod_{r=1}^{m} \frac{\delta_{r,J} + [r]_q}{[r+1]_q - \delta_{r+1,J}}.$$

The inequality (2.12) holds for $v = m + 1$, thus proving the result. □

Theorem 2.5. *If $\hbar \in \mathcal{T}_q^{\ell,J}(\lambda)$ then*

$$|a_v| \leq \frac{1}{[v]_q} \prod_{r=1}^{r-1} \frac{\delta_{r,J} + [r]_q}{[r+1]_q - \delta_{r+1,J}}, \quad \text{for } v \geq 2, \tag{2.13}$$

where $\delta_{r,J}$ is given by (1.5).

The proof follows by using Theorem 2.4 and (1.8).

3. (ρ, q)-neighborhoods for functions in the classes $\widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$ and $\mathcal{T}_q^{\ell,J}(\lambda)$

In the order to find some neighborhood results, we assume that $v = [v]_q$ in Definition 1.2 to get definition of neighborhood with q-derivative $\mathcal{N}_{q,\rho}^{\lambda}(h)$ and $\mathcal{N}_{q,\rho}^{\lambda}(e)$, where $[v]_q$ is given by Equation (1.3). In particular. For $v = \frac{([v]_q - \delta_{v,\ell}) + \left|[v]_q + \delta_{v,\ell} e^{-2i\lambda}\right|}{|1+e^{-2i\lambda}|}$ in Definition 1.2 to get definition of neighborhood for the classes $\widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$ and $\mathcal{T}_q^{\ell,J}(\lambda)$ which is $\mathcal{N}_{q,\rho}^{\ell,J}(\lambda; \hbar)$.

Theorem 3.1. *Let $\hbar \in \mathcal{N}_{q,1}(e)$, and defined by the form* (1.1)*, then*

$$\left| \frac{k \partial_q \hbar(k)}{\hbar_{\ell,J}(k)} - 1 \right| < 1, \tag{3.1}$$

where $\hbar_{\ell,J}$ is defined by (1.4).

Proof. Let $\hbar \in \mathcal{H}$, and $\partial_q \hbar(k) = k + \sum_{v=2}^{\infty} [v]_q a_v k^v$, $\hbar_{\ell,J}(k) = k + \sum_{v=2}^{\infty} \delta_{v,\ell} a_v k^v$, where $\delta_{v,\ell}$ is given by (1.5). Consider

$$\begin{aligned}
|k \partial_q \hbar(k) - \hbar_{\ell,J}(k)| &= \left| \sum_{v=2}^{\infty} ([v]_q - \delta_{v,\ell}) a_v k^{v-1} \right| \\
&< |k| \sum_{v=2}^{\infty} [v]_q |a_v| - \sum_{v=2}^{\infty} \delta_{v,\ell} |a_v| |k|^{v-1} \\
&= |k| - \sum_{v=2}^{\infty} \delta_{v,\ell} |a_v| |k|^{v-1} \\
&\leq |\hbar_{\ell,J}(k)|, \ k \in \Sigma.
\end{aligned}$$

This gives us the required result. □

Theorem 3.2. *Let $\hbar \in \mathcal{H}$, and for all complex number η, with $|\eta| < \rho$, if*

$$\frac{\hbar(k) + \eta k}{1 + \eta} \in \widetilde{\mathcal{S}}_q^{\ell,J}(\lambda). \tag{3.2}$$

Then

$$\mathcal{N}^{\ell,J}_{q, \frac{\rho|1+e^{-2i\lambda}|}{4}}(\lambda; \hbar) \subset \widetilde{\mathcal{S}}_q^{\ell,J}(\lambda).$$

Proof. Let $f \in \mathcal{N}^{\ell,J}_{q,\frac{\rho|1+e^{-2i\lambda}|}{4}}(\lambda;\hbar)$ and defined by $f(k) = k + \sum_{v=2}^{\infty} b_v k^v$. It is sufficient to prove that $f \in \widetilde{\mathcal{S}}^{\ell,J}_q(\lambda)$ to to prove the Theorem 3.2. We would prove this claim in next three steps.

We first note that Theorem 2.2 is equivalent to

$$\hbar \in \widetilde{\mathcal{S}}^{\ell,J}_q(\lambda) \Leftrightarrow \frac{1}{k}[(\hbar * t_\phi)(k)] \neq 0, \quad k \in \Sigma, \tag{3.3}$$

where

$$t_\phi(k) = k - \sum_{v=2}^{\infty} \frac{[v]_q - \delta_{v,\ell} - ([v]_q + \delta_{v,\ell} e^{-2i\lambda})e^{i\phi}}{(1+e^{-2i\lambda})e^{i\phi}} k^v, \tag{3.4}$$

and $0 \leq \phi < 2\pi$. We can write $t_\phi(k) = k - \sum_{v=2}^{\infty} t_v k^v$,
where

$$t_v = \frac{[v]_q - \delta_{v,\ell} - ([v]_q + \delta_{v,\ell} e^{-2i\lambda})e^{i\phi}}{(1+e^{-2i\lambda})e^{i\phi}}, \tag{3.5}$$

so that $|t_v| \leq \frac{4[v]_q}{|1+e^{-2i\lambda}|}$. Secondly we obtain that (3.2) is equivalent to

$$\left|\frac{\hbar(k) * t_\phi(k)}{k}\right| \geq \rho, \tag{3.6}$$

because, if $\hbar(k) = k + \sum_{v=2}^{\infty} a_v k^v \in \mathcal{H}$ and satisfy (3.2), then (3.3) is equivalent to

$$t_\phi \in \widetilde{\mathcal{S}}^{\ell,J}_q(\lambda) \Leftrightarrow \frac{1}{k}\left[\frac{\hbar(k) * t_\phi(k)}{1+\eta}\right] \neq 0, \qquad |\eta| < \rho.$$

Thirdly letting $f(k) = k + \sum_{v=2}^{\infty} b_v k^v$ we notice that

$$\begin{aligned}
\left|\frac{f(k) * t_\phi(k)}{k}\right| &= \left|\frac{\hbar(k) * t_\phi(k)}{k} + \frac{(f(k)-\hbar(k)) * t_\phi(k)}{k}\right| \\
&\geq \rho - \left|\frac{(f(k)-\hbar(k)) * t_\phi(k)}{k}\right| \text{ (by using (3.6))} \\
&= \rho - \left|\sum_{v=2}^{\infty} (b_v - a_v) t_v k^v\right| \\
&\geq \rho - |k| \sum_{v=2}^{\infty} \frac{4[v]_q}{|1+e^{-2i\lambda}|} |b_v - a_v| \\
&\geq \rho - \rho|w| > 0,
\end{aligned}$$

this prove that

$$\frac{f(k) * t_\phi(k)}{k} \neq 0, \qquad k \in \Sigma.$$

In view of our observations (3.3), it follows that $f \in \widetilde{\mathcal{S}}^{\ell,J}_q(\lambda)$. The theorem's proof is now complete. □

Theorem 3.3. *Let* $\hbar \in \widetilde{\mathcal{S}}^{\ell,J}_q(\lambda)$, *for* $\rho_1 < c$. *Then*

$$\mathcal{N}^{\ell,J}_{q,\rho_1}(\lambda;\hbar) \subset \widetilde{\mathcal{S}}^{\ell,J}_q(\lambda),$$

where $c \neq 0$ *with* $\left|\frac{(\hbar * t_\phi)(k)}{k}\right| \geq c$, $\rho_1 = \frac{\rho|1+e^{-2i\lambda}|}{4}$ *and* t_ϕ *is defined by* (3.4).

Proof. Let the function $m = k + \sum_{v=2}^{\infty} b_v k^v \in \mathcal{N}_{q,\rho_1}^{\ell,J}(\lambda;\hbar)$. It is enough to demonstrate that $\frac{(m*t_\phi)(k)}{k} \neq 0$ for Theorem 3.3's proof, where t_ϕ is given by (3.4). Consider

$$\left|\frac{m(k) * t_\phi(k)}{k}\right| \geq \left|\frac{h(k) * t_\phi(k)}{k}\right| - \left|\frac{(m(k) - h(k)) * t_\phi(k)}{k}\right|. \tag{3.7}$$

Since $\hbar \in \widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$, therefore applying Theorem 2.2, we obtain

$$\left|\frac{(\hbar * t_\phi)(k)}{k}\right| \geq c, \tag{3.8}$$

where c is a real value that is not zero and $k \in \Sigma$. Now

$$\begin{aligned}\left|\frac{(m(k) - \hbar(k)) * t_\phi(k)}{k}\right| &= \left|\sum_{v=2}^{\infty}(b_v - a_v)t_v k^v\right| \\ &\leq \sum_{v=2}^{\infty} \frac{|[v]_q - \delta_{v,\ell} - ([v]_q + \delta_{v,\ell}e^{-2i\lambda})e^{i\phi}|}{|1 + e^{-2i\lambda}|}|b_v - a_v| \\ &\leq \sum_{v=2}^{\infty} \frac{4[v]_q}{|1 + e^{-2i\lambda}|}|b_v - a_v| \\ &\leq \frac{\rho|1 + e^{-2i\lambda}|}{4} = \rho_1,\end{aligned} \tag{3.9}$$

using (3.8) and (3.9) in (3.7), we obtain

$$\left|\frac{m(k) * g(k)}{k}\right| \geq c - \rho_1 > 0,$$

where $\rho_1 < c$. This completes the proof. □

4. Conclusions

The centered (ℓ, J)-symmetrical functions in geometric function theory were the subject of this work. As a new topic emerged to take a more general approach and there are numerous uses for (ℓ, J)-symmetrical functions, including investigating fixed points, estimating the absolute values of particular integrals, and deriving conclusions of the Cartan's uniqueness theorem variety and motivated by the recent applications of the q-calculus, we have applied the two concepts for classes of λ-spirallike functions to introduce and study the classes $\widetilde{\mathcal{S}}_q^{\ell,J}(\lambda)$ and $\mathcal{T}_q^{\ell,J}(\lambda)$. We investigate a convolution conditions and coefficient estimates. Furthermore, these results are used to find sufficient condition, coefficient estimates investigate related neighborhood results. The idea used in this paper can easily be implemented to define several classes with different image domains. The opportunities for research using symmetric q-calculus, Janowski class or the basic q-hypergeometric functions in several diverse areas.

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Conflict of interest

The authors declare no conflict of interest.

References

1. F. Jackson, On q-functions and a certain difference operator, *Transactions of the Royal Society of Edinburgh*, **46** (1909), 253–281. http://dx.doi.org/10.1017/S0080456800002751
2. F. Jackson, On q-definite integrals, *Pure Applied Mathematics*, **41** (1910), 193–203.
3. M. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, *Complex Variables, Theory and Application: An International Journal*, **14** (1990), 77–84. http://dx.doi.org/10.1080/17476939008814407
4. M. Naeem, S. Hussain, S. Khan, T. Mahmood, M. Darus, Z. Shareef, Janowski type q-convex and q-close-to-convex functions associated with q-conic domain, *Mathematics*, **8** (2020), 440. http://dx.doi.org/10.3390/math8030440
5. H. Srivastava, M. Tahir, B. Khan, Q. Ahmad, N. Khan, Some general classes of q-starlike functions associated with the Janowski functions, *Symmetry*, **11** (2019), 292. http://dx.doi.org/10.3390/sym11020292
6. F. Alsarari, S. Alzahrani, Convolution properties of q-Janowski-type functions associated with (x, y)-symmetrical functions, *Symmetry*, **14** (2022), 1406. http://dx.doi.org/10.3390/sym14071406
7. M. Govindaraj, S. Sivasubramanian, On a class of analytic functions related to conic domains involving q-calculus, *Anal. Math.*, **43** (2017), 475–487. http://dx.doi.org/10.1007/s10476-017-0206-5
8. B. Khan, G. Liu, T. Shaba, S. Araci, N. Khan, M. Khan, Applications of q-derivative operator to the subclass of bi-univalent functions involving q-Chebyshev polynomials, *J. Math.*, **2022** (2022), 8162182. http://dx.doi.org/10.1155/2022/8162182
9. H. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. Sci.*, **44** (2020), 327–344. http://dx.doi.org/10.1007/s40995-019-00815-0
10. H. Srivastava, M. Arif, M. Raza, Convolution properties of meromorphically harmonic functions defined by a generalized convolution q-derivative operator, *AIMS Mathematics*, **6** (2021), 5869–5885. http://dx.doi.org/10.3934/math.2021347
11. M. Raza, H. Srivastava, M. Arif, K. Ahmed, Coefficient estimates for a certain family of analytic functions involving a q-derivative operator, *Ramanujan J.*, **55** (2021), 53–71. http://dx.doi.org/10.1007/s11139-020-00338-y
12. M. Ul-Haq, M. Raza, M. Arif, Q. Khan, H. Tang, q-analogue of differential subordinations, *Mathematics*, **7** (2019), 724. http://dx.doi.org/10.3390/math7080724
13. P. Liczberski, J. Połubiński, On (j, k)-symmetrical functions, *Math. Bohem.*, **120** (1995), 13–28. http://dx.doi.org/10.21136/MB.1995.125897

14. F. Alsarari, Certain subclass of Janowski functions associated with symmetrical functions, *Ital. J. Pure Appl. Math.*, **46** (2021), 91–100.

15. P. Gochhayat, A. Prajapat, Geometric properties on (j,k)-symmetric functions related to starlike and convex function, *Commun. Korean Math. Soc.*, **37** (2022), 455–472.

16. F. Al-Sarari, S. Latha, T. Bulboacă, On Janowski functions associated with (n,m)-symmetrical functions, *J. Taibah Univ. Sci.*, **13** (2019), 972–978. http://dx.doi.org/10.1080/16583655.2019.1665487

17. F. Al-Sarari, S. Latha, B. Frasin, A note on starlike functions associated with symmetric points, *Afr. Mat.*, **29** (2018), 945–953. http://dx.doi.org/10.1007/s13370-018-0593-1

18. K. Sakaguchi, On a certain univalent mapping, *J. Math. Soc. Japan*, **11** (1959), 72–75. http://dx.doi.org/10.2969/jmsj/01110072

19. A. Goodman, Univalent functions and nonanalytic curves, *Proc. Am. Math. Soc.*, **8** (1957), 598–601. http://dx.doi.org/10.1090/S0002-9939-1957-0086879-9

20. S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Am. Math. Soc.*, **81** (1981), 521–527.

21. K. Padmanabhan, M. Ganesan, Convolution conditions for certain classes of analytic functions, *Indian J. Pure Appl. Math.*, **15** (1984), 777–780.

22. H. Silverman, E. Silvia, D. Telage, Convolution conditions for convexity, starlikeness and spiral-likeness, *Math. Z.*, **162** (1978), 125–130. http://dx.doi.org/10.1007/BF01215069