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Research article

Finite-time stochastic synchronization of fuzzy bi-directional associative memory neural networks with Markovian switching and mixed time delays via intermittent quantized control

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Abstract: We are concerned in this paper with the finite-time synchronization problem for fuzzy bi-directional associative memory neural networks with Markovian switching, discrete-time delay in leakage terms, continuous-time and infinitely distributed delays in transmission terms. After detailed analysis, we come up with an intermittent quantized control for the concerned bi-directional associative memory neural network. By designing an elaborate Lyapunov-Krasovskii functional, we prove under certain additional conditions that the controlled network is stochastically synchronizable in finite time: The 1st moment of every trajectory of the error network system associated to the concerned controlled network tends to zero as time approaches a finite instant (the settling time) which is given explicitly, and remains to be zero constantly thereupon. In the meantime, we present a numerical example to illustrate that the synchronization control designed in this paper is indeed effective. Since the concerned fuzzy network includes Markovian jumping and several types of delays simultaneously, and it can be synchronized in finite time by our suggested control, as well as the suggested intermittent control is quantized which could reduce significantly the control cost, the theoretical results in this paper are rich in mathematical implication and have wide potential applicability in the real world.

Keywords: finite-time synchronization; fuzzy bi-directional associative memory neural networks; mixed time delays; Markovian jumping; intermittent quantized control
Mathematics Subject Classification: 93E15, 28E10, 34K20, 34K37, 34K50, 60H10

1. Introduction

With a desire to generalize a single-layer auto-associative Hebbian correlator to a two-layer pattern-matched hetero-associative circuits, Kosko designed the celebrated bi-directional associative memory neural networks (BAMNs); see [1–6]. In last several decades, BAMNs have been applied successfully in classification, associative memory, parallel computation, combinatorial optimization, signal processing, pattern recognization, image processing, etc.; see [5–7]. Successful application of BAMNs in such wide areas relies essentially on their stability or synchronizability. And therefore, extensive attensions have been paid to the study of stability, synchronizability and other dynamics of various BAMNs; see [6–14] and the vast references therein. In this paper, we shall investigate further the synchronization problem associated to BAMNs.

Since it would cost time to communicate information between neurons, time delays are inevitable in neural network models originated from real world applications. As pointed in [14, 15], delays could change the stability of dynamical systems, render dynamical systems to produce periodic oscillations or chaotic phenomenon, and so on. This makes it more challenging and interesting to study stabilization/synchronization problem for BAMNs with delays. Cao and Wan [11] exploited the so-called matrix measure technique to obtain a synchronization criterion for an inertial BAMN with time delays. Inspired partially by results in [11], Li and Li [12] obtained some new results concerning the synchronization problem for a time-delayed BAMN which is not inertial. Sader, Abdurahman and Jiang [13] designed a nonlinear feedback control for a special class of BAMNs, and proved that these controlled BAMNs are synchronizable at a general decay rate. For more inspiring results concerning stability of time-delayed BAMNs, the interested readers could consult [8, 16–20] as well as the references therein.

In the real world, uncertainty is unavoidable in the transmission of information through neural nodes. By reading [21] and the related references therein, we can conclude that fuzzy logic could play important roles in dealing with uncertainty. Zhang and Wu [21] investigated the finite time synchronization problem for a class of Takagi-Sugeno fuzzy complex networks. Except for Takagi-Sugeno logic, there is another fuzzy logic which is widely used in constructing neural network models, namely, the fuzzy "AND" ($\wedge$) and "OR" ($\vee$) operation reasoning. Under certain conditions, experts proved that fuzzy neural networks could approximate a large collection of nonlinear functions to any desired degree of accuracy; see [22]. In the last two decades, fuzzy BAMNs have also been well studied for their synchronizability, and a large number of papers on synchronization problem for fuzzy BAMNs have been published in recent years. Among the vast references in this respect, we would like to share [23], in which a class of BAMNs including fuzzy logic was investigated and interesting synchronization results on the concerning BAMNs were obtained via using the LMI (linear matrix inequalities) approach.

As indicated in [15, 24], the synaptic transmission in nervous systems can be considered as a noisy process brought on by random fluctuations from the release of neurotransmitters or other probabilistic factors. In other words, here is, aside from fuzzy uncertainty, some other uncertainty occurring in the transmission of information through neural nodes that can be modeled by special stochastic process, such as (time homogeneous/inhomogeneous) Markovian chain, Wiener process (Brownian motion), Lévy process, and so forth. Compared with other frequently used stochastic process, Markovian chain has, in a certain sense, the simplest structure. And therefore, many interesting synchronization criterion have been presented for Markovian switched neural networks (including BAMNs) in recent years; see [21, 25–27] and the references therein.

Thanks to the wide applicability, it is a hot topic to design control to synchronize neural networks in finite time in recent years; see [20, 21, 25, 27–31] and the references therein. Jia et al. [29]

designed adaptive sliding mode control for a class of uncertain fractional-order delayed memristive neural networks, and proved that the obtained controlled networks are synchronizable in finite time. Cheng et al. [30] proved that delayed memristive neural networks can be finite-time synchronized via adaptive aperiodically intermittent adjustment strategy. By reviewing the afore-mentioned references, we are inspired by the results to be interested in designing control to synchronize, in finite time, fuzzy BAMNs with Markovian jumping and several types of time delays. To improve the applicability of our theoretical results and inspired by [25, 32], we seek to design appropriate intermittent quantized control for our concerned network. As indicated in [25], it would certainly cut down the control cost and communication resources by using intermittent quantized control to synchronize neural networks in finite time. The idea to realize our goal in this paper is enlightened by the the afore-mentioned references, besides, [33–38] and the references therein help us a lot to find the appropriate way to prove rigrously the suggested control is indeed effective in synchronizing our concerned network in finite time. Zhai et al. [33] shared intermittent control which can synchronize a class of stochastic complex networks with delays. Zhou et al. [34] and Liu et al. [35] provided two types of self-triggered intermittent control to synchronize complex network and hybrid delayed multi-links systems, respectively. In [36, 37], the author group developed quantized control to synchronize a variety of inertial neural networks. Our contributions in this paper are summarized as follows:

(i) Intermittent quantized control is first designed successfully and proved to synchronize effectively in finite time fuzzy BAMNs with Markovian jumping, discrete-time delay in leakage terms, continuous-time and infinitely distributed delays in transmission terms. In comparison with [11–13, 17, 18, 20, 28], our concerned network includes simultaneously fuzzy uncertainty, random uncertainty and a variety of time delays of different nature and thus has wider potential applicability. The idea of applying intermittent quantized control would contribute towards cutting down control cost and communication resources in the real world.

(ii) Several novel criteria are established to guarantee the finite-time synchronizability of our concerned fuzzy network, and the convergence settling time is computed explicitly. Additionally, an illustrative example is solved numerically to justify the effectiveness of the suggested synchronization control and the correctness of the criteria established to guarantee the finite-time synchronizability. The main tool used in proving our main results is a Lyapunov-Krasovskii functional, which differs dramatically from the ones utilized in [25, 32]. As in [25], the 1st moment of trajectories of the error network system associated to our concerned network is chosen in proving the correctness of the criteria established to guarantee the finite-time synchronizability. As alluded in [25], this could reduce in a certain degree the conservatism of our finite-time synchronization criteria.

Notational conventions: Throughout this paper, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denotes the totality of real numbers, the interval $[0, +\infty)$ and the interval $(-\infty, 0]$, respectively; $D^+ f$ denotes the right upper Dini derivative of the given function f with respect to the independent variable t; $(\mathbb{R}, \mathscr{L}, dt)$ denotes the usual Lebesgue measure space; $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ (or $(\Omega, \mathscr{F}, \mathbb{F}, d\mathbb{P})$) denotes a complete filtered probability space, in which the filtration $\mathbb{F} = \{\mathcal{F}_t;\ t \in \mathbb{R}^+\}$ is assumed to satisfy the usual condition: $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathscr{F}$; and $\mathbb{F}$ is right-continuous in the sense that $\cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$, $t \in \mathbb{R}^+$; "$\mathbb{P}$ almost surely" is abbreviated as $\mathbb{P}$-a.s.; $\mathbb{E}X$ denotes the mathematical expectation of X, where X is an arbitrarily given random variable on Ω; $(\Omega \times \mathbb{R}, \mathscr{L} \otimes \mathscr{F}, d\mathbb{P} \times dt)$ denotes the product measure space of $(\mathbb{R}, \mathscr{L}, dt)$ and $(\Omega, \mathscr{F}, d\mathbb{P})$; For every pair A, $B \in \mathscr{F}$, $\mathbb{P}(B|A)$ designates the conditional probability of the event B given the event A;

$\{\gamma_t\}_{t\in\mathbb{R}^+}$ denotes an $\mathbb{F}$-adapted time homogeneous Markovian chain whose state space Ξ is finite and whose infinitesimal generator is denoted by $\Pi = (\pi_{\xi\tilde{\xi}})$, that is, for every pair $\xi, \tilde{\xi}$ in Ξ, it holds that

$$\mathbb{P}(\gamma_{t+\Delta t} = \tilde{\xi}|\gamma_t = \xi) = \mathbb{P}(\gamma_{\Delta t} = \tilde{\xi}|\gamma_0 = \xi) = \delta_{\xi\tilde{\xi}} + \pi_{\xi\tilde{\xi}}\Delta t + o(\Delta t),\ \text{as } \Delta t \to 0^+,\ \forall\, t \in \mathbb{R}^+,$$

where $\delta_{\xi\tilde{\xi}}$ is the celebrated Kronecker delta symbol, more precisely, $\delta_{\xi\tilde{\xi}} = 1$ if ξ coincides with $\tilde{\xi}$, and $\delta_{\xi\tilde{\xi}} = 0$, otherwise. By definition, $\Pi = (\pi_{\xi\tilde{\xi}})$ is required to satisfy $\pi_{\xi\tilde{\xi}} \geqslant 0$ whenever $\xi \neq \tilde{\xi}$, and

$$-\pi_{\xi\xi} = \sum_{\tilde{\xi}\in\Pi\setminus\{\xi\}} \pi_{\xi\tilde{\xi}} > 0,\ \forall\, \xi \in \Pi.$$

2. Formulation of the problem and the main results

We are concerned with in this paper the following class of fuzzy BAMNs with several types of time delays in both leakage terms and transmission terms:

$$\begin{cases} \dot{x}_i(t) = -\, l_{1i}x_i(t-\tau_{1i}) + \sum_{j=1}^{n} a_{\gamma_t 11ij} f_{11j}(y_j(t)) \\ \qquad + \sum_{j=1}^{n} a_{\gamma_t 12ij} f_{12j}(y_j(t-\sigma_{1j}(t))) \\ \qquad + \bigvee_{j=1}^{n} a_{\gamma_t 13ij} \int_{-\infty}^{t} K_{11j}(t-s) f_{13j}(y_j(s))ds \\ \qquad + \bigwedge_{j=1}^{n} a_{\gamma_t 14ij} \int_{-\infty}^{t} K_{12j}(t-s) f_{14j}(y_j(s))ds \\ \qquad + X_i(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ i = 1, \ldots, m, \\ \dot{y}_j(t) = -\, l_{2j}y_j(t-\tau_{2j}) + \sum_{i=1}^{m} a_{\gamma_t 21ji} f_{21i}(x_i(t)) \\ \qquad + \sum_{i=1}^{m} a_{\gamma_t 22ji} f_{22i}(x_i(t-\sigma_{2i}(t))) \\ \qquad + \bigvee_{i=1}^{m} a_{\gamma_t 23ji} \int_{-\infty}^{t} K_{21i}(t-s) f_{23i}(x_i(s))ds \\ \qquad + \bigwedge_{i=1}^{m} a_{\gamma_t 24ji} \int_{-\infty}^{t} K_{22i}(t-s) f_{24i}(x_i(s))ds \\ \qquad + Y_j(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ j = 1, \ldots, n, \end{cases} \tag{2.1}$$

where the stochastic processes $\{X_i(t)\}$ and $\{Y_j(t)\}$, required to be $\mathbb{F}$-adapted, are given by

$$\begin{cases} X_i(t) = I_i(t) + \sum_{j=1}^{n} \kappa_{11ij}(t) w_{11ij}(t) + \bigvee_{j=1}^{n} \kappa_{12ij}(t) w_{12ij}(t) \\ \qquad + \bigwedge_{j=1}^{n} \kappa_{13ij}(t) w_{13ij}(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ i = 1, \ldots, m, \\ Y_j(t) = J_j(t) + \sum_{i=1}^{m} \kappa_{21ji}(t) w_{21ji}(t) + \bigvee_{i=1}^{m} \kappa_{22ji}(t) w_{22ji}(t) \\ \qquad + \bigwedge_{i=1}^{m} \kappa_{23ji}(t) w_{23ij}(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ j = 1, \ldots, n. \end{cases} \tag{2.2}$$

In (2.1) and (2.2), $l_{1i} > 0$, $l_{2j} > 0$, $\tau_{1i} > 0$ and $\tau_{2j} > 0$ are constants; $x_i(t)$ and $y_j(t)$, required to be $\mathbb{F}$-adapted and $\mathbb{P}$ almost surely continuous in t, are called state trajectories of BAMNs (2.1); the connection coefficients (or connection weights) $a_{\gamma_t 11ij}$, $a_{\gamma_t 12ij}$, $a_{\gamma_t 13ij}$, $a_{\gamma_t 14ij}$, $a_{\gamma_t 21ji}$, $a_{\gamma_t 22ji}$, $a_{\gamma_t 23ji}$ and $a_{\gamma_t 24ji}$ are real constants; the activation functions $f_{11j}(u)$, $f_{12j}(u)$, $f_{13j}(u)$, $f_{14j}(u)$, $f_{21i}(u)$, $f_{22i}(u)$, $f_{23i}(u)$ and $f_{24i}(u)$ are Lipschitz continuous real-valued functions on $\mathbb{R}$; the delay kernels $K_{11j}(t)$, $K_{12j}(t)$, $K_{21i}(t)$ and $K_{22i}(t)$ are nonnegative-valued functions which are locally Lebesgue integrable in $\mathbb{R}^+$; $X_i(t)$ and $Y_j(t)$ could be viewed as disturbances; $i = 1, \ldots, m$, $j = 1, \ldots, n$.

The response system controlled by $U_i(t)$ and $V_j(t)$ reads:

$$\begin{cases} \dot{\tilde{x}}_i(t) = -l_{1i}\tilde{x}_i(t-\tau_{1i}) + \sum_{j=1}^{n} a_{\gamma_t 11ij} f_{11j}(\tilde{y}_j(t)) \\ \qquad + \sum_{j=1}^{n} a_{\gamma_t 12ij} f_{12j}(\tilde{y}_j(t-\sigma_{1j}(t))) \\ \qquad + \bigvee_{j=1}^{n} a_{\gamma_t 13ij} \int_{-\infty}^{t} K_{11j}(t-s) f_{13j}(\tilde{y}_j(s)) ds \\ \qquad + \bigwedge_{j=1}^{n} a_{\gamma_t 14ij} \int_{-\infty}^{t} K_{12j}(t-s) f_{14j}(\tilde{y}_j(s)) ds \\ \qquad + X_i(t) - U_i(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ i = 1, \ldots, m, \\ \dot{\tilde{y}}_j(t) = -l_{2j}\tilde{y}_j(t-\tau_{2j}) + \sum_{i=1}^{m} a_{\gamma_t 21ji} f_{21i}(\tilde{x}_i(t)) \\ \qquad + \sum_{i=1}^{m} a_{\gamma_t 22ji} f_{22i}(\tilde{x}_i(t-\sigma_{2i}(t))) \\ \qquad + \bigvee_{i=1}^{m} a_{\gamma_t 23ji} \int_{-\infty}^{t} K_{21i}(t-s) f_{23i}(\tilde{x}_i(s)) ds \\ \qquad + \bigwedge_{i=1}^{m} a_{\gamma_t 24ji} \int_{-\infty}^{t} K_{22i}(t-s) f_{24i}(\tilde{x}_i(s)) ds \\ \qquad + Y_j(t) - V_j(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ j = 1, \ldots, n. \end{cases} \tag{2.3}$$

Proposition 2.1. *Let $x_{i0}, y_{j0} : \Omega \times (-\infty, 0] \to \mathbb{R}$ be $\mathscr{F} \otimes \mathscr{L}$ measurable, and suppose that $x_{i0}(t)$ and $y_{j0}(t)$ are $\mathscr{F}_0$ measurable for all $t \in (-\infty, 0]$ and assume*

$$\begin{cases} \sup\limits_{t\in(-\infty,0]} \mathbb{E}|x_{i0}(t)| < +\infty, \\ \sup\limits_{t\in(-\infty,0]} \mathbb{E}|y_{j0}(t)| < +\infty, \end{cases}$$

$i = 1, \ldots, m,\ j = 1, \ldots, n$. Then, (2.1) *admits a unique state trajectory*

$$(x_1(t), \ldots, x_m(t); y_1(t), \ldots, y_n(t))$$

satisfying the initial condition

$$\begin{cases} x_i = x_{i0},\ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0],\ i = 1, \ldots, m, \\ y_j = y_{j0},\ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0],\ j = 1, \ldots, n. \end{cases} \tag{2.4}$$

Remark 2.1. *By Proposition 2.1, for the given initial data $\tilde{x}_{i0}(t)$ and $\tilde{y}_{j0}(t)$ ($i = 1, \ldots, m,\ j = 1, \ldots, n$), the response network system* (2.3) *subsequent by*

$$\begin{cases} \tilde{x}_i = \tilde{x}_{i0},\ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0],\ i = 1, \ldots, m, \\ \tilde{y}_j = \tilde{y}_{j0},\ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0],\ j = 1, \ldots, n, \end{cases} \tag{2.5}$$

admits a unique state trajectory

$$(\tilde{x}_1(t), \ldots, \tilde{x}_m(t); \tilde{y}_1(t), \ldots, \tilde{y}_n(t)),$$

where the initial data $\tilde{x}_{i0}(t)$ and $\tilde{y}_{j0}(t)$ satisfy the same conditions as that obeyed by $x_{i0}(t)$ and $y_{j0}(t)$ in Proposition 2.1, $i = 1, \ldots, m,\ j = 1, \ldots, n$.

Definition 2.1. *The drive network system* (2.1) *and the response network system* (2.3) *are said to be synchronized in finite time provided that there exists a $T > 0$ such that*

$$\begin{cases} \left.\begin{aligned} &\lim_{t\to T^-} \mathbb{E}|x_i(t) - \tilde{x}_i(t)| = 0, \\ &x_i(t) = \tilde{x}_i(t),\ t \in [T, +\infty),\ \mathbb{P}\text{-a.s.} \end{aligned}\right\} \quad i = 1, \ldots, m, \\ \left.\begin{aligned} &\lim_{t\to T^-} \mathbb{E}|y_j(t) - \tilde{y}_j(t)| = 0, \\ &y_j(t) = \tilde{y}_j(t),\ t \in [T, +\infty),\ \mathbb{P}\text{-a.s.} \end{aligned}\right\} \quad j = 1, \ldots, n. \end{cases} \tag{2.6}$$

Now we are in a position to introduce the intermittent quantized controls that would be used to synchronize the drive-response network system (2.1)–(2.3). Let $\{t_k\}_{k=0}^{\infty}$ be a strictly increasing sequence in $\mathbb{R}^+$ such that $t_0 = 0$ and that

$$\lim_{k\to\infty} t_k = +\infty.$$

The controls $U_i(t)$ and $V_j(t)$ are required to satisfy: For every $k \in \mathbb{N}_0$,

$$U_i(t) = V_j(t) = 0,\ t \in [t_{2k+1}, t_{2k+2}),\ \mathbb{P}\text{-a.s.,} \tag{2.7}$$

$$U_i(t) = -k_{1i}(t)(x_i(t) - \tilde{x}_i(t)) - \Upsilon\,\mathrm{sgn}(q(x_i(t) - \tilde{x}_i(t))),\ t \in [t_{2k}, t_{2k+1}),\ \mathbb{P}\text{-a.s.}, \tag{2.8}$$

$$V_j(t) = -k_{2j}(t)(y_i(t) - \tilde{y}_i(t)) - \Upsilon\,\mathrm{sgn}(q(y_i(t) - \tilde{y}_i(t))),\ t \in [t_{2k}, t_{2k+1}),\ \mathbb{P}\text{-a.s.}, \tag{2.9}$$

$$k_{1i}(t) = \hat{k}_{\gamma_t 1i}(1 + \check{k}_{1i}(t)),\ t \in [t_{2k}, t_{2k+1}),\ \mathbb{P}\text{-a.s.}, \tag{2.10}$$

$$k_{2j}(t) = \hat{k}_{\gamma_t 2j}(1 + \check{k}_{2j}(t)),\ t \in [t_{2k}, t_{2k+1}),\ \mathbb{P}\text{-a.s.}, \tag{2.11}$$

in which q is the so-called logarithmic quantizer, that is, an odd function mapping $\mathbb{R}$ into Λ obeying the rule $q(v) = \aleph_k = \aleph_0\rho^k$ if $v \in (\frac{\aleph_k}{1+\theta}, \frac{\aleph_k}{1-\theta}]$ for a certain $k \in \mathbb{Z}$, where $\theta = \frac{1-\rho}{1+\rho}$, $\aleph_0$ is a sufficiently large positive number to be specified later, and

$$\Lambda = \{\pm\aleph_k;\ \aleph_k = \aleph_0\rho^k,\ k \in \mathbb{Z}\};$$

$\Upsilon > 0$, $\hat{k}_{\gamma_t 1i} > 0$, $\hat{k}_{\gamma_t 2j} > 0$, $\check{k}_{1i}(t) \in [-\theta, \theta)$, $\check{k}_{2j}(t) \in [-\theta, \theta)$; $i = 1, \ldots, m$, $j = 1, \ldots, n$.

To summarize, for every $k \in \mathbb{N}_0$,

$$U_i(t) = -\hat{k}_{\gamma_t 1i}(1 + \check{k}_{1i}(t))(x_i(t) - \tilde{x}_i(t)) - \Upsilon\,\mathrm{sgn}(q(x_i(t) - \tilde{x}_i(t))),\ t \in [t_{2k}, t_{2k+1}),\ \mathbb{P}\text{-a.s.}, \tag{2.12}$$

$$V_j(t) = -\hat{k}_{\gamma_t 2j}(1 + \check{k}_{2j}(t))(y_i(t) - \tilde{y}_i(t)) - \Upsilon\,\mathrm{sgn}(q(y_i(t) - \tilde{y}_i(t))),\ t \in [t_{2k}, t_{2k+1}),\ \mathbb{P}\text{-a.s.} \tag{2.13}$$

We are now ready to record some results which is necessary in the proof of our main results.

Definition 2.2. *Let N be a positive integer. Given $V(x) : \mathbb{R}^N \to \mathbb{R}$. $V(x)$ is said to be a C-regular function provided that (i) $V(x)$ is regular in $\mathbb{R}^N$; (ii) $V(x)$ is positive definite in $\mathbb{R}^N$: $V(\mathbf{0}) = 0$, $V(x) > 0$ for all $x \in \mathbb{R}^N \setminus \{\mathbf{0}\}$; and (iii) $V(x)$ is coercive in the sense that*

$$\lim_{|x| \to +\infty} f(x) = +\infty.$$

Lemma 2.1. *(See [25]) Let $V(x) : \mathbb{R}^N \to \mathbb{R}$ be a C-regular function, and $x(t) : I \to \mathbb{R}^N$ be absolutely continuous where I is an interval (bounded or unbounded). Then $V(x(t))$ is absolutely continuous in I and it holds that: For every selection $\eta(t)$ in $\partial V(x(t))$, the Clarke generalized gradient of $V(x(t))$ at $x(t)$, it holds that*

$$D^+V(x(t)) = \eta(t)x'(t),\ t \in I \setminus \{\sup I\}.$$

Remark 2.2. *It is ready to verify that the well-known absolute value function $V(x) = |x|$, $x \in \mathbb{R}$, is C-regular, and to check that in this situation*

$$\partial V(x) = \begin{cases} \{-1\} = -1 & \text{if } x \in (-\infty, 0), \\ [-1, 1] & \text{if } x = 0, \\ \{1\} = 1 & \text{if } x \in (0, +\infty). \end{cases}$$

In the sequel, we denote $\mathfrak{C}(x) = \partial V(x)$, $x \in \mathbb{R}$, in which $V(x) = |x|$.

Lemma 2.2. *(See [25]) Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function. Suppose that $\Psi(0) > 0$ and that either*

$$D^+\Psi(t) \leqslant -\beta\Psi(t) - \alpha$$

holds if $t \in [t_{2k}, t_{2k+1})$ for some $k \in \mathbb{N}_0$, or

$$D^+\Psi(t) \leqslant \eta\Psi(t)$$

holds if $t \in [t_{2k+1}, t_{2k+2})$ *for some* $k \in \mathbb{N}_0$*, where* α*,* β *and* η *are all given positive constants. If* $\chi_k > 1$ *holds for all* $k \in \mathbb{N}_0$*, then*

$$\lim_{t \to T^-} \Psi(t) = 0$$

and

$$\Psi(t) = 0, \ \forall\, t \in [T, +\infty),$$

where

$$\begin{aligned}
\chi_k &= \frac{\beta(t_{2k+1} - t_{2k})}{\eta(t_{2k+2} - t_{2k+1})}, \\
T &= t_{2\tilde{k}} + \frac{1}{\beta}\left[\ln(\frac{\beta\Psi(0)}{\alpha} + 1) - \beta\sum_{k=0}^{\tilde{k}-1}(1 - \frac{1}{\chi_k})(t_{2k+2} - t_{2k+1})\right], \\
\tilde{k} &= \max\left\{k \in \mathbb{N}_0;\ \ln(\frac{\beta\Psi(0)}{\alpha} + 1) - \beta\sum_{i=0}^{k-1}(1 - \frac{1}{\chi_i})(t_{2i+2} - t_{2i+1}) > 0\right\}.
\end{aligned}$$

Assumption 2.1. $0 \leqslant L_{11j}, L_{12j}, L_{13j}, L_{14j}, L_{21i}, L_{22i}, L_{23i}, L_{24i} < +\infty$ with

$$\begin{aligned}
L_{11j} &= \sup_{u \neq v,\ u,v \in \mathbb{R}} |\frac{f_{11j}(u) - f_{11j}(v)}{u - v}|, \quad L_{12j} = \sup_{u \neq v,\ u,v \in \mathbb{R}} |\frac{f_{12j}(u) - f_{12j}(v)}{u - v}|, \\
L_{13j} &= \sup_{u \neq v,\ u,v \in \mathbb{R}} |\frac{f_{13j}(u) - f_{13j}(v)}{u - v}|, \quad L_{14j} = \sup_{u \neq v,\ u,v \in \mathbb{R}} |\frac{f_{14j}(u) - f_{14j}(v)}{u - v}|,
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
L_{21i} &= \sup_{u \neq v,\ u,v \in \mathbb{R}} |\frac{f_{21i}(u) - f_{21i}(v)}{u - v}|, \quad L_{22i} = \sup_{u \neq v,\ u,v \in \mathbb{R}} |\frac{f_{22i}(u) - f_{22i}(v)}{u - v}|, \\
L_{23i} &= \sup_{u \neq v,\ u,v \in \mathbb{R}} |\frac{f_{23i}(u) - f_{23i}(v)}{u - v}|, \quad L_{24i} = \sup_{u \neq v,\ u,v \in \mathbb{R}} |\frac{f_{24i}(u) - f_{24i}(v)}{u - v}|,
\end{aligned} \tag{2.15}$$

$i = 1, \ldots, m,\ j = 1, \ldots, n.$

Assumption 2.2. $\sigma_{1j}(t)$ and $\sigma_{2i}(t)$ are absolutely continuous, and $0 \leqslant \sigma_{1j}(t), \sigma_{2i}(t) < t$, $0 \leqslant \bar{\sigma}_{1j}$, $\bar{\sigma}_{2i} < +\infty$, $0 \leqslant \hat{\sigma}_{1j}, \hat{\sigma}_{2i} < 1$ with

$$\bar{\sigma}_{1j} = \sup_{t \in \mathbb{R}^+} \sigma_{1j}(t), \tag{2.16}$$

$$\bar{\sigma}_{2i} = \sup_{t \in \mathbb{R}^+} \sigma_{2i}(t), \tag{2.17}$$

$$\hat{\sigma}_{1j} = \operatorname{ess\,sup}_{t \in \mathbb{R}^+} \dot{\sigma}_{1j}(t), \tag{2.18}$$

$$\hat{\sigma}_{2i} = \operatorname{ess\,sup}_{t \in \mathbb{R}^+} \dot{\sigma}_{2i}(t), \tag{2.19}$$

$i = 1, \ldots, m,\ j = 1, \ldots, n.$

Assumption 2.3. There exists a $\beta^* > 0$ such that for every $\beta \in (-\infty, \beta^*)$, it holds that $0 \leqslant \bar{\check{K}}_{\beta 11j}, \bar{\check{K}}_{\beta 12j}$, $\bar{\check{K}}_{\beta 21j}, \bar{\check{K}}_{\beta 22j} < +\infty$ where

$$\begin{aligned}
\bar{\check{K}}_{\beta 11j} &= \int_0^{+\infty} \check{K}_{\beta 11j}(t)dt, \quad \bar{\check{K}}_{\beta 12j} = \int_0^{+\infty} \check{K}_{\beta 12j}(t)dt, \\
\bar{\check{K}}_{\beta 21j} &= \int_0^{+\infty} \check{K}_{\beta 21j}(t)dt, \quad \bar{\check{K}}_{\beta 22j} = \int_0^{+\infty} \check{K}_{\beta 22j}(t)dt,
\end{aligned}$$

with

$$\check{K}_{\beta 11j}(t) = K_{11j}(t)e^{\beta t}, \quad \check{K}_{\beta 12j}(t) = K_{12j}(t)e^{\beta t},$$
$$\check{K}_{\beta 21j}(t) = K_{21j}(t)e^{\beta t}, \quad \check{K}_{\beta 22j}(t) = K_{22j}(t)e^{\beta t},$$

$t \in \mathbb{R}^+$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Theorem 2.1. *Suppose that Assumptions 2.1–2.3 hold true. If there exists a* $\beta \in (0, \beta^*)$ *(see 2.3), a* $\rho \in (0, 1)$*, a* Υ *and some* $p_{\xi 1i}$ *along with* $p_{\xi 2j}$*, such that*

$$M_{\xi 1i} + \beta p_{\xi 1i} \leqslant 0, \ i = 1, \ldots, m, \ \xi \in \Xi, \tag{2.20}$$

$$M_{\xi 2j} + \beta p_{\xi 2j} \leqslant 0, \ j = 1, \ldots, n, \ \xi \in \Xi, \tag{2.21}$$

$$\chi_k = \frac{\beta(t_{2k+1} - t_{2k})}{\eta(t_{2k+2} - t_{2k+1})} > 1, \ k \in \mathbb{N}_0, \tag{2.22}$$

then the drive-response network system (2.1)–(2.3) *is synchronizable in finite time. More precisely, for every state trajectory* $(x_1(t), \ldots, x_m(t); y_1(t), \ldots, y_n(t))$ *of the drive network system* (2.1) *and every state trajectory* $(\tilde{x}_1(t), \ldots, \tilde{x}_m(t); \tilde{y}_1(t), \ldots, \tilde{y}_n(t))$ *of the response network system* (2.3)*, the assertion* (2.6) *in Definition 2.1 holds with*

$$\alpha = \Upsilon \min_{\xi \in \Xi} \left(\sum_{i=1}^{m} p_{\xi 1i} + \sum_{j=1}^{n} p_{\xi 2j}\right), \tag{2.23}$$

$$\begin{aligned} M_{\xi 1i} = &\sum_{\tilde{\xi} \in \Xi} \pi_{\xi \tilde{\xi}} p_{\tilde{\xi} 1i} + p_{\xi 1i} l_{1i} e^{\beta \tau_{1i}} + \sum_{j=1}^{n} p_{\xi 2j} L_{21i} |a_{\xi 21ji}| \\ &+ \sum_{j=1}^{n} \frac{p_{\xi 2j} |a_{\xi 22ji}| L_{22i} e^{\beta \bar{\sigma}_{2i}}}{1 - \hat{\sigma}_{2i}} + \sum_{j=1}^{n} p_{\xi 2j} |a_{\xi 23ji}| \bar{\check{K}}_{\beta 21i} L_{23i} \\ &+ \sum_{j=1}^{n} p_{\xi 2j} |a_{\xi 24ji}| \bar{\check{K}}_{\beta 22i} L_{24i} - p_{\xi 1i} \hat{k}_{\xi 1i} (1 - \theta), \end{aligned} \tag{2.24}$$

$$\begin{aligned} M_{\xi 2j} = &\sum_{\tilde{\xi} \in \Xi} \pi_{\xi \tilde{\xi}} p_{\tilde{\xi} 2j} + p_{\xi 2j} l_{2j} e^{\beta \tau_{2j}} + \sum_{i=1}^{m} p_{\xi 1i} L_{11j} |a_{\xi 11ij}| \\ &+ \sum_{i=1}^{m} \frac{p_{\xi 1i} |a_{\xi 12ij}| L_{12j} e^{\beta \bar{\sigma}_{1j}}}{1 - \hat{\sigma}_{1j}} + \sum_{i=1}^{m} p_{\xi 1i} |a_{\xi 13ij}| \bar{\check{K}}_{\beta 11j} L_{13j} \\ &+ \sum_{i=1}^{m} p_{\xi 1i} |a_{\xi 14ij}| \bar{\check{K}}_{\beta 12j} L_{14j} - p_{\xi 2j} \hat{k}_{\xi 2j} (1 - \theta), \end{aligned} \tag{2.25}$$

$$T = t_{2\tilde{k}} + \frac{1}{\beta} \left[\ln\left(\frac{\beta \varsigma}{\alpha} + 1\right) - \beta \sum_{k=0}^{\tilde{k}-1} \left(1 - \frac{1}{\chi_k}\right)(t_{2k+2} - t_{2k+1}) \right], \tag{2.26}$$

$$\tilde{k} = \max \left\{ k \in \mathbb{N}_0; \ \ln\left(\frac{\beta \varsigma}{\alpha} + 1\right) - \beta \sum_{i=0}^{k-1} \left(1 - \frac{1}{\chi_i}\right)(t_{2i+2} - t_{2i+1}) > 0 \right\}, \tag{2.27}$$

$$
\begin{aligned}
\varsigma =& \sum_{i=1}^{m} p_{\xi 1i}|x_i(0)-\tilde{x}_i(0)| + \sum_{j=1}^{n} p_{\xi 2j}|y_j(0)-\tilde{y}_j(0)| \\
&+ \sum_{i=1}^{m} p_{\xi 1i} l_{1i} e^{\beta\tau_{1i}} \int_{-\tau_{1i}}^{0} e^{\beta s}|x_i(s)-\tilde{x}_i(s)|ds \\
&+ \sum_{j=1}^{n} p_{\xi 2j} l_{2j} e^{\beta\tau_{2j}} \int_{-\tau_{2j}}^{0} e^{\beta s}|y_j(s)-\tilde{y}_j(s)|ds \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{p_{\xi 2j}|a_{\xi 22ji}|L_{22i}e^{\beta\bar{\sigma}_{2i}}}{1-\hat{\sigma}_{2i}} \int_{-\sigma_{2i}(0)}^{0} e^{\beta s}|x_i(s)-\tilde{x}_i(s)|ds \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{m} \frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1-\hat{\sigma}_{1j}} \int_{-\sigma_{1j}(0)}^{0} e^{\beta s}|y_j(s)-\tilde{y}_j(s)|ds \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 23ji}|L_{23i} \int_{0}^{+\infty} \check{K}_{\beta 21i}(s) \int_{-s}^{0} e^{\beta\tilde{s}}|x_i(\tilde{s})-\tilde{x}_i(\tilde{s})|d\tilde{s}ds \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 24ji}|L_{24i} \int_{0}^{+\infty} \check{K}_{\beta 22i}(s) \int_{-s}^{0} e^{\beta\tilde{s}}|x_i(\tilde{s})-\tilde{x}_i(\tilde{s})|d\tilde{s}ds \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 13ij}|L_{13j} \int_{0}^{+\infty} \check{K}_{\beta 11j}(s) \int_{-s}^{0} e^{\beta\tilde{s}}|y_j(\tilde{s})-\tilde{y}_j(\tilde{s})|d\tilde{s}ds \\
&+ \sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 14ij}|L_{14j} \int_{0}^{+\infty} \check{K}_{\beta 12j}(s) \int_{-s}^{0} e^{\beta\tilde{s}}|y_j(\tilde{s})-\tilde{y}_j(\tilde{s})|d\tilde{s}ds,
\end{aligned} \tag{2.28}
$$

$$
\eta = \max(\max_{\xi\in\Xi}\max_{i=1}^{m}\frac{\tilde{M}_{\xi 1i}}{p_{\xi 1i}}, \max_{\xi\in\Xi}\max_{j=1}^{n}\frac{\tilde{M}_{\xi 2j}}{p_{\xi 2j}}), \tag{2.29}
$$

$$
\begin{aligned}
\tilde{M}_{\xi 1i} =& \sum_{\tilde{\xi}\in\Xi}\pi_{\xi\tilde{\xi}}p_{\tilde{\xi}1i} + p_{\xi 1i}l_{1i}e^{\beta\tau_{1i}} + \sum_{j=1}^{n} p_{\xi 2j}L_{21i}|a_{\xi 21ji}| \\
&+ \sum_{j=1}^{n}\frac{p_{\xi 2j}|a_{\xi 22ji}|L_{22i}e^{\beta\bar{\sigma}_{2i}}}{1-\hat{\sigma}_{2i}} + \sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 23ji}|\bar{\check{K}}_{\beta 21i}L_{23i} \\
&+ \sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 24ji}|\bar{\check{K}}_{\beta 22i}L_{24i},
\end{aligned} \tag{2.30}
$$

$$
\begin{aligned}
\tilde{M}_{\xi 2j} =& \sum_{\tilde{\xi}\in\Xi}\pi_{\xi\tilde{\xi}}p_{\tilde{\xi}2j} + p_{\xi 2j}l_{2j}e^{\beta\tau_{2j}} + \sum_{i=1}^{m} p_{\xi 1i}L_{11j}|a_{\xi 11ij}| \\
&+ \sum_{i=1}^{m}\frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1-\hat{\sigma}_{1j}} + \sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 13ij}|\bar{\check{K}}_{\beta 11j}L_{13j} \\
&+ \sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 14ij}|\bar{\check{K}}_{\beta 12j}L_{14j},
\end{aligned} \tag{2.31}
$$

$i = 1, \ldots, m,\ j = 1, \ldots, n,\ \xi \in \Xi$.

3. Proof of the main results

Lemma 3.1. *Let N be a given positive integer and let $(\mu_1, \mu_2, \ldots, \mu_N)^\top \in \mathbb{R}^N$. Then for every pair $(x_1, x_2, \ldots, x_N)^\top$ and $(y_1, y_2, \ldots, y_N)^\top$ of vectors in $\mathbb{R}^N$, it holds that*

$$\begin{cases} |\bigvee_{k=1}^{N} \mu_k x_k - \bigvee_{k=1}^{N} \mu_k y_k| \leqslant \sum_{k=1}^{N} |\mu_k||x_k - y_k|, \\ |\bigwedge_{k=1}^{N} \mu_k x_k - \bigwedge_{k=1}^{N} \mu_k y_k| \leqslant \sum_{k=1}^{N} |\mu_k||x_k - y_k|. \end{cases}$$

Proof of Theorem 2.1. It is readily to see that the synchronizability of the drive-response network system (2.1)–(2.3) is equivalent to the stability of the error network system

$$\begin{cases} \dot{u}_i(t) = -l_{1i}u_i(t-\tau_{1i}) + \sum_{j=1}^{n} a_{\gamma_t 11ij}\hat{f}_{11j}(v_j(t)) + \sum_{j=1}^{n} a_{\gamma_t 12ij}\hat{f}_{12j}(v_j(t-\sigma_{1j}(t))) \\ \quad + \bigvee_{j=1}^{n} a_{\gamma_t 13ij} \int_{-\infty}^{t} K_{11j}(t-s) f_{13j}(\tilde{y}_j(s) + v_j(s))ds \\ \quad - \bigvee_{j=1}^{n} a_{\gamma_t 13ij} \int_{-\infty}^{t} K_{11j}(t-s) f_{13j}(\tilde{y}_j(s))ds \\ \quad + \bigwedge_{j=1}^{n} a_{\gamma_t 14ij} \int_{-\infty}^{t} K_{12j}(t-s) f_{14j}(\tilde{y}_j(s) + v_j(s))ds \\ \quad - \bigwedge_{j=1}^{n} a_{\gamma_t 14ij} \int_{-\infty}^{t} K_{12j}(t-s) f_{14j}(\tilde{y}_j(s))ds \\ \quad + U_i(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ i = 1, \ldots, m, \\ \dot{v}_j(t) = -l_{2j}v_j(t-\tau_{2j}) + \sum_{i=1}^{m} a_{\gamma_t 21ji}\hat{f}_{21i}(u_i(t)) + \sum_{i=1}^{m} a_{\gamma_t 22ji}\hat{f}_{22i}(u_i(t-\sigma_{2i}(t))) \\ \quad + \bigvee_{i=1}^{m} a_{\gamma_t 23ji} \int_{-\infty}^{t} K_{21i}(t-s) f_{23i}(\tilde{x}_i(s) + u_i(s))ds \\ \quad - \bigvee_{i=1}^{m} a_{\gamma_t 23ji} \int_{-\infty}^{t} K_{21i}(t-s) f_{23i}(\tilde{x}_i(s))ds \\ \quad + \bigwedge_{i=1}^{m} a_{\gamma_t 24ji} \int_{-\infty}^{t} K_{22i}(t-s) f_{24i}(\tilde{x}_i(s) + u_i(s))ds \\ \quad - \bigwedge_{i=1}^{m} a_{\gamma_t 24ji} \int_{-\infty}^{t} K_{22i}(t-s) f_{24i}(\tilde{x}_i(s))ds \\ \quad + V_j(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ j = 1, \ldots, n, \end{cases} \tag{3.1}$$

in which

$$u_i(t) = x_i(t) - \tilde{x}_i(t), \tag{3.2}$$

$$v_j(t) = y_j(t) - \tilde{y}_j(t), \tag{3.3}$$

$$\begin{aligned}\hat{f}_{11j}(v_j(t)) =& f_{11j}(y_j(t)) - f_{11j}(\tilde{y}_j(t))\\ =& f_{11j}(\tilde{y}_j(t) + v_j(t)) - f_{11j}(\tilde{y}_j(t)),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\hat{f}_{12j}(v_j(t - \sigma_{1j}(t))) =& f_{12j}(y_j(t - \sigma_{1j}(t))) - f_{12j}(\tilde{y}_j(t - \sigma_{1j}(t)))\\ =& f_{12j}(\tilde{y}_j(t - \sigma_{1j}(t)) + v_j(t - \sigma_{1j}(t))) - f_{12j}(\tilde{y}_j(t - \sigma_{1j}(t))),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\hat{f}_{21i}(u_i(t)) =& f_{21i}(\tilde{x}_i(t) + u_i(t)) - f_{21i}(\tilde{x}_i(t))\\ =& f_{21i}(x_i(t)) - f_{21i}(\tilde{x}_i(t)),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\hat{f}_{22i}(u_i(t - \sigma_{2i}(t))) =& f_{22i}(x_i(t - \sigma_{2i}(t))) - f_{22i}(\tilde{x}_i(t - \sigma_{2i}(t)))\\ =& f_{22i}(\tilde{x}_i(t - \sigma_{2i}(t)) + u_i(t - \sigma_{2i}(t))) - f_{22i}(\tilde{x}_i(t - \sigma_{2i}(t))),\end{aligned} \tag{3.7}$$

$i = 1, \ldots, m,\ j = 1, \ldots, n.$

Let us write

$$V(t) = \mathbb{E}\mathcal{V}_{\gamma_t}(t),\ t \in \mathbb{R}^+, \tag{3.8}$$

$$\mathcal{V}_\xi(t) = \mathcal{V}_{\xi 1}(t) + \mathcal{V}_{\xi 2}(t) + \mathcal{V}_{\xi 3}(t) + \mathcal{V}_{\xi 4}(t),\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ \xi \in \varXi, \tag{3.9}$$

$$\mathcal{V}_{\xi 1}(t) = \sum_{i=1}^{m} p_{\xi 1i}|u_i(t)| + \sum_{j=1}^{n} p_{\xi 2j}|v_j(t)|,\ t \in \mathbb{R},\ \mathbb{P}\text{-a.s.},\ \xi \in \varXi, \tag{3.10}$$

$$\begin{aligned}\mathcal{V}_{\xi 2}(t) =& \sum_{i=1}^{m} p_{\xi 1i} l_{1i} e^{\beta\tau_{1i}} \int_{t-\tau_{1i}}^{t} e^{-\beta(t-s)}|u_i(s)|ds\\ &+ \sum_{j=1}^{n} p_{\xi 2j} l_{2j} e^{\beta\tau_{2j}} \int_{t-\tau_{2j}}^{t} e^{-\beta(t-s)}|v_j(s)|ds,\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ \xi \in \varXi,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathcal{V}_{\xi 3}(t) =& \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{p_{\xi 2i}|a_{\xi 22ji}|L_{22i}e^{\beta\bar{\sigma}_{2i}}}{1 - \hat{\sigma}_{2i}} \int_{t-\sigma_{2i}(t)}^{t} e^{-\beta(t-s)}|u_i(s)|ds\\ &+ \sum_{j=1}^{n}\sum_{i=1}^{m} \frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1 - \hat{\sigma}_{1j}} \int_{t-\sigma_{1j}(t)}^{t} e^{-\beta(t-s)}|v_j(s)|ds,\ t \in \mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ \xi \in \varXi,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\mathcal{V}_{\xi 4}(t) =& \sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 23ji}|L_{23i} \int_0^{+\infty} \check{K}_{\beta 21i}(s) \int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds\\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 24ji}|L_{24i} \int_0^{+\infty} \check{K}_{\beta 22i}(s) \int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds\\ &+ \sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 13ij}|L_{13j} \int_0^{+\infty} \check{K}_{\beta 11j}(s) \int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\\ &+ \sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 14ij}|L_{14j} \int_0^{+\infty} \check{K}_{\beta 12j}(s) \int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds,\end{aligned} \tag{3.13}$$

$t \in \mathbb{R}^+$, $\mathbb{P}$-a.s., $\xi \in \Xi$. As in [25,32], we introduce the weak infinitesimal operator $\mathcal{L}$ for every stochastic process $\mathcal{X}(t)$ (having actually certain regularity in time variable t) defined in an interval I:

$$\mathbb{E}\mathcal{L}\mathcal{X}(t) = D^+\mathbb{E}\mathcal{X}(t),\ t \in I \setminus \{\sup I\}.$$

In light of (3.8), we have

$$D^+V(t) = \mathbb{E}\mathcal{L}\mathcal{V}_{\gamma_t}(t),\ t \in \mathbb{R}^+.$$

Thanks to Lemma 2.1, Remark 2.2, and the definition (3.10) of $\mathcal{V}_{\xi 1}(t)$, we have

$$\begin{aligned}
\mathcal{L}\mathcal{V}_{\xi 1}(t) = {} & \sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\dot{u}_i(t) + \sum_{i=1}^{m}\sum_{\tilde{\xi}\in\Xi} \pi_{\xi\tilde{\xi}} p_{\tilde{\xi}1i}|u_i(t)| \\
& + \sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\dot{v}_j(t) + \sum_{j=1}^{n}\sum_{\tilde{\xi}\in\Xi} \pi_{\xi\tilde{\xi}} p_{\tilde{\xi}2j}|v_j(t)| \\
= {} & \sum_{i=1}^{m}\sum_{\tilde{\xi}\in\Xi} \pi_{\xi\tilde{\xi}} p_{\tilde{\xi}1i}|u_i(t)| + \sum_{j=1}^{n}\sum_{\tilde{\xi}\in\Xi} \pi_{\xi\tilde{\xi}} p_{\tilde{\xi}2j}|v_j(t)| \\
& - \sum_{i=1}^{m} p_{\xi 1i} l_{1i}\eta_{u_i}(t)u_i(t-\tau_{1i}) + \sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\sum_{j=1}^{n} a_{\xi 11ij}\hat{f}_{11j}(v_j(t)) \\
& + \sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\sum_{j=1}^{n} a_{\xi 12ij}\hat{f}_{12j}(v_j(t-\sigma_{1j}(t))) \\
& + \Pi_{\xi 11}(t) + \Pi_{\xi 12}(t) - \sum_{i=1}^{m} p_{\xi 1i}\hat{k}_{\xi 1i}(1+\check{k}_{1i}(t))\eta_{u_i}(t)u_i(t) \\
& - \Upsilon\sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\,\mathrm{sgn}(q(u_i(t))) - \sum_{j=1}^{n} p_{\xi 2j} l_{2j}\eta_{v_j}(t)v_j(t-\tau_{2j}) \\
& + \sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\sum_{i=1}^{m} a_{\xi 21ji}\hat{f}_{21i}(u_i(t)) \\
& + \sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\sum_{i=1}^{m} a_{\xi 22ji}\hat{f}_{22i}(u_i(t-\sigma_{2i}(t))) \\
& + \Pi_{\xi 21}(t) + \Pi_{\xi 22}(t) - \sum_{j=1}^{n} p_{\xi 2j}\hat{k}_{\xi 2j}(1+\check{k}_{2j}(t))\eta_{v_j}(t)v_j(t) \\
& - \Upsilon\sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\,\mathrm{sgn}(q(v_j(t))),\ t \in [t_{2k}, t_{2k+1}],\ \mathbb{P}\text{-a.s., } \xi \in \Xi,
\end{aligned} \tag{3.14}$$

in which $\eta_{u_i}(t)$ is an arbitrarily given selection in $\mathfrak{C}(u_i(t))$, and $\eta_{v_j}(t)$ is an arbitrarily given selection in $\mathfrak{C}(v_j(t))$, $i = 1, \ldots, m$, $j = 1, \ldots, n$; see Remark 2.2 for the precise definition of the multi-valued function $\mathfrak{C}(\cdot)$; and $\Pi_{\xi 11}(t)$, $\Pi_{\xi 12}(t)$, $\Pi_{\xi 21}(t)$ as well as $\Pi_{\xi 22}(t)$ is

$$\begin{aligned}\Pi_{\xi 11}(t) = &\sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\bigvee_{j=1}^{n} a_{\xi 13ij}\int_{-\infty}^{t} K_{11j}(t-s)f_{13j}(\tilde{y}_j(s)+v_j(s))ds\\ &-\sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\bigvee_{j=1}^{n} a_{\xi 13ij}\int_{-\infty}^{t} K_{11j}(t-s)f_{13j}(\tilde{y}_j(s))ds,\ t\in\mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\Pi_{\xi 12}(t) = &\sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\bigwedge_{j=1}^{n} a_{\xi 14ij}\int_{-\infty}^{t} K_{12j}(t-s)f_{14j}(\tilde{y}_j(s)+v_j(s))ds\\ &-\sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\bigwedge_{j=1}^{n} a_{\xi 14ij}\int_{-\infty}^{t} K_{12j}(t-s)f_{14j}(\tilde{y}_j(s))ds,\ t\in\mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\end{aligned} \tag{3.16}$$

$$\begin{aligned}\Pi_{\xi 21}(t) = &\sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\bigvee_{i=1}^{m} a_{\xi 23ji}\int_{-\infty}^{t} K_{21i}(t-s)f_{23i}(\tilde{x}_i(s)+u_i(s))ds\\ &-\sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\bigvee_{i=1}^{m} a_{\xi 23ji}\int_{-\infty}^{t} K_{21i}(t-s)f_{23i}(\tilde{x}_i(s))ds,\ t\in\mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\end{aligned} \tag{3.17}$$

$$\begin{aligned}\Pi_{\xi 22}(t) = &\sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\bigwedge_{i=1}^{m} a_{\xi 24ji}\int_{-\infty}^{t} K_{22i}(t-s)f_{24i}(\tilde{x}_i(s)+u_i(s))ds\\ &-\sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\bigwedge_{i=1}^{m} a_{\xi 24ji}\int_{-\infty}^{t} K_{22i}(t-s)f_{24i}(\tilde{x}_i(s))ds,\ t\in\mathbb{R}^+,\ \mathbb{P}\text{-a.s.}\end{aligned} \tag{3.18}$$

By the definition of the logarithmic quantizer q, we have

$$\mathrm{sgn}(q(x)) = \mathrm{sgn}(x),\ x\in\mathbb{R}. \tag{3.19}$$

By the definition of $\mathfrak{C}(\cdot)$ (see Remark 2.2 for the details), we have

$$\eta_{u_i}(t)u_i(t) = |u_i(t)|,\ t\in\mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ i=1,\dots,m, \tag{3.20}$$

and

$$\eta_{v_j}(t)v_j(t) = |v_j(t)|,\ t\in\mathbb{R}^+,\ \mathbb{P}\text{-a.s.},\ j=1,\dots,n. \tag{3.21}$$

This, together (3.19) and (3.20) implies

$$\begin{aligned}\eta_{u_i}(t)\,\mathrm{sgn}(q(u_i(t))) =&\eta_{u_i}(t)\,\mathrm{sgn}(u_i(t))\\ =&\eta_{v_j}(t)\,\mathrm{sgn}(q(v_j(t)))\\ =&\eta_{v_j}(t)\,\mathrm{sgn}(v_j(t)) = 1\end{aligned} \tag{3.22}$$

whenever $u_i(t)v_j(t)\neq 0$, $i=1,\dots,m$, $j=1,\dots,n$.

Thanks to the definition of $\mathfrak{C}(\cdot)$ (see Remark 2.2), we have by using direct computation

$$
\begin{aligned}
& -\sum_{i=1}^{m} p_{\xi 1i} l_{1i} \eta_{u_i}(t) u_i(t-\tau_{1i}) \\
\leqslant & \sum_{i=1}^{m} p_{\xi 1i} l_{1i} |u_i(t-\tau_{1i})| \\
= & \sum_{i=1}^{m} p_{\xi 1i} l_{1i} \left(e^{\beta\tau_{1i}} |u_i(t)| - e^{\beta\tau_{1i}} D^{+} \int_{t-\tau_{1i}}^{t} e^{-\beta(t-s)} |u_i(s)| ds - \beta e^{\beta\tau_{1i}} \int_{t-\tau_{1i}}^{t} e^{-\beta(t-s)} |u_i(s)| ds \right) \\
= & \sum_{i=1}^{m} p_{\xi 1i} l_{1i} e^{\beta\tau_{1i}} |u_i(t)| - \mathcal{L} \sum_{i=1}^{m} p_{\xi 1i} l_{1i} e^{\beta\tau_{1i}} \int_{t-\tau_{1i}}^{t} e^{-\beta(t-s)} |u_i(s)| ds \\
& -\beta \sum_{i=1}^{m} p_{\xi 1i} l_{1i} e^{\beta\tau_{1i}} \int_{t-\tau_{1i}}^{t} e^{-\beta(t-s)} |u_i(s)| ds, \ t \in \mathbb{R}^{+}, \ \mathbb{P}\text{-a.s.}, \ i = 1, \ldots, m. \qquad (3.23)
\end{aligned}
$$

Mimick the steps in (3.23), to obtain

$$
\begin{aligned}
& -\sum_{j=1}^{n} p_{\xi 2j} l_{2j} \eta_{v_j}(t) v_j(t-\tau_{2j}) \\
\leqslant & \sum_{j=1}^{n} p_{\xi 2j} l_{2j} e^{\beta\tau_{2j}} |v_j(t)| - \mathcal{L} \sum_{j=1}^{n} p_{\xi 2j} l_{2j} e^{\beta\tau_{2j}} \int_{t-\tau_{2j}}^{t} e^{-\beta(t-s)} |v_j(s)| ds \\
& -\beta \sum_{j=1}^{n} p_{\xi 2j} l_{2j} e^{\beta\tau_{2j}} \int_{t-\tau_{2j}}^{t} e^{-\beta(t-s)} |v_j(s)| ds, \ t \in \mathbb{R}^{+}, \ \mathbb{P}\text{-a.s.}, \ j = 1, \ldots, n. \qquad (3.24)
\end{aligned}
$$

Thanks to the definition of $\mathfrak{C}(\cdot)$ (see Remark 2.2), (3.4), and Assumption 2.1 (especially (2.14)), we have by

$$
\begin{aligned}
& \sum_{i=1}^{m} p_{\xi 1i} \eta_{u_i}(t) \sum_{j=1}^{n} a_{\xi 11ij} \hat{f}_{11j}(v_j(t)) \\
\leqslant & \sum_{i=1}^{m} p_{\xi 1i} \sum_{j=1}^{n} |a_{\xi 11ij}| |\hat{f}_{11j}(v_j(t))| \\
\leqslant & \sum_{i=1}^{m} p_{\xi 1i} \sum_{j=1}^{n} L_{11j} |a_{\xi 11ij}| |v_j(t)| \\
= & \sum_{j=1}^{n} \sum_{i=1}^{m} p_{\xi 1i} L_{11j} |a_{\xi 11ij}| |v_j(t)|, \ t \in \mathbb{R}, \ \mathbb{P}\text{-a.s.} \qquad (3.25)
\end{aligned}
$$

Owing to (3.6) and Assumption 2.1 (especially (2.15)), take similar steps as in (3.25), to obtain

$$
\begin{aligned}
& \sum_{j=1}^{n} p_{\xi 2j} \eta_{v_j}(t) \sum_{i=1}^{m} a_{\xi 21ji} \hat{f}_{21i}(u_i(t)) \\
\leqslant & \sum_{i=1}^{m} \sum_{j=1}^{n} p_{\xi 2j} L_{21i} |a_{\xi 21ji}| |u_i(t)|, \ t \in \mathbb{R}, \ \mathbb{P}\text{-a.s.} \qquad (3.26)
\end{aligned}
$$

Utilize the definition of $\mathfrak{C}(\cdot)$ (see Remark 2.2), Assumption 2.1 (especially (2.14)), Assumption 2.2 (especially (2.16) and (2.18)), and some routine but tedious calculations, to arrive at

$$\begin{aligned}
&\sum_{i=1}^{m} p_{\xi 1i}\eta_{u_i}(t)\sum_{j=1}^{n} a_{\xi 12ij}\hat{f}_{12j}(v_j(t-\sigma_{1j}(t)))\\
&\leqslant \sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 12ij}|L_{12j}|v_j(t-\sigma_{1j}(t))|\\
&\leqslant \sum_{j=1}^{n}\sum_{i=1}^{m} \frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1-\hat{\sigma}_{1j}}\bigg(|v_j(t)| - D^{+}\int_{t-\sigma_{1j}(t)}^{t} e^{-\beta(t-s)}|v_j(s)|ds\\
&\quad -\beta\int_{t-\sigma_{1j}(t)}^{t} e^{-\beta(t-s)}|v_j(s)|ds\bigg)\\
&= \sum_{j=1}^{n}\sum_{i=1}^{m} \frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1-\hat{\sigma}_{1j}}|v_j(t)|\\
&\quad -\mathcal{L}\sum_{j=1}^{n}\sum_{i=1}^{m} \frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1-\hat{\sigma}_{1j}}\int_{t-\sigma_{1j}(t)}^{t} e^{-\beta(t-s)}|v_j(s)|ds\\
&\quad -\beta\sum_{j=1}^{n}\sum_{i=1}^{m} \frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1-\hat{\sigma}_{1j}}\int_{t-\sigma_{1j}(t)}^{t} e^{-\beta(t-s)}|v_j(s)|ds,\ t\in\mathbb{R}^{+},\ \mathbb{P}\text{-a.s.} \qquad (3.27)
\end{aligned}$$

In view of Assumption 2.1 (especially (2.15)) and Assumption 2.2 (especially (2.17) and (2.19)), we have immediately by mimicking steps in (3.27)

$$\begin{aligned}
&\sum_{j=1}^{n} p_{\xi 2j}\eta_{v_j}(t)\sum_{i=1}^{m} a_{\xi 22ji}\hat{f}_{22i}(u_i(t-\sigma_{2i}(t)))\\
&\leqslant \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{p_{\xi 2j}|a_{\xi 22ji}|L_{22i}e^{\beta\bar{\sigma}_{2i}}}{1-\hat{\sigma}_{2i}}|u_i(t)|\\
&\quad -\mathcal{L}\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{p_{\xi 2j}|a_{\xi 22ji}|L_{22i}e^{\beta\bar{\sigma}_{2i}}}{1-\hat{\sigma}_{2i}}\int_{t-\sigma_{2i}(t)}^{t} e^{-\beta(t-s)}|u_i(s)|ds\\
&\quad -\beta\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{p_{\xi 2j}|a_{\xi 22ji}|L_{22i}e^{\beta\bar{\sigma}_{2i}}}{1-\hat{\sigma}_{2i}}\int_{t-\sigma_{2i}(t)}^{t} e^{-\beta(t-s)}|u_i(s)|ds,\ t\in\mathbb{R}^{+},\ \mathbb{P}\text{-a.s.} \qquad (3.28)
\end{aligned}$$

By Lemma 3.1, we have directly

$$\begin{aligned}
\mathit{\Pi}_{\xi 11}(t) &\leqslant \sum_{i=1}^{m} p_{\xi 1i}\bigg|\bigvee_{j=1}^{n} a_{\xi 13ij}\int_{-\infty}^{t} K_{11j}(t-s)f_{13j}(\tilde{y}_j(s)+v_j(s))ds\\
&\quad -\bigvee_{j=1}^{n} a_{\xi 13ij}\int_{-\infty}^{t} K_{11j}(t-s)f_{13j}(\tilde{y}_j(s))ds\bigg|\\
&\leqslant \sum_{i=1}^{m} p_{\xi 1i}\sum_{j=1}^{n}|a_{\xi 13ij}|\Big|\int_{-\infty}^{t} K_{11j}(t-s)\hat{f}_{13j}(v_j(s))ds\Big|,\ t\in\mathbb{R}^{+},\ \mathbb{P}\text{-a.s.},\ \xi\in\mathit{\Xi}, \qquad (3.29)
\end{aligned}$$

where

$$\hat{f}_{13j}(v_j(s)) = f_{13j}(\tilde{y}_j(s) + v_j(s)) - f_{13j}(\tilde{y}_j(s)),\ t \in \mathbb{R},\ \mathbb{P}\text{-a.s.},\ j = 1, \ldots, n,$$

which, together with Assumption 2.1 (especially (2.14)), implies

$$|\hat{f}_{13j}(v_j(s))| \leqslant L_{13j}|v_j(s)|,\ t \in \mathbb{R},\ \mathbb{P}\text{-a.s.},\ j = 1, \ldots, n.$$

This, together with Assumption 2.3 and some tedious computations, implies

$$\begin{aligned}
&|\int_{-\infty}^{t} K_{11j}(t-s)\hat{f}_{13j}(v_j(s))ds| \\
\leqslant& L_{13j}\int_{-\infty}^{t} K_{11j}(t-s)|v_j(s)|ds \\
=& L_{13j}\int_{0}^{+\infty} K_{11j}(s)|v_j(t-s)|ds \\
=& L_{13j}\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)e^{-\beta s}|v_j(t-s)|ds \\
=& L_{13j}\left(\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)ds|v_j(t)| - D^{+}\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\right. \\
&\left. -\beta\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\right) \\
=& L_{13j}\left(\bar{\check{K}}_{\beta 11j}|v_j(t)| - \mathcal{L}\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\right. \\
&\left. -\beta\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\right),\ t \in \mathbb{R}^{+},\ \mathbb{P}\text{-a.s.},\ j = 1, \ldots, n.
\end{aligned}$$

This, together with (3.29), implies

$$\begin{aligned}
\varPi_{\xi 11}(t) \leqslant& \sum_{i=1}^{m} p_{\xi 1i}\sum_{j=1}^{n}|a_{\xi 13ij}|L_{13j}\left(\bar{\check{K}}_{\beta 11j}|v_j(t)| - \mathcal{L}\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\right. \\
&\left. -\beta\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\right) \\
=& \sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 13ij}|\bar{\check{K}}_{\beta 11j}L_{13j}|v_j(t)| \\
&-\mathcal{L}\sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 13ij}|L_{13j}\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds \\
&-\beta\sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 13ij}|L_{13j}\int_{0}^{+\infty} \check{K}_{\beta 11j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds,
\end{aligned} \tag{3.30}$$

$t \in \mathbb{R}^+$, $\mathbb{P}$-a.s., $\xi \in \Xi$. By analogy with (3.30), we can prove also

$$\begin{aligned}\Pi_{\xi 12}(t) \leqslant & \sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 14ij}|\bar{\check{K}}_{\beta 12j}L_{14j}|v_j(t)| \\ & - \mathcal{L}\sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 14ij}|L_{14j}\int_0^{+\infty}\check{K}_{\beta 12j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds \\ & - \beta\sum_{j=1}^{n}\sum_{i=1}^{m} p_{\xi 1i}|a_{\xi 14ij}|L_{14j}\int_0^{+\infty}\check{K}_{\beta 12j}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds, \end{aligned} \tag{3.31}$$

$t \in \mathbb{R}^+$, $\mathbb{P}$-a.s., $\xi \in \Xi$. By using Lemma 3.1, combining Assumption 2.1 (especially (2.15)) along with Assumption 2.3, and mimicking the steps in proving (3.30) and (3.31), we can prove

$$\begin{aligned}\Pi_{\xi 21}(t) \leqslant & \sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 23ji}|\bar{\check{K}}_{\beta 21i}L_{23i}|u_i(t)| \\ & - \mathcal{L}\sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 23ji}|L_{23i}\int_0^{+\infty}\check{K}_{\beta 21i}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds \\ & - \beta\sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 23ji}|L_{23i}\int_0^{+\infty}\check{K}_{\beta 21i}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds, \end{aligned} \tag{3.32}$$

$t \in \mathbb{R}^+$, $\mathbb{P}$-a.s., $\xi \in \Xi$. Taking similar steps in proving (3.32), we can prove

$$\begin{aligned}\Pi_{\xi 22}(t) \leqslant & \sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 24ji}|\bar{\check{K}}_{\beta 22i}L_{24i}|u_i(t)| \\ & - \mathcal{L}\sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 24ji}|L_{24i}\int_0^{+\infty}\check{K}_{\beta 22i}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds \\ & - \beta\sum_{i=1}^{m}\sum_{j=1}^{n} p_{\xi 2j}|a_{\xi 24ji}|L_{24i}\int_0^{+\infty}\check{K}_{\beta 22i}(s)\int_{t-s}^{t} e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds, \end{aligned} \tag{3.33}$$

$t \in \mathbb{R}^+$, $\mathbb{P}$-a.s., $\xi \in \Xi$. Plug (3.23)–(3.28) and (3.30)–(3.33) into (3.14), to yield

$$\begin{aligned}\mathcal{L}\mathcal{V}_{\xi 1}(t) \leqslant & \sum_{i=1}^{m} M_{\xi 1i}|u_i(t)| + \sum_{j=1}^{n} M_{\xi 2j}|v_j(t)| \\ & - \mathcal{L}\sum_{i=1}^{m} p_{\xi 1i}l_{1i}e^{\beta\tau_{1i}}\int_{t-\tau_{1i}}^{t} e^{-\beta(t-s)}|u_i(s)|ds \\ & - \beta\sum_{i=1}^{m} p_{\xi 1i}l_{1i}e^{\beta\tau_{1i}}\int_{t-\tau_{1i}}^{t} e^{-\beta(t-s)}|u_i(s)|ds \\ & - \mathcal{L}\sum_{j=1}^{n}\sum_{i=1}^{m} \frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1-\hat{\sigma}_{1j}}\int_{t-\sigma_{1j}(t)}^{t} e^{-\beta(t-s)}|v_j(s)|ds \end{aligned}$$

$$
\begin{aligned}
&-\beta\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{p_{\xi 1i}|a_{\xi 12ij}|L_{12j}e^{\beta\bar{\sigma}_{1j}}}{1-\hat{\sigma}_{1j}}\int_{t-\sigma_{1j}(t)}^{t}e^{-\beta(t-s)}|v_j(s)|ds\\
&-\mathcal{L}\sum_{j=1}^{n}\sum_{i=1}^{m}p_{\xi 1i}|a_{\xi 13ij}|L_{13j}\int_{0}^{+\infty}\check{K}_{\beta 11j}(s)\int_{t-s}^{t}e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\\
&-\beta\sum_{j=1}^{n}\sum_{i=1}^{m}p_{\xi 1i}|a_{\xi 13ij}|L_{13j}\int_{0}^{+\infty}\check{K}_{\beta 11j}(s)\int_{t-s}^{t}e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\\
&-\mathcal{L}\sum_{j=1}^{n}\sum_{i=1}^{m}p_{\xi 1i}|a_{\xi 14ij}|L_{14j}\int_{0}^{+\infty}\check{K}_{\beta 12j}(s)\int_{t-s}^{t}e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\\
&-\beta\sum_{j=1}^{n}\sum_{i=1}^{m}p_{\xi 1i}|a_{\xi 14ij}|L_{14j}\int_{0}^{+\infty}\check{K}_{\beta 12j}(s)\int_{t-s}^{t}e^{-\beta(t-\tilde{s})}|v_j(\tilde{s})|d\tilde{s}ds\\
&-\mathcal{L}\sum_{j=1}^{n}p_{\xi 2j}l_{2j}e^{\beta\tau_{2j}}\int_{t-\tau_{2j}}^{t}e^{-\beta(t-s)}|v_j(s)|ds\\
&-\beta\sum_{j=1}^{n}p_{\xi 2j}l_{2j}e^{\beta\tau_{2j}}\int_{t-\tau_{2j}}^{t}e^{-\beta(t-s)}|v_j(s)|ds\\
&-\mathcal{L}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_{\xi 2i}|a_{\xi 22ji}|L_{22i}e^{\beta\bar{\sigma}_{2i}}}{1-\hat{\sigma}_{2i}}\int_{t-\sigma_{2i}(t)}^{t}e^{-\beta(t-s)}|u_i(s)|ds\\
&-\beta\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_{\xi 2i}|a_{\xi 22ji}|L_{22i}e^{\beta\bar{\sigma}_{2i}}}{1-\hat{\sigma}_{2i}}\int_{t-\sigma_{2i}(t)}^{t}e^{-\beta(t-s)}|u_i(s)|ds\\
&-\mathcal{L}\sum_{i=1}^{m}\sum_{j=1}^{n}p_{\xi 2j}|a_{\xi 23ji}|L_{23i}\int_{0}^{+\infty}\check{K}_{\beta 21i}(s)\int_{t-s}^{t}e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds\\
&-\beta\sum_{i=1}^{m}\sum_{j=1}^{n}p_{\xi 2j}|a_{\xi 23ji}|L_{23i}\int_{0}^{+\infty}\check{K}_{\beta 21i}(s)\int_{t-s}^{t}e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds\\
&-\mathcal{L}\sum_{i=1}^{m}\sum_{j=1}^{n}p_{\xi 2j}|a_{\xi 24ji}|L_{24i}\int_{0}^{+\infty}\check{K}_{\beta 22i}(s)\int_{t-s}^{t}e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds\\
&-\beta\sum_{i=1}^{m}\sum_{j=1}^{n}p_{\xi 2j}|a_{\xi 24ji}|L_{24i}\int_{0}^{+\infty}\check{K}_{\beta 22i}(s)\int_{t-s}^{t}e^{-\beta(t-\tilde{s})}|u_i(\tilde{s})|d\tilde{s}ds\\
&-\Upsilon(\sum_{i=1}^{m}p_{\xi 1i}+\sum_{j=1}^{n}p_{\xi 2j}),\ t\in[t_{2k},t_{2k+1}],\ \mathbb{P}\text{-a.s.},
\end{aligned}
$$

or equivalently, to yield

$$
\begin{aligned}
\mathcal{L}\mathcal{V}_{\xi 1}(t)\leqslant&\sum_{i=1}^{m}M_{\xi 1i}|u_i(t)|+\sum_{j=1}^{n}M_{\xi 2j}|v_j(t)|\\
&-\mathcal{L}(\mathcal{V}_{\xi 2}(t)+\mathcal{V}_{\xi 3}(t)+\mathcal{V}_{\xi 4}(t))\\
&-\beta(\mathcal{V}_{\xi 2}(t)+\mathcal{V}_{\xi 3}(t)+\mathcal{V}_{\xi 4}(t))-\alpha,\ t\in[t_{2k},t_{2k+1}],\ \mathbb{P}\text{-a.s.},
\end{aligned}
\tag{3.34}
$$

where α, $M_{\xi 1i}$, $M_{\xi 2j}$, $\mathcal{V}_{\xi 2}(t)$, $\mathcal{V}_{\xi 3}(t)$ and $\mathcal{V}_{\xi 4}(t)$ are given by (2.23), (2.24), (2.25), (3.11), (3.12) and (3.13), respectively; $i = 1, \ldots, m$, $j = 1, \ldots, n$, $\xi \in \Xi$. Thanks to (2.20) and (2.21), in view of (3.9), we deduce from (3.34) that

$$\mathcal{L}\mathcal{V}_{\xi}(t) \leqslant -\beta\mathcal{V}_{\xi}(t) - \alpha, \ t \in [t_{2k}, t_{2k+1}], \ k \in \mathbb{N}_0, \ \mathbb{P}\text{-a.s.}$$

This, together with (3.8), implies immediately

$$D^{+}V(t) \leqslant -\beta V(t) - \alpha, \ t \in [t_{2k}, t_{2k+1}], \ k \in \mathbb{N}_0. \tag{3.35}$$

Taking similar steps as in deriving (3.34), we could get

$$\begin{aligned}\mathcal{L}\mathcal{V}_{\xi 1}(t) \leqslant & \sum_{i=1}^{m} \tilde{M}_{\xi 1i}|u_i(t)| + \sum_{j=1}^{n} \tilde{M}_{\xi 2j}|v_j(t)| \\ & - \beta(\mathcal{V}_{\xi 2}(t) + \mathcal{V}_{\xi 3}(t) + \mathcal{V}_{\xi 4}(t)), \ t \in [t_{2k+1}, t_{2k+2}], \ \mathbb{P}\text{-a.s.,}\end{aligned}$$

which, together with some tedious calculations, implies

$$\mathcal{L}\mathcal{V}_{\xi}(t) \leqslant \eta(\sum_{i=1}^{m} p_{\xi 1i}|u_i(t)| + \sum_{j=1}^{n} p_{\xi 2j}|v_j(t)|) \leqslant \eta\mathcal{V}_{\xi}(t), \ t \in [t_{2k+1}, t_{2k+2}], \ k \in \mathbb{N}_0, \ \mathbb{P}\text{-a.s.,}$$

where η, $\tilde{M}_{\xi 1i}$ and $\tilde{M}_{\xi 2j}$ are given as in (2.29), (2.30) and (2.31), respectively; $i = 1, \ldots, m$, $j = 1, \ldots, n$, $\xi \in \Xi$. This, together with (3.8), implies

$$D^{+}V(t) \leqslant \eta V(t), \ t \in [t_{2k+1}, t_{2k+2}], \ k \in \mathbb{N}_0. \tag{3.36}$$

In light of (3.35), (3.36) and (2.22), we deduce by applying Lemma 2.2 that

$$\lim_{t \to T^{-}} V(t) = 0$$

and

$$V(t) = 0, \ \forall\, t \in [T, +\infty),$$

where T is given by (2.26) alongside with (2.27) and (2.28). But in light of (3.2), (3.3) and (3.8)–(3.13), we have

$$\begin{aligned}& \min_{\xi \in \Xi,\ 1 \leqslant i \leqslant m} p_{\xi 1i} \sum_{i=1}^{m} \mathbb{E}|x_i(t) - \tilde{x}_i(t)| + \min_{\xi \in \Xi,\ 1 \leqslant j \leqslant n} p_{\xi 2j} \sum_{j=1}^{n} \mathbb{E}|y_j(t) - \tilde{y}_j(t)| \\ \leqslant & \mathbb{E} \sum_{i=1}^{m} p_{\gamma_t 1i}|x_i(t) - \tilde{x}_i(t)| + \mathbb{E} \sum_{j=1}^{n} p_{\gamma_t 2j}|y_j(t) - \tilde{y}_j(t)| \\ \leqslant & V(t), \ t \in \mathbb{R}^{+}.\end{aligned}$$

This, together with $\min_{\xi \in \Xi,\ 1 \leqslant i \leqslant m} p_{\xi 1i} > 0$ and $\min_{\xi \in \Xi,\ 1 \leqslant j \leqslant n} p_{\xi 2j} > 0$ which follow from the related assumption, implies immediately that the proof is complete. □

4. An illustrative example

In this section, we shall conduct numerical simulations to show the validity of the synchronization criteria (see Theorem 2.1) of this paper. We consider here the following BAMN:

$$
\begin{cases}
\dot{x}(t) = -1.5508x(t-1) + \dfrac{a_{\gamma_t 1111}(|y_1(t)+1| - |y_1(t)-1|)}{100} \\
\qquad + \dfrac{a_{\gamma_t 1112}(|y_2(t)+1| - |y_2(t)-1|)}{100} \\
\qquad + \dfrac{a_{\gamma_t 1211}}{100} y_1(t - \dfrac{t}{2+t}) + \dfrac{a_{\gamma_t 1212}}{100} y_2(t - \dfrac{t}{2+t}) \\
\qquad + \left(\dfrac{a_{\gamma_t 1311}}{100} \displaystyle\int_{-\infty}^{t} e^{-100(t-s)} y_1(s)ds\right) \bigvee \left(\dfrac{a_{\gamma_t 1312}}{100} \displaystyle\int_{-\infty}^{t} e^{-100(t-s)} y_2(s)ds\right) \\
\qquad + \left(\dfrac{a_{\gamma_t 1411}}{100} \displaystyle\int_{-\infty}^{t} e^{-100(t-s)} y_1(s)ds\right) \bigwedge \left(\dfrac{a_{\gamma_t 1412}}{100} \displaystyle\int_{-\infty}^{t} e^{-100(t-s)} y_2(s)ds\right) \\
\qquad + \sin t, \ t \in \mathbb{R}^+, \ \mathbb{P}\text{-a.s.}, \\
\dot{y}_1(t) = -0.7879 y_1(t-2) + \dfrac{a_{\gamma_t 2111}(|x(t)+1| - |x(t)-1|)}{100} + \dfrac{a_{\gamma_t 2211}}{100} x(t - \dfrac{t}{2+t}) \\
\qquad + \dfrac{a_{\gamma_t 2311}}{100} \displaystyle\int_{-\infty}^{t} e^{-100(t-s)} x(s)ds + \sin 2t, \ t \in \mathbb{R}^+, \ \mathbb{P}\text{-a.s.}, \\
\dot{y}_2(t) = -1.5708 y_2(t-1) + \dfrac{a_{\gamma_t 2121}(|x(t)+1| - |x(t)-1|)}{100} + \dfrac{a_{\gamma_t 2221}}{100} x(t - \dfrac{t}{2+t}) \\
\qquad + \dfrac{a_{\gamma_t 2321}}{100} \displaystyle\int_{-\infty}^{t} e^{-100(t-s)} x(s)ds + \sin 3t, \ t \in \mathbb{R}^+, \ \mathbb{P}\text{-a.s.},
\end{cases}
\tag{4.1}
$$

in which

$$
(a_{i111j}) = \begin{pmatrix} 8 & 6 \\ 3 & 7 \\ 2 & 5 \end{pmatrix}, \qquad (a_{i121j}) = \begin{pmatrix} 2 & 9 \\ 3 & 4 \\ 8 & 1 \end{pmatrix},
$$

$$
(a_{i131j}) = \begin{pmatrix} 2 & 7 \\ 3 & 5 \\ 8 & 4 \end{pmatrix}, \qquad (a_{i141j}) = \begin{pmatrix} 7 & 8 \\ 2 & 9 \\ 6 & 1 \end{pmatrix},
$$

$$
(a_{i2j11}) = \begin{pmatrix} 1 & 9 & 5 \\ 5 & 2 & 3 \\ 8 & 6 & 4 \end{pmatrix}, \quad (a_{i2j21}) = \begin{pmatrix} 3 & 2 & 5 \\ 9 & 7 & 1 \\ 4 & 6 & 8 \end{pmatrix}.
$$

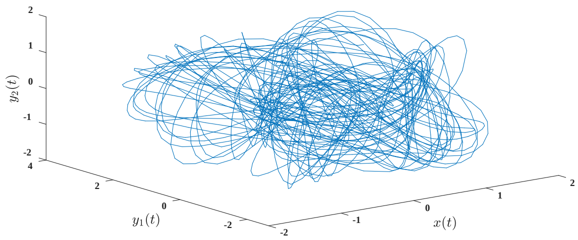
We assume here that the Markovian chain γ_t takes $\Xi = \{1, 2, 3\}$ as its state space, and takes the following Metzler matrix as its infinitesimal generator:

$$
(\pi_{\xi\tilde{\xi}}) = \begin{pmatrix} -5 & 1 & 4 \\ 6 & -8 & 2 \\ 7 & 3 & -10 \end{pmatrix}.
$$

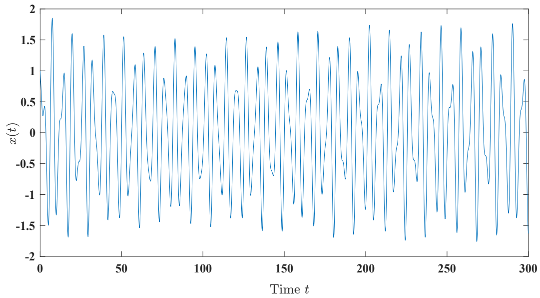
By utilizaing MATLAB, we can simulate numerically the trajectory, denoted by $(x(t), y_1(t), y_2(t))$ henceforth, of network system (4.1) supplemented by

$$\begin{cases} x(t) = e^t, \ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0], \\ y_1(t) = e^{5t}, \ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0], \\ y_2(t) = e^{2t}, \ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0]. \end{cases}$$

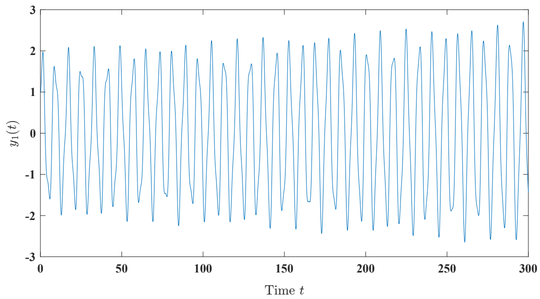
As the simulation result (see Figure 1) indicates, the network system (4.1) itself lacks stable equilibrium points, periodic trajectories and general trajectories (see especially $y_1(t)$ and the phase portrait).



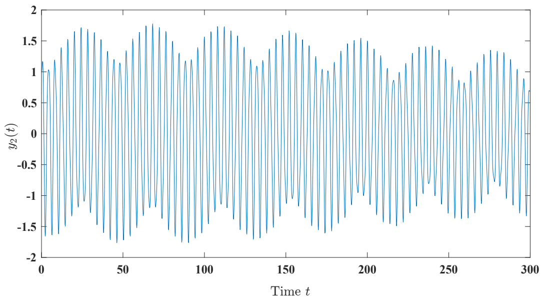
a. Simulation of the phase portrait of the network system (4.1)



b. x component of the state trajectory $(x(t), y_1(t), y_2(t))$ of the network system (4.1)



c. y_1 component of the state trajectory $(x(t), y_1(t), y_2(t))$ of the network system (4.1)



d. y_2 component of the state trajectory $(x(t), y_1(t), y_2(t))$ of the network system (4.1)

Figure 1. Numerical simulation of dynamics of the network system (4.1).

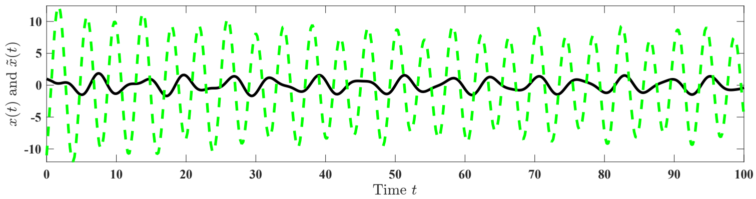
This means that: For two different trajectories $(\mathfrak{a}(t), \mathfrak{b}_1(t), \mathfrak{b}_2(t))$ and $(\tilde{\mathfrak{a}}(t), \tilde{\mathfrak{b}}_1(t), \tilde{\mathfrak{b}}_2(t))$, there exists no $0 < T \leqslant +\infty$ such that

$$\begin{cases} \lim_{t\to T} \mathbb{E}(|\mathfrak{a}(t) - \tilde{\mathfrak{a}}(t)| + |\mathfrak{b}_1(t) - \tilde{\mathfrak{b}}_1(t)| + |\mathfrak{b}_2(t) - \tilde{\mathfrak{b}}_2(t)|) = 0, & \text{if } T = +\infty, \\ \mathfrak{a}(t) = \tilde{\mathfrak{a}}(t),\ \mathfrak{b}_1(t) = \tilde{\mathfrak{b}}_1(t),\ \mathfrak{b}_2(t) = \tilde{\mathfrak{b}}_2(t),\ t > T,\ \mathbb{P}\text{-a.s.}, & \text{if } T \in (0, +\infty). \end{cases}$$

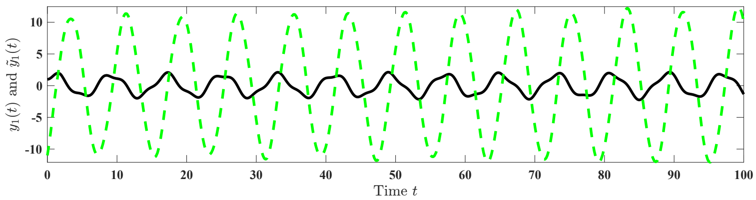
Actually, we conduct numerically, by using MATLAB, comparison between the trajectory $(x(t), y_1(t), y_2(t))$ and the trajectory $(\tilde{x}(t), \tilde{y}_1(t), \tilde{y}_2(t))$ of the network system (4.1) supplemented by

$$\begin{cases} \tilde{x}(t) = -10 - \cos t,\ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0], \\ \tilde{y}_1(t) = -10 - \cos 5t,\ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0], \\ \tilde{y}_2(t) = -10 - \cos 2t,\ d\mathbb{P} \times dt\text{-a.e. in } \Omega \times (-\infty, 0]. \end{cases}$$

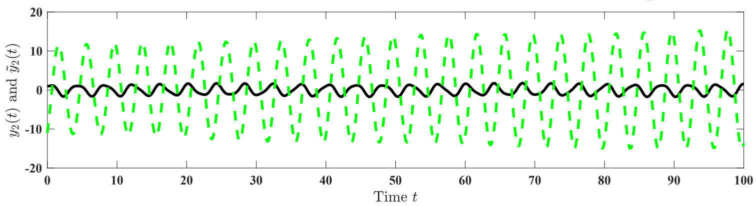
The simulation result (see Figure 2) reveals that: The trajectory $(\tilde{x}(t), \tilde{y}_1(t), \tilde{y}_2(t))$ does not approach the trajectory $(x(t), y_1(t), y_2(t))$ as time t tends to a finite/infinite time instant.



a. Evolution of $x(t)$ and the uncontrolled $\tilde{x}(t)$ with respect to time t



b. Evolution of $y_1(t)$ and the uncontrolled $\tilde{y}_1(t)$ with respect to time t



c. Evolution of $y_2(t)$ and the uncontrolled $\tilde{y}_2(t)$ with respect to time t

Figure 2. Comparison of two different state trajectories of the network system (4.1) (the solid curves representing $(x(t), y_1(t), y_2(t))$, while the dashed curves representing $(\tilde{x}(t), \tilde{y}_1(t), \tilde{y}_2(t))$).

Illuminated by the results in Theorem 2.1, to design control to synchronize the network system (4.1), we introduce an infinite sequence $\{t_n\}_{n=0}^{\infty}$ in $\mathbb{R}^+$ by

$$t_n = \sum_{j=0}^{[\frac{n}{2}]-1} \frac{1}{j+1} + \frac{1 + (-1)^{n+1}}{6([\frac{n}{2}] + 1)},\ n \in \mathbb{N}_0, \tag{4.2}$$

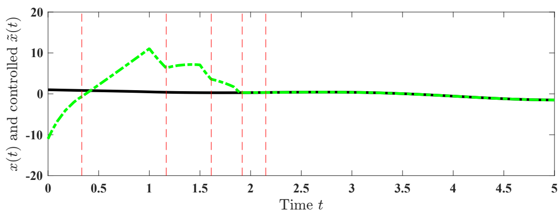
where $[x]$ denotes, here and hereafter, the greatest integer which does not exceed x, $x \in \mathbb{R}$. After careful analysis, we can conclude that: The sequence $\{t_n\}_{n=0}^{\infty}$ is strictly increasing, and satisfies the following properties:

$$t_{2k} = \sum_{j=0}^{k-1} \frac{1}{j+1} \text{ for } k \in \mathbb{N}_0,$$

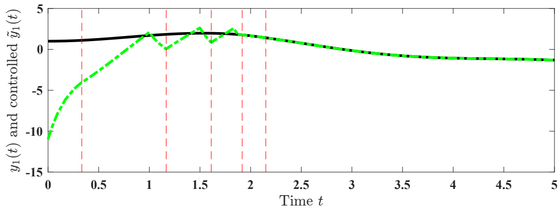
$$t_{2k+1} = \sum_{j=0}^{k-1} \frac{1}{j+1} + \frac{1}{3(k+1)} = \frac{2}{3} t_{2k} + \frac{1}{3} t_{2k+2} \text{ for } k \in \mathbb{N}_0,$$

$$t_0 = 0, \ \lim_{n\to\infty} t_n = +\infty, \ \frac{t_{2k+1} - t_{2k}}{t_{2k+2} - t_{2k+1}} = \frac{1}{2} \text{ for } k \in \mathbb{N}_0.$$

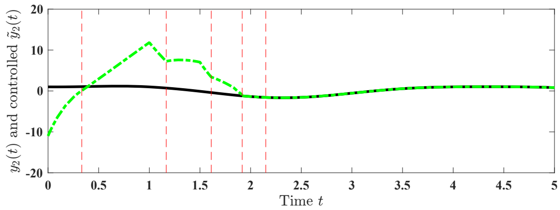
Enlightened by Theorem 2.1, we choose the control (2.12) and (2.13) as the candidate synchronization control for the network system (4.1). Numerical simulation based on MATLAB yields: The the control (2.12) and (2.13), with $\hat{k}_{\xi 11} = \hat{k}_{\xi 21} = \hat{k}_{\xi 22} = 17.3914$ ($\xi = 1, 2, 3$) and $\Upsilon = 25.9463$, can render the trajectory $(\tilde{x}(t), \tilde{y}_1(t), \tilde{y}_2(t))$ to "arrive at" the trajectory $(x(t), y_1(t), y_2(t))$ before the time instant $T = 1.9167$, and to coincide with $(x(t), y_1(t), y_2(t))$ thereupon; see Figure 3.



a. Evolution of $x(t)$ and the controlled $\tilde{x}(t)$ with respect to time t



b. Evolution of $y_1(t)$ and the controlled $\tilde{y}_1(t)$ with respect to time t



c. Evolution of $y_2(t)$ and the controlled $\tilde{y}_2(t)$ with respect to time t

Figure 3. Comparison of two different state trajectories of the drive network (4.1) and the controlled response network system (the solid curves representing the trajectory $(x(t), y_1(t), y_2(t))$ of the drive network system (4.1), the two-dashed curves representing the trajectory $(\tilde{x}(t), \tilde{y}_1(t), \tilde{y}_2(t))$ of the response network, and the vertical straight lines representing the instants that the control is paused).

5. Conclusions

We addressed the synchronization problem for a class of fuzzy BAMNs with Markovian switching in this paper. In comparision with the studies in the existing references, the concerned BAMNs in our paper include simultaneously discrete-time delay in leakage (in other words, forgetting) terms, continuous-time and infinitely distributed delays, fuzzy logic, as well as Markovian jumping in transmission terms (see (2.1) for the detailed information). This certainly provides more realistic models in applications, but brings us more difficulties in designing control to synchronize the concerned network system (2.1) in finite time. For the network system (2.1), we designed an intermittent quantized control. By coming up with a clever Lyapunov-Krasovskii functional, we proved under certain conditions that the controlled network system is stochastically synchronizable in finite time, more precisely, the 1st moments of trajectories of the error network system (3.1) of the drive network system (2.1) and the response network system (2.3) approach zero at finite time and remain to be zero thereupon. The main ingredient in proving our main results is a novel Lyapunov-Krasovskii functional, which can be adapted to deal with finite-time synchronization problem for BAMNs with time-varying leakage coefficients and transmission coefficients which generalize slightly our concerned network system (2.1).

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Conflict of interest

The authors declare that they have no conflicts of interest.

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