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Research article

Existence results for a coupled system of (k, φ)-Hilfer fractional differential equations with nonlocal integro-multi-point boundary conditions

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Abstract: In this paper, we investigate the existence and uniqueness of solutions to a nonlinear coupled systems of (k, φ)-Hilfer fractional differential equations supplemented with nonlocal integro-multi-point boundary conditions. We make use of the Banach contraction mapping principle to obtain the uniqueness result, while the existence results are proved with the aid of Krasnosel'skiǐ's fixed point theorem and Leray-Schauder alternative for the given problem. Examples demonstrating the application of the abstract results are also presented. Our results are of quite general nature and specialize in several new results for appropriate values of the parameters β_1, β_2, and the function φ involved in the problem at hand.

Keywords: (k, φ)-Hilfer fractional derivative; Riemann-Liouville fractional derivative; Caputo fractional derivative; existence; uniqueness; fixed point theorems
Mathematics Subject Classification: 34A08, 34B10

1. Introduction

Fractional differential systems appear in the mathematical models associated with several physical phenomena such as synchronization of chaotic systems [1–3], BAM neural networks with time varying delays [4], HIV-immune system with memory [5], anomalous diffusion [6], ecological effects [7], co-infection of malaria and HIV/AIDS [8], etc. One can find some theoretical results on fractional differential equations involving Riemann–Liouville, Caputo, Hadamard, and Hilfer type fractional derivative operators in the articles [9–18] and the references cited therein. For a detailed update on fractional calculus, we refer the reader to the books [19–27]. More recently, the concept of (k, φ)-Hilfer fractional derivative operator was introduced in [28], which received significant attention as it specializes in some known fractional derivative operators under a suitable choice of the parameters involved in its definition. In [29], the authors discussed the existence of solutions for a (k, φ)-Hilfer fractional boundary value problem with nonlocal integro-multi-point boundary conditions. A nonlocal coupled system for (k, φ)-Hilfer fractional differential equations with nonlocal multi-point boundary conditions were studied in [30].

In this paper, we introduce and study a new coupled system of (k, φ)-Hilfer fractional differential equations supplemented with nonlocal integral multi-point coupled boundary conditions given by

$$\begin{cases} {}^{k,H}D^{\alpha_1,\beta_1;\varphi}x(t) = f(t, x(t), y(t)), \quad t \in (a, b], \\ {}^{k,H}D^{\alpha_2,\beta_2;\varphi}y(t) = g(t, x(t), y(t)), \quad t \in (a, b], \\ x(a) = 0, \quad \displaystyle\int_a^b \varphi'(s)x(s)ds = \sum_{j=1}^{m} \eta_j\, y(\xi_j), \\ y(a) = 0, \quad \displaystyle\int_a^b \varphi'(s)y(s)ds = \sum_{i=1}^{n} \theta_i x(z_i), \end{cases} \tag{1.1}$$

where ${}^{k,H}D^{\alpha_1,\beta_1;\varphi}$, ${}^{k,H}D^{\alpha_2,\beta_2;\varphi}$ denote the (k, φ)-Hilfer fractional derivative operators of orders $\alpha_1, \alpha_2 \in (1, 2)$, and parameters $\beta_1, \beta_2 \in [0, 1]$, respectively, $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\eta_j, \theta_i \in \mathbb{R}$, $a < \xi_j, z_i < b$, $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, k$ and φ is an increasing function with $\varphi'(t) \neq 0$ for all $t \in [a, b]$.

Here we emphasize that the importance of nonlocal conditions can be understood in the sense that such conditions are used to model the peculiarities occurring inside the domain of physical and chemical processes as the classical initial and boundary conditions fail to cater to this situation. The present problem is motivated by useful applications of nonlocal boundary data in petroleum exploitation, thermodynamics, elasticity, wave propagation, etc., for instance, see [31, 32] and the details therein. For some recent theoretical works on nonlocal integral boundary value problems, we refer the reader to the articles [33–35].

The objective of the present research is to develop the existence theory for a class of nonlocal integral multi-point coupled boundary value problems involving (k, φ)-Hilfer fractional differential operators of different orders with the aid of the standard tools of the fixed point theory. The proposed study is important and useful in view of the wider scope of the (k, φ)-Hilfer fractional operators. It is imperative to mention that the (k, φ)-Hilfer fractional differential system considered in the problem (1.1) is of more general form and takes some special forms by fixing the values of φ and $\beta_i, i = 1, 2$. For instance the (k, φ)-Hilfer fractional differential system in (1.1) corresponds to (i) (k, φ)-Riemann-Liouville fractional differential system for $\beta_1 = 0 = \beta_2$; (ii) k-Riemann-Liouville fractional differential system for $\varphi(t) = t$; (iii) k-Hilfer-Katugampola fractional differential system for $\varphi(t) = t^\rho$; (iv) k-Katugampola fractional differential system for $\varphi(t) = t^\rho$ and $\beta_1 = 0 = \beta_2$; (v) k-Caputo-Katugampola fractional differential system when $\varphi(t) = t^\rho$, $\beta_1 = 1 = \beta_2$; (vi) k-Hilfer-Hadamard fractional differential system for $\varphi(t) = \log t$; (vii) k-Hadamard fractional differential system for $\varphi(t) = \log t, \beta_1 = 0 = \beta_2$; and (viii) k-Caputo-Hadamard fractional differential system for $\varphi(t) = \log t, \beta_1 = 1 = \beta_2$. For further details, see the paper [36].

The structure of the rest of the paper is as follows. Some definitions and lemmas are outlined in Section 2. An auxiliary result is also proved, which is used to transform the given nonlinear system into a fixed point problem. The main results, based on Banach's contraction mapping principle, the Leray-Schauder alternative, and Krasnosel'skiĭ's fixed-point theorem, are presented in Section 3. Section 4 contains the numerical examples illustrating our theoretical results.

2. Preliminaries

Definition 2.1. *[37] Let $h \in L^1([a,b],\mathbb{R})$, $k > 0$ and $\varphi : [a,b] \to \mathbb{R}$ is an increasing function with $\varphi'(t) \neq 0$ for all $t \in [a,b]$. Then the (k,φ)-Riemann-Liouville fractional integral of order $\alpha > 0$ $(\alpha \in \mathbb{R})$ of the function f is given by*

$$ {}^{k}I^{\alpha;\varphi}_{a+} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_{a+}^{\theta} \varphi'(s)(\varphi(t) - \varphi(s))^{\frac{\alpha}{k}-1} f(s)ds. $$

Definition 2.2. *[28] Let $\alpha, k \in \mathbb{R}^+ = (0,\infty)$, $\beta \in [0,1]$, φ is an increasing function such that $\varphi \in C^n([a,b],\mathbb{R})$, $\varphi'(t) \neq 0, t \in [a,b]$ and $f \in C^n([a,b],\mathbb{R})$. Then the (k,φ)-Hilfer fractional derivative of the function h of order α and type β, is defined by*

$$ {}^{k,H}D^{\alpha,\beta;\varphi} f(t) = I^{\beta(nk-\alpha);\varphi}_{a+} \left(\frac{k}{\varphi'(t)}\frac{d}{dt}\right)^n {}^{k}I^{(1-\beta)(nk-\alpha);\varphi}_{a+} f(t), \quad n = \left\lceil \frac{\alpha}{k} \right\rceil. $$

Lemma 2.1. *[28]* **(i)** *Let $\mu, k \in \mathbb{R}^+$ and $n = \left\lceil \frac{\mu}{k} \right\rceil$. Assume that $f \in C^n([a,b],\mathbb{R})$ and ${}^{k}I^{nk-\mu;\varphi}_{a+} h \in C^n([a,b],\mathbb{R})$. Then*

$$ {}^{k}I^{\mu;\varphi}\left({}^{k,RL}D^{\mu;\varphi} f(t)\right) = f(t) - \sum_{j=1}^{n} \frac{(\varphi(t)-\varphi(a))^{\frac{\mu}{k}-j}}{\Gamma_k(\mu - jk + k)} \left[\left(\frac{k}{\varphi'(t)}\frac{d}{dt}\right)^{n-j} {}^{k}I^{nk-\mu;\varphi}_{a+} f(t)\right]_{t=a}. $$

(ii) *Let $\alpha_1, k \in \mathbb{R}^+$ with $\alpha_1 < k, \beta_1 \in [a,b]$ and $p = \alpha_1 + \beta_1(k-\alpha_1)$. Then*

$$ {}^{k}I^{p;\varphi}\left({}^{k,RL}D^{p;\varphi} f\right)(t) = {}^{k}I^{\alpha_1;\varphi}\left({}^{k,H}D^{\alpha_1,\beta_1;\varphi} f\right)(t), \quad f \in C^n([a,b],\mathbb{R}). $$

Now, we prove an auxiliary result concerning a linear variant of the system (1.1).

Lemma 2.2. *Let $a < b, k > 0$, $1 < \alpha_1, \alpha_2 \leq 2, \beta_1, \beta_2 \in [0,1]$, $p = \alpha_1 + \beta_1(2k-\alpha_1)$, $q = \alpha_2 + \beta_2(2k-\alpha_2)$, $f \in C^2([a,b],\mathbb{R})$ and $A \neq 0$. Then, the unique solution of the nonlocal (k,ϕ)-Hilfer integro-multi-point system*

$$ \begin{cases} {}^{k,H}D^{\alpha_1,\beta_1;\varphi} x(t) = h_1(t), \quad t \in (a,b], \\ {}^{k,H}D^{\alpha_2,\beta_2;\varphi} y(t) = h_2(t), \quad t \in (a,b], \\ x(a) = 0, \quad \displaystyle\int_a^b \varphi'(s)x(s)ds = \sum_{j=1}^{m} \eta_j\, y(\xi_j), \\ y(a) = 0, \quad \displaystyle\int_a^b \varphi'(s)y(s)ds = \sum_{i=1}^{n} \theta_i x(z_i), \end{cases} \tag{2.1} $$

is given by

$$x(t) = {}^kI^{\alpha_1;\varphi}h_1(t) + \frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-1}}{A\Gamma_k(p)}\Bigg[A_2\left(\sum_{i=1}^{n}\theta_i\,{}^kI^{\alpha_1;\varphi}h_1(z_i) - \int_a^b \varphi'(s)\,{}^kI^{\alpha_2;\varphi}h_2(s)ds\right)$$
$$+A_4\left(\sum_{j=1}^{m}\eta_j\,{}^kI^{\alpha_2;\varphi}h_2(\xi_j) - \int_a^b \varphi'(s)\,{}^kI^{\alpha_1;\varphi}h_1(s)ds\right)\Bigg], \tag{2.2}$$

and

$$y(t) = {}^kI^{\alpha_2;\varphi}h_2(t) + \frac{(\varphi(t)-\varphi(a))^{\frac{q}{k}-1}}{A\Gamma_k(q)}\Bigg[A_1\left(\sum_{i=1}^{n}\theta_i\,{}^kI^{\alpha_1;\varphi}h_1(z_i) - \int_a^b \varphi'(s)\,{}^kI^{\alpha_2;\varphi}h_2(s)ds\right)$$
$$+A_3\left(\sum_{j=1}^{m}\eta_j\,{}^kI^{\alpha_2;\varphi}h_2(\xi_j) - \int_a^b \varphi'(s)\,{}^kI^{\alpha_1;\varphi}h_1(s)ds\right)\Bigg], \tag{2.3}$$

where

$$A_1 = k\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}}}{\Gamma(p+k)}, \qquad A_2 = \sum_{j=1}^{m}\eta_j\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{q}{k}-1}}{\Gamma_k(q)},$$
$$A_3 = \sum_{i=1}^{n}\theta_i\frac{(\varphi(z_i)-\varphi(a))^{\frac{p}{k}-1}}{\Gamma_k(p)}, \quad A_4 = k\frac{(\varphi(b)-\varphi(a))^{\frac{q}{k}}}{\Gamma(q+k)}, \tag{2.4}$$

with

$$A = A_1A_4 - A_2A_3. \tag{2.5}$$

Proof. Let (x, y) be a solution of the system (2.1). Operating fractional integral operators ${}^kI^{\alpha_1;\varphi}$ and ${}^kI^{\alpha_2;\varphi}$ on both sides of the first and second equations in (2.1) respectively, and using Lemma 2.1, we obtain

$$x(t) = {}^kI^{\alpha_1;\varphi}h_1(t) + c_0\frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-1}}{\Gamma_k(p)} + c_1\frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-2}}{\Gamma_k(p-k)}, \tag{2.6}$$

and

$$y(t) = {}^kI^{\alpha_2;\varphi}h_2(t) + d_0\frac{(\varphi(t)-\varphi(a))^{\frac{q}{k}-1}}{\Gamma_k(q)} + d_1\frac{(\varphi(t)-\varphi(a))^{\frac{q}{k}-2}}{\Gamma_k(q-k)}, \tag{2.7}$$

where

$$c_0 = \Big[\Big(\frac{k}{\varphi'(t)}\frac{d}{dt}\Big)\,{}^kI^{2k-p;\varphi}x(t)\Big]_{t=a}, \quad c_1 = \Big[{}^kI^{2k-p;\varphi}x(t)\Big]_{t=a},$$
$$d_0 = \Big[\Big(\frac{k}{\varphi'(t)}\frac{d}{dt}\Big)\,{}^kI^{2k-q;\varphi}y(t)\Big]_{t=a}, \quad d_1 = \Big[{}^kI^{2k-q;\varphi}y(t)\Big]_{t=a}.$$

Combining the conditions $x(a) = 0$ and $y(a) = 0$ with (2.6) and (2.7), we get $c_1 = 0$ and $d_1 = 0$, since $\frac{p}{k} - 2 < 0$ and $\frac{q}{k} - 2 < 0$. Using the conditions $\int_a^b \varphi'(s)x(s)ds = \sum_{j=1}^{m}\eta_j y(\xi_j)$, and $\int_a^b \varphi'(s)y(s)ds = \sum_{i=1}^{n}\theta_i x(z_i)$ in (2.6) and (2.7) after inserting $c_1 = 0$ and $d_1 = 0$, we get

$$\int_a^b \varphi'(s)\,{}^kI^{\alpha_1;\varphi}h_1(s)ds + c_0\int_a^b \varphi'(s)\frac{(\varphi(s)-\varphi(a))^{\frac{p}{k}-1}}{\Gamma_k(p)}ds$$

$$= \sum_{j=1}^{m}\eta_j\, {}^kI^{\alpha_2;\varphi}h_2(\xi_j) + d_0\sum_{j=1}^{m}\eta_j\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{q}{k}-1}}{\Gamma_k(q)},$$

$$\int_a^b \varphi'(s)\, {}^kI^{\alpha_2;\varphi}h_2(s)ds + d_0\int_a^b \varphi'(s)\frac{(\varphi(s)-\varphi(a))^{\frac{q}{k}-1}}{\Gamma_k(q)}ds$$
$$= \sum_{i=1}^{n}\theta_i\, {}^kI^{\alpha_1;\varphi}h_1(z_i) + c_0\sum_{i=1}^{n}\theta_i\frac{(\varphi(z_i)-\varphi(a))^{\frac{p}{k}-1}}{\Gamma_k(p)}.$$

In view of the notation (2.4), the above system of equations takes the form:

$$\begin{aligned} A_1c_0 - A_2d_0 &= \sum_{j=1}^{m}\eta_j\, {}^kI^{\alpha_2;\varphi}h_2(\xi_j) - \int_a^b \varphi'(s)\, {}^kI^{\alpha_1;\varphi}h_1(s)ds,\\ -A_3c_0 + A_4d_0 &= \sum_{i=1}^{n}\theta_i\, {}^kI^{\alpha_1;\varphi}h_1(z_i) - \int_a^b \varphi'(s)\, {}^kI^{\alpha_2;\varphi}h_2(s)ds. \end{aligned} \tag{2.8}$$

Solving the system (2.8) for c_0 and d_0, we find that

$$\begin{aligned} c_0 &= \frac{1}{A}\Bigg[A_2\Bigg(\sum_{i=1}^{n}\theta_i\, {}^kI^{\alpha_1;\varphi}h_1(z_i) - \int_a^b \varphi'(s)\, {}^kI^{\alpha_2;\varphi}h_2(s)ds\Bigg)\\ &\quad +A_4\Bigg(\sum_{j=1}^{m}\eta_j\, {}^kI^{\alpha_2;\varphi}h_2(\xi_j) - \int_a^b \varphi'(s)\, {}^kI^{\alpha_1;\varphi}h_1(s)ds\Bigg)\Bigg],\\ d_0 &= \frac{1}{A}\Bigg[A_1\Bigg(\sum_{i=1}^{n}\theta_i\, {}^kI^{\alpha_1;\varphi}h_1(z_i) - \int_a^b \varphi'(s)\, {}^kI^{\alpha_2;\varphi}h_2(s)ds\Bigg)\\ &\quad +A_3\Bigg(\sum_{j=1}^{m}\eta_j\, {}^kI^{\alpha_2;\varphi}h_2(\xi_j) - \int_a^b \varphi'(s)\, {}^kI^{\alpha_1;\varphi}h_1(s)ds\Bigg)\Bigg]. \end{aligned}$$

Replacing c_0, c_1, d_0, and d_1 by their respective values (found above) in (2.6) and (2.7), we obtain the solution (2.2) and (2.3). The converse follows by direct computation. Hence the proof is complete.

Finally, we summarize the fixed point theorems used to prove the main results in this paper. X is a Banach space in each theorem.

Lemma 2.3. (Banach fixed point theorem *[38]). Let D be a closed set in X and $T : D \to D$ satisfies*

$$|Tu - Tv| \le \lambda|u - v|, \ \ \textit{for some } \lambda \in (0,1), \ \textit{and for all } u, v \in D.$$

Then T admits one fixed point in D.

Lemma 2.4. *(Leray-Schauder alternative [39]). Let the set Ω be closed bounded convex in X and O an open set contained in Ω with $0 \in O$. Then, for the continuous and compact $T : \bar{U} \to \Omega$, either:*

($\mathfrak{a}$) *T admits a fixed point in $\bar{U}$, or*
($\mathfrak{aa}$) *There exists $\mathfrak{u} \in \partial U$ and $\mu \in (0,1)$ with $\mathfrak{u} = \mu T(\mathfrak{u})$.*

Lemma 2.5. *(Krasnosel'skiĭ fixed point theorem [40]). Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be operators such that (i) $Ax+By \in M$ where $x, y \in M$, (ii) A is compact and continuous and (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

3. Existence and uniqueness results

For a Banach space $X = C([a,b],\mathbb{R})$ endowed with the norm $\|x\| = \max\{|x(t)|, t \in [a,b]\}$, it is well-known that the product space $(X \times X, \|(x,y)\|)$ is a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 2.2, we define an operator $\mathcal{T} : X \times X \to X \times X$ by

$$\mathcal{T}(x,y)(t) = \begin{pmatrix} \mathcal{T}_1(x,y)(t) \\ \mathcal{T}_2(x,y)(t) \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned}\mathcal{T}_1(x,y)(t) &= {}^kI^{\alpha_1;\varphi} f(t,x(t),y(t)) + \frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-1}}{A\Gamma_k(p)}\Bigg[A_2\Big(\sum_{i=1}^{n}\theta_i\,{}^kI^{\alpha_1;\varphi} f(z_i,x(z_i),y(z_i)) \\ &\quad -\int_a^b \varphi'(s)\,{}^kI^{\alpha_2;\varphi} g(s,x(s),y(s))ds\Big) + A_4\Big(\sum_{j=1}^{m}\eta_j\,{}^kI^{\alpha_2;\varphi} g(\xi_j,x(\xi_j),y(\xi_j)) \\ &\quad -\int_a^b \varphi'(s)\,{}^kI^{\alpha_1;\varphi} f(s,x(s),y(s))ds\Big)\Bigg], \quad t \in [a,b],\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\mathcal{T}_2(x,y)(t) &= {}^kI^{\alpha_2;\varphi} g(t,x(t),y(t)) + \frac{(\varphi(t)-\varphi(a))^{\frac{q}{k}-1}}{A\Gamma_k(q)}\Bigg[A_1\Big(\sum_{i=1}^{n}\theta_i\,{}^kI^{\alpha_1;\varphi} f(z_i,x(z_i),y(z_i)) \\ &\quad -\int_a^b \varphi'(s)\,{}^kI^{\alpha_2;\varphi} g(s,x(s),y(s))ds\Big) + A_3\Big(\sum_{j=1}^{m}\eta_j\,{}^kI^{\alpha_2;\varphi} g(\xi_j,x(\xi_j),y(\xi_j)) \\ &\quad -\int_a^b \varphi'(s)\,{}^kI^{\alpha_1;\varphi} f(s,x(s),y(s))ds\Big)\Bigg] \quad t \in [a,b].\end{aligned} \tag{3.3}$$

For computational convenience, we introduce the notation:

$$\begin{aligned}\Omega_1 &= \frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} + \frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Big(A_2\sum_{i=1}^{n}|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \\ &\quad +A_4\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}\Big), \\ \Omega_2 &= \frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Big(A_2\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)} + A_4\sum_{j=1}^{m}|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\Big), \\ \Omega_3 &= \frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)} + \frac{(\varphi(t)-\varphi(a))^{\frac{q}{k}-1}}{A\Gamma_k(q)}\Big(A_1\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)}\end{aligned}$$

$$+A_3\sum_{j=1}^{m}|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\Big),$$

$$\Omega_4 = \frac{(\varphi(b)-\varphi(a))^{\frac{q}{k}-1}}{A\Gamma_k(q)}\Big(A_1\sum_{i=1}^{n}|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}+A_3\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}\Big),$$

$$\Omega_1^* = \Omega_1-\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}, \quad \Omega_3^*=\Omega_3-\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}. \tag{3.4}$$

In the following theorem, we prove the existence of a unique solution to the nonlocal (k,φ)-Hilfer fractional system (1.1) via Banach's contraction mapping principle (Lemma 2.3).

Theorem 3.1. *Assume that*

(H_1) *There exist constants $m_i, n_i, i=1,2$ such that, for all $t\in[a,b]$ and $u_i, v_i\in\mathbb{R}, i=1,2$,*

$$|f(t,u_1,u_2)-f(t,v_1,v_2)|\le m_1|u_1-v_1|+m_2|u_2-v_2|,$$
$$|g(t,u_1,u_2)-g(t,v_1,v_2)|\le n_1|u_1-v_1|+n_2|u_2-v_2|.$$

Then, the nonlocal (k,φ)-Hilfer fractional system (1.1) *has a unique solution on $[a,b]$, provided that*

$$(\Omega_1+\Omega_4)(m_1+m_2)+(\Omega_2+\Omega_3)(n_1+n_2)<1, \tag{3.5}$$

where $\Omega_i, i=1,2,3,4$ are given in (3.4).

Proof. Let us verify the hypothesis of Banach's contraction mapping principle [38] in the following two steps:

(i) $\mathcal{T}B_r\subset B_r$, where the operator $\mathcal{T}$ is defined by (3.1) and $B_r=\{(x,y)\in X\times X:\|(x,y)\|\le r\}$ with

$$r\ge\frac{(\Omega_1+\Omega_4)N_1+(\Omega_2+\Omega_3)N_2}{1-[(\Omega_1+\Omega_4)(m_1+m_2)+(\Omega_2+\Omega_3)(n_1+n_2)]},$$

$N_1=\sup_{t\in[a,b]}f(t,0,0)<\infty$, $N_2=\sup_{t\in[a,b]}g(t,0,0)<\infty$.

(ii) The operator $\mathcal{T}$ is a contraction.

To establish (i), let $(x,y)\in B_r$. Then, we have

$$\begin{aligned}|\mathcal{T}_1(x,t)(t)| \le\; & {}^kI^{\alpha_1;\varphi}(|f(t,x(t),y(t))-f(t,0,0)|+|f(t,0,0)|)\\ &+\frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Big[A_2\Big(\sum_{i=1}^{n}|\theta_i|\,{}^kI^{\alpha_1;\varphi}(|f(z_i,x(z_i),y(z_i))-f(z_i,0,0)|\\ &+|f(z_i,0,0)|)+\int_a^b\varphi'(s)\,{}^kI^{\alpha_2;\varphi}(|g(s,x(s),y(s))-g(s,0,0)|+|g(s,0,0)|)ds\Big)\\ &+A_4\Big(\sum_{j=1}^{m}|\eta_j|\,{}^kI^{\alpha_2;\varphi}(|g(\xi_j,x(\xi_j),y(\xi_j))-g(\xi_j,0,0)|+|g(\xi_j,0,0)|)\\ &+\int_a^b\varphi'(s)\,{}^kI^{\alpha_1;\varphi}(|f(s,x(s),y(s))-f(s,0,0)|+|f(s,0,0)|)ds\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
&\leq {}^{k}I^{\alpha_1;\varphi}(m_1\|x\|+m_2\|y\|+N_1) \\
&\quad+\frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg[A_2\Big(\sum_{i=1}^{n}|\theta_i|\,{}^{k}I^{\alpha_1;\varphi}(m_1\|x\|+m_2\|y\|+N_1)(z_i) \\
&\quad+\int_a^b\varphi'(s)\,{}^{k}I^{\alpha_2;\varphi}(n_1\|x\|+n_2\|y\|+N_2)ds\Big) \\
&\quad+A_4\Big(\sum_{j=1}^{m}|\eta_j|\,{}^{k}I^{\alpha_2;\varphi}(n_1\|x\|+n_2\|y\|+N_2)(\xi_j) \\
&\quad+\int_a^b\varphi'(s)\,{}^{k}I^{\alpha_1;\varphi}(m_1\|x\|+m_2\|y\|+N_1)ds\Big)\Bigg] \\
&\leq \frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}(m_1\|x\|+m_2\|y\|+N_1) \\
&\quad+\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg[A_2\Big(\sum_{i=1}^{n}|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}(m_1\|x\|+m_2\|y\|+N_1) \\
&\quad+\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)}(n_1\|x\|+n_2\|y\|+N_2)ds\Big) \\
&\quad+A_4\Big(\sum_{j=1}^{m}|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}(n_1\|x\|+n_2\|y\|+N_2) \\
&\quad+\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}(m_1\|x\|+m_2\|y\|+N_1)ds\Big)\Bigg] \\
&= (m_1\|x\|+m_2\|y\|+N_1)\Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \\
&\quad+\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Big(A_2\sum_{i=1}^{n}|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}+A_4\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}\Big)\Bigg\} \\
&\quad+(n_1\|x\|+n_2\|y\|+N_2)\Bigg\{\frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Big(A_2\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)} \\
&\quad+A_4\sum_{j=1}^{m}|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\Big)\Bigg\} \\
&= (m_1\|x\|+m_2\|y\|+N_1)\Omega_1+(n_1\|x\|+n_2\|y\|+N_2)\Omega_2 \\
&= (m_1\Omega_1+n_1\Omega_2)\|x\|+(m_2\Omega_1+n_2\Omega_2)\|y\|+\Omega_1N_1+\Omega_2N_2 \\
&\leq (m_1\Omega_1+n_1\Omega_2+m_2\Omega_1+n_2\Omega_2)r+\Omega_1N_1+\Omega_2N_2.
\end{aligned}
$$

Similarly, one can obtain that

$$|\mathcal{T}_2(x,y)(t)| \leq (m_1\Omega_4+n_1\Omega_3+m_2\Omega_4+n_2\Omega_3)r+\Omega_4N_1+\Omega_3N_2.$$

Consequently, using the preceding inequalities, we have

$$\|\mathcal{T}(x,y)\| \;=\; \|\mathcal{T}_1(x,y)\|+\|\mathcal{T}_2(x,y)\|$$

$$
\begin{aligned}
&= \sup_{t\in[a,b]} |\mathcal{T}_1(x,y)(t)| + \sup_{t\in[a,b]} |\mathcal{T}_2(x,y)(t)| \\
&\le [(\Omega_1+\Omega_4)(m_1+m_2)+(\Omega_2+\Omega_3)(n_1+n_2)]\, r + (\Omega_1+\Omega_4)N_1 + (\Omega_2+\Omega_3)N_2 \\
&\le r,
\end{aligned}
$$

which implies that $\mathcal{T}B_r \subset B_r$ as $(x,y)\in B_r$ is an arbitrary element.

Next, we prove (ii). For $(x_1,y_1),(x_2,y_2)\in X\times X$, and for any $t\in[a,b]$, we get

$$
\begin{aligned}
& |\mathcal{T}_1(x_2,y_2)(t) - \mathcal{T}_1(x_1,y_1)(t)| \\
\le\ & {}^kI^{\alpha_1;\varphi}|f(t,x_2(t),y_2(t)) - f(t,x_1(t),y_1(t))| \\
& + \frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg[A_2\Bigg(\sum_{i=1}^{n}|\theta_i|\,{}^kI^{\alpha_1;\varphi}|f(z_i,x_2(z_i),y_2(z_i)) - f(z_i,x_1(z_i),y_1(z_i))| \\
& + \int_a^b \varphi'(s)\,{}^kI^{\alpha_2;\varphi}|g(s,x_2(s),y_2(s)) - g(s,x_1(s),y_1(s))|ds\Bigg) \\
& + A_4\Bigg(\sum_{j=1}^{m}|\eta_j|\,{}^kI^{\alpha_2;\varphi}|g(\xi_j,x_2(\xi_j),y_2(\xi_j)) - g(\xi_j,x_1(\xi_j),y_1(\xi_j))| \\
& + \int_a^b \varphi'(s)\,{}^kI^{\alpha_1;\varphi}|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|ds\Bigg)\Bigg] \\
\le\ & (m_1\|x_2-x_1\| + m_2\|y_2-y_1\|)\Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \\
& + \frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg(A_2\sum_{i=1}^{n}|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} + A_4\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}\Bigg)\Bigg\} \\
& + (n_1\|x_2-x_1\| + n_2\|y_2-y_1\|)\Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg(A_2\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)}ds \\
& + A_4\sum_{j=1}^{m}|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\Bigg)\Bigg\} \\
=\ & (m_1\|x_2-x_1\| + m_2\|y_2-y_1\|)\Omega_1 + (n_1\|x_2-x_1\| + n_2\|y_2-y_1\|)\Omega_2 \\
=\ & (m_1\Omega_1 + n_1\Omega_2)\|x_2-x_1\| + (m_2\Omega_1 + n_2\Omega_2)\|y_2-y_1\|,
\end{aligned}
$$

and consequently we obtain

$$
\|\mathcal{T}_1(x_2,y_2) - \mathcal{T}_1(x_1,y_1)\| \le (m_1\Omega_1 + n_1\Omega_2 + m_2\Omega_1 + n_2\Omega_2)(\|x_2-x_1\| + \|y_2-y_1\|). \tag{3.6}
$$

Similarly, one can find that

$$
\|\mathcal{T}_2(x_2,y_2) - \mathcal{T}_2(x_1,y_1)\| \le (m_1\Omega_4 + n_1\Omega_3 + m_2\Omega_4 + n_2\Omega_3)(\|x_2-x_1\| + \|y_2-y_1\|). \tag{3.7}
$$

From (3.6) and (3.7), it follows that

$$
\|\mathcal{T}(x_2,y_2)(t) - \mathcal{T}(x_1,y_1)\| \le ((\Omega_1+\Omega_4)(m_1+m_2) + (\Omega_2+\Omega_3)(n_1+n_2))\,(\|x_2-x_1\| + \|y_2-y_1\|),
$$

which shows that the operator $\mathcal{T}$ is a contraction by virtue of the condition (3.5). Therefore, the conclusion of Banach's contraction mapping principle applies and hence the nonlocal (k,φ)-Hilfer fractional system (1.1) has a unique solution on $[a,b]$.

Now, we prove two existence results for the nonlocal (k,φ)-Hilfer fractional system (1.1) by using Leray-Schauder alternative (Lemma 2.4) and Kraslosel'skiĭ's fixed point (Lemma 2.5).

Theorem 3.2. *Let us assume that the continuous functions $f, g : [a,b]\times\mathbb{R}^2 \to \mathbb{R}$ satisfy the following condition:*

(H_2) *There exist real constants $k_i,\ \nu_i\ \geq 0$ $(i = 1,2)$ and $k_0 > 0, \nu_0 > 0$ such that $\forall w_i \in \mathbb{R}$ $(i = 1,2)$,*

$$|f(t,w_1,w_2)| \leq k_0 + k_1|w_1| + k_2|w_2|,\ \ |g(t,w_1,w_2)| \leq \nu_0 + \nu_1|w_1| + \nu_2|w_2|. \tag{3.8}$$

Then, there exists at least one solution for the nonlocal (k,φ)-Hilfer fractional system (1.1) *on* $[a,b]$, *if*

$$(\Omega_1+\Omega_4)k_1 + (\Omega_2+\Omega_3)\nu_1 < 1 \ \ \text{and}\ \ (\Omega_1+\Omega_4)k_2 + (\Omega_2+\Omega_3)\nu_2 < 1,$$

where $\Omega_i, i = 1,2,3,4$ are given in (3.4).

Proof. Observe that continuity of f and g implies that of the operator $\mathcal{T}$. Next we show that the operator $\mathcal{T}$ is uniformly bounded. Let $E \subset X\times X$ be any bounded set. Then, there exist positive constants L_1 and L_2 such that

$$|f(t,x(t),y(t)| \leq L_1, \quad |g(t,x(t),y(t)| \leq L_2, \ \ \forall (x,y)\in E.$$

Then, for any $(x,y)\in E$, we have

$$\begin{aligned}|\mathcal{T}_1(x,y)(t)| \ \leq\ & \Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} + \frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg(A_2\sum_{i=1}^{n}|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)} \\ & + A_4\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}\Bigg)\Bigg\}L_1 \\ & + \Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg(A_2\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)} \\ & + A_4\sum_{j=1}^{m}|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\Bigg)\Bigg\}L_2,\end{aligned}$$

which yields

$$\|\mathcal{T}_1(x,y)\| \leq \Omega_1L_1 + \Omega_2L_2.$$

Likewise, one can get

$$\|\mathcal{T}_2(x,y)\| \leq \Omega_3L_1 + \Omega_4L_2.$$

Therefore, from the foregoing inequalities, we have

$$\|\mathcal{T}(x,y)\| = \|\mathcal{T}_1(x,y)\| + \|\mathcal{T}_2(x,y)\| \leq (\Omega_1+\Omega_3)L_1 + (\Omega_2+\Omega_4)L_2,$$

which shows that the operator $\mathcal{T}$ is uniformly bounded.

To establish that the operator $\mathcal{T}$ is equicontinuous, we take $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. Then we have

$$|\mathcal{T}_1(x(t_2),y(t_2)) - \mathcal{T}_1(x(t_1),y(t_1))|$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma_k(\alpha_1)}\bigg|\int_a^{t_1}\varphi'(s)[(\varphi(t_2)-\varphi(s))^{\frac{\alpha_1}{k}-1}-(\varphi(t_1)-\varphi(s))^{\frac{\alpha_1}{k}-1}]f(s,x(s),y(s))ds\\
&\quad+\int_{t_1}^{t_2}\varphi'(s)(\varphi(t_2)-\varphi(s))^{\frac{\alpha_1}{k}-1}f(s,x(s),y(s))ds\bigg|\\
&\quad+\frac{(\varphi(t_2)-\varphi(a))^{\frac{p}{k}-1}-(\varphi(t_1)-\varphi(a))^{\frac{p}{k}-1}}{A\Gamma_k(p)}\\
&\quad\times\Bigg[A_2\Bigg(\sum_{i=1}^{n}|\theta_i|\,{}^kI^{\alpha_1;\varphi}|f(z_i,x(z_i),y(z_i))|+\int_a^b\varphi'(s)\,{}^kI^{\alpha_2;\varphi}|g(s,x(s),y(s))|ds\Bigg)\\
&\quad+A_4\Bigg(\sum_{j=1}^{m}|\eta_j|\,{}^kI^{\alpha_2;\varphi}|g(\xi_j,x(\xi_j),y(\xi_j))|+\int_a^b\varphi'(s)\,{}^kI^{\alpha_1;\varphi}|f(s,x(s),y(s))|ds\Bigg)\Bigg]\\
&\le \frac{L_1}{\Gamma_k(\alpha_1+k)}[2(\varphi(t_2)-\varphi(t_1))^{\frac{\alpha_1}{k}}+|(\varphi(t_2)-\varphi(a))^{\frac{\alpha_1}{k}}-(\varphi(t_1)-\varphi(a))^{\frac{\alpha_1}{k}}|]\\
&\quad+\frac{(\varphi(t_2)-\varphi(a))^{\frac{p}{k}-1}-(\varphi(t_1)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\\
&\quad\times\Bigg[A_2\Bigg(\sum_{i=1}^{n}|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}L_1+\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)}L_2\Bigg)\\
&\quad+A_4\Bigg(\sum_{j=1}^{m}|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}L_2+\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}L_1\Bigg)\Bigg],
\end{aligned}
$$

which is independent of (x, y) and tends to zero as $t_2 - t_1 \to 0$. Therefore, $\mathcal{T}_1(x, y)$ is equicontinuous. Analogously, we can prove that $\mathcal{T}_2(x, y)$ is equicontinuous. Consequently the operator $\mathcal{T}(x, y)$ is completely continuous.

Lastly, it will be shown that the set $\mathcal{E} = \{(x, y) \in X \times X : (x, y) = \lambda\mathcal{T}(x, y), 0 \le \lambda \le 1\}$ is bounded. For this, let $(x, y) \in \mathcal{E}$, then $(x, y) = \lambda\mathcal{T}(x, y)$. For any $t \in [a, b]$, we have

$$x(t) = \lambda\mathcal{T}_1(x, y)(t), \quad y(t) = \lambda\mathcal{T}_2(x, y)(t).$$

Then

$$
\begin{aligned}
|x(t)| &\le \Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}+\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg(A_2\sum_{i=1}^{n}|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\\
&\quad+A_4\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}\Bigg)\Bigg\}(k_0+k_1\|x\|+\|y\|)\\
&\quad+\Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Bigg(A_2\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)}\\
&\quad+A_4\sum_{j=1}^{m}|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\Bigg)\Bigg\}(\nu_0+\nu_1\|x\|+\nu_2\|y\|),
\end{aligned}
$$

and

$$
|y(t)| \le \Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}+\frac{(\varphi(b)-\varphi(a))^{\frac{q}{k}-1}}{A\Gamma_k(q)}\Bigg(A_1\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)}
$$

$$+A_3 \sum_{j=1}^{m} |\eta_j| \frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\Bigg)\Bigg\}(\nu_0+\nu_1\|x\|+\nu_2\|y\|)$$
$$+\Bigg\{\frac{(\varphi(b)-\varphi(a))^{\frac{q}{k}-1}}{A\Gamma_k(q)}\Bigg(A_1 \sum_{i=1}^{n} |\theta_i| \frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}$$
$$+\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}\Bigg)\Bigg\}(k_0+k_1\|x\|+\|y\|).$$

Consequently, we get

$$\|x\| \le (k_0+k_1\|x\|+\|y\|)\Omega_1+(\nu_0+\nu_1\|x\|+\nu_2\|y\|)\Omega_2,$$
$$\|y\| \le (k_0+k_1\|x\|+\|y\|)\Omega_4+(\nu_0+\nu_1\|x\|+\nu_2\|y\|)\Omega_3,$$

and hence

$$\begin{aligned}\|x\|+\|y\| \quad \le \quad & (\Omega_1+\Omega_4)k_0+((\Omega_2+\Omega_3)\nu_0+(\Omega_1+\Omega_4)k_1+(\Omega_2+\Omega_3)\nu_1)\|x\| \\ & +(\Omega_1+\Omega_4)k_2+(\Omega_2+\Omega_3)\nu_2)\|y\|.\end{aligned}$$

Therefore,

$$\|(x,y)\| \le \frac{(\Omega_1+\Omega_4)k_0+(\Omega_2+\Omega_3)\nu_0}{M_0},$$

where M_0 is defined by

$$M_0=\min\{1-[(\Omega_1+\Omega_4)k_1+(\Omega_2+\Omega_3)\nu_1], 1-[(\Omega_1+\Omega_4)k_2+(\Omega_2+\Omega_3)\nu_2]\}>0,$$

which leads to the fact that $\mathcal{E}$ is bounded. Thus, by Leray-Schauder alternative [39], the nonlocal (k,φ)-Hilfer fractional system (1.1) has at least one solution on $[a,b]$. The proof is complete.

Our next result is based on Krasnosel'skiĭ's fixed point theorem [40].

Theorem 3.3. *Suppose that* (H_1) *and the following condition hold:*

(H_3) *There exist continuous nonnegative functions P and* $Q \in C([a,b],\mathbb{R}^+)$ *satisfying*

$$|f(t,x,y)| \le P(t), \quad |g(t,x,y)| \le Q(t), \ \text{ for each } \ (t,x,y) \in [a,b]\times\mathbb{R}\times\mathbb{R}.$$

Then, the nonlocal (k,φ)*-Hilfer fractional system* (1.1) *has at least one solution on* $[a,b]$, *if*

$$(\Omega_1^*+\Omega_4)(m_1+m_2)+(\Omega_2+\Omega_3^*)(n_1+n_2)<1, \tag{3.9}$$

where $\Omega_i, i=2,4$ *and* $\Omega_i^*, i=1,3$ *are given in* (3.4).

Proof. We decompose the operator $\mathcal{T}$ defined by (3.1) into four operators $\mathcal{T}_{1,1}, \mathcal{T}_{1,2}, \mathcal{T}_{2,1}$ and $\mathcal{T}_{2,2}$ as follows:

$$\begin{aligned}\mathcal{T}_{1,1}(x,y)(t) &= {}^kI^{\alpha_1;\varphi}f(t,x(t),y(t)), \quad t\in[a,b],\\ \mathcal{T}_{1,2}(x,y)(t) &= \frac{(\varphi(t)-\varphi(a))^{\frac{p}{k}-1}}{A\Gamma_k(p)}\Bigg[A_2\Bigg(\sum_{i=1}^{n}\theta_i\,{}^kI^{\alpha_1;\varphi}f(z_i,x(z_i),y(z_i))\end{aligned}$$

$$
\begin{aligned}
&\quad -\int_a^b \varphi'(s)\,{}^kI^{\alpha_2;\varphi}g(s,x(s),y(s))ds\Big)\\
&\quad +A_4\Big(\sum_{j=1}^m \eta_j\,{}^kI^{\alpha_2;\varphi}g(\xi_j,x(\xi_j),y(\xi_j))\\
&\quad -\int_a^b \varphi'(s)\,{}^kI^{\alpha_1;\varphi}f(s,x(s),y(s))ds\Big)\Big],\quad t\in[a,b],\\
\mathcal{T}_{2,1}(x,y)(t) &= {}^kI^{\alpha_2;\varphi}g(t,x(t),y(t)),\quad t\in[a,b],\\
\mathcal{T}_{2,2}(x,y)(t) &= \frac{(\varphi(t)-\varphi(a))^{\frac{q}{k}-1}}{A\Gamma_k(q)}\Big[A_1\Big(\sum_{i=1}^n \theta_i\,{}^kI^{\alpha_1;\varphi}f(z_i,x(z_i),y(z_i))\\
&\quad -\int_a^b \varphi'(s)\,{}^kI^{\alpha_2;\varphi}g(s,x(s),y(s))ds\Big)\\
&\quad +A_3\Big(\sum_{j=1}^m \eta_j\,{}^kI^{\alpha_2;\varphi}g(\xi_j,x(\xi_j),y(\xi_j))\\
&\quad -\int_a^b \varphi'(s)\,{}^kI^{\alpha_1;\varphi}f(s,x(s),y(s))ds\Big)\Big],\quad t\in[a,b].
\end{aligned}
$$

It is clear that $\mathcal{T}_1=\mathcal{T}_{1,1}+\mathcal{T}_{1,2}$, $\mathcal{T}_2=\mathcal{T}_{2,1}+\mathcal{T}_{2,2}$. Let us introduce the closed ball: $B_\rho=\{(x,y)\in X\times X:\|(x,y)\|\le\rho\}$ with $\rho\ge(\Omega_1+\Omega_3)\|P\|+(\Omega_2+\Omega_4)\|Q\|$. For any $x=(x_1,x_2), y=(y_1,y_2)\in B_\rho$, working as in Theorem 3.2, we have

$$|\mathcal{T}_{1,1}(x_1,y_2)(t)+\mathcal{T}_{1,2}(y_1,y_2)(t)|\le\Omega_1\|P\|+\Omega_2\|Q\|.$$

Similarly, one can get

$$|\mathcal{T}_{2,1}(x_1,y_2)(t)+\mathcal{T}_{2,2}(y_1,y_2)(t)|\le\Omega_3\|P\|+\Omega_4\|Q\|.$$

Thus, we obtain

$$\|\mathcal{T}_1x+\mathcal{T}_2y\|\le(\Omega_1+\Omega_3)\|P\|+(\Omega_2+\Omega_4)\|\|Q\|<\rho,$$

which shows that $\mathcal{T}_1x+\mathcal{T}_2y\in B_\rho$.

To establish that the operator $(\mathcal{T}_{1,2},\mathcal{T}_{2,2})$ is a contraction, let (x_1,x_2), $(y_1,y_2)\in B_\rho$. Then, as in the proof of Theorem (3.1), we can get

$$
\begin{aligned}
&|\mathcal{T}_{1,2}(x_2,y_2)(t)-\mathcal{T}_{1,2}(x_1,y_1)(t)|\\
\le\;& (m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)\Big\{\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\\
&\times\Big(A_2\sum_{i=1}^n|\theta_i|\frac{(\varphi(z_i)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}+A_4\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}+1}}{\Gamma_k(\alpha_1+2k)}\Big)\Big\}\\
&+(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\Big\{\frac{(\varphi(b)-\varphi(a))^{\frac{p}{k}-1}}{|A|\Gamma_k(p)}\Big(A_2\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}+1}}{\Gamma_k(\alpha_2+2k)}\\
&+A_4\sum_{j=1}^m|\eta_j|\frac{(\varphi(\xi_j)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= (m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)\Omega_1^*+(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\Omega_2 \\
&= (m_1\Omega_1^*+n_1\Omega_2)\|x_2-x_1\|+(m_2\Omega_1^*+n_2\Omega_2)\|y_1-y_2\|, \qquad (3.10)
\end{aligned}
$$

and

$$
|\mathcal{T}_{2,2}(x_1,y_1)(t)-\mathcal{T}_{2,2}(x_2,y_2)(t)| \le (m_1\Omega_4+n_1\Omega_3^*)\|x_2-x_1\|+(m_2\Omega_4+n_2\Omega_3^*)\|y_2-y_1\|. \qquad (3.11)
$$

From (3.10) and (3.11), we have

$$
\begin{aligned}
&\|(\mathcal{T}_{1,2},\mathcal{T}_{2,2})(x_1,y_1)-(\mathcal{T}_{1,2},\mathcal{T}_{2,2})(x_2,y_2)\| \\
\le\ & \Big\{(\Omega_1^*+\Omega_4)(m_1+m_2)+(\Omega_2+\Omega_3^*)(n_1+n_2)\Big\}(\|x_1-x_2\|+\|y_1-y_2\|),
\end{aligned}
$$

which shows that $(\mathcal{T}_{1,2},\mathcal{T}_{2,2})$ is a contraction by means of the condition (3.9). Notice that the operator $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is continuous since the functions f and g are continuous. Further, $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is uniformly bounded on B_ρ, since

$$
\|\mathcal{T}_{1,1}(x,y)\| \le \frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\|P\|,
$$

and

$$
\|\mathcal{T}_{2,1}(x,y)\| \le \frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\|Q\|.
$$

Hence, we obtain

$$
\|(\mathcal{T}_{1,1},\mathcal{T}_{2,1})(x,y)\| \le \frac{(\varphi(b)-\varphi(a))^{\frac{\varphi}{k}}}{\Gamma_k(\varphi+k)}\|P\|+\frac{(\varphi(b)-\varphi(a))^{\frac{\alpha_2}{k}}}{\Gamma_k(\alpha_2+k)}\|Q\|,
$$

which implies that the set $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})B_\rho$ is uniformly bounded. Next, it will be shown that that the set $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})B_\rho$ is equicontinuous. For any $(x,y)\in B_\rho$ and $t_1,t_2\in[a,b]$ with $t_1<t_2$, we have

$$
\begin{aligned}
&|\mathcal{T}_{1,1}(x,y)(t_2)-\mathcal{T}_{1,1}(x,y)(t_1)| \\
\le\ & \frac{1}{\Gamma_k(\alpha_1)}\bigg|\int_a^{t_1}\varphi'(s)[(\varphi(t_2)-\varphi(s))^{\frac{\alpha_1}{k}-1}-(\varphi(t_1)-\varphi(s))^{\frac{\alpha_1}{k}-1}]f(s,x(s),y(s))ds \\
&+\int_{t_1}^{t_2}\varphi'(s)(\varphi(t_2)-\varphi(s))^{\frac{\alpha_1}{k}-1}f(s,x(s),y(s))ds\bigg| \\
\le\ & \frac{\|P\|}{\Gamma_k(\alpha_1+k)}[2(\varphi(t_2)-\varphi(t_1))^{\frac{\alpha_1}{k}}+|(\varphi(t_2)-\varphi(a))^{\frac{\alpha_1}{k}}-(\varphi(t_1)-\varphi(a))^{\frac{\alpha_1}{k}}|],
\end{aligned}
$$

which tends to zero as $t_1\to t_2$, independently of $(x,y)\in B_\rho$. Similarly, one can find that $|\mathcal{T}_{2,1}(x,y)(t_2)-\mathcal{T}_{2,1}(x,y)(t_1)|\to 0$ as $t_1\to t_2$ independently of $(x,y)\in B_\rho$. Thus, $|(\mathcal{T}_{1,1},\mathcal{T}_{2,1})(x,y)(t_2)-(\mathcal{T}_{1,1},\mathcal{T}_{2,1})(x,y)(t_1)|$ tends to zero, as $t_1\to t_2$, and hence $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is equicontinuous. Consequently, the operator $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is compact on B_ρ, by the Arzelá-Ascoli theorem. Hence, the nonlocal (k,φ)-Hilfer fractional system (1.1) has at least one solution on $[a,b]$ by Krasnosel'skiĭ's fixed point theorem.

4. Illustrative examples

In this section, we present examples illustrating the applicability of our main results.

Consider the following nonlocal coupled system of (k,φ)-Hilfer fractional differential equations:

$$\begin{cases} {}^{\frac{5}{4},H}\mathcal{D}^{\frac{3}{2},\frac{1}{3};t^3e^{-t}}x(t) = f(t,x(t),y(t)), & t \in \left(\frac{1}{9},\frac{17}{9}\right], \\ {}^{\frac{5}{4},H}\mathcal{D}^{\frac{5}{3},\frac{2}{3};t^3e^{-t}}y(t) = g(t,x(t),y(t)), & t \in \left(\frac{1}{9},\frac{17}{9}\right], \\ x\left(\frac{1}{9}\right) = 0, \qquad \int_{\frac{1}{9}}^{\frac{17}{9}} \varphi'(s)x(s)ds = \frac{1}{11}y\left(\frac{2}{9}\right) + \frac{1}{13}y\left(\frac{4}{9}\right) + \frac{1}{15}y\left(\frac{5}{9}\right), \\ y\left(\frac{1}{9}\right) = 0 \qquad \int_{\frac{1}{9}}^{\frac{17}{9}} \varphi'(s)y(s)ds = \frac{1}{12}x\left(\frac{7}{9}\right) + \frac{1}{14}x\left(\frac{8}{9}\right) + \frac{1}{16}x\left(\frac{11}{9}\right), \end{cases} \tag{4.1}$$

where $k = 5/4$, $\alpha_1 = 3/2$, $\beta_1 = 1/3$, $\alpha_2 = 5/3$, $\beta_2 = 2/3$, $\varphi(t) = t^3e^{-t}$, $a = 1/9$, $b = 17/9$, $m = 3$, $\eta_1 = 1/11$, $\eta_2 = 1/13$, $\eta_3 = 1/15$, $\xi_1 = 2/9$, $\xi_2 = 4/9$, $\xi_3 = 5/9$, $n = 3$, $\theta_1 = 1/12$, $\theta_2 = 1/14$, $\theta_3 = 1/16$, $z_1 = 7/9$, $z_2 = 8/9$, $z_3 = 11/9$. Then, with the aid of the given data, we find that $p = 11/6$, $q = 20/9$, $\Gamma_{\frac{5}{4}}(p) \approx 0.9828090918$, $\Gamma_{\frac{5}{4}}(q) \approx 1.101113581$, $A_1 \approx 0.5933053129$, $A_2 \approx 0.01904961698$, $A_3 \approx 0.1295611400$, $A_4 \approx 0.4002998007$, $A \approx 0.2350319084$, $\Gamma_{\frac{5}{4}}(\alpha_1 + 5/4) \approx 1.440110329$, $\Gamma_{\frac{5}{4}}(\alpha_2 + 5/4) \approx 1.603221698$, $\Omega_1 \approx 1.171782826$, $\Omega_2 \approx 0.02369021066$, $\Omega_3 \approx 1.158588899$, $\Omega_4 \approx 0.2282160746$, $\Omega_1^* \approx 0.4623382179$, $\Omega_3^* \approx 0.5198025994$.

Illustration of Theorem 3.1.

Consider the nonlinear unbounded functions $f, g : [1/9, 17/9] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(t,x,y) = \frac{e^{-(9t-1)^2}}{14}\left(\frac{x^2 + 2|x|}{1+|x|}\right) + \frac{\tan^{-1}|y|}{9(t+1)} + \frac{1}{2}t, \tag{4.2}$$

$$g(t,x,y) = \frac{\sin^2(\pi t)}{5}\tan^{-1}|x| + \frac{1}{9(9t+1)^2}\left(\frac{y^2+|y|}{1+|y|}\right) + \frac{1}{4}t^2. \tag{4.3}$$

It is easy to find that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \frac{1}{7}|x_1 - x_2| + \frac{1}{10}|y_1 - y_2|, \tag{4.4}$$

and

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le \frac{1}{5}|x_1 - x_2| + \frac{1}{9}|y_1 - y_2|, \tag{4.5}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Choosing $m_1 = 1/7$, $m_2 = 1/10$, $n_1 = 1/5$ and $n_2 = 1/9$, we get $(\Omega_1 + \Omega_4)(m_1 + m_2) + (\Omega_2 + \Omega_3)(n_1 + n_2) \approx 0.7078199005 < 1$. Thus, by Theorem 3.1, we conclude that the nonlocal coupled system of (k,φ)-Hilfer fractional differential equations (4.1) with f and g defined in (4.4) and (4.5) respectively, has a unique solution on $[1/9, 17/9]$.

Illustration of Theorem 3.2.

Let the nonlinear functions $f, g : [1/9, 17/9] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(t,x,y) = \frac{1}{9t+1} + \frac{1}{8}e^{1-9t}\left(\frac{x^4}{1+|x|^3}\right) + \frac{|y|}{3}\cos^7|x|, \tag{4.6}$$

$$g(t,x,y) = \frac{1}{18t+1} + \frac{1}{9t+10}\tan^{-1}|x| + \frac{1}{4}e^{-(1-9t)^2}\left(\frac{y^4}{1+|y|^3}\right). \tag{4.7}$$

Then we have

$$|f(t,x,y)| \le \frac{1}{2} + \frac{1}{8}|x| + \frac{1}{3}|y| \quad \text{and} \quad |g(t,x,y)| \le \frac{1}{3} + \frac{1}{11}|x| + \frac{1}{4}|y|.$$

Setting $k_0 = 1/2$, $k_1 = 1/8$, $k_2 = 1/3$, $\nu_0 = 1/3$, $\nu_1 = 1/11$, $\nu_2 = 1/4$, we have $(\Omega_1 + \Omega_3)k_1 + (\Omega_2 + \Omega_4)\nu_1 \approx 0.3141970370 < 1$ and $(\Omega_1 + \Omega_3)k_2 + (\Omega_2 + \Omega_4)\nu_2 \approx 0.8397671463 < 1$. Therefore, by Theorem 3.2, the nonlocal coupled system of (k,φ)-Hilfer fractional differential equations (4.1) with f and g given by (4.6) and (4.7) respectively, has at least one solution on $[1/9, 17/9]$.

Illustration of Theorem 3.3.

Consider the nonlinear bounded functions $f, g : [1/9, 17/9] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ represented by

$$f(t,x,y) = \frac{\sin \pi t}{5}\left(\frac{|x|}{1+|x|}\right) + \frac{\cos^2|y|}{(9t+1)^2} + \frac{1}{6}, \tag{4.8}$$

$$g(t,x,y) = \frac{1}{3}\left(\frac{|x|\cos^4 \pi t}{1+|x|}\right) + \frac{\tan^{-1}|y|}{9t+3} + \frac{1}{11}. \tag{4.9}$$

Note that f and g are bounded as

$$|f(t,x,y)| \le \frac{1}{5}\sin \pi t + \frac{1}{(9t+1)^2} + \frac{1}{6}, \quad |g(t,x,y)| \le \frac{1}{3}\cos^4 \pi t + \frac{1}{9t+3} + \frac{1}{11}.$$

Also, we have that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \frac{1}{5}|x_1 - x_2| + \frac{1}{4}|y_1 - y_2|,$$

and

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le \frac{1}{3}|x_1 - x_2| + \frac{1}{4}|y_1 - y_2|.$$

Letting $m_1 = 1/5$, $m_2 = 1/4$, $n_1 = 1/3$ and $n_2 = 1/4$, we find that $(\Omega_1^* + \Omega_4)((1/5) + (1/4)) + (\Omega_2 + \Omega_3^*)((1/3) + (1/4)) \approx 0.6277869041 < 1$. Therefore, by Theorem 3.3, the nonlocal coupled system for (k,φ)-Hilfer fractional differential equations (4.1) with f and g defined by (4.8) and (4.9) respectively, has at least one solution on $[1/9, 17/9]$.

5. Conclusions

In this paper, we have presented the existence and uniqueness criteria for the solutions of a coupled system of (k,φ)-Hilfer fractional differential equations complemented with nonlocal coupled integro-multi-point boundary conditions. We established the desired results by means of the standard fixed point theorems after converting the given nonlinear problem into a fixed point problem. We also demonstrated the application of the obtained results by providing some numerical examples. It is imperative to note that our results are of more general nature and produce some new results as special cases. For example, the results for a coupled system of (k,φ)-Riemann-Liouville fractional differential equations follow by fixing $\beta_i = 0, i = 1,2$ in the obtained results. On the other hand, by taking $\beta_i = 1, i = 1,2$ in the results of this paper, we obtain the ones for a coupled system (k,φ)-Caputo fractional differential equations. Moreover, our results reduce to the ones for a system involving k-Hilfer-Hadamard fractional derivative operators and k-Hilfer-Katugampola fractional derivative operators by letting $\varphi(t) = \log t$ and $\varphi(t) = t^\rho$, respectively.

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Conflict of interest

The authors declare no conflict of interest.

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