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Research article

Norms of some operators between weighted-type spaces and weighted Lebesgue spaces

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Abstract: We calculate the norms of several concrete operators, mostly of some integral-type ones between weighted-type spaces of continuous functions on several domains. We also calculate the norm of an integral-type operator on some subspaces of the weighted Lebesgue spaces.

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1. Introduction

By $\mathbb{R}$ we denote the set of real numbers, by $\mathbb{R}_+$ the interval $[0,+\infty)$, the space of continuous functions on a set Ω we denote by $C(\Omega)$, whereas the space of continuously differentiable functions on Ω we denote by $C^1(\Omega)$. A vector $(x_1,\dots,x_n)\in\mathbb{R}^n$ we denote by x. If $c\in\mathbb{R}$, then by $\vec{c}$ we denote the vector $(c,c,\dots,c)$ (for example, $\vec{1}=(1,\dots,1)$). By $\langle x,y\rangle$ we denote the Euclidean inner product of vectors $x,y\in\mathbb{R}^n$, that is, $\langle x,y\rangle=\sum_{j=1}^n x_jy_j$. The Lebesgue measure on $\mathbb{R}^n$ we denote by $dV(x)$, whereas by $d\sigma(\zeta)$ we denote the surface measure on the unit sphere $\mathbb{S}\subset\mathbb{R}^n$. A function $w:\Omega\to\mathbb{R}$ is called a weight function or simply weight if it is positive and continuous. The class of all weights on Ω we denote by $W(\Omega)$.

Let $w\in W(\Omega)$. The weighted-type space $C_w(\Omega)$ consists of all $f\in C(\Omega)$ such that

$$\|f\|_w := \sup_{t\in\Omega} w(t)|f(t)| < +\infty. \tag{1.1}$$

By using a standard argument, which is applied to the space $C(\Omega)$, it is shown that $C_w(\Omega)$ is a Banach space. Various weighted-type spaces of continuous or analytic functions and operators on them have been investigated considerably for several decades (see, e.g., [1,2,12,19,25,32–35,38,42,43,48,50,51] and the related references therein).

Let $\mathcal{L}^p_w(\mathbb{R}^n) = \mathcal{L}^p_w$, where $p \geq 1$ and $w \in W(\mathbb{R}^n)$, be the weighted $\mathcal{L}^p$ space consisting of all measurable functions f such that

$$\|f\|_{\mathcal{L}^p_w} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dV(x) \right)^{1/p} < +\infty.$$

With the norm $\|\cdot\|_{\mathcal{L}^p_w}$ the space $\mathcal{L}^p_w$ is Banach.

Let X and Y be normed spaces, and $L : X \to Y$ be a linear operator. We say that the operator is bounded if there is $M \geq 0$, such that

$$\|Lx\|_Y \leq M\|x\|_X,$$

for every $x \in X$ ([6, 26, 27, 44, 45]).

Norm of the operator is defined by

$$\|L\|_{X\to Y} = \sup_{x\in B_X} \|Lx\|_Y,$$

where B_X denotes the unit ball in the space X.

Finding norms of linear operators is one of the basic problems in operator theory. Many classical results can be found in books and surveys on functional analysis, operator theory and inequalities (see, for example, [6, 7, 9, 10, 16, 23, 26, 27, 44, 45]; see also some of the original sources [13, 14, 28]). For some recent results in the topic, including some on multi-linear operators (for the definition and some examples see [52, p. 51–55]), see, for example, [4,9,11,20,33–36,38,40–42] and the related references therein.

Let u be a function defined on Ω. Then by M_u we denote the multiplication operator

$$M_u(f)(t) = u(t)f(t), \quad t \in \Omega, \tag{1.2}$$

where f is a function on Ω.

There has been some interest in the multiplication operators on spaces of functions [35, 49]. Motivated by some of our previous results on calculating and estimating norms of concrete operators and a problem in [45], here we present some formulas for norms of the multiplication and several integral-type operators between weighted-type spaces. We also calculate norm of an integral-type operator on some subspaces of $\mathcal{L}^p_w(\mathbb{R}^n)$ space. For various integral-type operators see, e.g., [3–5, 7–10, 16, 19–22, 24, 29–32, 37, 39–43, 46, 47]. Some of the formulas we have got long time ago, but have never published them. Some of the formulas could be matters of folklore, but we could not found references.

2. Main results

This section presents our main results and some analyses.

2.1. Multiplication operator between weighted-type spaces

The following result is a simple and basic one, and should be a matter of folklore. However, it is useful and instructive, because of which we give a proof.

Theorem 1. *Let $w_1, w_2 \in W(\Omega)$. Then the operator $M_u : C_{w_1}(\Omega) \to C_{w_1 w_2}(\Omega)$ is bounded if and only if*

$$u \in C_{w_2}(\Omega). \tag{2.1}$$

Moreover, if (2.1) *holds then*

$$\|M_u\|_{C_{w_1}(\Omega)\to C_{w_1w_2}(\Omega)} = \|u\|_{w_2}. \tag{2.2}$$

Proof. First, assume that condition (2.1) holds, that is, that

$$\|u\|_{w_2} < +\infty. \tag{2.3}$$

Then, we have

$$\|M_u(f)\|_{w_1w_2} = \sup_{t\in\Omega} w_1(t)w_2(t)|u(t)f(t)| \le \sup_{t\in\Omega} w_2(t)|u(t)| \sup_{t\in\Omega} w_1(t)|f(t)| = \|u\|_{w_2}\|f\|_{w_1}$$

from which by taking the supremum over the ball $B_{C_{w_1}(\Omega)}$ we get

$$\|M_u\|_{C_{w_1}(\Omega)\to C_{w_1w_2}(\Omega)} \le \|u\|_{w_2}. \tag{2.4}$$

From (2.3) and (2.4) the boundedness of the operator $M_u : C_{w_1}(\Omega) \to C_{w_1w_2}(\Omega)$ follows.

Now assume that the operator $M_u : C_{w_1}(\Omega) \to C_{w_1w_2}(\Omega)$ is bounded. Since w_1 is a positive continuous function we see that $1/w_1$ is also such a function. Note that

$$\|1/w_1\|_{w_1} = \sup_{t\in\Omega} w_1(t) \cdot \frac{1}{w_1(t)} = 1. \tag{2.5}$$

Further, we have

$$\|M_u(1/w_1)\|_{w_1w_2} = \sup_{t\in\Omega} w_1(t)w_2(t)\left|u(t)\frac{1}{w_1(t)}\right| = \|u\|_{w_2}. \tag{2.6}$$

From (2.5), (2.6) and the boundedness of the operator $M_u : C_{w_1}(\Omega) \to C_{w_1w_2}(\Omega)$ it follows that

$$\|u\|_{w_2} \le \|M_u\|_{C_{w_1}(\Omega)\to C_{w_1w_2}(\Omega)} < +\infty, \tag{2.7}$$

which means that (2.1) holds.

If condition (2.1) holds, then from the inequalities in (2.4) and (2.7), we immediately obtain formula (2.2). □

Remark 1. Note that the simple fact in (2.5) plays one of the decisive roles in finding the norm of the operator $M_u : C_{w_1}(\Omega) \to C_{w_1w_2}(\Omega)$. Related facts are very useful in finding norms of concrete operators acting from weighted-type spaces and will be also used further in this paper.

2.2. Appearance of an integral-type operator in differential equations

Consider the initial value problem

$$y'(t) = -\beta(t)y(t) + f(t), \tag{2.8}$$

$$y(0) = 0, \tag{2.9}$$

where $f, \beta \in C(\mathbb{R}_+)$.

By using the Euler multiplier $e^{\int_0^t \beta(\zeta)d\zeta}$ from (2.8) we have

$$\left(y(t)e^{\int_0^t \beta(\zeta)d\zeta}\right)' = f(t)e^{\int_0^t \beta(\zeta)d\zeta}.$$

By integrating the last relation and using condition (2.9), after some calculation, we obtain

$$y(t) = \int_0^t e^{\int_t^s \beta(\zeta)d\zeta} f(s)ds. \tag{2.10}$$

Note that formula (2.10) presents a linear operator, say, L which is defined as follows

$$y(t) = L(f)(t), \quad t \in \mathbb{R}_+,$$

and acts from $C(\mathbb{R}_+)$ into the subspace of $C^1(\mathbb{R}_+)$ consisting of all $g \in C^1(\mathbb{R}_+)$ such that $g(0) = 0$.

Consider the operator from $C_{w_1}(\mathbb{R}_+)$ to $C_{w_2}(\mathbb{R}_+)$. Using the definitions of the spaces $C_{w_1}(\mathbb{R}_+)$ and $C_{w_2}(\mathbb{R}_+)$, we have

$$\begin{aligned}\|L(f)\|_{w_2} &= \sup_{t\in\mathbb{R}_+} w_2(t) \left|\int_0^t e^{\int_t^s \beta(\zeta)d\zeta} f(s)ds\right| \\ &\le \|f\|_{w_1} \sup_{t\in\mathbb{R}_+} w_2(t) \int_0^t e^{\int_t^s \beta(\zeta)d\zeta} \frac{ds}{w_1(s)},\end{aligned}$$

from which it follows that

$$\|L\|_{C_{w_1}(\mathbb{R}_+)\to C_{w_2}(\mathbb{R}_+)} \le \sup_{t\in\mathbb{R}_+} w_2(t) \int_0^t e^{\int_t^s \beta(\zeta)d\zeta} \frac{ds}{w_1(s)}. \tag{2.11}$$

From (2.5) and since

$$\|L(1/w_1)\|_{w_2} = \sup_{t\in\mathbb{R}_+} w_2(t) \int_0^t e^{\int_t^s \beta(\zeta)d\zeta} \frac{ds}{w_1(s)},$$

we have

$$\|L\|_{C_{w_1}(\mathbb{R}_+)\to C_{w_2}(\mathbb{R}_+)} \ge \sup_{t\in\mathbb{R}_+} w_2(t) \int_0^t e^{\int_t^s \beta(\zeta)d\zeta} \frac{ds}{w_1(s)}. \tag{2.12}$$

From (2.11) and (2.12) we obtain

$$\|L\|_{C_{w_1}(\mathbb{R}_+)\to C_{w_2}(\mathbb{R}_+)} = \sup_{t\in\mathbb{R}_+} w_2(t) \int_0^t e^{\int_t^s \beta(\zeta)d\zeta} \frac{ds}{w_1(s)}. \tag{2.13}$$

From the analysis that we have just conduced it follows that the following result holds.

Theorem 2. *Let* $w_1, w_2 \in W(\mathbb{R}_+)$, $\beta \in C(\mathbb{R}_+)$ *and*

$$L(f)(t) = \int_0^t e^{\int_t^s \beta(\zeta)d\zeta} f(s)ds. \tag{2.14}$$

Then the operator $L : C_{w_1}(\mathbb{R}_+) \to C_{w_2}(\mathbb{R}_+)$ *is bounded if and only if*

$$M := \sup_{t\in\mathbb{R}_+} w_2(t) \int_0^t e^{\int_t^s \beta(\zeta)d\zeta} \frac{ds}{w_1(s)} < +\infty. \tag{2.15}$$

Moreover, if the operator is bounded then

$$||L||_{C_{w_1}(\mathbb{R}_+)\to C_{w_2}(\mathbb{R}_+)} = M.$$

Let

$$||f||_\delta := \sup_{t\in\mathbb{R}_+} e^{\delta t}|f(t)|,$$

where $\delta \in \mathbb{R}_+$, and let

$$C_\delta(\mathbb{R}_+) = \{f \in C(\mathbb{R}_+) : ||f||_\delta < +\infty\}.$$

The following example shows that for some functions w_1, w_2 and β the norm of the operator $L : C_{w_1}(\mathbb{R}_+) \to C_{w_2}(\mathbb{R}_+)$ can be explicitly calculated ([45, Problem 7.31]).

Corollary 1. *Let* $w_1(t) = e^{\alpha t}$, $\alpha \geq 0$, $w_2(t) = e^{\gamma t}$, $\beta(t) = \beta$, *and* $\beta > \alpha \geq \gamma$. *Then the operator* $L : C_\alpha(\mathbb{R}_+) \to C_\gamma(\mathbb{R}_+)$ *is bounded and the following statements hold.*

(a) If $\alpha = \gamma$, *then*

$$||L||_{C_\alpha(\mathbb{R}_+)\to C_\alpha(\mathbb{R}_+)} = \frac{1}{\beta - \alpha}. \tag{2.16}$$

(b) If $\alpha > \gamma$, *then*

$$||L||_{C_\alpha(\mathbb{R}_+)\to C_\gamma(\mathbb{R}_+)} = \left(\frac{(\alpha-\gamma)^{\alpha-\gamma}}{(\beta-\gamma)^{\beta-\gamma}}\right)^{\frac{1}{\beta-\alpha}}. \tag{2.17}$$

Proof. By Theorem 2, we have that formula (2.15) holds with $w_1(t) = e^{\alpha t}$, $w_2(t) = e^{\gamma t}$ and $\beta(t) = \beta$, that is,

$$||L||_{C_\alpha(\mathbb{R}_+)\to C_\gamma(\mathbb{R}_+)} = \sup_{t\in\mathbb{R}_+} e^{\gamma t} \int_0^t e^{\beta(s-t)} e^{-\alpha s} ds. \tag{2.18}$$

(a) Since $\alpha = \gamma$ from (2.18) we have

$$||L||_{C_\alpha(\mathbb{R}_+)\to C_\alpha(\mathbb{R}_+)} = \sup_{t\in\mathbb{R}_+} e^{(\alpha-\beta)t} \int_0^t e^{(\beta-\alpha)s} ds = \sup_{t\in\mathbb{R}_+} \frac{1 - e^{-(\beta-\alpha)t}}{\beta-\alpha} = \frac{1}{\beta-\alpha}.$$

(b) In this case from (2.18) we have

$$\|L\|_{C_\alpha(\mathbb{R}_+)\to C_\gamma(\mathbb{R}_+)} = \sup_{t\in\mathbb{R}_+} e^{(\gamma-\beta)t}\int_0^t e^{(\beta-\alpha)s}ds = \sup_{t\in\mathbb{R}_+} \frac{e^{(\gamma-\alpha)t}-e^{(\gamma-\beta)t}}{\beta-\alpha}. \tag{2.19}$$

Let $g(t) := e^{(\gamma-\alpha)t} - e^{(\gamma-\beta)t}$, then we have $g(0) = 0$, $\lim_{t\to+\infty} g(t) = 0$ (since $\gamma < \alpha < \beta$), and $g(t) = e^{(\gamma-\beta)t}(e^{(\beta-\alpha)t} - 1) \geq 0$, $t \in \mathbb{R}_+$. Since

$$g'(t) = (\gamma-\alpha)e^{(\gamma-\alpha)t} - (\gamma-\beta)e^{(\gamma-\beta)t}$$

we have that $g'(t) = 0$ if and only if

$$e^t = \left(\frac{\alpha-\gamma}{\beta-\gamma}\right)^{\frac{1}{\alpha-\beta}}.$$

Hence,

$$\sup_{t\in\mathbb{R}_+}(e^{(\gamma-\alpha)t} - e^{(\gamma-\beta)t}) = \left(\frac{\alpha-\gamma}{\beta-\gamma}\right)^{\frac{\gamma-\alpha}{\alpha-\beta}} - \left(\frac{\alpha-\gamma}{\beta-\gamma}\right)^{\frac{\gamma-\beta}{\alpha-\beta}} = \frac{\beta-\alpha}{\beta-\gamma}\left(\frac{\alpha-\gamma}{\beta-\gamma}\right)^{\frac{\alpha-\gamma}{\beta-\alpha}}$$

from which together with (2.19) and some calculation, formula (2.17) follows. □

2.3. An extension of Theorem 2 and its corollaries

Let

$$L(f)(t) = h(t)\int_0^{t_1}\cdots\int_0^{t_n} g(s)f(s)ds_1\cdots ds_n, \tag{2.20}$$

where $t = (t_1,\dots,t_n)$, $s = (s_1,\dots,s_n)$, $s_j, t_j \in \mathbb{R}_+$, $j = \overline{1,n}$, and $g, h \in C(\mathbb{R}^n_+)$.

The following theorem is an extension of Theorem 2.

Theorem 3. *Let $v, w, h, g \in W(\mathbb{R}^n_+)$ and operator L be given in* (2.20)*. Then the operator $L : C_w(\mathbb{R}^n_+) \to C_v(\mathbb{R}^n_+)$ is bounded if and only if*

$$\widetilde{M} := \sup_{t\in\mathbb{R}^n_+} v(t)h(t)\int_0^{t_1}\cdots\int_0^{t_n}\frac{g(s)}{w(s)}ds_1\cdots ds_n < +\infty, \tag{2.21}$$

and if it is bounded then the norm of the operator is equal to $\widetilde{M}$.

Proof. Assume that (2.21) holds. Then we have

$$\begin{aligned}\|L(f)\|_v &= \sup_{t\in\mathbb{R}^n_+} v(t)h(t)\left|\int_0^{t_1}\cdots\int_0^{t_n} g(s)f(s)ds_1\cdots ds_n\right| \\ &\leq \|f\|_w \sup_{t\in\mathbb{R}^n_+} v(t)h(t)\left|\int_0^{t_1}\cdots\int_0^{t_n}\frac{g(s)}{w(s)}ds_1\cdots ds_n\right|\end{aligned}$$

from which along with (2.21) the boundedness of the operator $L : C_w(\mathbb{R}^n_+) \to C_v(\mathbb{R}^n_+)$ follows. Moreover, we have

$$\|L\|_{C_w(\mathbb{R}^n_+)\to C_v(\mathbb{R}^n_+)} \leq \widetilde{M}. \tag{2.22}$$

If the operator $L : C_w(\mathbb{R}^n_+) \to C_v(\mathbb{R}^n_+)$ is bounded, then since the function $f_0(t) = \frac{1}{w(t)}$ belongs to $C_w(\mathbb{R}^n_+)$ and $\|f_0\|_w = 1$, we have

$$\|L\|_{C_w(\mathbb{R}^n_+)\to C_v(\mathbb{R}^n_+)} \geq \|L(f_0)\|_v = \sup_{t\in\mathbb{R}^n_+} v(t)h(t)\left|\int_0^{t_1}\cdots\int_0^{t_n}\frac{g(s)}{w(s)}ds_1\cdots ds_n\right|, \tag{2.23}$$

from which together with the boundedness of the operator $L : C_w(\mathbb{R}^n_+) \to C_v(\mathbb{R}^n_+)$ and positivity of functions g and w we obtain (2.21). From (2.22) and (2.23) we obtain

$$\|L\|_{C_w(\mathbb{R}^n_+)\to C_v(\mathbb{R}^n_+)} = \widetilde{M},$$

completing the proof. □

The following corollary is an extension of Corollary 1.

Corollary 2. *Let $v, w \in W(\mathbb{R}^n_+)$, $j = \overline{1,n}$, $\beta_j \in C(\mathbb{R}_+)$, $j = \overline{1,n}$, and*

$$L(f)(t) = \int_0^{t_1}\cdots\int_0^{t_n} e^{\sum_{j=1}^n \int_{t_j}^{s_j}\beta_j(\zeta_j)d\zeta_j} f(s)ds_1\cdots ds_n. \tag{2.24}$$

Then the operator $L : C_w(\mathbb{R}^n_+) \to C_v(\mathbb{R}^n_+)$ is bounded if and only if

$$\widehat{M} := \sup_{t\in\mathbb{R}^n_+} v(t)\int_0^{t_1}\cdots\int_0^{t_n} e^{\sum_{j=1}^n \int_{t_j}^{s_j}\beta_j(\zeta_j)d\zeta_j}\frac{ds_1\cdots ds_n}{w(s)} < +\infty. \tag{2.25}$$

Moreover, if the operator is bounded then

$$\|L\|_{C_w(\mathbb{R}^n_+)\to C_v(\mathbb{R}^n_+)} = \widehat{M}.$$

The following integral-type operator is a special case of operator (2.24)

$$\widetilde{L}(f)(t) = \int_0^{t_1}\cdots\int_0^{t_n} e^{\sum_{j=1}^n \beta_j(s_j-t_j)} f(s)ds_1\cdots ds_n. \tag{2.26}$$

Let $C_{\vec{\delta}}$, $\delta_j \geq 0$, $j = \overline{1,n}$, be the class of all $f \in C(\mathbb{R}^n_+)$ such that

$$\|f\|_{\vec{\delta}} = \sup_{t\in\mathbb{R}^n_+} e^{\sum_{j=1}^n \delta_j t_j}|f(t)| = \sup_{t\in\mathbb{R}^n_+} e^{\langle t,\vec{\delta}\rangle}|f(t)| < +\infty. \tag{2.27}$$

The following consequence of Corollary 2 is an ultimate extension of Corollary 1.

Corollary 3. *Let $w_j(t) = e^{\alpha_j t_j}$, $\beta_j(t) = \beta_j$, and $\beta_j > \alpha_j \geq \gamma_j$, $j = \overline{1,n}$. Then the operator $\widetilde{L} : C_{\vec{\alpha}}(\mathbb{R}^n_+) \to C_{\vec{\gamma}}(\mathbb{R}^n_+)$ is bounded and*

$$\|\widetilde{L}\|_{C_{\vec{\alpha}}(\mathbb{R}^n_+)\to C_{\vec{\gamma}}(\mathbb{R}^n_+)} = \prod_{\alpha_j\neq\gamma_j}\left(\frac{(\alpha_j-\gamma_j)^{\alpha_j-\gamma_j}}{(\beta_j-\gamma_j)^{\beta_j-\gamma_j}}\right)^{\frac{1}{\beta_j-\alpha_j}}\prod_{\alpha_j=\gamma_j}\left(\frac{1}{\beta_j-\alpha_j}\right). \tag{2.28}$$

Proof. By using (2.26), (2.27), Corollary 1 and Corollary 2, we have

$$\begin{aligned}\|\widetilde{L}\|_{C_{\vec{\alpha}}(\mathbb{R}^n_+)\to C_{\vec{\gamma}}(\mathbb{R}^n_+)} &= \sup_{t\in\mathbb{R}^n_+} e^{\sum_{j=1}^n \gamma_j t_j}\left|\int_0^{t_1}\cdots\int_0^{t_n} e^{\sum_{j=1}^n \beta_j(s_j-t_j)}\frac{ds_1\cdots ds_n}{\prod_{j=1}^n e^{\alpha_j s_j}}\right| \\ &= \prod_{j=1}^n \sup_{t_j\in\mathbb{R}_+} e^{(\gamma_j-\beta_j)t_j}\int_0^{t_j} e^{(\beta_j-\alpha_j)s_j}ds_j \\ &= \prod_{\alpha_j\neq\gamma_j}\left(\frac{(\alpha_j-\gamma_j)^{\alpha_j-\gamma_j}}{(\beta_j-\gamma_j)^{\beta_j-\gamma_j}}\right)^{\frac{1}{\beta_j-\alpha_j}}\prod_{\alpha_j=\gamma_j}\left(\frac{1}{\beta_j-\alpha_j}\right),\end{aligned}$$

as desired. □

Remark 2. The norm in formula (2.28) is achieved for the function

$$f_{\vec{\alpha}}(t) := e^{-\langle t,\vec{\alpha}\rangle}.$$

Indeed, we have $f_{\vec{\alpha}} \in C(\mathbb{R}^n_+)$,

$$\|f_{\vec{\alpha}}\|_{\vec{\alpha}} = 1, \tag{2.29}$$

and

$$\begin{aligned}\|\widetilde{L}(f_{\vec{\alpha}})\| &= \prod_{j=1}^n \sup_{t_j\in\mathbb{R}_+} e^{(\gamma_j-\beta_j)t_j}\int_0^{t_j} e^{(\beta_j-\alpha_j)s_j}ds_j \\ &= \prod_{\alpha_j\neq\gamma_j}\left(\frac{(\alpha_j-\gamma_j)^{\alpha_j-\gamma_j}}{(\beta_j-\gamma_j)^{\beta_j-\gamma_j}}\right)^{\frac{1}{\beta_j-\alpha_j}}\prod_{\alpha_j=\gamma_j}\left(\frac{1}{\beta_j-\alpha_j}\right),\end{aligned} \tag{2.30}$$

From (2.28)–(2.30) the claim follows.

2.4. An integral-type operator between weighted-type spaces

Let $g \in C([0,1)^n)$ and

$$T_g(f)(x) = \prod_{j=1}^n x_j \int_0^1\cdots\int_0^1 f(t_1x_1,\dots,t_nx_n)g(t_1x_1,\dots,t_nx_n)\prod_{j=1}^n dt_j, \tag{2.31}$$

where $x \in [0,1)^n$. The operator on the polydisk was studied in [32].

From now on, for the operator in (2.31) we use the notation

$$T_g(f)(x) = \prod_{j=1}^n x_j \int_0^1\cdots\int_0^1 f(t\cdot x)g(t\cdot x)\prod_{j=1}^n dt_j.$$

By $Q_{\vec{\gamma}}$ we denote the space of all $f \in C([0,1)^n)$ such that

$$\|f\|_{Q_{\vec{\gamma}}} = \sup_{x\in[0,1)^n}\prod_{j=1}^n (1-x_j)^{\gamma_j}|f(x)| < +\infty,$$

where $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$ is such that $\gamma_j > 0$, $j = \overline{1,n}$. The quantity $\|\cdot\|_{Q_{\vec{\gamma}}}$ is a norm on the space.

In the theorem which follows we estimate norm of the operator $T_g : Q_{\vec{\alpha}} \to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}$, under some conditions posed on the vectors $\vec{\alpha}$ and $\vec{\beta}$, and calculate it for a concrete function g.

Theorem 4. *Let $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^n_+$ be such that $\alpha_j + \beta_j > 1$, $j = \overline{1,n}$, and*

$$\|g\|_{Q_{\vec{\beta}}} < +\infty. \tag{2.32}$$

Then the operator $T_g : Q_{\vec{\alpha}} \to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}$ is bounded and

$$\|T_g\|_{Q_{\vec{\alpha}} \to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}} \le \frac{\|g\|_{Q_{\vec{\beta}}}}{\prod_{j=1}^n(\alpha_j + \beta_j - 1)}. \tag{2.33}$$

If additionally

$$g(x) = \prod_{j=1}^n \frac{1}{(1 - x_j)^{\beta_j}} \tag{2.34}$$

then

$$\|T_g\|_{Q_{\vec{\alpha}} \to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}} = \frac{1}{\prod_{j=1}^n(\alpha_j + \beta_j - 1)}. \tag{2.35}$$

Proof. Suppose that relation (2.32) holds. Let f be an arbitrary function in $Q_{\vec{\alpha}}$ and x be an arbitrary point in the cube $[0, 1)^n$. Then by using the definition of the spaces $Q_{\vec{\alpha}}$ and $Q_{\vec{\beta}}$, some known inequalities, as well as some calculations it follows that

$$\begin{aligned}
|T_g f(x)| \le& \prod_{j=1}^n x_j \int_0^1 \cdots \int_0^1 |f(t\cdot x) g(t\cdot x)| \prod_{j=1}^n dt_j \\
\le& \prod_{j=1}^n x_j \int_0^1 \cdots \int_0^1 \frac{|f(t\cdot x)| \prod_{j=1}^n (1 - t_j x_j)^{\alpha_j}}{\prod_{j=1}^n (1 - t_j x_j)^{\alpha_j+\beta_j}} |g(t\cdot x)| \prod_{j=1}^n (1 - t_j x_j)^{\beta_j} dt_j \\
\le& \|f\|_{Q_{\vec{\alpha}}} \|g\|_{Q_{\vec{\beta}}} \prod_{j=1}^n x_j \int_0^1 \cdots \int_0^1 \frac{dt_1 \cdots dt_n}{\prod_{j=1}^n (1 - t_j x_j)^{\alpha_j+\beta_j}} \\
=& \|f\|_{Q_{\vec{\alpha}}} \|g\|_{Q_{\vec{\beta}}} \prod_{j=1}^n \int_0^1 \frac{x_j dt_j}{(1 - t_j x_j)^{\alpha_j+\beta_j}} \\
=& \frac{\|f\|_{Q_{\vec{\alpha}}} \|g\|_{Q_{\vec{\beta}}}}{\prod_{j=1}^n(\alpha_j + \beta_j - 1)} \prod_{j=1}^n \frac{1 - (1 - x_j)^{\alpha_j+\beta_j-1}}{(1 - x_j)^{\alpha_j+\beta_j-1}},
\end{aligned}$$

from which it follows that

$$\prod_{j=1}^n (1 - x_j)^{\alpha_j+\beta_j-1} |T_g f(x)| \le \|f\|_{Q_{\vec{\alpha}}} \|g\|_{Q_{\vec{\beta}}} \prod_{j=1}^n \frac{1 - (1 - x_j)^{\alpha_j+\beta_j-1}}{\alpha_j + \beta_j - 1}, \tag{2.36}$$

for every $x \in [0,1)^n$ and $f \in Q_{\vec{\alpha}}$.

By taking the supremum in (2.36) over the set $[0,1)^n$, it follows that the following inequality holds

$$\|T_g(f)\|_{Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}} \leq \frac{\|g\|_{Q_{\vec{\beta}}}}{\prod_{j=1}^n(\alpha_j+\beta_j-1)}\|f\|_{Q_{\vec{\alpha}}}, \tag{2.37}$$

for every $f \in Q_{\vec{\alpha}}$.

By taking the supremum in (2.37) over the unit ball $B_{Q_{\vec{\beta}}}$ the boundedness of the operator $T_g : Q_{\vec{\alpha}} \to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}$ follows.

Moreover, from inequality (2.37) we obtain the following estimate for the norm of the operator

$$\|T_g\|_{Q_{\vec{\alpha}}\to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}} \leq \frac{\|g\|_{Q_{\vec{\beta}}}}{\prod_{j=1}^n(\alpha_j+\beta_j-1)}. \tag{2.38}$$

Now, assume that the operator $T_g : Q_{\vec{\alpha}} \to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}$ is bounded and that function g is defined as in (2.34).

Let

$$f_0(x) = \frac{1}{\prod_{j=1}^n(1-x_j)^{\alpha_j}}, \tag{2.39}$$

then

$$\|f_0\|_{Q_{\vec{\alpha}}} = 1. \tag{2.40}$$

By using (2.34), (2.39) and (2.40), as well as some standard calculations it follows that

$$\begin{aligned}
&\|T_g\|_{Q_{\vec{\alpha}}\to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}} \geq \|T_g(f_0)\|_{Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}}\\
&= \sup_{x\in[0,1)^n} \prod_{j=1}^n x_j(1-x_j)^{\alpha_j+\beta_j-1}\left|\int_0^1 \cdots \int_0^1 \frac{g(t\cdot x)}{\prod_{j=1}^n(1-t_jx_j)^{\alpha_j}} \prod_{j=1}^n dt_j\right|\\
&= \sup_{x\in[0,1)^n} \prod_{j=1}^n (1-x_j)^{\alpha_j+\beta_j-1} \int_0^1 \frac{x_j dt_j}{(1-t_jx_j)^{\alpha_j+\beta_j}}\\
&= \sup_{x\in[0,1)^n} \prod_{j=1}^n \frac{1-(1-x_j)^{\alpha_j+\beta_j-1}}{\alpha_j+\beta_j-1}\\
&= \prod_{j=1}^n \frac{1}{\alpha_j+\beta_j-1}.
\end{aligned} \tag{2.41}$$

From (2.38), (2.41), and since in this case $\|g\|_{Q_{\vec{\beta}}} = 1$, we have

$$\|T_g\|_{Q_{\vec{\alpha}}\to Q_{\vec{\alpha}+\vec{\beta}-\vec{1}}} = \frac{1}{\prod_{j=1}^n(\alpha_j+\beta_j-1)},$$

finishing the proof of the theorem. □

Generally speaking operator (2.31) can be considered on functions defined on any set of the form

$$\prod_{j=1}^{n}[0,c_j) \quad \text{or} \quad \prod_{j=1}^{n}[0,c_j], \tag{2.42}$$

where $c_j \in [0,+\infty]$, $j=\overline{1,n}$, and where we exclude the case $\prod_{j=1}^{n}[0,+\infty]$.

Our next result considers the boundedness of operator (2.31) between such spaces.

Theorem 5. *Let $u, v, w \in W(I)$, $g \in C_v(I)$, where the set I has one of the forms in* (2.42). *If*

$$\sup_{x\in I} u(x)\int_0^{x_1}\cdots\int_0^{x_n}\frac{ds_1\cdots ds_n}{w(s)v(s)} < +\infty, \tag{2.43}$$

then the operator $T_g : C_w(I) \to C_u(I)$ is bounded.

If additionally

$$g(x) = \frac{1}{v(x)} \tag{2.44}$$

then

$$\|T_{1/v}\|_{C_w(I)\to C_u(I)} = \sup_{x\in I} u(x)\int_0^{x_1}\cdots\int_0^{x_n}\frac{ds_1\cdots ds_n}{w(s)v(s)}. \tag{2.45}$$

Proof. Using the definitions of the spaces $C_w(I)$ and $C_u(I)$, and the change of variables $s_j = x_j t_j$, $j=\overline{1,n}$, we have

$$\begin{aligned}
|T_g f(x)| &= \left|\prod_{j=1}^{n} x_j \int_0^1\cdots\int_0^1 f(t\cdot x)g(t\cdot x)\prod_{j=1}^{n} dt_j\right| \\
&= \left|\prod_{j=1}^{n} x_j \int_0^1\cdots\int_0^1 \frac{w(t\cdot x)f(t\cdot x)v(t\cdot x)g(t\cdot x)}{w(t\cdot x)v(t\cdot x)}\prod_{j=1}^{n} dt_j\right| \\
&\le \prod_{j=1}^{n} x_j \int_0^1\cdots\int_0^1 \frac{\|f\|_w\|g\|_v}{w(t\cdot x)v(t\cdot x)}\prod_{j=1}^{n} dt_j \\
&= \int_0^{x_1}\cdots\int_0^{x_n}\frac{\|f\|_w\|g\|_v}{w(s)v(s)}\prod_{j=1}^{n} ds_j,
\end{aligned} \tag{2.46}$$

for every $x \in I$ and $f \in C_w(I)$.

Multiplying (2.46) by $u(x)$, then taking the supremum over the set I we have

$$\sup_{x\in I} u(x)|T_g f(x)| \le \|f\|_w\|g\|_v \sup_{x\in I} u(x)\int_0^{x_1}\cdots\int_0^{x_n}\frac{ds_1\cdots ds_n}{w(s)v(s)}$$

from which it follows that

$$\|T_g\|_{C_w(I)\to C_u(I)} \le \|g\|_v \sup_{x\in I} u(x)\int_0^{x_1}\cdots\int_0^{x_n}\frac{ds_1\cdots ds_n}{w(s)v(s)}. \tag{2.47}$$

Using the assumption $g \in C_v(I)$, (2.43) and (2.47) the boundedness of $T_g : C_w(I) \to C_u(I)$ follows.
If (2.44) holds, then

$$\|g\|_v = 1. \tag{2.48}$$

Now, note that for $\widetilde{f_0}(x) = \frac{1}{w(x)}$ we have

$$\|\widetilde{f_0}\|_w = 1. \tag{2.49}$$

Further, we have

$$\begin{aligned}
\|T_{1/v}(\widetilde{f_0})\|_u &= \sup_{x\in I} u(x)|T_{1/v}(\widetilde{f_0})(x)| \\
&= \sup_{x\in I} u(x)\left|\prod_{j=1}^{n} x_j \int_0^1 \cdots \int_0^1 \widetilde{f_0}(t\cdot x)g(t\cdot x)\prod_{j=1}^{n} dt_j\right| \\
&= \sup_{x\in I} u(x)\left|\prod_{j=1}^{n} x_j \int_0^1 \cdots \int_0^1 \frac{dt_1\cdots dt_n}{w(t\cdot x)v(t\cdot x)}\right| \\
&= \sup_{x\in I} u(x)\int_0^{x_1} \cdots \int_0^{x_n} \frac{ds_1\dots ds_n}{w(s)v(s)}.
\end{aligned} \tag{2.50}$$

From (2.49) and (2.50) we obtain

$$\sup_{x\in I} u(x)\int_0^{x_1} \cdots \int_0^{x_n} \frac{ds_1\dots ds_n}{w(s)v(s)} \le \|T_{1/v}\|_{C_w(I)\to C_u(I)}. \tag{2.51}$$

Combining (2.47), (2.48) and (2.50) we get (2.45). □

Remark 3. Note that in the case

$$w(x) = e^{\langle x,\vec{\alpha}\rangle} \quad \text{and} \quad v(x) = e^{\langle x,\vec{\beta}\rangle},$$

we have

$$\begin{aligned}
|T_g f(x)| &= \left|\prod_{j=1}^{n} x_j \int_0^1 \cdots \int_0^1 f(t\cdot x)g(t\cdot x)\prod_{j=1}^{n} dt_j\right| \\
&= \left|\prod_{j=1}^{n} x_j \int_0^1 \cdots \int_0^1 \frac{e^{\langle t\cdot x,\vec{\alpha}\rangle} f(t\cdot x)e^{\langle t\cdot x,\vec{\beta}\rangle} g(t\cdot x)}{e^{\langle t\cdot x,\vec{\alpha}\rangle} e^{\langle t\cdot x,\vec{\beta}\rangle}}\prod_{j=1}^{n} dt_j\right| \\
&\le \prod_{j=1}^{n} x_j \int_0^1 \cdots \int_0^1 \frac{\|f\|_{\vec{\alpha}}\|g\|_{\vec{\beta}}}{e^{\langle t\cdot x,\vec{\alpha}+\vec{\beta}\rangle}}\prod_{j=1}^{n} dt_j \\
&= \|f\|_{\vec{\alpha}}\|g\|_{\vec{\beta}}\prod_{j=1}^{n}\int_0^{x_j} e^{-(\alpha_j+\beta_j)s_j}ds_j \\
&= \|f\|_{\vec{\alpha}}\|g\|_{\vec{\beta}}\prod_{j=1}^{n}\frac{1-e^{-(\alpha_j+\beta_j)x_j}}{\alpha_j+\beta_j},
\end{aligned} \tag{2.52}$$

from which by taking the supremum in (2.52) over the set $\mathbb{R}^n_+$ it follows that

$$\|T_g f\|_\infty \le \|f\|_{\vec{\alpha}}\|g\|_{\vec{\beta}}.$$

Here, as usual

$$\|h\|_\infty = \sup_{x\in\mathbb{R}^n_+} |h(x)|,$$

the standard supremum norm.

2.5. *Another integral-type operator between weighted-type spaces*

Let $g \in C([0,1)^n)$ and

$$\widehat{T}_g(f)(x) = \int_0^1 \cdots \int_0^1 f(t_1x_1,\dots,t_nx_n)g(t_1,\dots,t_n)\prod_{j=1}^n dt_j, \tag{2.53}$$

where $x \in \mathbb{R}^n$. From now on, for the operator in (2.53) we use the notation

$$\widehat{T}_g(f)(x) = \int_0^1 \cdots \int_0^1 f(t\cdot x)g(t)\prod_{j=1}^n dt_j.$$

Let $u \in W(\mathbb{R}^n)$ and

$$\|f\|_u = \sup_{x\in\mathbb{R}^n} u(x)|f(x)|.$$

The following theorem holds.

Theorem 6. *Let $g \in C([0,1)^n)$, $g(x) \ge 0$, $x \in \mathbb{R}^n$, $u, v \in W(\mathbb{R}^n)$, such that*

$$u(t\cdot x) = \prod_{j=1}^n t_j^{\alpha_j} u(x), \tag{2.54}$$

for some $\alpha_j \in \mathbb{R}_+$, $j = \overline{1,n}$.

Then the operator $\widehat{T}_g : C_u(\mathbb{R}^n) \to C_v(\mathbb{R}^n)$ is bounded if and only if

$$\sup_{x\in\mathbb{R}^n\setminus\{\vec{0}\}} \frac{v(x)}{u(x)} \int_0^1 \cdots \int_0^1 \frac{g(t)}{\prod_{j=1}^n t_j^{\alpha_j}} \prod_{j=1}^n dt_j < \infty. \tag{2.55}$$

Moreover, if the operator $\widehat{T}_g : C_u(\mathbb{R}^n) \to C_v(\mathbb{R}^n)$ is bounded then

$$\|\widehat{T}_g\|_{C_u(\mathbb{R}^n)\to C_v(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n\setminus\{\vec{0}\}} \frac{v(x)}{u(x)} \int_0^1 \cdots \int_0^1 \frac{g(t)}{\prod_{j=1}^n t_j^{\alpha_j}} \prod_{j=1}^n dt_j. \tag{2.56}$$

Proof. Assume that (2.55) holds. Let $f \in C_u(\mathbb{R}^n)$. Then by using the definition of the norm in $C_u(\mathbb{R}^n)$ and (2.54) we have

$$\begin{aligned} |\widehat{T}_g f(x)| \leq& \int_0^1 \cdots \int_0^1 |f(t\cdot x)g(t)| \prod_{j=1}^n dt_j \\ \leq& \|f\|_u \int_0^1 \cdots \int_0^1 \frac{g(t)}{u(t\cdot x)} \prod_{j=1}^n dt_j \\ =& \frac{\|f\|_u}{u(x)} \int_0^1 \cdots \int_0^1 \frac{g(t)}{\prod_{j=1}^n t_j^{\alpha_j}} \prod_{j=1}^n dt_j, \end{aligned}$$

from which it follows that

$$v(x)|\widehat{T}_g f(x)| \leq \|f\|_u \frac{v(x)}{u(x)} \int_0^1 \cdots \int_0^1 \frac{g(t)}{\prod_{j=1}^n t_j^{\alpha_j}} \prod_{j=1}^n dt_j. \tag{2.57}$$

By taking the supremum in (2.57) over the set $\mathbb{R}^n \setminus \{\vec{0}\}$, it follows that the following inequality holds

$$\|\widehat{T}_g(f)\|_v \leq \|f\|_u \sup_{x\in\mathbb{R}^n\setminus\{\vec{0}\}} \frac{v(x)}{u(x)} \int_0^1 \cdots \int_0^1 \frac{g(t)}{\prod_{j=1}^n t_j^{\alpha_j}} \prod_{j=1}^n dt_j. \tag{2.58}$$

By taking the supremum in (2.58) over the unit ball $B_{C_u(\mathbb{R}^n)}$ the boundedness of the operator $\widehat{T}_g : C_u(\mathbb{R}^n) \to C_v(\mathbb{R}^n)$ follows. Moreover, we have

$$\|\widehat{T}_g\|_{C_u(\mathbb{R}^n)\to C_v(\mathbb{R}^n)} \leq \sup_{x\in\mathbb{R}^n\setminus\{\vec{0}\}} \frac{v(x)}{u(x)} \int_0^1 \cdots \int_0^1 \frac{g(t)}{\prod_{j=1}^n t_j^{\alpha_j}} \prod_{j=1}^n dt_j. \tag{2.59}$$

Now assume that the operator $\widehat{T}_g : C_u(\mathbb{R}^n) \to C_v(\mathbb{R}^n)$ is bounded. Let

$$\widehat{f_0}(x) = \frac{1}{u(x)}. \tag{2.60}$$

Then

$$\|\widehat{f_0}\|_u = 1. \tag{2.61}$$

By using (2.54), (2.60) and (2.61), as well as some standard calculations it follows that

$$\begin{aligned} \|\widehat{T}_g\|_{C_u(\mathbb{R}^n)\to C_v(\mathbb{R}^n)} \geq& \|\widehat{T}_g(\widehat{f_0})\|_v \\ =& \sup_{x\in\mathbb{R}^n\setminus\{\vec{0}\}} v(x)\left|\int_0^1 \cdots \int_0^1 \frac{g(t)}{u(x\cdot t)} \prod_{j=1}^n dt_j\right| \\ =& \sup_{x\in\mathbb{R}^n\setminus\{\vec{0}\}} \frac{v(x)}{u(x)} \int_0^1 \cdots \int_0^1 \frac{g(t)}{\prod_{j=1}^n t_j^{\alpha_j}} \prod_{j=1}^n dt_j, \end{aligned} \tag{2.62}$$

from which (2.55) follows.

If the operator $\widehat{T}_g : C_u(\mathbb{R}^n) \to C_v(\mathbb{R}^n)$ is bounded then from (2.59) and (2.62) we get (2.56), finishing the proof of the theorem. □

The following theorem is proved similar to Theorem 6, so we omit the proof.

Theorem 7. *Let* $g \in C[0,1)$*,* $g(t) \geq 0$*,* $t \in \mathbb{R}$*,* $u, v \in W(\mathbb{R}^n)$*, such that*

$$u(tx) = t^{\alpha} u(x), \tag{2.63}$$

for some $\alpha > 0$ *and every* $t \in [0,1)$ *and* $x \in \mathbb{R}^n$, *and*

$$\widehat{L_g}(f)(x) = \int_0^1 f(tx) g(t) dt. \tag{2.64}$$

Then the operator $\widehat{L_g} : C_u(\mathbb{R}^n) \to C_v(\mathbb{R}^n)$ *is bounded if and only if*

$$\sup_{x \in \mathbb{R}^n \setminus \{\vec{0}\}} \frac{v(x)}{u(x)} \int_0^1 \frac{g(t)}{t^{\alpha}} < +\infty. \tag{2.65}$$

Moreover, if the operator $\widehat{L_g} : C_u(\mathbb{R}^n) \to C_v(\mathbb{R}^n)$ *is bounded then*

$$\|\widehat{L_g}\|_{C_u \to C_v} = \sup_{x \in \mathbb{R}^n \setminus \{\vec{0}\}} \frac{v(x)}{u(x)} \int_0^1 \frac{g(t)}{t^{\alpha}}.$$

Example 1. Let

$$u(x) = \|x\|_p \quad \text{and} \quad v(x) = \|x\|_q,$$

where $1 \leq \min\{p, q\} \leq \max\{p, q\} < +\infty$ and for $r \geq 1$

$$\|x\|_r = \left(\sum_{j=1}^n |x_j|^r \right)^{1/r}.$$

Since all the norms on a finite-dimensional linear space are equivalent (here the linear space is $\mathbb{R}^n$), we have that there are positive constants C_1 and C_2 such that

$$C_1 \|x\|_q \leq \|x\|_p \leq C_2 \|x\|_q.$$

Hence, we have

$$\sup_{x \in \mathbb{R}^n \setminus \{\vec{0}\}} \frac{v(x)}{u(x)} \leq \frac{1}{C_1} < +\infty.$$

Note also that in this case we have

$$u(tx) = t u(x).$$

Hence, to guaranty the boundedness of the operator $\widehat{L_g} : C_u(\mathbb{R}^n) \to C_v(\mathbb{R}^n)$ in this case, the corresponding condition in (2.65) holds if the function g satisfies the condition

$$\int_0^1 \frac{g(t)}{t} dt < +\infty.$$

2.6. On a Hardy integral operator

Let $\widehat{\mathcal{L}}^p_w(\mathbb{R}^n) = \widehat{\mathcal{L}}^p_w$ be a linear subspace of $\mathcal{L}^p_w$ containing constant functions, and such that the integral means

$$M_p^p(f,r) = \int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta)$$

are non-increasing for each $f \in \widehat{\mathcal{L}}^p_w$.

Example 2. An example of such a space consists of all harmonic functions on $\mathbb{R}^n$ [18, 31], for which the integral means are nondecreasing functions (see, e.g., [17]; for one-dimensional case see [26]).

Theorem 8. *Let μ be a nonnegative Borel measure on the interval $[0,1]$, $w \in W(\mathbb{R}^n)$ be a radial function such that*

$$\int_{\mathbb{R}^n} w(x)dV(x) = 1, \tag{2.66}$$

and

$$L_\mu(f)(x) = \int_0^1 f(tx)d\mu(t). \tag{2.67}$$

Then the operator $L_\mu : \widehat{\mathcal{L}}^p_w(\mathbb{R}^n) \to \widehat{\mathcal{L}}^p_w(\mathbb{R}^n)$ is bounded if and only if

$$\int_0^1 d\mu(t) < +\infty. \tag{2.68}$$

Moreover, if the operator $L_\mu : \widehat{\mathcal{L}}^p_w(\mathbb{R}^n) \to \widehat{\mathcal{L}}^p_w(\mathbb{R}^n)$ is bounded then

$$\|L_\mu\|_{\widehat{\mathcal{L}}^p_w(\mathbb{R}^n)\to\widehat{\mathcal{L}}^p_w(\mathbb{R}^n)} = \int_0^1 d\mu(t). \tag{2.69}$$

Proof. First assume that (2.68) holds. By using Minkowski's integral inequality (see, e.g., [16, 30]), polar coordinates (see, e.g., [18] or [26, p.150]), the assumption that w is radial, i.e., $w(r\zeta) = w(r)$, $x = r\zeta \in \mathbb{R}^n$, and the monotonicity of the integral means, we have

$$\begin{aligned}
\|L_\mu(f)\|_{\widehat{\mathcal{L}}^p_w} &= \left(\int_{\mathbb{R}^n} \left|\int_0^1 f(tx)d\mu(t)\right|^p w(x)dV(x)\right)^{1/p} \\
&\le \int_0^1 \left(\int_{\mathbb{R}^n} |f(tx)|^p w(x)dV(x)\right)^{1/p} d\mu(t) \\
&= \int_0^1 \left(\int_0^{+\infty}\int_{\mathbb{S}} |f(tr\zeta)|^p d\sigma(\zeta) w(r) r^{n-1} dr\right)^{1/p} d\mu(t) \\
&\le \int_0^1 \left(\int_0^{+\infty}\int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta) w(r) r^{n-1} dr\right)^{1/p} d\mu(t) \\
&= \|f\|_{\widehat{\mathcal{L}}^p_w} \int_0^1 d\mu(t),
\end{aligned}$$

from which it follows that

$$\|L_\mu\|_{\widehat{\mathcal{L}}^p_w \to \widehat{\mathcal{L}}^p_w} \le \int_0^1 d\mu(t). \tag{2.70}$$

Now, assume that the operator $L_\mu : \widehat{\mathcal{L}}^p_w(\mathbb{R}^n) \to \widehat{\mathcal{L}}^p_w(\mathbb{R}^n)$ is bounded. Note that from (2.66) we have

$$\|1\|_{\widehat{\mathcal{L}}^p_w} = 1.$$

On the other hand, by the definition of the space $\widehat{\mathcal{L}}^p_w$, we have $\widehat{f_0}(x) \equiv 1 \in \widehat{\mathcal{L}}^p_w$. From this and since

$$\|L_\mu(\widehat{f_0})\|_{\widehat{\mathcal{L}}^p_w} = \int_0^1 d\mu(t)$$

we get

$$\int_0^1 d\mu(t) \le \|L_\mu\|_{\widehat{\mathcal{L}}^p_w \to \widehat{\mathcal{L}}^p_w}. \tag{2.71}$$

If the operator $L_\mu : \widehat{\mathcal{L}}^p_w(\mathbb{R}^n) \to \widehat{\mathcal{L}}^p_w(\mathbb{R}^n)$ is bounded, then from (2.70) and (2.71) we get (2.69). □

Remark 4. The operator in (2.67) is a Hardy integral-type operator [15].

3. Conclusions

Here we calculate the norms of several concrete operators between weighted-type spaces of continuous functions on several domains, as well as the norm of an integral-type operator on some subspaces of the weighted Lebesgue spaces. Several methods, ideas and tricks, which could be used in some other settings, are presented.

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Conflict of interest

The author declares that he has no competing interest.

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