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Research article

Bipolar complex fuzzy semigroups

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Abstract: The notion of the bipolar complex fuzzy set (BCFS) is a fundamental notion to be considered for tackling tricky and intricate information. Here, in this study, we want to expand the notion of BCFS by giving a general algebraic structure for tackling bipolar complex fuzzy (BCF) data by fusing the conception of BCFS and semigroup. Firstly, we investigate the bipolar complex fuzzy (BCF) sub-semigroups, BCF left ideal (BCFLI), BCF right ideal (BCFRI), BCF two-sided ideal (BCFTSI) over semigroups. We also introduce bipolar complex characteristic function, positive (ω,η)-cut, negative (ϱ,σ)-cut, positive and $\big((\omega,\eta),(\varrho,\sigma)\big)$-cut. Further, we study the algebraic structure of semigroups by employing the most significant concept of BCF set theory. Also, we investigate numerous classes of semigroups such as right regular, left regular, intra-regular, and semi-simple, by the features of the bipolar complex fuzzy ideals. After that, these classes are interpreted concerning BCF left ideals, BCF right ideals, and BCF two-sided ideals. Thus, in this analysis, we portray that for a semigroup Ş and for each BCFLI $\mathrm{M}_1=\left(\lambda_{P-\mathrm{M}_1},\lambda_{N-\mathrm{M}_1}\right)=\left(\lambda_{RP-\mathrm{M}_1}+\iota\,\lambda_{IP-\mathrm{M}_1},\lambda_{RN-\mathrm{M}_1}+\iota\,\lambda_{IN-\mathrm{M}_1}\right)$ and BCFRI $\mathrm{M}_2=\left(\lambda_{P-\mathrm{M}_2},\lambda_{N-\mathrm{M}_2}\right)=\left(\lambda_{RP-\mathrm{M}_2}+\iota\,\lambda_{IP-\mathrm{M}_2},\lambda_{RN-\mathrm{M}_2}+\iota\,\lambda_{IN-\mathrm{M}_2}\right)$ over Ş, $\mathrm{M}_1\cap\mathrm{M}_2=\mathrm{M}_1\odot\mathrm{M}_2$ if and only if Ş is a regular semigroup. At last, we introduce regular, intra-regular semigroups and show that $\mathrm{M}_1\cap\mathrm{M}_2\preccurlyeq\mathrm{M}_1\odot\mathrm{M}_2$ for each BCFLI $\mathrm{M}_1=\left(\lambda_{P-\mathrm{M}_1},\lambda_{N-\mathrm{M}_1}\right)=\left(\lambda_{RP-\mathrm{M}_1}+\iota\,\lambda_{IP-\mathrm{M}_1},\lambda_{RN-\mathrm{M}_1}+\iota\,\lambda_{IN-\mathrm{M}_1}\right)$ and for each BCFRI $\mathrm{M}_2=\left(\lambda_{P-\mathrm{M}_2},\lambda_{N-\mathrm{M}_2}\right)=\left(\lambda_{RP-\mathrm{M}_2}+\iota\,\lambda_{IP-\mathrm{M}_2},\lambda_{RN-\mathrm{M}_2}+\iota\,\lambda_{IN-\mathrm{M}_2}\right)$ over Ş if and only if a semigroup Ş is regular and intra-regular.

Keywords: bipolar complex fuzzy set; bipolar complex fuzzy sub-semigroups; bipolar complex fuzzy left (right) ideals
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1. Introduction

With the advancement of the world, the ambiguity and uncertainty in the life of human beings were increasing and an expert or decision-analyst couldn't handle such sort of ambiguities and uncertainties by employing the theory of crisp set. Thus, Zadeh [1] diagnosed the fuzzy set theory (FST) and its elementary results in 1965 to cope with such sort of ambiguities and uncertainties by changing the two-point set $\{0, 1\}$ to the unit interval $[0, 1]$. The FST holds a supportive grade which contains in $[0, 1]$. The FST attracted numerous scholars from almost every field of science and they did research and utilized the FST in their respective fields. Rosenfeld [2] firstly employed the FST in the environment of groups to structured fuzzy groups. Kuroki [3–6] interpreted fuzzy semigroups (FSG), bi-ideal in semigroups, and fuzzy ideal. The fuzzy ideals and bi-ideals in FSGs were also presented by Dib and Galhum [7]. The fuzzy identities with application to FSGs were established by Budimirovic et al. [8]. The generalized fuzzy interior ideals and fuzzy regular sub-semigroup were given in [9,10] respectively. The fuzzy bi-ideals, fuzzy radicals, and fuzzy prime ideals of ordered semigroups are presented in [11,12]. Kehayopulu and Tsingelis [13] and Xie and Tang [14] presented the concept of regular and intra-regular ordered semigroups. Khan et al. [15] explored certain characterizations of intra-regular semigroups. Jaradat and Al-Husban [16] investigated multi-fuzzy group spaces.

The conception of bipolar fuzzy (BF) set is one of the generalizations of FST, as FST is unable to cover the negative opinion or negative supportive grade of human beings. Thus, Zhang [17] initiated the BF set theory (BFST) to cover both positive and negative opinions of human beings by enlarging the range of FST ($[0, 1]$) to the BFST $([0, 1], [-1, 0])$. The BFST holds a positive supportive grade (PSG) which contains in $[0, 1]$ and negative supportive grade (NSG) which contains in $[-1, 0]$. Kim et al. [18] initiated BCFST in semigroups. Kang and Kang [19] explored BFST applied to sub-semigroups with the operations of semigroups. BFST in Γ-semigroups was interpreted by Majumder [20]. The certain properties of BF sub-semigroups of a semigroup are presented in [21,22]. Chinnadurai and Arulmozhi [23] described the characterization of BF ideals in ordered Γ-semigroups. BF abundant semigroups by Li et al. [24]. Ban et al. [25] initiated BF ideals with operation in semigroups. Gaketem and Khamrot [26] presented BF weakly interior ideals. The generalized BF interior ideals in ordered semigroups were interpreted by Ibrar et al. [27]. The BF graph was discussed in [28–30]. Mahmood [31] diagnosed a new approach to the bipolar soft set. Akram et al. [32] presented a characterization of BF soft Γ-semigroups. Deli and Karaaslan [33] defined bipolar FPSS theory. Various researchers expand the conception of BFS such as Deli et al. [34] investigated bipolar neutrosophic sets (BNS), Deli and Subas [35] introduced bipolar neutrosophic refined sets, Ali et al. [46] investigated bipolar neutrosophic soft sets.

The FST and BFST merely cope with the ambiguities and uncertainties which are in one dimension but unable to cope with 2^{nd} dimension which is the phase term. Thus, Ramot et al. [37] diagnosed the theory of complex FS (CFS) by transforming the range of FST ($[0, 1]$) to the unit circle in a complex plane. In the CFS theory (CFST) Ramot et al. [37] added the phase term in the supportive grade. After that, Tamir et al. [38] diagnosed the CFST in the cartesian structure by transforming the range from the unit circle to the unit square of the complex plane. Al-Husban and Salleh [39] presented complex fuzzy (CF) groups that rely on CF space. Alolaiyan et al. [40] the conception of CF subgroups. The above-discussed theories have their drawbacks, for instance, FST can't cover the negative opinion, BFST can't cover the 2^{nd} dimension and CFST can't cover the negative opinion. Thus to cover all these drawbacks Mahmood and Ur Rehman [41] introduced the theory of the BCF set. BCF set covers the PSG which contains in $[0,1]+\iota\,[0,1]$ (real part contains in $[0,1]$ and unreal part contains in $[0,1]$) and NSG which contains in $[-1,0]+\iota\,[-1,0]$ (real part contains in $[-1,0]$ and unreal part contains in $[-1,0]$). The theory of the BCF set has a great mathematical structure that generalizes the FST, BFST, and CFST, for example, a CEO of a company wants to install a new air conditioning system in a company's head office. For this he has to observe four aspects i.e., positive effect on the office's environment, the positive response of the employees, the extra burden on the company expenditures, and the negative response of the employees. No prevailing theories except the BCF set can model such kinds of information. A lot of researchers worked on the theory of BCF set for instance Al-Husban et al. [42] investigated the properties for BCFS. Mahmood et al. [43] diagnosed Hamacher aggregation operators (AOs), Mahmood and Ur Rehman [44] explored Dombi AOs, Mahmood et al. [45] AOs. The BCF soft set was diagnosed by Mahmood et al. [46].

The conception of a semigroup is a prosperous area of modern algebra. It is obvious from the name that semigroup is the modification of the conception of the group, since a semigroup not requires to contain elements that have inverses. In the earlier stages, a lot of researchers work on semigroup from the perspective of ring and group. The conception of semigroup may be assumed as the effective offspring of ring theory because the ring theory provides some insight into how to create the notion of ideals in the semigroup. Moreover, the conception of a semigroup is an influential approach and has been utilized by numerous scholars and employed in various areas such as mathematical biology, control theory, nonlinear dynamical systems, stochastic processes, etc. Because of the importance of semigroup, various scholars modified this concept to introduce novel notions such as fuzzy semigroup [3–6], intuitionistic fuzzy semigroup [47], bipolar fuzzy semigroup [19], etc. The concept of fuzzy semigroup has various application such as fuzzy languages, theory fuzzy coding, etc., that shows the importance of fuzzy algebraic structure and their modifications. In recent years, numerous authors generalized the conception of fuzzy algebraic structures and employed genuine-life dilemmas in various areas of science. What would happen if someone working on automata theory and trying to solve a problem and for that he/she needs a BCF algebraic structure (i.e., BCF semigroup) but until now there is no such structure in the literature. Therefore inspired by this here in this analysis we employ the theory of the BCF set to the algebraic structures of semigroups:

- To describe BCF sub-semigroup, BCFLI, BCFRI, and BCFTSI.
- To introduce numerous classes of semigroups for instance, right regular, left regular, intra-regular, and semi-simple, by the features of the bipolar complex fuzzy ideals. In addition, these classes are interpreted in relation to BCFLIs, BCFRIs, and BCFTSIs.
- To show that, for a semigroup $Ş$ and for each BCFLI $\mathrm{M}_1=\left(\lambda_{P-\mathrm{M}_1},\lambda_{N-\mathrm{M}_1}\right)=\left(\lambda_{RP-\mathrm{M}_1}+\iota\,\lambda_{IP-\mathrm{M}_1},\lambda_{RN-\mathrm{M}_1}+\iota\,\lambda_{IN-\mathrm{M}_1}\right)$ and BCFRI $\mathrm{M}_2=\left(\lambda_{P-\mathrm{M}_2},\lambda_{N-\mathrm{M}_2}\right)=\left(\lambda_{RP-\mathrm{M}_2}+\iota\,\lambda_{IP-\mathrm{M}_2},\lambda_{RN-\mathrm{M}_2}+\iota\,\lambda_{IN-\mathrm{M}_2}\right)$ over $Ş$, $\mathrm{M}_1\cap\mathrm{M}_2=\mathrm{M}_1\odot\mathrm{M}_2$ if and only if $Ş$ is a regular semigroup.
- To interpret regular, intra-regular semigroups and show that $\mathrm{M}_1\cap\mathrm{M}_2\preccurlyeq\mathrm{M}_1\odot\mathrm{M}_2$ for each BCFLI $\mathrm{M}_1=\left(\lambda_{P-\mathrm{M}_1},\lambda_{N-\mathrm{M}_1}\right)=\left(\lambda_{RP-\mathrm{M}_1}+\iota\,\lambda_{IP-\mathrm{M}_1},\lambda_{RN-\mathrm{M}_1}+\iota\,\lambda_{IN-\mathrm{M}_1}\right)$ and for each BCFRI $\mathrm{M}_2=\left(\lambda_{P-\mathrm{M}_2},\lambda_{N-\mathrm{M}_2}\right)=\left(\lambda_{RP-\mathrm{M}_2}+\iota\,\lambda_{IP-\mathrm{M}_2},\lambda_{RN-\mathrm{M}_2}+\iota\,\lambda_{IN-\mathrm{M}_2}\right)$ over $Ş$ if and only if a semigroup $Ş$ is regular and intra-regular.

The introduced conceptions are an advancement of the fuzzy set (FS), bipolar fuzzy set (BFS), and complex FS (CFS) in the environment of semigroups and from the introduced notions we can easily achieve these conceptions in the environment of FS, BFS, and CFS.

The quick assessment of the composition of this analysis: In Section 2, we studied, the fundamental concepts such as FS, fuzzy sub-semigroup, BF set, BF set sub-semigroup, BCF set and its related concepts In Section 3, we introduced the BCF sub-semigroup, BCFLI, BCFRI, BCFTSI, bipolar complex characteristic function, positive (ω,η)-cut, negative (ϱ,σ)-cut, positive and $\big((\omega,\eta),(\varrho,\sigma)\big)$-cut. Further, we also discuss their related theorems. In Section 4, we provided the characterizations of various categories of semigroups such as semi-simple, intra-regular, left, right ideals, and regular by the properties of BCF ideals (BCFIs). Additionally, we describe these in terms of BCFLIs, and BCFRIs. The conclusion is presented in Section 5.

2. Preliminaries

The fundamental concepts such as FS, fuzzy sub-semigroup, BF set, BF set sub-semigroup, BCF set, and its related concepts are reviewed in this section. we will take $Ş$ as a semigroup in this analysis.

Definition 1. [1] A mathematical shape

$$\mathrm{M}=\{(x,\lambda_{\mathrm{M}}(x))\mid x\in\mathfrak{X}\}$$

is known as FS on $\mathfrak{X}$. Seemingly, $\lambda_{\mathrm{M}}(x):\mathfrak{X}\to[0,1]$ called the supportive grade.

Definition 2. [3] Suppose an FS $\mathrm{M}=\lambda_{\mathrm{M}}(x)$ over $Ş$, then M is said to be a fuzzy sub-semigroup of $Ş$ if $\forall\, x,y\in Ş$,

$$\lambda_{\mathrm{M}}(xy)\geq\min\{\lambda_{\mathrm{M}}(x),\lambda_{\mathrm{M}}(y)\}.$$

Definition 3. [3] Suppose an FS $\mathrm{M}=\lambda_{\mathrm{M}}(x)$ over $Ş$, then M is said to be fuzzy left (right) ideal of $Ş$ if $\forall\, x,y\in Ş$,

$$\lambda_{\mathrm{M}}(xy)\geq\lambda_{\mathrm{M}}(y)\big(\lambda_{\mathrm{M}}(xy)\geq\lambda_{\mathrm{M}}(x)\big).$$

M is said to be a two-sided ideal if it is both fuzzy left ideal and fuzzy right ideal.

Definition 4. [17] A mathematical shape

$$\mathrm{M}=\{(x,\lambda_{P-\mathrm{M}}(x),\lambda_{N-\mathrm{M}}(x))\mid x\in\mathfrak{X}\}$$

is known as the BF set. Seemingly, $\lambda_{P-\mathrm{M}}(x):\mathfrak{X}\to[0,1]$ and $\lambda_{N-\mathrm{M}}(x):\mathfrak{X}\to[0,1]$, called the positive supportive grade and the negative supportive grade.

Definition 5. [18] Suppose a BF set $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})$ over $Ş$, then M is said to be BF sub-semigroup of $Ş$ if $\forall\, x,y\in Ş$,

(1) $\lambda_{P-\mathrm{M}}(xy)\geq\min\{\lambda_{P-\mathrm{M}}(x),\lambda_{P-\mathrm{M}}(y)\}$,

(2) $\lambda_{N-\mathrm{M}}(xy)\leq\max\{\lambda_{P-\mathrm{M}}(x),\lambda_{P-\mathrm{M}}(y)\}$.

Definition 6. [18] Suppose a BF set $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})$ over $Ş$, then M is said to be BF left (right) ideal of $Ş$ if $\forall\, x,y\in Ş$,

(1) $\lambda_{P-\mathrm{M}}(xy)\geq\lambda_{P-\mathrm{M}}(y)\big(\lambda_{P-\mathrm{M}}(xy)\geq\lambda_{P-\mathrm{M}}(x)\big)$,

(2) $\lambda_{N-\mathrm{M}}(xy)\leq\lambda_{N-\mathrm{M}}(y)\big(\lambda_{N-\mathrm{M}}(xy)\leq\lambda_{N-\mathrm{M}}(x)\big)$.

Definition 7. [41] A mathematical shape

$$\mathrm{M}=\{(x,\lambda_{P-\mathrm{M}}(x),\lambda_{N-\mathrm{M}}(x))\mid x\in\mathfrak{X}\}.$$

BCF set on $\mathfrak{X}$ is known as BCF set. Seemingly, $\lambda_{P-\mathrm{M}}(x)=\lambda_{RP-\mathrm{M}}(x)+\iota\,\lambda_{IP-\mathrm{M}}(x)$ and $\lambda_{N-\mathrm{M}}(x)=\lambda_{RN-\mathrm{M}}(x)+\iota\,\lambda_{IN-\mathrm{M}}(x)$, called the positive supportive grade and negative supportive grade with $\lambda_{RP-\mathrm{M}}(x),\lambda_{IP-\mathrm{M}}(x)\in[0,1]$ and $\lambda_{RN-\mathrm{M}}(x),\lambda_{IN-\mathrm{M}}(x)\in[-1,0]$. In this analysis, the structure of the

BCF set will be considered as $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\,\lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\,\lambda_{IN-\mathrm{M}})$.

Definition 8. [41] For two BCF set $\mathrm{M}_1 = \left(\lambda_{P-\mathrm{M}_1}, \lambda_{N-\mathrm{M}_1}\right) = \left(\lambda_{RP-\mathrm{M}_1} + \iota\,\lambda_{IP-\mathrm{M}_1}, \lambda_{RN-\mathrm{M}_1} + \iota\,\lambda_{IN-\mathrm{M}_1}\right)$ and $\mathrm{M}_2 = \left(\lambda_{P-\mathrm{M}_2}, \lambda_{N-\mathrm{M}_2}\right) = \left(\lambda_{RP-\mathrm{M}_2} + \iota\,\lambda_{IP-\mathrm{M}_2}, \lambda_{RN-\mathrm{M}_2} + \iota\,\lambda_{IN-\mathrm{M}_2}\right)$, we have

(1) $\mathrm{M}_1^C = \left(1 - \lambda_{RP-\mathrm{M}_1} + \iota\left(1 - \lambda_{RP-\mathrm{M}_1}\right), -1 - \lambda_{RN-\mathrm{M}_1} + \iota\left(-1 - \lambda_{IN-\mathrm{M}_1}\right)\right)$,

(2) $\mathrm{M}_1 \cup \mathrm{M}_2 = \begin{pmatrix} \max\left(\lambda_{RP-\mathrm{M}_1}, \lambda_{RP-\mathrm{M}_2}\right) + \iota\,\max\left(\lambda_{IP-\mathrm{M}_1}, \lambda_{IP-\mathrm{M}_2}\right), \\ \min\left(\lambda_{RN-\mathrm{M}_1}, \lambda_{RN-\mathrm{M}_2}\right) + \iota\,\min\left(\lambda_{IN-\mathrm{M}_1}, \lambda_{IN-\mathrm{M}_2}\right) \end{pmatrix}$,

(3) $\mathrm{M}_1 \cap \mathrm{M}_2 = \begin{pmatrix} \min\left(\lambda_{RP-\mathrm{M}_1}, \lambda_{RP-\mathrm{M}_2}\right) + \iota\,\min\left(\lambda_{IP-\mathrm{M}_1}, \lambda_{IP-\mathrm{M}_2}\right), \\ \max\left(\lambda_{RN-\mathrm{M}_1}, \lambda_{RN-\mathrm{M}_2}\right) + \iota\,\max\left(\lambda_{IN-\mathrm{M}_1}, \lambda_{IN-\mathrm{M}_2}\right) \end{pmatrix}$.

3. BCF sub-semigroups and ideals

In this section, we are going to introduce the BCF sub-semigroup, BCFLI, BCFRI, BCFTSI, bipolar complex characteristic function, positive (ω, η)-cut, negative (ϱ, σ)-cut, positive and $\left((\omega, \eta), (\varrho, \sigma)\right)$-cut. Further, we also discuss their related theorems. Throughout this analysis, for two BCF set $\mathrm{M}_1 = \left(\lambda_{P-\mathrm{M}_1}, \lambda_{N-\mathrm{M}_1}\right) = \left(\lambda_{RP-\mathrm{M}_1} + \iota\,\lambda_{IP-\mathrm{M}_1}, \lambda_{RN-\mathrm{M}_1} + \iota\,\lambda_{IN-\mathrm{M}_1}\right)$ and $\mathrm{M}_2 = \left(\lambda_{P-\mathrm{M}_2}, \lambda_{N-\mathrm{M}_2}\right) = \left(\lambda_{RP-\mathrm{M}_2} + \iota\,\lambda_{IP-\mathrm{M}_2}, \lambda_{RN-\mathrm{M}_2} + \iota\,\lambda_{IN-\mathrm{M}_2}\right)$, $\mathrm{M}_1 \preccurlyeq \mathrm{M}_2$ if $\lambda_{P-\mathrm{M}_1} \leq \lambda_{P-\mathrm{M}_2}$ and $\lambda_{N-\mathrm{M}_1} \geq \lambda_{N-\mathrm{M}_2}$ that is, $\lambda_{RP-\mathrm{M}_1} \leq \lambda_{RP-\mathrm{M}_2}$, $\lambda_{IP-\mathrm{M}_1} \leq \lambda_{IP-\mathrm{M}_2}$ and $\lambda_{RN-\mathrm{M}_1} \geq \lambda_{RN-\mathrm{M}_2}$, $\lambda_{IN-\mathrm{M}_1} \geq \lambda_{IN-\mathrm{M}_2}$.

Definition 8. Suppose a BCF set $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\,\lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\,\lambda_{IN-\mathrm{M}})$ over Ş, then M is known as BCF sub-semigroup of Ş if $\forall$ x, y $\in$ Ş,

(1) $\lambda_{P-\mathrm{M}}(xy) \geq \min\{\lambda_{P-\mathrm{M}}(x), \lambda_{P-\mathrm{M}}(y)\} \Rightarrow \lambda_{RP-\mathrm{M}}(xy) \geq \min\{\lambda_{RP-\mathrm{M}}(x), \lambda_{RP-\mathrm{M}}(y)\}$ and $\lambda_{IP-\mathrm{M}}(xy) \geq \min\{\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(y)\}$,

(2) $\lambda_{N-\mathrm{M}}(xy) \leq \max\{\lambda_{P-\mathrm{M}}(x), \lambda_{P-\mathrm{M}}(y)\} \Rightarrow \lambda_{RN-\mathrm{M}}(xy) \leq \max\{\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(y)\}$ and $\lambda_{IN-\mathrm{M}}(xy) \leq \max\{\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(y)\}$.

Example 1. Suppose a semigroup Ş $= \{e, x_1, x_2, x_3, x_4\}$ interpreted as Table 1:

Table 1. The Cayley table of Ş of Example 1.

.	e	x_1	x_2	x_3	x_4
e	e	e	e	e	e
x_1	e	e	e	e	e
x_2	e	e	x_2	x_3	x_4
x_3	e	e	x_2	x_3	x_4
x_4	e	e	x_2	x_3	x_4

Next, define a BCF subset $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\,\lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\,\lambda_{IN-\mathrm{M}})$ over Ş as

$$\mathrm{M} = \left\{ \begin{matrix} \left(e, (0.9 + \iota\,0.87, -0.23 - \iota\,0.25)\right), \left(x_1, (0.7 + \iota\,0.75, -0.33 - \iota\,0.36)\right), \\ \left(x_2, (0.5 + \iota\,0.62, -0.6 - \iota\,0.3)\right), \left(x_3, (0.5 + \iota\,0.62, -0.6 - \iota\,0.3)\right), \\ \left(x_4, (0.5 + \iota\,0.62, -0.6 - \iota\,0.3)\right), \end{matrix} \right\}$$

then, for e, x $\in$ Ş we have

(1) We have

$\lambda_{RP-\mathrm{M}}(ex_1) = \lambda_{RP-\mathrm{M}}(e) = 0.9$ and $\min\{\lambda_{RP-\mathrm{M}}(e), \lambda_{RP-\mathrm{M}}(x_1)\} = \min\{0.9, 0.7\} = 0.7$
$\Rightarrow \lambda_{RP-\mathrm{M}}(ex_1) \geq \min\{\lambda_{RP-\mathrm{M}}(e), \lambda_{RP-\mathrm{M}}(x_1)\}$,
$\lambda_{IP-\mathrm{M}}(ex_1) = \lambda_{IP-\mathrm{M}}(e) = 0.87$ and $\min\{\lambda_{IP-\mathrm{M}}(e), \lambda_{IP-\mathrm{M}}(x_1)\} = \min\{0.87, 0.75\} = 0.75$
$\Rightarrow \lambda_{IP-\mathrm{M}}(ex_1) \geq \min\{\lambda_{IP-\mathrm{M}}(e), \lambda_{IP-\mathrm{M}}(x_1)\} \Rightarrow \lambda_{P-\mathrm{M}}(ex_1) \geq \min\{\lambda_{P-\mathrm{M}}(e), \lambda_{P-\mathrm{M}}(x_1)\}$.

(2) Next,

$\lambda_{RN-M}(ex_1) = \lambda_{RN-M}(e) = -0.23$ and $\max\{\lambda_{RN-M}(e), \lambda_{RN-M}(x_1)\} = \max\{-0.23, -0.33\} = -0.23$

$\Rightarrow \lambda_{RN-M}(ex_1) \leq \max\{\lambda_{RN-M}(e), \lambda_{RN-M}(x_1)\}$,

$\lambda_{IN-M}(ex_1) = \lambda_{IN-M}(e) = -0.25$ and $\max\{\lambda_{IN-M}(e), \lambda_{IN-M}(x_1)\} = \max\{-0.25, -0.36\} = -0.25$

$\Rightarrow \lambda_{IN-M}(ex_1) \leq \max\{\lambda_{IN-M}(e), \lambda_{In-M}(x_1)\} \Rightarrow \lambda_{N-M}(ex_1) \leq \max\{\lambda_{N-M}(e), \lambda_{N-M}(x_1)\}$.

The remaining elements of $Ş$ can verify similarly. Thus M is a BCF sub-semigroup.

Definition 9. Suppose two BCF sets $M_1 = (\lambda_{P-M_1}, \lambda_{N-M_1}) = (\lambda_{RP-M_1} + \iota\,\lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\,\lambda_{IN-M_1})$ and $M_2 = (\lambda_{P-M_2}, \lambda_{N-M_2}) = (\lambda_{RP-M_2} + \iota\,\lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\,\lambda_{IN-M_2})$ over $Ş$, then the product of $M_1 \circledcirc M_2$ is described as

$$M_1 \circledcirc M_2 = (\lambda_{P-M_1} \circ \lambda_{P-M_2}, \lambda_{N-M_1} \circ \lambda_{N-M_2})$$

$$= (\lambda_{RP-M_1} \circ \lambda_{RP-M_2} + \iota\,\lambda_{IP-M_1} \circ \lambda_{IP-M_2}, \lambda_{RN-M_1} \circ \lambda_{RN-M_2} + \iota\,\lambda_{IN-M_1} \circ \lambda_{IN-M_2})$$

where,

$$(\lambda_{RP-M_1} \circ \lambda_{RP-M_2})(x) = \begin{cases} \sup\limits_{x=yz}\left\{\min\left(\lambda_{RP-M_1}(y), \lambda_{RP-M_2}(z)\right)\right\} & if\ x = yz\ for\ some\ y, z \in Ş \\ 0 & otherwise \end{cases},$$

$$(\lambda_{IP-M_1} \circ \lambda_{IP-M_2})(x) = \begin{cases} \sup\limits_{x=yz}\left\{\min\left(\lambda_{IP-M_1}(y), \lambda_{IP-M_2}(z)\right)\right\} & if\ x = yz\ for\ some\ y, z \in Ş \\ 0 & otherwise \end{cases},$$

$$(\lambda_{RN-M_1} \circ \lambda_{RN-M_2})(x) = \begin{cases} \inf\limits_{x=yz}\left\{\max\left(\lambda_{RN-M_1}(y), \lambda_{RN-M_2}(z)\right)\right\} & if\ x = yz\ for\ some\ y, z \in Ş \\ 0 & otherwise \end{cases},$$

$$(\lambda_{IN-M_1} \circ \lambda_{IN-M_2})(x) = \begin{cases} \inf\limits_{x=yz}\left\{\max\left(\lambda_{IN-M_1}(y), \lambda_{IN-M_2}(z)\right)\right\} & if\ x = yz\ for\ some\ y, z \in Ş \\ 0 & otherwise \end{cases}.$$

Remark 1. Clearly, the operation "$\circledcirc$" is associative.

Theorem 1. Suppose that $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\,\lambda_{IP-M}, \lambda_{RN-M} + \iota\,\lambda_{IN-M})$ is a BCF set over $Ş$, then $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\,\lambda_{IP-M}, \lambda_{RN-M} + \iota\,\lambda_{IN-M})$ is said to be BCF sub-semigroup of $Ş$ if and only if $M \circledcirc M \preccurlyeq M$.

Proof. Suppose that $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\,\lambda_{IP-M}, \lambda_{RN-M} + \iota\,\lambda_{IN-M})$ is a BCF sub-semigroup over $Ş$ and $x \in Ş$, if $\lambda_{RP-M} \circ \lambda_{RP-M} = 0, \lambda_{IP-M} \circ \lambda_{IP-M} = 0, \lambda_{RN-M} \circ \lambda_{RN-M} = 0$, and $\lambda_{IN-M} \circ \lambda_{IN-M} = 0$, then clearly, $M \circledcirc M \preccurlyeq M$. Otherwise there are elements $y, z \in Ş$ s.t $x = yz$, then

$$(\lambda_{RP-M} \circ \lambda_{RP-M})(x) = \sup_{x=yz}\{\min(\lambda_{RP-M}(y), \lambda_{RP-M}(z))\}$$

$$\leq \sup_{x=yz}\{\lambda_{RP-M}(yz)\} = \lambda_{RP-M}(x)$$

and

$$(\lambda_{IP-M} \circ \lambda_{IP-M})(x) = \sup_{x=yz}\{\min(\lambda_{IP-M}(y), \lambda_{IP-M}(z))\}$$

$$\leq \sup_{x=yz}\{\lambda_{IP-M}(yz)\} = \lambda_{IP-M}(x).$$

Next,

$$(\lambda_{RN-M} \circ \lambda_{RN-M})(x) = \inf_{x=yz}\{\max(\lambda_{RN-M}(y), \lambda_{RN-M}(z))\}$$

$$\geq \inf_{x=yz}\{\lambda_{RN-M}(yz)\} = \lambda_{RN-M}(x)$$

and

$$(\lambda_{IN-M} \circ \lambda_{IN-M})(x) = \inf_{x=yz}\{\max(\lambda_{IN-M}(y), \lambda_{IN-M}(z))\}$$

$$\geq \inf_{x=yz}\{\lambda_{IN-M}(yz)\} = \lambda_{IN-M}(x).$$

Thus, $(\lambda_{RP-M} \circ \lambda_{RP-M})(x) \leq \lambda_{RP-M}(x)$, $(\lambda_{IP-M} \circ \lambda_{IP-M})(x) \leq \lambda_{IP-M}(x) \Rightarrow (\lambda_{P-M} \circ \lambda_{P-M})(x) \leq \lambda_{P-M}(x)$ and $(\lambda_{RN-M} \circ \lambda_{RN-M})(x) \geq \lambda_{RN-M}(x)$, $(\lambda_{IN-M} \circ \lambda_{IN-M})(x) \geq \lambda_{IN-M}(x) \Rightarrow (\lambda_{N-M} \circ \lambda_{N-M})(x) \geq \lambda_{N-M}(x)$. Consequently, $M \circledcirc M \preccurlyeq M$.
Conversely, let $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\, \lambda_{IP-M}, \lambda_{RN-M} + \iota\, \lambda_{IN-M})$ is a BCF set over $Ş$ such that $M \circledcirc M \preccurlyeq M$ and $x, y, z \in Ş$ such that $x = yz$. Then

$$\lambda_{P-M}(yz) = \lambda_{P-M}(x) = \lambda_{RP-M}(x) + \iota\, \lambda_{IP-M}(x).$$

Now take

$$\lambda_{RP-M}(x) \geq (\lambda_{RP-M} \circ \lambda_{RP-M})(x) = \sup_{x=yz}\{\min(\lambda_{RP-M}(y), \lambda_{RP-M}(z))\}$$

$$\geq \min(\lambda_{RP-M}(y), \lambda_{RP-M}(z))$$

and

$$\lambda_{IP-M}(x) \geq (\lambda_{IP-M} \circ \lambda_{IP-M})(x) = \sup_{x=yz}\{\min(\lambda_{IP-M}(y), \lambda_{P-M}(z))\}$$

$$\geq \min(\lambda_{IP-M}(y), \lambda_{IP-M}(z))$$

$$\Rightarrow \lambda_{P-M}(yz) \geq \min(\lambda_{P-M}(y), \lambda_{P-M}(z)),$$

similarly,

$$\lambda_{N-M}(yz) = \lambda_{N-M}(x) = \lambda_{RN-M}(x) + \iota\, \lambda_{IN-M}(x).$$

Now take

$$\lambda_{RN-M}(x) \leq (\lambda_{RN-M} \circ \lambda_{RM-M})(x) = \inf_{x=yz}\{\max(\lambda_{RN-M}(y), \lambda_{RN-M}(z))\}$$

$$\leq \max(\lambda_{RN-M}(y), \lambda_{RN-M}(z)),$$

and

$$\lambda_{IN-M}(x) \leq (\lambda_{IN-M} \circ \lambda_{RM-M})(x) = \inf_{x=yz}\{\max(\lambda_{IN-M}(y), \lambda_{IN-M}(z))\}$$

$$\leq \max(\lambda_{IN-M}(y), \lambda_{IN-M}(z))$$

$$\Rightarrow \lambda_{N-M}(yz) \leq \max(\lambda_{N-M}(y), \lambda_{N-M}(z)).$$

This implies that M is a BCF sub-semigroup over $Ş$.

Following we are going to describe the BCF left (right) ideal.

Definition 10. Suppose a BCF set $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})=(\lambda_{RP-\mathrm{M}}+\iota\,\lambda_{IP-\mathrm{M}},\lambda_{RN-\mathrm{M}}+\iota\,\lambda_{IN-\mathrm{M}})$ over Ş, then

(1) M is known as BCF left ideal (BCFLI) of Ş if $\forall\, x, y \in$ Ş

1) $\lambda_{P-\mathrm{M}}(xy)\geq\lambda_{P-\mathrm{M}}(y) \Rightarrow \lambda_{RP-\mathrm{M}}(xy)\geq\lambda_{RP-\mathrm{M}}(y)$ and $\lambda_{IP-\mathrm{M}}(xy)\geq\lambda_{IP-\mathrm{M}}(y)$;
2) $\lambda_{N-\mathrm{M}}(xy)\leq\lambda_{N-\mathrm{M}}(y) \Rightarrow \lambda_{RN-\mathrm{M}}(xy)\leq\lambda_{RN-\mathrm{M}}(y)$ and $\lambda_{IN-\mathrm{M}}(xy)\leq\lambda_{IN-\mathrm{M}}(y)$.

(2) M is known as the BCF right ideal (BCFRI) of Ş if $\forall\, x, y \in$ Ş

1) $\lambda_{P-\mathrm{M}}(xy)\geq\lambda_{P-\mathrm{M}}(x) \Rightarrow \lambda_{RP-\mathrm{M}}(xy)\geq\lambda_{RP-\mathrm{M}}(x)$ and $\lambda_{IP-\mathrm{M}}(xy)\geq\lambda_{IP-\mathrm{M}}(x)$;
2) $\lambda_{N-\mathrm{M}}(xy)\leq\lambda_{N-\mathrm{M}}(x) \Rightarrow \lambda_{RN-\mathrm{M}}(xy)\leq\lambda_{RN-\mathrm{M}}(x)$ and $\lambda_{IN-\mathrm{M}}(xy)\leq\lambda_{IN-\mathrm{M}}(x)$.

(3) M is known as BCF two-sided ideal (BCFTSI) (BCF ideal) if it is both BCFLI and BCFRI.

Remark 2. It is evident that each BCFLI, BCFRI, and BCFTSI over Ş is a BCF sub-semigroup. But the converse is not valid.

Example 2.

(1) The BCF sub-semigroup $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})=(\lambda_{RP-\mathrm{M}}+\iota\,\lambda_{IP-\mathrm{M}},\lambda_{RN-\mathrm{M}}+\iota\,\lambda_{IN-\mathrm{M}})$ over Ş in Example 1 is not a BCFLI, because

$$\lambda_{RN-\mathrm{M}}(ex_1)=\lambda_{RN-\mathrm{M}}(e)=-0.23 \text{ and } \lambda_{RN-\mathrm{M}}(x_1)=-0.33,$$

thus,

$$\lambda_{RN-\mathrm{M}}(ex_1)\nleq\lambda_{RN-\mathrm{M}}(x_1)\Rightarrow\lambda_{N-\mathrm{M}}(ex_1)\nleq\lambda_{N-\mathrm{M}}(x_1),$$

and not BCFRI because

$$\lambda_{RN-\mathrm{M}}(x_1e)=\lambda_{RN-\mathrm{M}}(e)=-0.23 \text{ and } \lambda_{RN-\mathrm{M}}(x_1)=-0.33,$$

thus,

$$\lambda_{RN-\mathrm{M}}(x_1e)\nleq\lambda_{RN-\mathrm{M}}(x_1)\Rightarrow\lambda_{N-\mathrm{M}}(x_1e)\nleq\lambda_{N-\mathrm{M}}(x_1).$$

Hence, M is also not a BCFTSI.

(2) Consider the semigroup Ş of Example 1 and a BCF subset $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})=(\lambda_{RP-\mathrm{M}}+\iota\,\lambda_{IP-\mathrm{M}},\lambda_{RN-\mathrm{M}}+\iota\,\lambda_{IN-\mathrm{M}})$ over Ş as

$$\mathrm{M}=\left\{\begin{matrix}(e,(0.9+\iota\,0.87,-0.6-\iota\,0.3)),(x_1,(0.7+\iota\,0.75,-0.33-\iota\,0.36)),\\(x_2,(0.5+\iota\,0.62,-0.23-\iota\,0.25)),(x_3,(0.5+\iota\,0.62,-0.23-\iota\,0.25)),\\(x_4,(0.5+\iota\,0.62,-0.23-\iota\,0.25))\end{matrix}\right\}$$

then, M is BCFLI, BCFRI, and BCFTSI over Ş.

The below-given theorem explains that the BCF set $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})=(\lambda_{RP-\mathrm{M}}+\iota\,\lambda_{IP-\mathrm{M}},\lambda_{RN-\mathrm{M}}+\iota\,\lambda_{IN-\mathrm{M}})$ of Ş is a BCFLI (BCFRI) over Ş if and only if Ş $\circledcirc$ M $\preccurlyeq$ M (M $\circledcirc$ Ş $\preccurlyeq$ M).

Theorem 2. Suppose that $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})=(\lambda_{RP-\mathrm{M}}+\iota\,\lambda_{IP-\mathrm{M}},\lambda_{RN-\mathrm{M}}+\iota\,\lambda_{IN-\mathrm{M}})$ is a BCF set over Ş, then

(1) $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})=(\lambda_{RP-\mathrm{M}}+\iota\,\lambda_{IP-\mathrm{M}},\lambda_{RN-\mathrm{M}}+\iota\,\lambda_{IN-\mathrm{M}})$ is a BCFLI over Ş if and only if Ş $\circledcirc$ M $\preccurlyeq$ M;

(2) $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})=(\lambda_{RP-\mathrm{M}}+\iota\,\lambda_{IP-\mathrm{M}},\lambda_{RN-\mathrm{M}}+\iota\,\lambda_{IN-\mathrm{M}})$ is a BCFRI over Ş if and only if M $\circledcirc$ Ş $\preccurlyeq$ M;

(3) $\mathrm{M}=(\lambda_{P-\mathrm{M}},\lambda_{N-\mathrm{M}})=(\lambda_{RP-\mathrm{M}}+\iota\,\lambda_{IP-\mathrm{M}},\lambda_{RN-\mathrm{M}}+\iota\,\lambda_{IN-\mathrm{M}})$ is a BCFTSI over Ş if and only if Ş $\circledcirc$ M $\preccurlyeq$ M and M $\circledcirc$ Ş $\preccurlyeq$ M,

holds.

Proof. 1. Suppose that $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCFLI over Ş and $x \in$ Ş, if $\lambda_{RP-Ş} \circ \lambda_{RP-\mathrm{M}} = 0, \lambda_{IP-Ş} \circ \lambda_{IP-\mathrm{M}} = 0, \lambda_{RN-Ş} \circ \lambda_{RN-\mathrm{M}} = 0$, and $\lambda_{IN-Ş} \circ \lambda_{IN-\mathrm{M}} = 0$, then clearly, Ş $\circledcirc$ M $\preccurlyeq$ M. Otherwise there are elements $y, z \in$ Ş s.t $x = yz$, then

$$\left(\lambda_{RP-Ş} \circ \lambda_{RP-\mathrm{M}}\right)(x) = \sup_{x=yz}\left\{\min\left(\lambda_{RP-Ş}(y), \lambda_{RP-\mathrm{M}}(z)\right)\right\} = \sup_{x=yz}\left\{\min\left(1, \lambda_{RP-\mathrm{M}}(z)\right)\right\}$$

$$= \sup_{x=yz}\{\lambda_{RP-\mathrm{M}}(z)\} \leq \sup_{x=yz}\{\lambda_{RP-\mathrm{M}}(yz)\} = \lambda_{RP-\mathrm{M}}(x),$$

and

$$\left(\lambda_{IP-Ş} \circ \lambda_{IP-\mathrm{M}}\right)(x) = \sup_{x=yz}\left\{\min\left(\lambda_{IP-Ş}(y), \lambda_{IP-\mathrm{M}}(z)\right)\right\} = \sup_{x=yz}\left\{\min\left(1, \lambda_{IP-\mathrm{M}}(z)\right)\right\}$$

$$= \sup_{x=yz}\{\lambda_{IP-\mathrm{M}}(z)\} \leq \sup_{x=yz}\{\lambda_{IP-\mathrm{M}}(yz)\} = \lambda_{IP-\mathrm{M}}(x).$$

Next,

$$\left(\lambda_{RN-Ş} \circ \lambda_{RN-\mathrm{M}}\right)(x) = \inf_{x=yz}\left\{\max\left(\lambda_{RN-Ş}(y), \lambda_{RN-\mathrm{M}}(z)\right)\right\} = \inf_{x=yz}\left\{\max\left(-1, \lambda_{RN-\mathrm{M}}(z)\right)\right\}$$

$$= \inf_{x=yz}\{\lambda_{RN-\mathrm{M}}(z)\} \geq \inf_{x=yz}\{\lambda_{RN-\mathrm{M}}(yz)\} = \lambda_{RN-\mathrm{M}}(x),$$

and

$$\left(\lambda_{IN-Ş} \circ \lambda_{IN-\mathrm{M}}\right)(x) = \inf_{x=yz}\left\{\max\left(\lambda_{IN-Ş}(y), \lambda_{IN-\mathrm{M}}(z)\right)\right\} = \inf_{x=yz}\left\{\max\left(-1, \lambda_{IN-\mathrm{M}}(z)\right)\right\}$$

$$= \inf_{x=yz}\{\lambda_{IN-\mathrm{M}}(z)\} \geq \inf_{x=yz}\{\lambda_{IN-\mathrm{M}}(yz)\} = \lambda_{IN-\mathrm{M}}(x).$$

Thus,
$\left(\lambda_{RP-Ş} \circ \lambda_{RP-\mathrm{M}}\right)(x) \leq \lambda_{RP-\mathrm{M}}(x)$, $\left(\lambda_{IP-Ş} \circ \lambda_{IP-\mathrm{M}}\right)(x) \leq \lambda_{IP-\mathrm{M}}(x)$
$\Rightarrow \left(\lambda_{P-Ş} \circ \lambda_{P-\mathrm{M}}\right)(x) \leq \lambda_{P-\mathrm{M}}(x)$ and $\left(\lambda_{RN-Ş} \circ \lambda_{RN-\mathrm{M}}\right)(x) \geq \lambda_{RN-\mathrm{M}}(x)$, $\left(\lambda_{IN-Ş} \circ \lambda_{IN-\mathrm{M}}\right)(x) \geq \lambda_{IN-\mathrm{M}}(x) \Rightarrow \left(\lambda_{N-Ş} \circ \lambda_{N-\mathrm{M}}\right)(x) \geq \lambda_{N-\mathrm{M}}(x)$. Consequently, Ş $\circledcirc$ M $\preccurlyeq$ M.
Conversely, let $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCF set over Ş such that Ş $\circledcirc$ M $\preccurlyeq$ M and $x, y, z \in$ Ş such that $x = yz$. Then

$$\lambda_{P-\mathrm{M}}(yz) = \lambda_{P-\mathrm{M}}(x) = \lambda_{RP-\mathrm{M}}(x) + \iota\, \lambda_{IP-\mathrm{M}}(x).$$

Now take

$$\lambda_{RP-\mathrm{M}}(x) \geq \left(\lambda_{RP-Ş} \circ \lambda_{RP-\mathrm{M}}\right)(x) = \sup_{x=yz}\left\{\min\left(\lambda_{RP-Ş}(y), \lambda_{RP-\mathrm{M}}(z)\right)\right\}$$

$$= \sup_{x=yz}\left\{\min\left(1, \lambda_{RP-\mathrm{M}}(z)\right)\right\} \geq \min\left(1, \lambda_{RP-\mathrm{M}}(z)\right) = \lambda_{RP-\mathrm{M}}(z)$$

$$\Rightarrow \lambda_{RP-\mathrm{M}}(yz) \geq \lambda_{RP-\mathrm{M}}(z)$$

and

$$\lambda_{IP-\mathrm{M}}(x) \geq \left(\lambda_{IP-Ş} \circ \lambda_{IP-\mathrm{M}}\right)(x) = \sup_{x=yz}\left\{\min\left(\lambda_{IP-Ş}(y), \lambda_{IP-\mathrm{M}}(z)\right)\right\}$$

$$= \sup_{x=yz}\left\{\min\left(1, \lambda_{IP-\mathrm{M}}(z)\right)\right\} \geq \min\left(1, \lambda_{IP-\mathrm{M}}(z)\right) = \lambda_{IP-\mathrm{M}}(z)$$

$$\Rightarrow \lambda_{IP-\mathrm{M}}(\text{yz}) \geq \lambda_{iP-\mathrm{M}}(\text{z}),$$

similarly,

$$\lambda_{N-\mathrm{M}}(\text{yz}) = \lambda_{N-\mathrm{M}}(\text{x}) = \lambda_{RN-\mathrm{M}}(\text{x}) + \iota\, \lambda_{IN-\mathrm{M}}(\text{x}).$$

Now take

$$\lambda_{RN-\mathrm{M}}(\text{x}) \leq \left(\lambda_{RN-\text{Ş}} \circ \lambda_{RN-\mathrm{M}}\right)(\text{x}) = \inf_{\text{x}=\text{yz}} \left\{\max\left(\lambda_{RN-\text{Ş}}(\text{y}), \lambda_{RN-\mathrm{M}}(\text{z})\right)\right\}$$

$$= \inf_{\text{x}=\text{yz}}\left\{\max\left(-1, \lambda_{RN-\mathrm{M}}(\text{z})\right)\right\} \leq \max\left(-1, \lambda_{RN-\mathrm{M}}(\text{z})\right) = \lambda_{RN-\mathrm{M}}(\text{z})$$

$$\Rightarrow \lambda_{RN-\mathrm{M}}(\text{yz}) \leq \lambda_{RN-\mathrm{M}}(\text{z})$$

and

$$\lambda_{IN-\mathrm{M}}(\text{x}) \leq \left(\lambda_{IN-\text{Ş}} \circ \lambda_{IN-\mathrm{M}}\right)(\text{x}) = \inf_{\text{x}=\text{yz}} \left\{\max\left(\lambda_{IN-\text{Ş}}(\text{y}), \lambda_{IN-\mathrm{M}}(\text{z})\right)\right\}$$

$$= \inf_{\text{x}=\text{yz}}\left\{\max\left(-1, \lambda_{IN-\mathrm{M}}(\text{z})\right)\right\} \leq \max\left(-1, \lambda_{IN-\mathrm{M}}(\text{z})\right) = \lambda_{IN-\mathrm{M}}(\text{z})$$

$$\Rightarrow \lambda_{IN-\mathrm{M}}(\text{yz}) \leq \lambda_{IN-\mathrm{M}}(\text{z}).$$

This implies that M is a BCFLI over Ş.
The proof of 2 and 3 is likewise the proof of 1, so we are omitting the proof here.

Definition 11. Suppose a BCF set $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ over Ş, then

(1) For each $\omega, \eta \in [0,1]$ the set $\mathcal{P}(\lambda_{P-\mathrm{M}}, (\omega,\eta)) = \{\text{x} \in \text{Ş}: \lambda_{RP-\mathrm{M}} \geq \omega \text{ and } \lambda_{IP-\mathrm{M}} \geq \eta\}$ is known as positive (ω,η)-cut of M.

(2) For each $\varrho, \sigma \in [-1,0]$ the set $\mathcal{N}(\lambda_{N-\mathrm{M}}, (\varrho,\sigma)) = \{\text{x} \in \text{Ş}: \lambda_{RN-\mathrm{M}} \leq \varrho \text{ and } \lambda_{IN-\mathrm{M}} \leq \sigma\}$ is known as negative (ϱ,σ)-cut of M.

(3) The set $\mathcal{PN}\left(\mathrm{M}, \left((\omega,\eta),(\varrho,\sigma)\right)\right) = \mathcal{P}(\lambda_{P-\mathrm{M}}, (\omega,\eta)) \cap \mathcal{N}(\lambda_{N-\mathrm{M}}, (\varrho,\sigma))$ is known as the $\left((\omega,\eta),(\varrho,\sigma)\right)$-cut of M.

Theorem 3. Suppose a BCF set $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ over Ş, then

(1) For each $\omega, \eta \in [0,1]$, $\varrho, \sigma \in [-1,0]$, the non-empty set $\mathcal{PN}\left(\mathrm{M}, \left((\omega,\eta),(\varrho,\sigma)\right)\right)$ is a sub-semigroup of Ş if and only if $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCF sub-semigroup over Ş;

(2) For each $\omega, \eta \in [0,1]$, $\varrho, \sigma \in [-1,0]$, the non-empty set $\mathcal{PN}\left(\mathrm{M}, \left((\omega,\eta),(\varrho,\sigma)\right)\right)$ is a left ideal of Ş if and only if $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCFLI over Ş;

(3) For each $\omega, \eta \in [0,1]$, $\varrho, \sigma \in [-1,0]$, the non-empty set $\mathcal{PN}\left(\mathrm{M}, \left((\omega,\eta),(\varrho,\sigma)\right)\right)$ is a right ideal of Ş if and only if $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCFRI over Ş;

(4) For each $\omega, \eta \in [0,1]$, $\varrho, \sigma \in [-1,0]$, the non-empty set $\mathcal{PN}\left(\mathrm{M}, \left((\omega,\eta),(\varrho,\sigma)\right)\right)$ is a two-sided ideal of Ş if and only if $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCFTSI over Ş,

holds.

Proof. 1. Suppose that $\mathcal{PN}\left(\mathrm{M},\ ((\omega,\eta),(\varrho,\sigma))\right)$ is a sub-semigroup over Ş, $x, y \in$ Ş, and $\omega = \min(\lambda_{RP-\mathrm{M}}(x), \lambda_{RP-\mathrm{M}}(y))$ and $\eta = \min(\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(y))$. Evidently, $\lambda_{RP-\mathrm{M}}(x) \geq \min(\lambda_{RP-\mathrm{M}}(x), \lambda_{RP-\mathrm{M}}(y)) = \omega$, $\lambda_{RP-\mathrm{M}}(y) \geq \min(\lambda_{RP-\mathrm{M}}(x), \lambda_{RP-\mathrm{M}}(y)) = \omega$, $\lambda_{IP-\mathrm{M}}(x) \geq \min(\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(y)) = \eta$ and $\lambda_{IP-\mathrm{M}}(y) \geq \min(\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(y)) = \eta$. Similarly, suppose $\varrho = \max(\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(y))$ and $\sigma = \max(\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(y))$. Evidently, $\lambda_{RN-\mathrm{M}}(x) \leq \max(\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(y)) = \varrho$, $\lambda_{RN-\mathrm{M}}(y) \leq \max(\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(y)) = \varrho$, $\lambda_{IN-\mathrm{M}}(x) \leq \max(\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(y)) = \sigma$ and $\lambda_{IN-\mathrm{M}}(y) \leq \max(\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(y)) = \sigma$ which implies that $x, y \in \mathcal{PN}\left(\mathrm{M},\ ((\omega,\eta),(\varrho,\sigma))\right)$. As $\mathcal{PN}\left(\mathrm{M},\ ((\omega,\eta),(\varrho,\sigma))\right)$ is a sub-semigroup over Ş, so $xy \in \mathcal{PN}\left(\mathrm{M},\ ((\omega,\eta),(\varrho,\sigma))\right)$. Thus, $\lambda_{RP-\mathrm{M}}(xy) \geq \omega = \min(\lambda_{RP-\mathrm{M}}(x), \lambda_{RP-\mathrm{M}}(y))$, $\lambda_{IP-\mathrm{M}}(xy) \geq \eta = \min(\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(y))$, $\lambda_{RN-\mathrm{M}}(xy) \leq \varrho = \max(\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(y))$, $\lambda_{IN-\mathrm{M}}(xy) \leq \sigma = \max(\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(y))$. Consequently, $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}})$ is a BCF sub-semigroup over Ş.

Conversely, let $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}})$ is a BCF sub-semigroup over Ş and $x, y \in$ Ş such that $x, y \in \mathcal{PN}\left(\mathrm{M},\ ((\omega,\eta),(\varrho,\sigma))\right)\ \forall\ \omega,\eta \in [0,1]$, $\varrho,\sigma \in [-1,0]$. Since $\lambda_{RP-\mathrm{M}}(x) \geq \omega$, $\lambda_{RP-\mathrm{M}}(y) \geq \omega$ $\lambda_{IP-\mathrm{M}}(x) \geq \eta$, $\lambda_{IP-\mathrm{M}}(y) \geq \eta$, $\lambda_{RN-\mathrm{M}}(x) \leq \varrho$, $\lambda_{RN-\mathrm{M}}(y) \leq \varrho$, $\lambda_{IN-\mathrm{M}}(x) \leq \sigma$, and $\lambda_{IN-\mathrm{M}}(y) \leq \sigma$. Hence, $\lambda_{RP-\mathrm{M}}(xy) \geq \min(\lambda_{RP-\mathrm{M}}(x), \lambda_{RP-\mathrm{M}}(y)) \geq \omega$, $\lambda_{IP-\mathrm{M}}(xy) \geq \min(\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(y)) \geq \eta$, $\lambda_{RN-\mathrm{M}}(xy) \leq \max(\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(y)) \leq \varrho$, and $\lambda_{IN-\mathrm{M}}(xy) \leq \max(\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(y)) \leq \sigma$. Thus, $xy \in \mathcal{PN}\left(\mathrm{M},\ ((\omega,\eta),(\varrho,\sigma))\right)$ and $\mathcal{PN}\left(\mathrm{M},\ ((\omega,\eta),(\varrho,\sigma))\right)$ is a sub-semigroup of Ş.

The rest are the same as 1.

Definition 12. The bipolar complex characteristic function of a subset $\mathfrak{Q}$ of Ş, is indicated by $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ and demonstrated as

$$\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x) = \begin{cases} 1 + \iota\, 1 & \text{if } x \in \mathfrak{Q} \\ 0 + \iota\, 0, & \text{otherwise} \end{cases},$$

$$\lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x) = \begin{cases} -1 - \iota\, 1 & \text{if } x \in \mathfrak{Q} \\ 0 + \iota\, 0, & \text{otherwise} \end{cases}.$$

Remark 3. We observe that Ş can be taken as a BCF set of itself and write $\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x) = \lambda_{P-Ş}(x)$ and $\lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x) = \lambda_{N-Ş}(x)$.

Theorem 4. Suppose that $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ is a bipolar complex characteristic function over Ş, then

(1) $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ is a BCF sub-semigroup over Ş if and only if $\mathfrak{Q}$ is a sub-semigroup of Ş;

(2) $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ is a BCFLI over Ş if and only if $\mathfrak{Q}$ is a left idea of Ş;

(3) $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ is a BCFRI over Ş if and only if $\mathfrak{Q}$ is a right ideal of Ş;

(4) $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ is a BCFTSI over Ş if and only if $\mathfrak{Q}$ is a two-sided ideal of Ş,

holds.

Proof. Suppose that $\mathfrak{Q}$ is a sub-semigroup of Ş and let $x, y \in \mathfrak{Q}$, then
$\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x) = 1 + \iota\, 1 = \lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(y)$ and $\lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x) = -1 - \iota\, 1 = \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(y)$
as $xy \in \mathfrak{Q}$, thus,

$$\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(xy) = 1 + \iota\, 1 = \min(1 + \iota\, 1, 1 + \iota\, 1) = \min\left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x), \lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(y)\right)$$

and

$$\lambda_{N-M^{\mathfrak{Q}}}(\text{x}\text{y}) = -1 - \iota\, 1 = \max(-1 - \iota\, 1, -1 - \iota\, 1) = \max\left(\lambda_{N-M^{\mathfrak{Q}}}(\text{x}), \lambda_{N-M^{\mathfrak{Q}}}(\text{y})\right).$$

Next if $\text{x} \notin \mathfrak{Q}$ or $\text{y} \notin \mathfrak{Q}$ then
$\lambda_{P-M^{\mathfrak{Q}}}(\text{x}) = 0 + \iota\, 0$ or $\lambda_{P-M^{\mathfrak{Q}}}(\text{y}) = 0 + \iota\, 0$ and $\lambda_{N-M^{\mathfrak{Q}}}(\text{x}) = 0 + \iota\, 0$ or $\lambda_{N-M^{\mathfrak{Q}}}(\text{y}) = 0 + \iota\, 0$

$$\lambda_{P-M^{\mathfrak{Q}}}(\text{x}\text{y}) \geq 0 + \iota\, 0 = \min\left(\lambda_{P-M^{\mathfrak{Q}}}(\text{x}), \lambda_{P-M^{\mathfrak{Q}}}(\text{y})\right)$$

and

$$\lambda_{N-M^{\mathfrak{Q}}}(\text{x}\text{y}) \leq 0 + \iota\, 0 = \max\left(\lambda_{N-M^{\mathfrak{Q}}}(\text{x}), \lambda_{N-M^{\mathfrak{Q}}}(\text{y})\right).$$

Thus, $M^{\mathfrak{Q}} = \left(\lambda_{P-M^{\mathfrak{Q}}}, \lambda_{N-M^{\mathfrak{Q}}}\right)$ is a BCF sub-semigroup over Ş.
Conversely, let $M^{\mathfrak{Q}} = \left(\lambda_{P-M^{\mathfrak{Q}}}, \lambda_{N-M^{\mathfrak{Q}}}\right)$ is a BCF sub-semigroup over Ş and $\text{x} \in$ Ş such that $\text{x} \in \mathfrak{Q}$. Thus we have
$\lambda_{P-M^{\mathfrak{Q}}}(\text{x}) = 1 + \iota\, 1$ and $\lambda_{N-M^{\mathfrak{Q}}}(\text{x}) = -1 - \iota\, 1$
$\Rightarrow \text{x} \in \mathcal{PN}\left(M, \left((1,1),(-1,-1)\right)\right)$. Let $\text{y} \in$ Ş such that $\text{y} \in \mathcal{PN}\left(M, \left((1,1),(-1,-1)\right)\right)$. This shows that $\lambda_{RP-M^{\mathfrak{Q}}}(\text{x}) \geq 1$, $\lambda_{IP-M^{\mathfrak{Q}}}(\text{x}) \geq 1$ and $\lambda_{RN-M^{\mathfrak{Q}}}(\text{x}) \leq -1$, $\lambda_{IN-M^{\mathfrak{Q}}}(\text{x}) \leq -1$, and so $\text{y} \in \mathfrak{Q}$. Hence $\mathfrak{Q} = \mathcal{PN}\left(M, \left((1,1),(-1,-1)\right)\right)$. By Theorem 3 we obtained that $\mathfrak{Q}$ is a sub-semigroup of Ş.

Lemma 1. For two BCF set $M^{\mathfrak{Q}} = \left(\lambda_{P-M^{\mathfrak{Q}}}, \lambda_{N-M^{\mathfrak{Q}}}\right)$ and $M^{\mathfrak{P}} = \left(\lambda_{P-M^{\mathfrak{P}}}, \lambda_{N-M^{\mathfrak{P}}}\right)$ over Ş, then
(1) $M^{\mathfrak{Q}} \cap M^{\mathfrak{P}} = M^{\mathfrak{Q} \cap \mathfrak{P}}$;
(2) $M^{\mathfrak{Q}} \circledcirc M^{\mathfrak{P}} = M^{\mathfrak{Q}\mathfrak{P}}$,
holds

Proof. Omitted.

Theorem 5. Suppose that $M_1 = \left(\lambda_{P-M_1}, \lambda_{N-M_1}\right) = \left(\lambda_{RP-M_1} + \iota\, \lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\, \lambda_{IN-M_1}\right)$ and $M_2 = \left(\lambda_{P-M_2}, \lambda_{N-M_2}\right) = \left(\lambda_{RP-M_2} + \iota\, \lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\, \lambda_{IN-M_2}\right)$ are two BCF sets over Ş, then

(1) Assume that $M_1 = \left(\lambda_{P-M_1}, \lambda_{N-M_1}\right) = \left(\lambda_{RP-M_1} + \iota\, \lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\, \lambda_{IN-M_1}\right)$ and $M_2 = \left(\lambda_{P-M_2}, \lambda_{N-M_2}\right) = \left(\lambda_{RP-M_2} + \iota\, \lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\, \lambda_{IN-M_2}\right)$ are two BCF sub-semigroup over Ş, then $M_1 \cap M_2$ is a BCF sub-semigroup over Ş;
(2) Assume that $M_1 = \left(\lambda_{P-M_1}, \lambda_{N-M_1}\right) = \left(\lambda_{RP-M_1} + \iota\, \lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\, \lambda_{IN-M_1}\right)$ and $M_2 = \left(\lambda_{P-M_2}, \lambda_{N-M_2}\right) = \left(\lambda_{RP-M_2} + \iota\, \lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\, \lambda_{IN-M_2}\right)$ are two BCFLIs over Ş, then $M_1 \cap M_2$ is a BCFLI over Ş;
(3) Assume that $M_1 = \left(\lambda_{P-M_1}, \lambda_{N-M_1}\right) = \left(\lambda_{RP-M_1} + \iota\, \lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\, \lambda_{IN-M_1}\right)$ and $M_2 = \left(\lambda_{P-M_2}, \lambda_{N-M_2}\right) = \left(\lambda_{RP-M_2} + \iota\, \lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\, \lambda_{IN-M_2}\right)$ are two BCFRIs over Ş, then $M_1 \cap M_2$ is a BCFRI over Ş;
(4) Assume that $M_1 = \left(\lambda_{P-M_1}, \lambda_{N-M_1}\right) = \left(\lambda_{RP-M_1} + \iota\, \lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\, \lambda_{IN-M_1}\right)$ and $M_2 = \left(\lambda_{P-M_2}, \lambda_{N-M_2}\right) = \left(\lambda_{RP-M_2} + \iota\, \lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\, \lambda_{IN-M_2}\right)$ are two BCFTSIs over Ş, then $M_1 \cap M_2$ is a BCFTSI over Ş,

holds.

Proof. 1. For any $\text{x}, \text{y} \in$ Ş, we have

$$\left(\lambda_{P-M_1} \cap \lambda_{P-M_2}\right)(\text{x}\text{y}) = \min\left(\lambda_{RP-M_1}(\text{x}\text{y}), \lambda_{RP-M_2}(\text{x}\text{y})\right) + \iota\, \min\left(\lambda_{IP-M_1}(\text{x}\text{y}), \lambda_{IP-M_2}(\text{x}\text{y})\right).$$

Now take

$$\min\left(\lambda_{RP-M_1}(xy), \lambda_{RP-M_2}(xy)\right)$$
$$\geq \min\left(\min\left(\lambda_{RP-M_1}(x), \lambda_{RP-M_1}(y)\right), \min\left(\lambda_{RP-M_2}(x), \lambda_{RP-M_2}(y)\right)\right)$$
$$= \min\left(\min\left(\lambda_{RP-M_1}(x), \lambda_{RP-M_2}(x)\right), \min\left(\lambda_{RP-M_1}(y), \lambda_{RP-M_2}(y)\right)\right)$$
$$= \min\left(\left(\lambda_{RP-M_1} \cap \lambda_{RP-M_2}\right)(x), \left(\lambda_{RP-M_1} \cap \lambda_{RP-M_2}\right)(y)\right),$$

and

$$\min\left(\lambda_{IP-M_1}(xy), \lambda_{IP-M_2}(xy)\right) \geq \min\left(\min\left(\lambda_{IP-M_1}(x), \lambda_{IP-M_1}(y)\right), \min\left(\lambda_{IP-M_2}(x), \lambda_{IP-M_2}(y)\right)\right)$$
$$= \min\left(\min\left(\lambda_{IP-M_1}(x), \lambda_{IP-M_2}(x)\right), \min\left(\lambda_{IP-M_1}(y), \lambda_{IP-M_2}(y)\right)\right)$$
$$= \min\left(\left(\lambda_{IP-M_1} \cap \lambda_{IP-M_2}\right)(x), \left(\lambda_{IP-M_1} \cap \lambda_{IP-M_2}\right)(y)\right)$$
$$\Rightarrow \left(\lambda_{P-M_1} \cap \lambda_{P-M_2}\right)(xy) \geq \min\left(\left(\lambda_{P-M_1} \cap \lambda_{P-M_2}\right)(x), \left(\lambda_{P-M_1} \cap \lambda_{P-M_2}\right)(y)\right).$$

Similarly,

$$\left(\lambda_{N-M_1} \cap \lambda_{N-M_2}\right)(xy) = \max\left(\lambda_{RN-M_1}(xy), \lambda_{RN-M_2}(xy)\right) + \iota \max\left(\lambda_{IN-M_1}(xy), \lambda_{IN-M_2}(xy)\right).$$

Now take

$$\max\left(\lambda_{RN-M_1}(xy), \lambda_{RN-M_2}(xy)\right)$$
$$\leq \max\left(\max\left(\lambda_{RN-M_1}(x), \lambda_{RN-M_1}(y)\right), \max\left(\lambda_{RN-M_2}(x), \lambda_{RN-M_2}(y)\right)\right)$$
$$= \max\left(\max\left(\lambda_{RN-M_1}(x), \lambda_{RN-M_2}(x)\right), \max\left(\lambda_{RN-M_1}(y), \lambda_{RN-M_2}(y)\right)\right)$$
$$= \max\left(\left(\lambda_{RN-M_1} \cap \lambda_{RN-M_2}\right)(x), \left(\lambda_{RN-M_1} \cap \lambda_{RN-M_2}\right)(y)\right),$$

and

$$\max\left(\lambda_{IN-M_1}(xy), \lambda_{IN-M_2}(xy)\right)$$
$$\leq \max\left(\max\left(\lambda_{IN-M_1}(x), \lambda_{IN-M_1}(y)\right), \max\left(\lambda_{IN-M_2}(x), \lambda_{IN-M_2}(y)\right)\right)$$
$$= \max\left(\max\left(\lambda_{IN-M_1}(x), \lambda_{IN-M_2}(x)\right), \max\left(\lambda_{IN-M_1}(y), \lambda_{IN-M_2}(y)\right)\right)$$
$$= \max\left(\left(\lambda_{IN-M_1} \cap \lambda_{IN-M_2}\right)(x), \left(\lambda_{IN-M_1} \cap \lambda_{IN-M_2}\right)(y)\right)$$
$$\Rightarrow \left(\lambda_{N-M_1} \cap \lambda_{N-M_2}\right)(xy) \leq \max\left(\left(\lambda_{N-M_1} \cap \lambda_{N-M_2}\right)(x), \left(\lambda_{N-M_1} \cap \lambda_{N-M_2}\right)(y)\right).$$

Thus, $M_1 \cap M_2$ is a BCF sub-semigroup over Ş.
The proofs of parts 2–4 are likewise part 1.
Theorem 6. Suppose a BCFRI $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\,\lambda_{IP-M}, \lambda_{RN-M} + \iota\,\lambda_{IN-M})$ over Ş, then $M \cup (Ş \circledcirc M)$ is a BCFTSI over Ş.
Proof. As Ş is a BCFLI, so

$$Ş \circledcirc \left(M \cup \left((Ş \circledcirc M)\right)\right) = (Ş \circledcirc M) \cup (Ş \circledcirc Ş \circledcirc M)$$

$$\preccurlyeq (Ş \circledcirc M) \cup (Ş \circledcirc M) = Ş \circledcirc M \preccurlyeq M \cup (Ş \circledcirc M).$$

This shows that $M \cup (Ş \circledcirc M)$ is a BCFLI over Ş. Now

$$\left(M \cup \left((Ş \circledcirc M)\right)\right) \circledcirc Ş = (M \circledcirc Ş) \cup (Ş \circledcirc M \circledcirc Ş)$$

$$\preccurlyeq M \cup (Ş \circledcirc M).$$

This shows that $M \cup (Ş \circledcirc M)$ is a BCFRI over Ş. Thus $M \cup (Ş \circledcirc M)$ is a BCFTSI over Ş.

Corollary 1. Suppose a BCFLI $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\, \lambda_{IP-M}, \lambda_{RN-M} + \iota\, \lambda_{IN-M})$ over Ş, then $M \cup (M \circledcirc Ş)$ is a BCFTSI over Ş.

4. Describing regular semigroups

Here, we provide the characterizations of various categories of semigroups such as semi-simple, intra-regular, left, right ideals, and regular by the properties of BCF ideals (BCFIs). We also describe these in terms of BCFLIs, and BCFRIs. For better understanding, remember that an element $x \in Ş$ is known as regular if $\exists$ an element $y \in Ş$ s.t $x = xyx$. If each element of Ş is regular then Ş is known as regular semigroup. An element $e \in Ş$ is known as idempotent if $e.e = e$.

Theorem 7. Each BCFI over a regular semigroup Ş is idempotent.

Proof. Assume that $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\, \lambda_{IP-M}, \lambda_{RN-M} + \iota\, \lambda_{IN-M})$ is a BCFI over regular semigroup Ş, then by employing Theorem (2 part (3)), we get

$$M \circledcirc M \preccurlyeq Ş \circledcirc M \preccurlyeq M.$$

Now let $x \in Ş$. Then as Ş is a regular semigroup, $\exists$ an element $y \in Ş$ s.t $x = xyx$, hence

$$(\lambda_{RP-M} \circ \lambda_{RP-M})(x) = \sup_{x=ab}\left\{\min\left(\lambda_{RP-M}(a), \lambda_{RP-M}(b)\right)\right\}$$

$$\geq \min\left(\lambda_{RP-M}(xy), \lambda_{RP-M}(x)\right)$$

$$\geq \min\left(\lambda_{RP-M}(x), \lambda_{RP-M}(x)\right) = \lambda_{RP-M}(x)$$

and

$$(\lambda_{IP-M} \circ \lambda_{IP-M})(x) = \sup_{x=ab}\left\{\min\left(\lambda_{IP-M}(a), \lambda_{IP-M}(b)\right)\right\}$$

$$\geq \min\left(\lambda_{IP-M}(xy), \lambda_{IP-M}(x)\right)$$

$$\geq \min\left(\lambda_{IP-M}(x), \lambda_{IP-M}(x)\right) = \lambda_{IP-M}(x).$$

This means that $(\lambda_{P-M} \circ \lambda_{P-M})(x) \geq \lambda_{P-M}(x)$. Next,

$$(\lambda_{RN-M} \circ \lambda_{RN-M})(x) = \inf_{x=ab}\left\{\max\left(\lambda_{RN-M}(a), \lambda_{RN-M}(b)\right)\right\}$$

$$\leq \max\left(\lambda_{RN-M}(xy), \lambda_{RN-M}(x)\right)$$

$$\leq \max\left(\lambda_{RN-M}(x), \lambda_{RN-M}(x)\right) = \lambda_{RN-M}(x)$$

and

$$(\lambda_{IN-\mathrm{M}} \circ \lambda_{IN-\mathrm{M}})(x) = \inf_{x=ab}\{\max(\lambda_{IN-\mathrm{M}}(a), \lambda_{IN-\mathrm{M}}(b))\}$$

$$\leq \max(\lambda_{IN-\mathrm{M}}(xy), \lambda_{IN-\mathrm{M}}(x))$$

$$\leq \max(\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(x)) = \lambda_{IN-\mathrm{M}}(x).$$

This means that $(\lambda_{N-\mathrm{M}} \circ \lambda_{N-\mathrm{M}})(x) \leq \lambda_{N-\mathrm{M}}(x)$. Hence, $\mathrm{M} \circledcirc \mathrm{M} = \mathrm{M}$, thus $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}})$ is idempotent.

Theorem 8. For a semigroup $\mathcal{S}$,

(1) $\mathcal{S}$ is a regular semigroup;

(2) For each BCFLI $\mathrm{M}_1 = (\lambda_{P-\mathrm{M}_1}, \lambda_{N-\mathrm{M}_1}) = (\lambda_{RP-\mathrm{M}_1} + \iota\,\lambda_{IP-\mathrm{M}_1}, \lambda_{RN-\mathrm{M}_1} + \iota\,\lambda_{IN-\mathrm{M}_1})$ and BCFRI $\mathrm{M}_2 = (\lambda_{P-\mathrm{M}_2}, \lambda_{N-\mathrm{M}_2}) = (\lambda_{RP-\mathrm{M}_2} + \iota\,\lambda_{IP-\mathrm{M}_2}, \lambda_{RN-\mathrm{M}_2} + \iota\,\lambda_{IN-\mathrm{M}_2})$ over $\mathcal{S}$, $\mathrm{M}_1 \cap \mathrm{M}_2 = \mathrm{M}_1 \circledcirc \mathrm{M}_2$,

are equivalent.

Proof. $1 \Rightarrow 2$. Suppose that $\mathrm{M}_1 = (\lambda_{P-\mathrm{M}_1}, \lambda_{N-\mathrm{M}_1}) = (\lambda_{RP-\mathrm{M}_1} + \iota\,\lambda_{IP-\mathrm{M}_1}, \lambda_{RN-\mathrm{M}_1} + \iota\,\lambda_{IN-\mathrm{M}_1})$ and $\mathrm{M}_2 = (\lambda_{P-\mathrm{M}_2}, \lambda_{N-\mathrm{M}_2}) = (\lambda_{RP-\mathrm{M}_2} + \iota\,\lambda_{IP-\mathrm{M}_2}, \lambda_{RN-\mathrm{M}_2} + \iota\,\lambda_{IN-\mathrm{M}_2})$ are BCFLI and BCFRI over $\mathcal{S}$ respectively, then by employing Theorem (2 part (3)), we have that

$$\mathrm{M}_1 \circledcirc \mathrm{M}_2 \preccurlyeq \mathrm{M}_2 \circledcirc \mathrm{M}_2 \text{ and } \mathrm{M}_1 \circledcirc \mathrm{M}_2 \preccurlyeq \mathrm{M}_1 \circledcirc \mathcal{S} \preccurlyeq \mathrm{M}_1,$$

so,

$$\mathrm{M}_1 \circledcirc \mathrm{M}_2 \preccurlyeq \mathrm{M}_1 \cap \mathrm{M}_2.$$

Next, assume that $x \in \mathcal{S}$ and as $\mathcal{S}$ is regular semigroup, $\exists\ y \in \mathcal{S}$ s.t $x = xyx$. Therefore we have

$$(\lambda_{RP-\mathrm{M}} \circ \lambda_{RP-\mathrm{M}})(x) = \sup_{x=ab}\{\min(\lambda_{RP-\mathrm{M}}(a), \lambda_{RP-\mathrm{M}}(b))\}$$

$$\geq \min(\lambda_{RP-\mathrm{M}}(xy), \lambda_{RP-\mathrm{M}}(x))$$

$$\geq \min(\lambda_{RP-\mathrm{M}}(x), \lambda_{RP-\mathrm{M}}(x)) = \lambda_{RP-\mathrm{M}}(x)$$

and

$$(\lambda_{IP-\mathrm{M}} \circ \lambda_{IP-\mathrm{M}})(x) = \sup_{x=ab}\{\min(\lambda_{IP-\mathrm{M}}(a), \lambda_{IP-\mathrm{M}}(b))\}$$

$$\geq \min(\lambda_{IP-\mathrm{M}}(xy), \lambda_{IP-\mathrm{M}}(x))$$

$$\geq \min(\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(x)) = \lambda_{IP-\mathrm{M}}(x).$$

This means that $(\lambda_{P-\mathrm{M}} \circ \lambda_{P-\mathrm{M}})(x) \geq \lambda_{P-\mathrm{M}}(x)$. Next,

$$(\lambda_{RN-\mathrm{M}} \circ \lambda_{RN-\mathrm{M}})(x) = \inf_{x=ab}\{\max(\lambda_{RN-\mathrm{M}}(a), \lambda_{RN-\mathrm{M}}(b))\}$$

$$\leq \max(\lambda_{RN-\mathrm{M}}(xy), \lambda_{RN-\mathrm{M}}(x))$$

$$\leq \max(\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(x)) = \lambda_{RN-\mathrm{M}}(x)$$

and

$$(\lambda_{IN-\mathrm{M}} \circ \lambda_{IN-\mathrm{M}})(x) = \inf_{x=ab}\{\max(\lambda_{IN-\mathrm{M}}(a), \lambda_{IN-\mathrm{M}}(b))\}$$

$$\leq \max(\lambda_{IN-\mathrm{M}}(xy), \lambda_{IN-\mathrm{M}}(x))$$

$$\leq \max(\lambda_{IN-M}(x), \lambda_{IN-M}(x)) = \lambda_{IN-M}(x).$$

Thus, $M_1 \circledcirc M_2 \succcurlyeq M_1$ and consequently, $M_1 \cap M_2 = M_1 \circledcirc M_2$.
$2 \Rightarrow 1$. Suppose that $\mathfrak{U}_1$ is any left ideal of $Ş$ and $\mathfrak{U}_2$ is any right ideal of $Ş$, then by employing Theorem 4, we get that $M^{\mathfrak{U}_1} = (\lambda_{P-M^{\mathfrak{U}_1}}, \lambda_{N-M^{\mathfrak{U}_1}})$ be a BCFRI and $M^{\mathfrak{U}_2} = (\lambda_{P-M^{\mathfrak{U}_2}}, \lambda_{N-M^{\mathfrak{U}_2}})$ be a BCFLI over $Ş$. Now by employing Lemma 1, we get

$$(\lambda_{P-M^{\mathfrak{U}_1\mathfrak{U}_2}})(x) = (\lambda_{P-M^{\mathfrak{U}_1}} \circ \lambda_{P-M^{\mathfrak{U}_2}})(x)$$
$$= (\lambda_{P-M^{\mathfrak{U}_1}} \wedge \lambda_{P-M^{\mathfrak{U}_2}})(x) = (\lambda_{P-M^{\mathfrak{U}_1 \cap \mathfrak{U}_2}})(x) = 1 + \iota\, 1.$$

Thus, $x \in \mathfrak{U}_1\mathfrak{U}_2$ and hence $\mathfrak{U}_1 \cap \mathfrak{U}_2 \subseteq \mathfrak{U}_1\mathfrak{U}_2$. Consequently, $\mathfrak{U}_1 \cap \mathfrak{U}_2 = \mathfrak{U}_1\mathfrak{U}_2$.
Before going to the next result, we recall that $Ş$ is known as left (right) zero if $\forall\, x, y \in Ş$, $xy = x$ ($xy = y$).

Theorem 9. Suppose that $Ş$ is a regular semigroup, then
(1) The family $\Psi(Ş)$ of all idempotents of $Ş$ makes a left (right) zero sub-semigroup of $Ş$,
(2) For each BCFLI (BCFRI) $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\, \lambda_{IP-M}, \lambda_{RN-M} + \iota\, \lambda_{IN-M})$ over $Ş$, $\lambda_{P-M}(x) = \lambda_{P-M}(y) \Rightarrow \lambda_{RP-M}(x) = \lambda_{RP-M}(y)$ and $\lambda_{IP-M}(x) = \lambda_{IP-M}(y)$, and $\lambda_{N-M}(x) = \lambda_{N-M}(y) \Rightarrow \lambda_{RN-M}(x) = \lambda_{RN-M}(y)$ and $\lambda_{IN-M}(x) = \lambda_{IN-M}(y)\ \forall\, x, y \in Ş$.
are equivalent.

Proof. $1 \Rightarrow 2$. Suppose that $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\, \lambda_{IP-M}, \lambda_{RN-M} + \iota\, \lambda_{IN-M})$ is a BCFLI on $Ş$ and $x, y \in Ş$ such that $x, y \in \Psi(Ş)$, then as 1 holds so we have that $xy = x$ and $yx = y$ and

$$\lambda_{RP-M}(x) = \lambda_{RP-M}(xy) \geq \lambda_{RP-M}(y)$$

and,

$$\lambda_{RP-M}(y) = \lambda_{RP-M}(yx) \geq \lambda_{RP-M}(x).$$

Next, we have

$$\lambda_{IP-M}(x) = \lambda_{IP-M}(xy) \geq \lambda_{IP-M}(y)$$

and,

$$\lambda_{IP-M}(y) = \lambda_{IP-M}(yx) \geq \lambda_{IP-M}(x).$$

This implies that $\lambda_{P-M}(x) = \lambda_{P-M}(y)$. Likewise one can show that $\lambda_{N-M}(x) = \lambda_{N-M}(y)$.
$2 \Rightarrow 1$. As $Ş$ is a regular semigroup and $\Psi(Ş)$ is non-empty. Hence by utilizing Theorem (4 part (2)) we get that bipolar complex characteristic function $M^{Ş_y} = (\lambda_{P-M^{Ş_y}}, \lambda_{N-M^{Ş_y}})$ of the left ideal $Ş_y$ is a BCFLI on $Ş$. Consequently, $(\lambda_{N-M^{Ş_y}})(x) = (\lambda_{N-M^{Ş_y}})(y) = -1 - \iota\, 1$ and so $x \in Ş_y$. Therefore, for some $a \in Ş$, $x = ay = a(yy) = (ay)y = xy$. Consequently, $\Psi(Ş)$ is a left zero sub-semigroup on $Ş$. Likewise one can prove for right zero.
Before going to the next result, we recall that, if for every $x \in Ş$ $\exists$ $y \in Ş$ such that $x = x^2 y$ then $Ş$ is known as right (left) regular.

Theorem 10. Suppose a semigroup $Ş$, then
(1) $Ş$ is left (right) regular;
(2) For each BCFRI (BCFLI) $M = (\lambda_{P-M}, \lambda_{N-M}) = (\lambda_{RP-M} + \iota\, \lambda_{IP-M}, \lambda_{RN-M} + \iota\, \lambda_{IN-M})$ over $Ş$, $\lambda_{P-M}(x) = \lambda_{P-M}(x^2) \Rightarrow \lambda_{RP-M}(x) = \lambda_{RP-M}(x^2)$ and $\lambda_{IP-M}(x) = \lambda_{IP-M}(x^2)$, and $\lambda_{N-M}(x) = \lambda_{N-M}(x^2) \Rightarrow \lambda_{RN-M}(x) = \lambda_{RN-M}(x^2)$ and $\lambda_{IN-M}(x) = \lambda_{IN-M}(x^2)\ \forall\, x \in Ş$,
are equivalent.

Proof. $1 \Rightarrow 2$. Assume that $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCFLI over Ş and $x \in$ Ş, then as we know that Ş is left regular, so $\exists\ y \in$ Ş such that $x = yx^2$. Thus,

$$\lambda_{RP-\mathrm{M}}(x) = \lambda_{RP-\mathrm{M}}(yx^2) \geq \lambda_{RP-\mathrm{M}}(x^2)$$

and,

$$\lambda_{RP-\mathrm{M}}(x^2) \geq \lambda_{RP-\mathrm{M}}(x).$$

Next, we have

$$\lambda_{IP-\mathrm{M}}(x) = \lambda_{IP-\mathrm{M}}(yx^2) \geq \lambda_{IP-\mathrm{M}}(x^2)$$

and,

$$\lambda_{IP-\mathrm{M}}(x^2) \geq \lambda_{IP-\mathrm{M}}(x).$$

This implies that $\lambda_{P-\mathrm{M}}(x) = \lambda_{P-\mathrm{M}}(x^2)$. Likewise one can show that $\lambda_{N-\mathrm{M}}(x) = \lambda_{N-\mathrm{M}}(x^2)$.
$2 \Rightarrow 1$. Suppose $x \in$ Ş, then by Theorem (4 part (2)), we have that bipolar complex characteristic function $\mathrm{M}^{x^2 \cup Şx^2} = \left(\lambda_{P-\mathrm{M}^{x^2 \cup Şx^2}}, \lambda_{N-\mathrm{M}^{x^2 \cup Şx^2}}\right)$ of left ideal $x^2 \cup Şx^2$ of Ş is a BCFLI over Ş. As $x^2 \in x^2 \cup Şx^2$, so $\lambda_{N-\mathrm{M}^{x^2 \cup Şx^2}}(x) = \lambda_{N-\mathrm{M}^{x^2 \cup Şx^2}}(x^2) = -1 - \iota\, 1 \Rightarrow x \in x^2 \cup Şx^2$ and so, Ş is left-regular. One can prove likewise for right regular.

Before discussing the next definition we recall that a subset $\mathfrak{Q} \neq \emptyset$ of Ş is known as semiprime if $\forall\, x \in Ş, x^2 \in \mathfrak{Q} \Rightarrow x \in \mathfrak{Q}$.

Definition 13. A BCF set $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ over Ş is known as BCF semiprime if $\forall\, x \in$ Ş $\lambda_{P-\mathrm{M}}(x) \geq \lambda_{P-\mathrm{M}}(x^2) \Rightarrow \lambda_{RP-\mathrm{M}}(x) \geq \lambda_{RP-\mathrm{M}}(x^2)$ and $\lambda_{IP-\mathrm{M}}(x) \geq \lambda_{IP-\mathrm{M}}(x^2)$, and $\lambda_{N-\mathrm{M}}(x) \leq \lambda_{N-\mathrm{M}}(x^2) \Rightarrow \lambda_{RN-\mathrm{M}}(x) \leq \lambda_{RN-\mathrm{M}}(x^2)$ and $\lambda_{IN-\mathrm{M}}(x) \leq \lambda_{IN-\mathrm{M}}(x^2)$.

Theorem 11. Suppose $\mathfrak{Q} \neq \emptyset$ is a subset of Ş, then
(1) $\mathfrak{Q}$ is semiprime;
(2) The bipolar complex characteristic function $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ of $\mathfrak{Q}$ is a BCF semiprime set,
are equivalent.

Proof. $1 \Rightarrow 2$. Let $x \in$ Ş. If $x^2 \in \mathfrak{Q}$, $\Rightarrow$ $x \in \mathfrak{Q}$. Then, $\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x) = 1 + \iota\, 1 = \lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x^2)$ and $\lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x) = -1 - \iota\, 1 = \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x^2)$. If $x^2 \notin \mathfrak{Q}$, then $\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x^2) = 0 + \iota\, 0 \leq \lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x)$ and $\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}(x^2) = 0 + \iota\, 0 \geq \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x)$. Consequently, $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ is a BCFSP set.
$2 \Rightarrow 1$. Suppose $x \in$ Ş such that $x^2 \in \mathfrak{Q}$. As $\mathrm{M}^{\mathfrak{Q}} = \left(\lambda_{P-\mathrm{M}^{\mathfrak{Q}}}, \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}\right)$ is a BCFSP set, so $\lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x) \leq \lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x^2) = -1 - \iota\, 1$ and $\lambda_{N-\mathrm{M}^{\mathfrak{Q}}}(x) = -1 - \iota\, 1$, i.e. $x \in \mathfrak{Q}$. Therefore, $\mathfrak{Q}$ is a semiprime.

Theorem 12. For a BCF sub-semigroup $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ over Ş the following
(1) $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is BCFSP set on Ş.
(2) For each $x \in$ Ş, $\lambda_{P-\mathrm{M}}(x) \geq \lambda_{P-\mathrm{M}}(x^2) \Rightarrow \lambda_{RP-\mathrm{M}}(x) \geq \lambda_{RP-\mathrm{M}}(x^2)$ and $\lambda_{IP-\mathrm{M}}(x) \geq \lambda_{IP-\mathrm{M}}(x^2)$, and $\lambda_{N-\mathrm{M}}(x) \leq \lambda_{N-\mathrm{M}}(x^2) \Rightarrow \lambda_{RN-\mathrm{M}}(x) \leq \lambda_{RN-\mathrm{M}}(x^2)$ and $\lambda_{IN-\mathrm{M}}(x) \leq \lambda_{IN-\mathrm{M}}(x^2)$.

Proof. $1 \Rightarrow 2$. Suppose that $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCF semiprime set on Ş and $x \in$ Ş, then we get that

$$\lambda_{P-\mathrm{M}}(x) \geq \lambda_{P-\mathrm{M}}(x^2) \Rightarrow \lambda_{RP-\mathrm{M}}(x) \geq \lambda_{RP-\mathrm{M}}(x^2) \text{ and } \lambda_{IP-\mathrm{M}}(x) \geq \lambda_{IP-\mathrm{M}}(x^2),$$

and

$$\lambda_{N-\mathrm{M}}(x) \leq \lambda_{N-\mathrm{M}}(x^2) \Rightarrow \lambda_{RN-\mathrm{M}}(x) \leq \lambda_{RN-\mathrm{M}}(x^2) \text{ and } \lambda_{IN-\mathrm{M}}(x) \leq \lambda_{IN-\mathrm{M}}(x^2)$$

thus,

$$\lambda_{P-\mathrm{M}}(x^2) \geq \min\big(\lambda_{P-\mathrm{M}}(x), \lambda_{P-\mathrm{M}}(x)\big) = \lambda_{P-\mathrm{M}}(x)$$

$$\Rightarrow \lambda_{RP-\mathrm{M}}(x^2) \geq \min\big(\lambda_{RP-\mathrm{M}}(x), \lambda_{RP-\mathrm{M}}(x)\big) = \lambda_{RP-\mathrm{M}}(x) \text{ and}$$

$$\lambda_{IP-\mathrm{M}}(x^2) \geq \min\big(\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(x)\big) = \lambda_{IP-\mathrm{M}}(x),$$

and

$$\lambda_{N-\mathrm{M}}(x^2) \leq \max\big(\lambda_{N-\mathrm{M}}(x), \lambda_{N-\mathrm{M}}(x)\big) = \lambda_{N-\mathrm{M}}(x)$$

$$\Rightarrow \lambda_{RN-\mathrm{M}}(x^2) \leq \max\big(\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(x)\big) = \lambda_{RN-\mathrm{M}}(x) \text{ and}$$

$$\lambda_{IN-\mathrm{M}}(x^2) \leq \max\big(\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(x)\big) = \lambda_{IN-\mathrm{M}}(x).$$

Consequently, 2 holds. $2 \Rightarrow 1$ is obvious.
Before going to describe the next theorem, we recall the definition of intra-regular. If for every $x \in Ş$ $\exists\ y_1, y_2 \in Ş$ such that $x = y_1 x^2 y_2$.

Theorem 13. For $Ş$, the following
(1) $Ş$ is intra-regular;
(2) Each BCFTSI over $Ş$ is BCF semiprime,
are equivalent.

Proof. $1 \Rightarrow 2$. Assume that $\mathrm{M} = (\lambda_{P-\mathrm{M}}, \lambda_{N-\mathrm{M}}) = (\lambda_{RP-\mathrm{M}} + \iota\, \lambda_{IP-\mathrm{M}}, \lambda_{RN-\mathrm{M}} + \iota\, \lambda_{IN-\mathrm{M}})$ is a BCFTSI over $Ş$ and $x \in Ş$. As $Ş$ is intra-regular, so $\exists\ y_1, y_2 \in Ş$ such that $x = y_1 x^2 y_2$. Thus, we get

$$\lambda_{P-\mathrm{M}}(x) = \lambda_{P-\mathrm{M}}(y_1 x^2 y_2) \Rightarrow \lambda_{RP-\mathrm{M}}(x) = \lambda_{RP-\mathrm{M}}(y_1 x^2 y_2) \geq \lambda_{RP-\mathrm{M}}(x^2 y_2) \geq \lambda_{RP-\mathrm{M}}(x^2).$$

And $\lambda_{IP-\mathrm{M}}(x) = \lambda_{IP-\mathrm{M}}(y_1 x^2 y_2) \geq \lambda_{IP-\mathrm{M}}(x^2 y_2) \geq \lambda_{IP-\mathrm{M}}(x^2)$, thus

$$\lambda_{P-\mathrm{M}}(x) \geq \lambda_{P-\mathrm{M}}(x^2)$$

and

$$\lambda_{N-\mathrm{M}}(x) = \lambda_{N-\mathrm{M}}(y_1 x^2 y_2) \Rightarrow \lambda_{RN-\mathrm{M}}(x) = \lambda_{RN-\mathrm{M}}(y_1 x^2 y_2) \leq \lambda_{RN-\mathrm{M}}(x^2 y_2) \leq \lambda_{RN-\mathrm{M}}(x^2).$$

And $\lambda_{IN-\mathrm{M}}(x) = \lambda_{IN-\mathrm{M}}(y_1 x^2 y_2) \leq \lambda_{IN-\mathrm{M}}(x^2 y_2) \leq \lambda_{IN-\mathrm{M}}(x^2)$, thus

$$\lambda_{N-\mathrm{M}}(x) \leq \lambda_{N-\mathrm{M}}(x^2).$$

It follows that $\lambda_{P-\mathrm{M}}(x) = \lambda_{P-\mathrm{M}}(x^2)$ and $\lambda_{N-\mathrm{M}}(x) = \lambda_{N-\mathrm{M}}(x^2)$.
$2 \Rightarrow 1$. As 1 holds, so by Theorem (4 part (4)), we have that bipolar complex characteristic function $\mathrm{M}^{\Im[x^2]} = \left(\lambda_{P-\mathrm{M}^{\Im[x^2]}}, \lambda_{N-\mathrm{M}^{\Im[x^2]}}\right)$ of principal ideal $\Im[x^2] = x^2 \cup Şx^2 \cup x^2Ş \cup x^2Şx^2$ of $Ş$ is a BCFTSI over $Ş$. As $x^2 \in \Im[x^2]$, so $\lambda_{N-\mathrm{M}^{\Im[x^2]}}(x) = \lambda_{N-\mathrm{M}^{\Im[x^2]}}(x^2) = -1 - \iota\, 1 \Rightarrow x \in x^2 \cup Şx^2 \cup x^2Ş \cup x^2Şx^2$. $Ş$ is intra-regular. This completes the proof.

Theorem 14. For $Ş$, the following
(1) $Ş$ is intra-regular;
(2) $\mathrm{M}_1 \cap \mathrm{M}_2 \preccurlyeq \mathrm{M}_1 \circledcirc \mathrm{M}_2$ for each BCFLI $\mathrm{M}_1 = \left(\lambda_{P-\mathrm{M}_1}, \lambda_{N-\mathrm{M}_1}\right) = \left(\lambda_{RP-\mathrm{M}_1} + \iota\, \lambda_{IP-\mathrm{M}_1}, \lambda_{RN-\mathrm{M}_1} + \iota\, \lambda_{IN-\mathrm{M}_1}\right)$ and for each BCFRI $\mathrm{M}_2 = \left(\lambda_{P-\mathrm{M}_2}, \lambda_{N-\mathrm{M}_2}\right) = \left(\lambda_{RP-\mathrm{M}_2} + \iota\, \lambda_{IP-\mathrm{M}_2}, \lambda_{RN-\mathrm{M}_2} + \iota\, \lambda_{IN-\mathrm{M}_2}\right)$ over $Ş$,
are equivalent.

Proof. $1 \Rightarrow 2$. Suppose that $M_1 = \left(\lambda_{P-M_1}, \lambda_{N-M_1}\right) = \left(\lambda_{RP-M_1} + \iota\,\lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\,\lambda_{IN-M_1}\right)$ is a BCFLI and $M_2 = \left(\lambda_{P-M_2}, \lambda_{N-M_2}\right) = \left(\lambda_{RP-M_2} + \iota\,\lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\,\lambda_{IN-M_2}\right)$ is a BCFRI over Ş and $x \in$ Ş, then as Ş is intra-regular so $\exists\, y_1, y_2 \in$ Ş such that $x = y_1 x^2 y_2$. Thus,

$$\left(\lambda_{RP-M_1} \circ \lambda_{RP-M_2}\right)(x) = \sup_{x = z_1 z_2} \left\{\min\left(\lambda_{RP-M_1}(z_1), \lambda_{RP-M_2}(z_2)\right)\right\}$$

$$\geq \min\left(\lambda_{RP-M_1}(y_1 x), \lambda_{RP-M_2}(x z_2)\right)$$

$$\geq \min\left(\lambda_{RP-M_1}(x), \lambda_{RP-M_2}(x)\right) = \left(\lambda_{RP-M_1} \wedge \lambda_{RP-M_2}\right)(x)$$

and

$$\left(\lambda_{IP-M_1} \circ \lambda_{IP-M_2}\right)(x) = \sup_{x = z_1 z_2} \left\{\min\left(\lambda_{IP-M_1}(z_1), \lambda_{IP-M_2}(z_2)\right)\right\}$$

$$\geq \min\left(\lambda_{IP-M_1}(y_1 x), \lambda_{IP-M_2}(x z_2)\right)$$

$$\geq \min\left(\lambda_{IP-M_1}(x), \lambda_{IP-M_2}(x)\right) = \left(\lambda_{IP-M_1} \wedge \lambda_{IP-M_2}\right)(x).$$

Next,

$$\left(\lambda_{RN-M_1} \circ \lambda_{RN-M_2}\right)(x) = \inf_{x = z_1 z_2} \left\{\max\left(\lambda_{RN-M_1}(z_1), \lambda_{RN-M_2}(z_2)\right)\right\}$$

$$\leq \max\left(\lambda_{RN-M_1}(y_1 x), \lambda_{RN-M_2}(x z_2)\right)$$

$$\leq \max\left(\lambda_{RN-M_1}(x), \lambda_{RN-M_2}(x)\right) = \left(\lambda_{RN-M_1} \vee \lambda_{RN-M_2}\right)(x)$$

and

$$\left(\lambda_{IN-M_1} \circ \lambda_{IN-M_2}\right)(x) = \inf_{x = z_1 z_2} \left\{\max\left(\lambda_{IN-M_1}(z_1), \lambda_{IN-M_2}(z_2)\right)\right\}$$

$$\leq \max\left(\lambda_{IN-M_1}(y_1 x), \lambda_{IN-M_2}(x z_2)\right)$$

$$\leq \max\left(\lambda_{IN-M_1}(x), \lambda_{IN-M_2}(x)\right) = \left(\lambda_{IN-M_1} \vee \lambda_{IN-M_2}\right)(x).$$

Thus, we have $M_1 \cap M_2 \preccurlyeq M_1 \circledcirc M_2$.
$2 \Rightarrow 1$. Suppose that $\mathfrak{U}_1$ is any left ideal of Ş and $\mathfrak{U}_2$ is any right ideal of Ş, and $x \in$ Ş such that $x \in \mathfrak{U}_1 \cap \mathfrak{U}_2$, then $x \in \mathfrak{U}_1$ and $x \in \mathfrak{U}_2$, by Theorem 4 $M^{\mathfrak{U}_1} = \left(\lambda_{P-M^{\mathfrak{U}_1}}, \lambda_{N-M^{\mathfrak{U}_1}}\right)$ is a BCFLI and $M^{\mathfrak{U}_1} = \left(\lambda_{P-M^{\mathfrak{U}_1}}, \lambda_{N-M^{\mathfrak{U}_1}}\right)$ is a BCFRI over Ş. Now by Lemma 1, we obtain

$$\left(\lambda_{N-M^{\mathfrak{U}_1\mathfrak{U}_2}}\right)(x) = \left(\lambda_{N-M^{\mathfrak{U}_1}} \circ \lambda_{N-M^{\mathfrak{U}_2}}\right)(x)$$

$$\leq (\lambda_{N-M} \wedge \lambda_{N-M})(x) = \left(\lambda_{N-M^{\mathfrak{U}_1 \cap \mathfrak{U}_2}}\right)(x) = -1 - \iota\, 1.$$

Thus, we have $x \in \mathfrak{U}_1\mathfrak{U}_2$ and we get that $\mathfrak{U}_1 \cap \mathfrak{U}_2 \subseteq \mathfrak{U}_1\mathfrak{U}_2$. Consequently, Ş is intra-regular.
Theorem 15. For Şk, the following
(1) Ş is regular and intra-regular;

(2) $M_1 \cap M_2 \preccurlyeq (M_1 \circledcirc M_2) \cap (M_2 \circledcirc M_1)$ for each BCFRI $M_1 = (\lambda_{P-M_1}, \lambda_{N-M_1}) = (\lambda_{RP-M_1} + \iota\,\lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\,\lambda_{IN-M_1})$ and BCFRI $M_2 = (\lambda_{P-M_2}, \lambda_{N-M_2}) = (\lambda_{RP-M_2} + \iota\,\lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\,\lambda_{IN-M_2})$ over Ş,

are equivalent.

Proof. $1 \Rightarrow 2$. Suppose that $M_1 = (\lambda_{P-M_1}, \lambda_{N-M_1}) = (\lambda_{RP-M_1} + \iota\,\lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\,\lambda_{IN-M_1})$ is a BCFRI and $M_2 = (\lambda_{P-M_2}, \lambda_{N-M_2}) = (\lambda_{RP-M_2} + \iota\,\lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\,\lambda_{IN-M_2})$ is a BCFLI over Ş, then by employing Theorems 8 and 14 we have that

$M_1 \cap M_2 = M_2 \cap M_1 \preccurlyeq M_2 \circledcirc M_1$ and $M_1 \cap M_2 \preccurlyeq M_1 \circledcirc M_2$.

Thus,

$$M_1 \cap M_2 \preccurlyeq (M_1 \circledcirc M_2) \cap (M_2 \circledcirc M_1).$$

$2 \Rightarrow 1$. Suppose that $M_1 = (\lambda_{P-M_1}, \lambda_{N-M_1}) = (\lambda_{RP-M_1} + \iota\,\lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\,\lambda_{IN-M_1})$ is a BCFRI and $M_2 = (\lambda_{P-M_2}, \lambda_{N-M_2}) = (\lambda_{RP-M_2} + \iota\,\lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\,\lambda_{IN-M_2})$ is a BCFLI over Ş, then

$$M_1 \cap M_2 \preccurlyeq M_1 \cap M_2 \preccurlyeq (M_1 \circledcirc M_2) \cap (M_2 \circledcirc M_1) \preccurlyeq M_2 \circledcirc M_1.$$

Therefore, by employing Theorem 14 we get that Ş is intra-regular. Next,

$$(M_1 \circledcirc M_2) \preccurlyeq Ş \circledcirc M_2 \preccurlyeq M_2 \text{ and } (M_1 \circledcirc M_2) \preccurlyeq M_1 \circledcirc Ş \preccurlyeq M_1,$$

which implies that $M_1 \circledcirc M_2 \preccurlyeq M_1 \cap M_2$ and it always holds that $M_1 \cap M_2 \preccurlyeq M_1 \circledcirc M_2 \Rightarrow M_1 \cap M_2 = M_1 \circledcirc M_2$. Consequently, Ş is a regular semigroup.

Now we recall the conception of semi-simple before discussing the next theorem. If every two-sided ideal of Ş is idempotent then Ş is known as semi-simple.

Theorem 16. For Şk, the following

(1) Ş is semi-simple,

(2) Each BCFTSI on Ş is idempotent,

(3) $M_1 \cap M_2 \preccurlyeq M_1 \circledcirc M_2$ for each BCFTSIs $M_1 = (\lambda_{P-M_1}, \lambda_{N-M_1}) = (\lambda_{RP-M_1} + \iota\,\lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\,\lambda_{IN-M_1})$ and $M_2 = (\lambda_{P-M_2}, \lambda_{N-M_2}) = (\lambda_{RP-M_2} + \iota\,\lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\,\lambda_{IN-M_2})$ over Ş,

are equivalent.

Proof. $1 \Rightarrow 2$. Suppose that $M_1 = (\lambda_{P-M_1}, \lambda_{N-M_1}) = (\lambda_{RP-M_1} + \iota\,\lambda_{IP-M_1}, \lambda_{RN-M_1} + \iota\,\lambda_{IN-M_1})$ and $M_2 = (\lambda_{P-M_2}, \lambda_{N-M_2}) = (\lambda_{RP-M_2} + \iota\,\lambda_{IP-M_2}, \lambda_{RN-M_2} + \iota\,\lambda_{IN-M_2})$ are two BCFTSIs over Ş, by assumption

$$(M_1 \circledcirc M_2) \preccurlyeq Ş \circledcirc M_2 \preccurlyeq M_2 \text{ and } (M_1 \circledcirc M_2) \preccurlyeq M_1 \circledcirc Ş \preccurlyeq M_1,$$

which implies that $M_1 \circledcirc M_2 \preccurlyeq M_1 \cap M_2$. Next, let $x \in Ş$ and as Ş is semi-simple so $\exists\ y_1, y_2, z_1, z_2 \in Ş$ such that $x = (y_1 x y_2)(z_1 x z_2)$, thus

$$(\lambda_{RP-M} \circ \lambda_{RP-M})(x) = \sup_{x=ab}\{\min(\lambda_{RP-M}(a), \lambda_{RP-M}(b))\}$$

$$\geq \min(\lambda_{RP-M}(y_1 x y_2), \lambda_{RP-M}(z_1 x z_2))$$

$$\geq \min(\lambda_{RP-M}(x y_2), \lambda_{RP-M}(x z_2))$$

$$\geq \min(\lambda_{RP-M}(x), \lambda_{RP-M}(x)) = (\lambda_{RP-M} \wedge \lambda_{RP-M})(x)$$

and

$$(\lambda_{IP-\mathrm{M}} \circ \lambda_{IP-\mathrm{M}})(x) = \sup_{x=ab}\{\min(\lambda_{IP-\mathrm{M}}(a), \lambda_{IP-\mathrm{M}}(b))\}$$

$$\geq \min(\lambda_{IP-\mathrm{M}}(y_1xy_2), \lambda_{IP-\mathrm{M}}(z_1xz_2))$$

$$\geq \min(\lambda_{IP-\mathrm{M}}(xy_2), \lambda_{IP-\mathrm{M}}(xz_2))$$

$$\geq \min(\lambda_{IP-\mathrm{M}}(x), \lambda_{IP-\mathrm{M}}(x)) = (\lambda_{IP-\mathrm{M}} \wedge \lambda_{IP-\mathrm{M}})(x).$$

Thus, $(\lambda_{P-\mathrm{M}} \circ \lambda_{P-\mathrm{M}})(x) \geq (\lambda_{P-\mathrm{M}} \wedge \lambda_{P-\mathrm{M}})(x)$. Next,

$$(\lambda_{RN-\mathrm{M}} \circ \lambda_{RN-\mathrm{M}})(x) = \inf_{x=ab}\{\max(\lambda_{RN-\mathrm{M}}(a), \lambda_{RN-\mathrm{M}}(b))\}$$

$$\leq \max(\lambda_{RN-\mathrm{M}}(y_1xy_2), \lambda_{RN-\mathrm{M}}(z_1xz_2))$$

$$\leq \max(\lambda_{RN-\mathrm{M}}(xy_2), \lambda_{RN-\mathrm{M}}(xz_2))$$

$$\leq \max(\lambda_{RN-\mathrm{M}}(x), \lambda_{RN-\mathrm{M}}(x)) = (\lambda_{RN-\mathrm{M}} \vee \lambda_{RN-\mathrm{M}})(x)$$

and

$$(\lambda_{IN-\mathrm{M}} \circ \lambda_{IN-\mathrm{M}})(x) = \inf_{x=ab}\{\max(\lambda_{IN-\mathrm{M}}(a), \lambda_{IN-\mathrm{M}}(b))\}$$

$$\leq \max(\lambda_{IN-\mathrm{M}}(y_1xy_2), \lambda_{IN-\mathrm{M}}(z_1xz_2))$$

$$\leq \max(\lambda_{IN-\mathrm{M}}(xy_2), \lambda_{IN-\mathrm{M}}(xz_2))$$

$$\leq \max(\lambda_{IN-\mathrm{M}}(x), \lambda_{IN-\mathrm{M}}(x)) = (\lambda_{IN-\mathrm{M}} \vee \lambda_{IN-\mathrm{M}})(x).$$

Thus, $\mathrm{M}_1 \circledcirc \mathrm{M}_2 \preccurlyeq \mathrm{M}_1 \cap \mathrm{M}_2$ and so $\mathrm{M}_1 \circledcirc \mathrm{M}_2 = \mathrm{M}_1 \cap \mathrm{M}_2$.
$3 \Rightarrow 2$ is obvious.
$2 \Rightarrow 1$. Suppose that $x \in Ş$, then by employing Theorem (4 part (4)), we have that bipolar complex characteristic function $\mathrm{M}^{\Im[x]} = (\lambda_{P-\mathrm{M}^{\Im[x]}}, \lambda_{N-\mathrm{M}^{\Im[x]}})$ of principal ideal $\Im[x]$ of $Ş$ is a BCFTSI over $Ş$. By Lemma 1 we obtain

$$(\lambda_{N-\mathrm{M}^{\Im[x]\Im[x]}})(x) = (\lambda_{N-\mathrm{M}^{\Im[x]}} \circ \lambda_{N-\mathrm{M}^{\Im[x]}})(x)$$

$$\leq (\lambda_{N-\mathrm{M}^{\Im[x]}} \wedge \lambda_{N-\mathrm{M}^{\Im[x]}})(x) = (\lambda_{N-\mathrm{M}^{\Im[x]\cap\Im[x]}})(x) = -1 - \iota 1.$$

Since, $x \in \Im[x]\Im[x]\Im[x]$, we have

$$x \in (x \cup Şx \cup xŞ \cup ŞxŞ)(x \cup Şx \cup xŞ \cup ŞxŞ)(x \cup Şx \cup xŞ \cup ŞxŞ) \subseteq (ŞxŞ)(ŞxŞ).$$

Therefore, $Ş$ is semi-simple.

5. Conclusions

The conception of a semigroup is an influential approach and has been utilized by numerous scholars and employed in various areas. Due to the great significance of semigroup, numerous authors modified this concept to introduce novel notions such as fuzzy semigroup, bipolar fuzzy semigroup, etc. The concept of fuzzy semigroup has various applications such as fuzzy languages, theory fuzzy coding, etc. In recent years, numerous authors generalized the conception of fuzzy algebraic structures and employed genuine-life dilemmas in various areas of science. To keep in mind all this, and the research gap, in this analysis we investigated the algebraic structure of semigroups by employing the

BCF set. Firstly, we established BCF sub-semigroup, BCFLI, BCFRI, and BCFTSI over $\c{S}$ and then initiated their related theorem with proof. Further, we diagnosed bipolar complex characteristic function, positive (ω,η)-cut, negative (ϱ,σ)-cut, positive and $\big((\omega,\eta),(\varrho,\sigma)\big)$-cut and their associated results with proof. Secondly, we established various classes of semigroups such as intra-regular, left regular, right regular, and semi-simple, by the features of the BCF ideals and proved their related results. Also, these classes are interpreted in terms of BCFLIs, BCFRIs, and BCFTSIs. In this regard, we showed that, for a semigroup $\c{S}$, $\c{S}$ is a regular semigroup if and only if for each BCFLI $\mathrm{M}_1=\big(\lambda_{P-\mathrm{M}_1},\lambda_{N-\mathrm{M}_1}\big)=\big(\lambda_{RP-\mathrm{M}_1}+\iota\,\lambda_{IP-\mathrm{M}_1},\lambda_{RN-\mathrm{M}_1}+\iota\,\lambda_{IN-\mathrm{M}_1}\big)$ and BCFRI $\mathrm{M}_2=\big(\lambda_{P-\mathrm{M}_2},\lambda_{N-\mathrm{M}_2}\big)=\big(\lambda_{RP-\mathrm{M}_2}+\iota\,\lambda_{IP-\mathrm{M}_2},\lambda_{RN-\mathrm{M}_2}+\iota\,\lambda_{IN-\mathrm{M}_2}\big)$ over $\c{S}$, $\mathrm{M}_1\cap\mathrm{M}_2=\mathrm{M}_1\circledcirc\mathrm{M}_2$. Furthermore, we construed regular, intra-regular semigroup and showed that a semigroup $\c{S}$ is regular and intra-regular iff $\mathrm{M}_1\cap\mathrm{M}_2\preccurlyeq\mathrm{M}_1\circledcirc\mathrm{M}_2$ for each BCFLI $\mathrm{M}_1=\big(\lambda_{P-\mathrm{M}_1},\lambda_{N-\mathrm{M}_1}\big)=\big(\lambda_{RP-\mathrm{M}_1}+\iota\,\lambda_{IP-\mathrm{M}_1},\lambda_{RN-\mathrm{M}_1}+\iota\,\lambda_{IN-\mathrm{M}_1}\big)$ and for each BCFRI $\mathrm{M}_2=\big(\lambda_{P-\mathrm{M}_2},\lambda_{N-\mathrm{M}_2}\big)=\big(\lambda_{RP-\mathrm{M}_2}+\iota\,\lambda_{IP-\mathrm{M}_2},\lambda_{RN-\mathrm{M}_2}+\iota\,\lambda_{IN-\mathrm{M}_2}\big)$ over $\c{S}$. The introduced combination of BCFS and semigroup is the generalization of the fuzzy set (FS), bipolar fuzzy set (BFS), and complex FS (CFS) in the environment of semigroups and from the introduced notions we can easily achieve these conceptions.

In the future, we want to expand this research to BCF bi-ideals, BCF quasi-ideals, and BCF interior ideals. Further, we would like to review numerous notions like BCF soft sets [46], interval-valued neutrosophic SSs [48], and bipolar complex intuitionistic FS [49] and would try to fuse them with the notion of the semigroup.

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Conflict of interest

About the publication of this manuscript the authors declare that they have no conflict of interest.

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