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Research article

Existence results by Mönch's fixed point theorem for a tripled system of sequential fractional differential equations

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Abstract: In this paper, we study the existence of the solutions for a tripled system of Caputo sequential fractional differential equations. The main results are established with the aid of Mönch's fixed point theorem. The stability of the tripled system is also investigated via the Ulam-Hyer technique. In addition, an applied example with graphs of the behaviour of the system solutions with different fractional orders are provided to support the theoretical results obtained in this study.

Keywords: Caputo fractional derivative; tripled system; existence; mixed boundary conditions; H-U stability
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1. Introduction

Fractional calculus (FCs) was first employed in 1695 when L'Hpital's summarized his discoveries in a letter to Leibniz. (FCs) was studied by several twentieth-century authors, including Liouville, Grunwald, Letnikov, and Riemann. This field of mathematics, known as fractional differential equations (FDEs), was invented by mathematicians as a pure branch of mathematics with just a few applications in mathematics. Fractional differential equations appear naturally in a number of fields, such as physics, engineering, biophysics, blood flow phenomena, aerodynamics electroanalytical chemistry, biology, and economics. For more details, we refer the readers to [1–5] and many other references therein.

In [6], the authors studied the drug concentration in the blood model via Psi-Caputo fractional derivative, where the fractional model there shows more accurate results in estimating the drug concentration in the blood. And in [7], again the fractional modeling for the logistic population growth shows superiority over the ordinary one.

The majority of research on FDEs is based on fractional derivatives of the R-L and Caputo types. See [8–13]. Several studies have been conducted over the years to investigate how stability concepts such as Mittag-Lefler function, exponential, and Lyapunov stability apply to various types of dynamic systems. Ulam and Hyers, on the other hand, identified previously unknown types of stability known as Ulam-stability [14–19]. Hyers type of stability study contributes significantly to our understanding of chemical processes and fluid movement, as well as semiconductors, population dynamics, heat conduction, and elasticity. While others have reported results using other types of stability, Ulams group designed and implemented a type of stability for ordinary, fractional differential, and difference equations, see [20–22].

Tripled fractional boundary conditions are regulated by three related differential equations with three initial or boundary conditions. Despite popular belief, researchers have paid less attention to studies of tripled fractional systems. According to the authors observations, there is no analytical literature on the existence of tripled systems of SFDEs. This is true to the best of their knowledge. A tripled fractional boundary value problem is being investigated by a few researchers. In [23], nonlinear mappings in partially ordered complete metric spaces were only studied by Berinde and Borcut, who developed the idea of tripled fixed points. Karakaya et al. [24] studied tripled fixed points for a class of condensing operators in Banach spaces.

Recently, Subramanian et al. [18] investigated the existence and Hyers-Ulam type stability results fa or nonlinear coupled system of Caputo-Hadamard type FDEs with multi-point a non-local integral boundary conditions via the alternatives of Leray-Schauder, Banach fixed point theorems, H-U stable. Authors in [25], studied a nonlinear coupled system of three fractional differential equations with non-local coupled boundary conditions

$$\begin{cases} {}^{c}D^{\eta}_{a^{+}}u(\varpi) = \rho(\varpi, u(\varpi), x(\varpi), y(\varpi)), \quad 1 < \eta \leq 2, \varpi \in [a,b], \\ {}^{c}D^{\xi}_{a^{+}}x(\varpi) = \varphi(\varpi, u(\varpi), x(\varpi), y(\varpi)), \quad 1 < \xi \leq 2, \varpi \in [a,b], \\ {}^{c}D^{\zeta}_{a^{+}}y(\varpi) = \psi(\varpi, u(\varpi), x(\varpi), y(\varpi)), \quad 2 < \zeta \leq 3, \varpi \in [a,b], \\ u(a) = u_0, \qquad u(b) = \sum_{i=1}^{m} p_i x(\psi_i), \\ x(a) = 0, \qquad x(b) = \sum_{j=1}^{n} q_j y(\phi_j), \\ y(\xi_1) = 0, \qquad y(\xi_2) = 0, \qquad y(b) = \sum_{k=1}^{l} r_k u(\omega_k), \\ a < \xi_1 < \xi_2 < \psi_1 < \cdots < \psi_m < \phi_1 < \cdots < \phi_n < \omega_1 < \cdots < \omega_l < b, \end{cases}$$

where ${}^{c}D^{\chi}$ is a CFDs of order $\chi \in \{\eta, \xi, \zeta\}, \rho, \varphi, \psi : [a,b] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ are given functions, $p_i, q_j, r_k \in \mathcal{R}_e, i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, l$. The existence and uniqueness results for the system are proved via Leray-Schauder alternative and Banach's contraction mapping principle.

In [26], the authors investigated the existence and uniqueness of a solution for the tripled fractional systems with cyclic boundary conditions:

$$\begin{cases} {}^{c}\mathcal{D}^{\psi_k}\mathfrak{w}_k(\tau) = \mathfrak{f}_k(\tau, \mathfrak{w}(\tau)), \qquad 1 < \psi_k \leq 2, \\ \mathfrak{w}_k^{(j)}(0) = a_{k,j}\mathfrak{w}^{(j)}_{\varepsilon(k)}(\mathcal{T}), \qquad k = 1,2,3; j = 0,1. \end{cases}$$

Where ${}^{c}\mathcal{D}^{\psi_k}_0$ denotes the Caputo fractional derivatives (CFDs) of order ψ_k, $\tau \in J = [0, \mathcal{T}], \mathfrak{f}_k : J \times$

$\mathcal{R}_e^3 \to \mathcal{R}_e$ are continuous functions, $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3) \in \mathcal{R}_e^3, \varepsilon = (1, 2, 3)$ is a cyclic permutation, and $a_{k,j} \in k = 1, 2, 3, j = 0, 1$.

Recently, in 2022, the authors developed the existence theory for a new class of nonlinear coupled systems of sequential fractional differential equations supplemented with coupled, non-conjugate, Riemann-Stieltjes, integro-multipoint boundary conditions [27]:

$$\begin{cases} ({}^c\mathcal{D}^{\xi_1+1} + {}^c\mathcal{D}^{\xi_1})\Phi_1(\omega) = \mathcal{G}_1(\omega, \Phi_1(\omega), \Psi_1(\omega)), & 2 < \xi_1 < 3, \omega \in [0, 1], \\ ({}^c\mathcal{D}^{\zeta_1+1} + {}^c\mathcal{D}^{\zeta_1})\Psi_1(\omega) = \mathcal{G}_2(\omega, \Phi_1(\omega), \Psi_1(\omega)), & 2 < \zeta_1 < 3, \omega \in [0, 1], \end{cases} \tag{1.1}$$

subject to the coupled boundary conditions:

$$\begin{cases} \Phi_1(0) = 0, \Phi_1'(0) = 0,\ \Phi_1'(0) = 0,\ \Phi_1(1) = k\int_0^\rho \Psi_1(s)dAs + \sum_{i=1}^{n-2} \alpha_i \Psi_1(\sigma_i) + k_1 \int_v^1 \Psi_1(s)dA(s), \\ \Psi_1(0) = 0, \Psi_1'(0) = 0,\ \Psi_1'(0) = 0,\ \Psi_1(1) = h\int_0^\rho \Phi_1(s)dAs + \sum_{i=1}^{n-2} \beta_i \Phi_1(\sigma_i) + h_1 \int_v^1 \Phi_1(s)dA(s), \end{cases} \tag{1.2}$$

where ${}^c\mathcal{D}^P$ denotes the Caputo fractional derivative of order $P \in \xi_1, \zeta_1$, $0 < \rho < \sigma_i < v < 1$, $\mathcal{G}_1, \mathcal{G}_2 : [0, 1] \times \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ are given continuous functions, $k, k_1, h, h_1, \alpha_i, \beta_i \in \mathcal{R}, i = 1, 2, \cdots n-2$ and A is a function of bounded variation.

Motivated by the aforementioned works, the following system represents a unique class of tripled systems of SFDEs equipped with non-local multi-point coupled boundary conditions:

$$\begin{cases} ({}^c\mathfrak{D}^{\psi} + \varphi{}^c\mathfrak{D}^{\psi-1})\mathfrak{w}(\varpi) = \mathfrak{f}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)), \quad 2 < \psi \leq 3, \\ ({}^c\mathfrak{D}^{\phi} + \varphi{}^c\mathfrak{D}^{\phi-1})\mathfrak{v}(\varpi) = \mathfrak{g}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)), \quad 2 < \phi \leq 3, \\ ({}^c\mathfrak{D}^{\omega} + \varphi{}^c\mathfrak{D}^{\omega-1})\mathfrak{u}(\varpi) = \mathfrak{h}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)), \quad 3 < \omega \leq 4, \\ \mathfrak{w}(0) = 0, \qquad \mathfrak{w}'(0) = 0, \qquad \mathfrak{w}(\mathcal{T}) = \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \mathfrak{v}(\zeta_j) + \Pi_1 \mathcal{I}^{\varsigma} \mathfrak{v}(\vartheta), \\ \mathfrak{v}(0) = 0, \qquad \mathfrak{v}'(0) = 0, \qquad \mathfrak{v}(\mathcal{T}) = \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \mathfrak{u}(\zeta_j) + \Pi_2 \mathcal{I}^{\varrho} \mathfrak{u}(\vartheta), \\ \mathfrak{u}(0) = 0, \qquad \mathfrak{u}'(0) = 0, \qquad \mathfrak{u}''(0) = 0, \qquad \mathfrak{u}(\mathcal{T}) = \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \mathfrak{w}(\zeta_j) + \Pi_3 \mathcal{I}^{\delta} \mathfrak{w}(\vartheta), \end{cases} \tag{1.3}$$

where ${}^c\mathfrak{D}^{\chi}$ is a CFDs of order $\chi \in \{\psi, \phi, \omega\}$, $\mathfrak{f}, \mathfrak{g}, \mathfrak{h} : [0, \mathcal{T}] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ are given functions, $\xi_j, \nu_j, \sigma_j \in \mathcal{R}_e, j = 1, \ldots, k-2, 0 < \zeta_j, \vartheta < 1, \varphi > 0$ are non-negative real constants and $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Pi_1, \Pi_2, \Pi_3$ are real constants.

The originality and distinction of this work are summarized in employing mönchs fixed point theorem with the aid of the Kuratowski measure of non-compactness and Carathéodorys conditions, to verify the necessary conditions for the existence of the solution to the system of fractional and nonlinear equations of sequential type. This work also examines the stability of the solution for the proposed system of equations.

The following is the remainder of the article: The second section is dedicated to explaining the fundamental principles of fractional calculus and the associated definitions and lemmas. mönchs fixed point theorem is used in Section 3 to show existence results. In Section 4, the stability of Hyers-

Ulam solutions is examined, and a set of requirements are established that ensures the stability of these solutions. Section 5 provides some examples are provided applied example support the theoretical claims.

2. Preliminaries

This portion introduces basic fractional calculus concepts, definitions, and tentative results [1–3].

Definition 2.1. *[28] The left and right-sided generalized fractional integrals (GFIs) of $f \in \mathcal{Z}_b^q(c,d)$ of order $\xi > 0$ and $\rho < 0$ for $-\infty < c < \tau < d < \infty$, are are defined as follows:*

$$({}^{\rho}\mathcal{I}_{c+}^{\xi} f)(\tau) = \frac{\rho^{1-\xi}}{\omega(\xi)} \int_c^{\tau} \frac{s^{\rho-1}}{(\tau^{\rho} - s^{\rho})^{1-\xi}} f(\rho) d\rho, \tag{2.1}$$

$$({}^{\rho}\mathcal{I}_{d-}^{\zeta} f)(\tau) = \frac{\rho^{1-\xi}}{\omega(\xi)} \int_{\tau}^{d} \frac{s^{\rho-1}}{(s^{\rho} - \tau^{\rho})^{1-\xi}} f(\rho) d\rho. \tag{2.2}$$

Definition 2.2. *[28] The fractional integral of order ψ with the lower limit zero for a function k is defined as*

$$\mathcal{I}^{\psi} k(\tau) = \frac{1}{\Gamma(\psi)} \int_0^{\tau} \frac{k(\rho)}{(\tau - \rho)^{1-\psi}} ds, \tau > 0, \psi > 0, \tag{2.3}$$

provided the right-hand side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\psi) = \int_0^{\infty} \tau^{\psi-1} e^{-\tau} d\tau$.

Definition 2.3. *[28] The generalized fractional derivatives (GFDs) which are associated with GFIs (2.1) and (2.2) for $0 \le c < \tau < d < \infty$, are defined as follows:*

$$\begin{aligned} ({}^{\rho}\mathcal{D}_{c+}^{\xi} f)(\tau) &= \left(\tau^{1-\rho} \frac{d}{d\tau}\right)^n ({}^{\rho}\mathcal{I}_{c+}^{n-\xi} f)(\tau) \\ &= \frac{\rho^{\xi-n+1}}{\omega(n-\xi)} \left(\tau^{1-\rho} \frac{d}{d\tau}\right)^n \int_c^{\tau} \frac{s^{\rho-1}}{(\tau^{\rho} - s^{\rho})^{\xi-n+1}} f(\rho) d\rho, \end{aligned} \tag{2.4}$$

$$\begin{aligned} ({}^{\rho}\mathcal{D}_{d-}^{\xi} f)(\tau) &= \left(-\tau^{1-\rho} \frac{d}{d\tau}\right)^n ({}^{\rho}\mathcal{I}_{d-}^{n-\xi} f)(\tau) \\ &= \frac{\rho^{\xi-n+1}}{\omega(n-\xi)} \left(-\tau^{1-\rho} \frac{d}{d\tau}\right)^n \int_{\tau}^{d} \frac{s^{\rho-1}}{(\tau^{\rho} - s^{\rho})^{\xi-n+1}} f(\rho) d\rho, \end{aligned} \tag{2.5}$$

if the integrals exist.

Definition 2.4. *[28] The Riemann-Liouville fractional derivative of order $\psi > 0, n-1 < \psi < n, n \in \mathbb{N}$ is defined as*

$$D_{0+}^{\psi} k(\tau) = \frac{1}{\Gamma(n-\psi)} \left(\frac{d}{d\tau}\right)^n \int_0^{\tau} (\tau - \rho)^{n-\psi-1} k(\rho) d\rho, \tau > 0, \tag{2.6}$$

where the function k has absolutely continuous derivative up to order $(n-1)$.

Definition 2.5. *[28] The Caputo derivative of order $\psi \in [n-1, n)$ for a function $k : [0, \infty) \to (\mathbb{R})$ can be written as*

$$ {}^{c}D^{\psi}_{0+}k(\tau) = D^{\psi}_{0+}\left(k(\tau) - \sum_{m=0}^{n-1} \frac{\tau^m}{m!} f^{(m)}(0)\right), \tau > 0, n-1 < r < n. \tag{2.7}$$

Note that the CFDs of order $\psi \in [n-1, n)$ almost everywhere on $[0, \infty)$ if $k \in \mathcal{AC}^n([0, \infty), (\mathbb{R}))$.

Remark 2.1. *[28] If $k \in C^n[0, \infty)$, then*

$$ {}^{c}D^{\psi}_{0+}k(\tau) = \frac{1}{\Gamma(n-\psi)} \int_0^{\tau} \frac{k^n(\rho)}{(\tau-\rho)^{\psi+1-n}} ds = I^{n-\psi}k(n)(\tau), \tau > 0, n-1 < \psi < n.$$

Denote the Banach space of all continuous functions z from $[\mathfrak{a}, \mathcal{T}]$ into Q by $C([\mathfrak{a}, \mathcal{T}], Q)$, accompanied by the norm: $\|Z\| = \sup_{\mathfrak{a} \le \tau \le \mathcal{T}} \{z(\tau)\}$

Definition 2.6. *(See [29]). The Kuratowski measure of non-compactness $k(\cdot)$. Defined on bounded set $\mathcal{U}$ of Banach space Q is :*

$$k(\mathcal{U}) := \inf\{r > 0 : \mathcal{U} = \cup_{i=1}^{m} \mathcal{U}_i \ \textit{and diam } (\mathcal{U}_i) \le r, \ for \ 1 \le i \le m\}.$$

Lemma 2.1. *(See [29]). Given the Banach space Q with $\mathcal{U}, \mathcal{V}$ are two bounded proper subsets of Q, then the following properties hold true:*

(1) If $\mathcal{U} \subset \mathcal{V}$, then $k(\mathcal{U}) \le k(\mathcal{V})$;

(2) $k(\mathcal{U}) = k(\bar{\mathcal{U}}) = k(\overline{conv}\mathcal{U})$;

(3) $\mathcal{U}$ is relatively compact $k(\mathcal{U}) = 0$;

(4) $k(\delta\mathcal{U}) = |\delta| k(\mathcal{U}), \delta \in$;

(5) $k(\mathcal{U} \cup \mathcal{V}) = \max\{k(\mathcal{U}), k(\mathcal{V})\}$;

(6) $k(\mathcal{U} + \mathcal{V}) = k(\mathcal{U}) + k(\mathcal{V}), \mathcal{U} + \mathcal{V} = \{x | x = \mathfrak{u} + \mathfrak{v}, \mathfrak{u} \in \mathcal{U}, \mathfrak{v} \in \mathcal{V}\}$;

(7) $k(\mathcal{U} + y) = k(\mathcal{U}), \forall y \in Q$.

Lemma 2.2. *(See [30]). Given an equicontinuous and bounded set $\mathcal{S} \subset C([\mathfrak{a}, \mathcal{T}], Q)$, then the function $\varpi \rightarrowtail k(\mathcal{S}(\varpi))$ is continuous on $[\mathfrak{a}, \mathcal{T}]$, $k_C(\mathcal{S}) = \max_{\varpi \in [\mathfrak{a}, \mathcal{T}]} k(\mathcal{S}(\varpi))$, and*

$$k\left(\int_{\mathfrak{a}}^{\mathcal{T}} x(\tau) d\tau\right) \le \left(\int_{\mathfrak{a}}^{\mathcal{T}} (x(\tau)) d\tau\right), \mathcal{S}(\tau) = \{x(\tau) : x \in \mathcal{S}\}. \tag{2.8}$$

Definition 2.7. *(See [31]). Given the function $\Psi : [\mathfrak{a}, \mathcal{T}] \times Q \to Q, \Psi$ satisfy the Carathéodory's conditions, if the following conditions applies:*

$\Psi(\varpi, z)$ is measurable in ϖ for $z \in Q$;

$\Psi(\varpi, z)$ is continuous in $z \in Q$ for $\varpi \in [\mathfrak{a}, \mathcal{T}]$.

Theorem 2.1. *(mönch's fixed point theorem [32]). Given a bounded, closed, and convex subset $\Omega \subset Q$, such that $0 \in \Omega$, let also $\mathcal{T}$ be a continuous mapping of Ω into itself.*

If $\mathcal{S} = \overline{conv}\mathcal{T}(\mathcal{S})$, or $\mathcal{S} = \mathcal{T}(\mathcal{S}) \cup \{0\}$, then $k(\mathcal{S}) = 0$, satisfied $\forall\ \mathcal{S} \subset \Omega$, then $\mathcal{T}$ has a fixed point.

We are now ready to introduce an important lemma that we have discovered to solve the system.

Lemma 2.3. *Let* $\widetilde{\mathcal{G}_1}, \widetilde{\mathcal{G}_2}, \widetilde{\mathcal{G}_3} \in C[0,\mathcal{T}]$ *and* $\Delta \neq 0$. *Then the solution of the linear fractional differential system*

$$\begin{cases} ({}^c\mathfrak{D}^{\psi} + \varphi {}^c\mathfrak{D}^{\psi-1})\mathfrak{w}(\varpi) = \widetilde{\mathcal{G}_1}, \quad 2<\psi\leq 3, \\ ({}^c\mathfrak{D}^{\phi} + \varphi {}^c\mathfrak{D}^{\phi-1})\mathfrak{v}(\varpi) = \widetilde{\mathcal{G}_2}, \quad 2<\phi\leq 3, \\ ({}^c\mathfrak{D}^{\omega} + \varphi {}^c\mathfrak{D}^{\omega-1})\mathfrak{u}(\varpi) = \widetilde{\mathcal{G}_3}, \quad 3<\omega\leq 4, \\ \mathfrak{w}(0)=0, \qquad \mathfrak{w}'(0)=0, \qquad \mathfrak{w}(\mathcal{T}) = \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \mathfrak{v}(\zeta_j) + \Pi_1 \mathcal{I}^{\varsigma}\mathfrak{v}(\vartheta), \\ \mathfrak{v}(0)=0, \qquad \mathfrak{v}'(0)=0, \qquad \mathfrak{v}(\mathcal{T}) = \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \mathfrak{u}(\zeta_j) + \Pi_2 \mathcal{I}^{\varrho}\mathfrak{u}(\vartheta), \\ \mathfrak{u}(0)=0, \qquad \mathfrak{u}'(0)=0, \qquad \mathfrak{u}''(0)=0, \qquad \mathfrak{u}(\mathcal{T}) = \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \mathfrak{w}(\zeta_j) + \Pi_3 \mathcal{I}^{\delta}\mathfrak{w}(\vartheta), \end{cases} \tag{2.9}$$

it is provided by the following equations

$$\begin{aligned} &\mathfrak{w}(\varpi) \\ &= \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1 \sum_{j=1}^{k-2}\xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\widetilde{\mathcal{G}_2}(\tau)d\tau\right)d\rho \\ &+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\widetilde{\mathcal{G}_2}(m)dm\right)d\tau\right)d\rho \\ &-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\widetilde{\mathcal{G}_1}(\tau)d\tau\right)d\rho \\ &+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\widetilde{\mathcal{G}_3}(\tau)d\tau\right)d\rho \\ &+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\widetilde{\mathcal{G}_3}(m)dm\right)d\tau\right)d\rho \\ &-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\widetilde{\mathcal{G}_2}(\tau)d\tau\right)d\rho\Bigg\} \\ &+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\widetilde{\mathcal{G}_2}(\tau)d\tau\right)d\rho \\ &+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\widetilde{\mathcal{G}_2}(m)dm\right)d\tau\right)d\rho \\ &-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\widetilde{\mathcal{G}_1}(\tau)d\tau\right)d\rho\Bigg\} \\ &+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\widetilde{\mathcal{G}_1}(\tau)d\tau\right)d\rho \end{aligned}$$

$$
\begin{aligned}
&+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} \widetilde{\mathcal{G}_1}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \widetilde{\mathcal{G}_3}(\tau)d\tau \right) d\rho \Big\} \Big) \Big] \\
&+ \int_0^{\varpi} e^{-\varphi(\varpi-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \widetilde{\mathcal{G}_1}(\tau)d\tau \right) d\rho, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{v}(\varpi) =& \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\Delta} \Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5 \Bigg\{ \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \widetilde{\mathcal{G}_3}(\tau)d\tau \right) d\rho \\
&+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)} \widetilde{\mathcal{G}_3}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \widetilde{\mathcal{G}_2}(\tau)d\tau \right) d\rho \Bigg\} \\
&+\mathcal{E}_4\mathcal{E}_6 \Bigg\{ \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \widetilde{\mathcal{G}_2}(\tau)d\tau \right) d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \widetilde{\mathcal{G}_2}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \widetilde{\mathcal{G}_1}(\tau)d\tau \right) d\rho \Bigg\} \\
&+\mathcal{E}_1\mathcal{E}_4 \Bigg\{ \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \widetilde{\mathcal{G}_1}(\tau)d\tau \right) d\rho \\
&+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} \widetilde{\mathcal{G}_1}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \widetilde{\mathcal{G}_3}(\tau)d\tau \right) d\rho \Bigg\} \Bigg) \Bigg] \\
&+ \int_0^{\varpi} e^{-\varphi(\varpi-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \widetilde{\mathcal{G}_2}(\tau)d\tau \right) d\rho, \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{u}(\varpi) =& \frac{(\varphi^2\varpi^2 - 2\varphi\varpi + 2 - e^{-\varphi\varpi})}{\varphi^3\Delta} \\
&\times \Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6 \Bigg\{ \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \widetilde{\mathcal{G}_2}(\tau)d\tau \right) d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \widetilde{\mathcal{G}_2}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \widetilde{\mathcal{G}_1}(\tau)d\tau \right) d\rho \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\widetilde{\mathcal{G}_1}(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\widetilde{\mathcal{G}_1}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\widetilde{\mathcal{G}_3}(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\widetilde{\mathcal{G}_3}(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\widetilde{\mathcal{G}_3}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\widetilde{\mathcal{G}_2}(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\widetilde{\mathcal{G}_3}(\tau)d\tau\right)d\rho, \qquad (2.12)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{E}_1 =&\frac{(\varphi\mathcal{T}-1+e^{-\varphi\mathcal{T}})}{\varphi^2},\\
\mathcal{E}_2 =&\frac{1}{\varphi^2}\left[\Upsilon_1\sum_{j=1}^{k-2}\xi_j(\varphi\zeta_j-1+e^{-\varphi\zeta_j})+\Pi_1\int_0^{\vartheta}\frac{(\vartheta-\rho)^{\varsigma-1}}{\Gamma(\varsigma)}(\varphi s-1+e^{-\varphi s})d\rho\right],\\
\mathcal{E}_3 =&\frac{(\varphi\mathcal{T}-1+e^{-\varphi\mathcal{T}})}{\varphi^2},\\
\mathcal{E}_4 =&\frac{1}{\varphi^3}\left[\Upsilon_2\sum_{j=1}^{k-2}\nu_j(\varphi^2\zeta_j^{\,2}-2\varphi\zeta_j+2-e^{-\varphi\zeta_j})+\Pi_2\int_0^{\vartheta}\frac{(\vartheta-\rho)^{\varrho-1}}{\Gamma(\varrho)}(\varphi^2 s^2-2\varphi s+2-2e^{-\varphi s})d\rho\right],\\
\mathcal{E}_5 =&\frac{(\varphi^2\mathcal{T}^2-2\varphi\mathcal{T}+2-2e^{-\varphi\mathcal{T}})}{\varphi^3},\\
\mathcal{E}_6 =&\frac{1}{\varphi^2}\left[\Upsilon_3\sum_{j=1}^{k-2}\sigma_j(\varphi\zeta_j-1+e^{-\varphi\zeta_j})+\Pi_3\int_0^{\vartheta}\frac{(\vartheta-\rho)^{\delta-1}}{\Gamma(\delta)}(\varphi s-1+e^{-\varphi s})d\rho\right],\\
\Delta =&\mathcal{E}_1\mathcal{E}_3\mathcal{E}_5-\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6. \qquad (2.13)
\end{aligned}
$$

3. Main results

Let $\hat{\mathcal{J}} = C([0,\mathcal{T}],\mathcal{R}_e)$ be a Banach space endowed with the norm $\|\mathfrak{w}\| = \sup\{|\mathfrak{w}(\varpi)|, \varpi \in [0,\mathcal{T}]\}$. Then $(\hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}, \|(\mathfrak{w},\mathfrak{v},\mathfrak{u})\|_{\hat{\mathcal{J}}})$ is also a Banach space equipped with the norm $\|(\mathfrak{w},\mathfrak{v},\mathfrak{u})\|_{\hat{\mathcal{J}}} = \|\mathfrak{w}\| + \|\mathfrak{v}\| + \|\mathfrak{u}\|$, $\mathfrak{w},\mathfrak{v},\mathfrak{u} \in \hat{\mathcal{J}}$.

In view of Lemma 2.3, we define an operator $\mathfrak{F} : \hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}} \to \hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}$ by

$$\mathfrak{F}(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi)) = \mathfrak{F}_1(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi)),$$

$$\mathfrak{F}_2(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi)),$$
$$\mathfrak{F}_3(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi)),$$

where,

$$\begin{aligned}
&\mathfrak{F}_1(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi)) \qquad (3.1)\\
=&\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho,
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{F}_2(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta} \qquad (3.2)\\
&\times\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho
\end{aligned}$$

$$+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(m, \mathfrak{w}(m), \mathfrak{v}(m), \mathfrak{u}(m)) dm \right) d\tau \right) d\rho$$

$$- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho \Bigg\}$$

$$+\mathcal{E}_4\mathcal{E}_6 \Bigg\{ \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho$$

$$+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(m, \mathfrak{w}(m), \mathfrak{v}(m), \mathfrak{u}(m)) dm \right) d\tau \right) d\rho$$

$$- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho \Bigg\}$$

$$+\mathcal{E}_1\mathcal{E}_4 \Bigg\{ \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho$$

$$+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(m, \mathfrak{w}(m), \mathfrak{v}(m), \mathfrak{u}(m)) dm \right) d\tau \right) d\rho$$

$$- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho \Bigg\} \Bigg) \Bigg]$$

$$+ \int_0^{\varpi} e^{-\varphi(\varpi-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho,$$

$$\mathfrak{F}_3(\mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)) = \frac{(\varphi^2\varpi^2 - 2\varphi\varpi + 2 - e^{-\varphi\varpi})}{\varphi^3\Delta} \tag{3.3}$$

$$\times \Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6 \Bigg\{ \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho$$

$$+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(m, \mathfrak{w}(m), \mathfrak{v}(m), \mathfrak{u}(m)) dm \right) d\tau \right) d\rho$$

$$- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho \Bigg\}$$

$$+\mathcal{E}_1\mathcal{E}_3 \Bigg\{ \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho$$

$$+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(m, \mathfrak{w}(m), \mathfrak{v}(m), \mathfrak{u}(m)) dm \right) d\tau \right) d\rho$$

$$- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho \Bigg\}$$

$$+\mathcal{E}_2\mathcal{E}_6 \Bigg\{ \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(\tau, \mathfrak{w}(\tau), \mathfrak{v}(\tau), \mathfrak{u}(\tau)) d\tau \right) d\rho$$

$$+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(m, \mathfrak{w}(m), \mathfrak{v}(m), \mathfrak{u}(m)) dm \right) d\tau \right) d\rho$$

$$
\begin{aligned}
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))d\tau\right)d\rho.
\end{aligned}
$$

For easy computations, we set

$$
\begin{aligned}
\mathcal{P}_1 =&\frac{(\varphi-1+e^{-\varphi})}{\varphi^2\mathcal{E}_1}\\
&\times\Bigg[\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)\\
&+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\Bigg\}\Bigg]+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\psi)},\\
\mathcal{Q}_1 =&\frac{(\varphi-1+e^{-\varphi})}{\varphi^2\mathcal{E}_1}\Bigg[\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\Bigg\}\\
&+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\Bigg\}\\
&+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\Bigg],\\
\mathcal{O}_1 =&\frac{(\varphi-1+e^{-\varphi})}{\varphi^2\mathcal{E}_1}\Bigg[\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\Bigg\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\Bigg\}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\\
&+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\Bigg\}\Bigg],\\
\mathcal{P}_2 =&\frac{(\varphi-1+e^{-\varphi})}{\varphi^2\Delta}\Bigg[\mathcal{E}_4\mathcal{E}_6\Bigg\{\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}\Bigg\}+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)\\
&+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\Bigg\}\Bigg],\\
\mathcal{Q}_2 =&\frac{(\varphi-1+e^{-\varphi})}{\varphi^2\Delta}\Bigg[\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_5\Bigg\{\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\Bigg\}\Bigg]+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\phi)},\\
\mathcal{O}_2 =&\frac{(\varphi-1+e^{-\varphi})}{\varphi^2\Delta}\Bigg[\mathcal{E}_1\mathcal{E}_4\Bigg\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\Bigg\}+\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\\
&+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\Bigg\}\Bigg],\\
\mathcal{P}_3 =&\frac{(\varphi^2-2\varphi+2-2e^{-\varphi})}{\varphi^3\Delta}\Bigg[\mathcal{E}_3\mathcal{E}_6\Bigg\{\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}\Bigg\}+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)
\end{aligned}
$$

$$+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\Big\}\Big]\Big],$$

$$\begin{aligned}
Q_3=&\frac{(\varphi^2-2\varphi+2-2e^{-\varphi})}{\varphi^3\Delta}\Bigg[\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)\\
&+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\Bigg\}+\mathcal{E}_2\mathcal{E}_6\left\{\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\right\}\Bigg],\\
O_3=&\frac{(\varphi^2-2\varphi+2-2e^{-\varphi})}{\varphi^3\Delta}\Bigg[\mathcal{E}_1\mathcal{E}_3\left\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\right\}+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\\
&+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\Bigg\}\Bigg]+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\omega)}.
\end{aligned}\tag{3.4}$$

For the applicability of the Mönch's fixed point theorem of our proposed triple system of our proposed triple system given in (1.3), we set the following assumptions:

($\mathcal{H}_1$) Assume the functions $\mathfrak{f},\mathfrak{g},\mathfrak{h}:[0,\mathcal{T}]\times\mathcal{R}_e^3\to\mathcal{R}_e$ satisfy Carathéodory's conditions.

($\mathcal{H}_2$) There exist $\mathfrak{l}_{\mathfrak{f}},\mathfrak{l}_{\mathfrak{g}},\mathfrak{l}_{\mathfrak{h}}\in\mathcal{L}^{\infty}([0,\mathcal{T}]),\mathcal{R}_{e_+}$, and there exist a non-decreasing continuous functions $\partial_{\mathfrak{f}},\partial_{\mathfrak{g}},\partial_{\mathfrak{h}}:\mathcal{R}_{e_+}\to\mathcal{R}_{e_+}$, such that $\forall\varpi\in[0,\mathcal{T}]$ $\forall(\mathfrak{w},\mathfrak{v},\mathfrak{u})\in\hat{\mathcal{J}}^3$ we have

$$\begin{aligned}
\|\mathfrak{f}(\varpi,\mathfrak{w},\mathfrak{v},\mathfrak{u})\|&\le\mathfrak{l}_{\mathfrak{f}}(\varpi)\partial_{\mathfrak{f}}\left(\|\mathfrak{w}\|_\infty+\|\mathfrak{v}\|_\infty+\|\mathfrak{u}\|_\infty\right),\\
\|\mathfrak{g}(\varpi,\mathfrak{w},\mathfrak{v},\mathfrak{u})\|&\le\mathfrak{l}_{\mathfrak{g}}(\varpi)\partial_{\mathfrak{g}}\left(\|\mathfrak{w}\|_\infty+\|\mathfrak{v}\|_\infty+\|\mathfrak{u}\|_\infty\right),\\
\|\mathfrak{h}(\varpi,\mathfrak{w},\mathfrak{v},\mathfrak{u})\|&\le\mathfrak{l}_{\mathfrak{h}}(\varpi)\partial_{\mathfrak{h}}\left(\|\mathfrak{w}\|_\infty+\|\mathfrak{v}\|_\infty+\|\mathfrak{u}\|_\infty\right).
\end{aligned}$$

($\mathcal{H}_3$) let $\mathcal{S}\subset\hat{\mathcal{J}}^3$ be a bounded set, and $\forall\varpi\in[0,\mathcal{T}]$, then

$$\begin{aligned}
K(\mathfrak{f}(\varpi,\mathcal{S}))&\le\mathfrak{l}_{\mathfrak{f}}(\varpi)K(\mathcal{S}),\\
K(\mathfrak{g}(\varpi,\mathcal{S}))&\le\mathfrak{l}_{\mathfrak{g}}(\varpi)K(\mathcal{S}),\\
K(\mathfrak{h}(\varpi,\mathcal{S}))&\le\mathfrak{l}_{\mathfrak{h}}(\varpi)K(\mathcal{S}).
\end{aligned}$$

Theorem 3.1. *Assume that the assumptions* ($\mathcal{H}_1$), ($\mathcal{H}_2$), *and* ($\mathcal{H}_3$) *are satisfied. If*

$$\max\{\mathcal{P}_1\mathfrak{l}_{\mathfrak{f}}^*+\mathcal{P}_2\mathfrak{l}_{\mathfrak{g}}^*+\mathcal{P}_3\mathfrak{l}_{\mathfrak{h}}^*,Q_1\mathfrak{l}_{\mathfrak{f}}^*+Q_2\mathfrak{l}_{\mathfrak{g}}^*+Q_3\mathfrak{l}_{\mathfrak{h}}^*,O_1\mathfrak{l}_{\mathfrak{f}}^*+O_2\mathfrak{l}_{\mathfrak{g}}^*+O_3\mathfrak{l}_{\mathfrak{h}}^*\}<1,\tag{3.5}$$

where $\mathfrak{l}_{\mathfrak{f}}^*=\sup_{0\le\varpi\le\mathcal{T}}\mathfrak{l}_{\mathfrak{f}}(\varpi)$, $\mathfrak{l}_{\mathfrak{g}}^*=\sup_{0\le\varpi\le\mathcal{T}}\mathfrak{l}_{\mathfrak{g}}(\varpi)$, $\mathfrak{l}_{\mathfrak{h}}^*=\sup_{0\le\varpi\le\mathcal{T}}\mathfrak{l}_{\mathfrak{h}}(\varpi)$.

Then there exist at least one solution for the boundary value problem (1.3) on $[0,\mathcal{T}]$.

Proof. Introducing the following continuous operator $\mathcal{F}:\hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}\to\hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}$

$$\mathcal{F}(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))=\begin{pmatrix}\mathcal{F}_1(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))\\ \mathcal{F}_2(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))\\ \mathcal{F}_3(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))\end{pmatrix}.$$

According to the assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_2$), the operator $\mathcal{F}$ is well defined, then the following operator equation can be an equivalent equation to the fractional equation given by (3.1)–(3.3)

$$(\mathfrak{w},\mathfrak{v},\mathfrak{u})=\mathfrak{F}((\mathfrak{w},\mathfrak{v},\mathfrak{u})).\tag{3.6}$$

Subsequently, proving the existence of the solution to Eq (3.6) is equivalent to proving the existence of a solution to Eq (1.3).

Let $\mathcal{A}_{\Psi} = \{(\mathfrak{w}, \mathfrak{v}, \mathfrak{u}) \in \hat{\mathcal{J}}^3 : \|(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})_\infty\| \leq \Psi, \Psi > 0\}$, be a closed bounded convex ball in $\hat{\mathcal{J}}^3$ with

$$\Psi \geq \mathfrak{l}_{\mathfrak{f}}^{*}\partial_{\mathfrak{f}}(\Psi)\left[\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3\right] + \mathfrak{l}_{\mathfrak{g}}^{*}\partial_{\mathfrak{g}}(\Psi)\left[\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3\right] + \mathfrak{l}_{\mathfrak{h}}^{*}\partial_{\mathfrak{h}}(\Psi)\left[\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3\right].$$

For the possibility of applying mönch's fixed point theorem, we will proceed with the proof in the form of four steps and thus we achieve the desired goal by proving the existence of a solution to the Eq (1.3).

Step 1. $\mathcal{F}$ maps $\mathcal{A}_{\Psi}$, into itself.

For all $(\mathfrak{w}, \mathfrak{v}, \mathfrak{u}) \in \mathcal{A}_{\Psi}$, $\varpi \in [0, \mathcal{T}]$, we obtain

$$\begin{aligned}
&\|\mathfrak{F}_1(\mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi))\|_\infty \qquad (3.7)\\
=&\frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
\leq&\ \mathfrak{l}_{\mathfrak{f}}^{*}\partial_{\mathfrak{f}}(\Psi)\mathcal{P}_1 + \mathfrak{l}_{\mathfrak{g}}^{*}\partial_{\mathfrak{g}}(\Psi)\mathcal{P}_2 + \mathfrak{l}_{\mathfrak{h}}^{*}\partial_{\mathfrak{h}}(\Psi)\mathcal{P}_3, \qquad (3.8)
\end{aligned}$$

then

$$\|\mathfrak{F}_1(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))\|_\infty \le \mathfrak{l}^*_{\mathfrak{f}}\partial_{\mathfrak{f}}(\Psi)\mathcal{P}_1 + \mathfrak{l}^*_{\mathfrak{g}}\partial_{\mathfrak{g}}(\Psi)\mathcal{P}_2 + \mathfrak{l}^*_{\mathfrak{h}}\partial_{\mathfrak{h}}(\Psi)\mathcal{P}_3, \tag{3.9}$$

similarly,

$$\|\mathfrak{F}_2(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))\|_\infty \le \mathfrak{l}^*_{\mathfrak{f}}\partial_{\mathfrak{f}}(\Psi)\mathcal{Q}_1 + \mathfrak{l}^*_{\mathfrak{g}}\partial_{\mathfrak{g}}(\Psi)\mathcal{Q}_2 + \mathfrak{l}^*_{\mathfrak{h}}\partial_{\mathfrak{h}}(\Psi)\mathcal{Q}_3, \tag{3.10}$$

and

$$\|\mathfrak{F}_3(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))\|_\infty \le \mathfrak{l}^*_{\mathfrak{f}}\partial_{\mathfrak{f}}(\Psi)\mathcal{O}_1 + \mathfrak{l}^*_{\mathfrak{g}}\partial_{\mathfrak{g}}(\Psi)\mathcal{O}_2 + \mathfrak{l}^*_{\mathfrak{h}}\partial_{\mathfrak{h}}(\Psi)\mathcal{O}_3. \tag{3.11}$$

Consequently,

$$\begin{aligned}\|\mathfrak{F}(\mathfrak{w}(\varpi),\mathfrak{v}(\varpi),\mathfrak{u}(\varpi))\|_\infty \le& \mathfrak{l}^*_{\mathfrak{f}}\partial_{\mathfrak{f}}(\Psi)(\mathcal{P}_1+\mathcal{Q}_1+\mathcal{O}_1) + \mathfrak{l}^*_{\mathfrak{g}}\partial_{\mathfrak{g}}(\Psi)(\mathcal{P}_2+\mathcal{Q}_2+\mathcal{O}_2) + \mathfrak{l}^*_{\mathfrak{h}}\partial_{\mathfrak{h}}(\Psi)(\mathcal{P}_3+\mathcal{Q}_2+\mathcal{O}_3)\\ \le& \Psi. \end{aligned}\tag{3.12}$$

Hence, the operator $\mathcal{F}$ maps the ball $\mathcal{A}_\Psi$ into itself.

Step 2. The operator $\mathcal{F}$ is continuous.

Let $\{(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n)\} \in \mathcal{A}_\Psi$ such that $(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) \to (\mathfrak{w},\mathfrak{v},\mathfrak{u})$ as $n \to \infty$. We indicate that $\|\mathfrak{F}(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) - \mathfrak{F}(\mathfrak{w},\mathfrak{v},\mathfrak{u})\| \to 0$. Since functions $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ satisfy Carath*é*odory's conditions, we conclude that $\mathfrak{F}_1(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) \to \mathfrak{F}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})$, $\mathfrak{F}_2(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) \to \mathfrak{F}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})$, and $\mathfrak{F}_3(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) \to \mathfrak{F}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})$ as $n \to \infty$. Now, due to condition $(\mathcal{H}_2)$ and the Lebesgue dominated convergence theorem, we obtain that $\|\mathfrak{F}_1(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) - \mathfrak{F}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})\|$, $\|\mathfrak{F}_2(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) - \mathfrak{F}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})\|$, and $\|\mathfrak{F}_3(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) - \mathfrak{F}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})\| \to 0$. as $n \to \infty$. Consequently, $\|\mathfrak{F}(\mathfrak{w}_n,\mathfrak{v}_n,\mathfrak{u}_n) - \mathfrak{F}(\mathfrak{w},\mathfrak{v},\mathfrak{u})\|_\infty \to 0$, which implies that $\mathcal{F}$ is continuous on $\mathcal{A}_\Psi$.

Step 3. The operator $\mathcal{F}$ is equicontinuous.

Let $\varpi_1,\varpi_2 \in [0,\mathcal{T}]$ with $\varpi_1 < \varpi_2$. Then we have

$$\begin{aligned}&\|\mathfrak{F}_1((\mathfrak{w},\mathfrak{v},\mathfrak{u}))(\varpi_2) - \mathfrak{F}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})(\varpi_1)\|_\infty \le \frac{(\varphi\varpi_2 - \varphi\varpi_1 + e^{\varphi\varpi_2} - e^{\varphi\varpi_1})}{\varphi^2\mathcal{E}_1}\\
&\times\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\Bigg\}\end{aligned}$$

$$+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\left\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\right.$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho$$
$$\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\right\}$$
$$+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\left\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\right.$$
$$+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho$$
$$\left.\left.\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\right\}\right)\right]$$
$$+\left|\int_0^{\varpi_1}(e^{-\varphi(\varpi_2-\rho)}-e^{-\varphi(\varpi_1-\rho)})\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\right.$$
$$\left.+\int_{\varpi_2}^{\varpi_1}e^{-\varphi(\varpi_2-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\right|,$$

$$\leq\frac{(\varphi\varpi_2-\varphi\varpi_1+e^{\varphi\varpi_2}-e^{\varphi\varpi_1})}{\varphi^2\mathcal{E}_1}$$
$$\times\left\{\left[\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\left\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)\right.\right.\right.$$
$$\left.\left.+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\right\}\right]$$
$$+\left[\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}\right.$$
$$+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}$$
$$\left.+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\right]+\left[\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\left\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\right\}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\left\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\right.\right.$$
$$\left.\left.\left.+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\right\}\right]\wp_3\right\}$$
$$+\int_0^{\varpi_1}(e^{-\varphi(\varpi_2-\rho)}-e^{-\varphi(\varpi_1-\rho)})\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}du\right)d\rho+$$
$$\int_{\varpi_2}^{\varpi_1}e^{-\varphi(\varpi_2-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}du\right)d\rho\to 0,\ \text{ as }\ \varpi_1\to\varpi_2.$$

As $\varpi_2 \to \varpi_1$, we obtain that $\|\mathcal{F}_1((\mathfrak{w},\mathfrak{v},\mathfrak{u}))(\varpi_2)-\mathcal{F}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})(\varpi_1)\| \to 0$. Similarly,

$\|\mathcal{F}_2((\mathfrak{w},\mathfrak{v},\mathfrak{u}))(\varpi_2)-\mathcal{F}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})(\varpi_1)\| \to 0$, and $\|\mathcal{F}_3((\mathfrak{w},\mathfrak{v},\mathfrak{u}))(\varpi_2)-\mathcal{F}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})(\varpi_1)\| \to 0$ as $\varpi_2 \to \varpi_1$. Consequently, $\|\mathcal{F}((\mathfrak{w},\mathfrak{v},\mathfrak{u}))(\varpi_2)-\mathcal{F}(\mathfrak{w},\mathfrak{v},\mathfrak{u})(\varpi_1)\|_\infty \to 0$, as $\varpi_2 \to \varpi_1$. Thus $\mathcal{F}$ is equicontinuous.

Step 4. Finally, we need to satisfy the mönch's hypothesis, so we let $\mathcal{U}=\mathcal{U}_1\cap\mathcal{U}_2\cap\mathcal{U}_3$, where $\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3\subseteq\mathcal{A}_\psi$. Moreover, $\mathcal{U}_1,\mathcal{U}_2$, and $\mathcal{U}_3$ are assumed to be bounded and equicontinuous, such that $\mathcal{U}_1\subset\overline{conv}(\mathcal{F}_1(\mathcal{U}_1)\cup\{0\})$, $\mathcal{U}_2\subset\overline{conv}(\mathcal{F}_2(\mathcal{U}_2)\cup\{0\})$ and $\mathcal{U}_3\subset\overline{conv}(\mathcal{F}_3(\mathcal{U}_3)\cup\{0\})$, thus the functions $\mathcal{W}_1(\varpi)=K(\mathcal{U}_1(\varpi))$,$\mathcal{W}_2(\varpi)=K(\mathcal{U}_2(\varpi))$, and $\mathcal{W}_3(\varpi)=K(\mathcal{U}_3(\varpi))$ are continuous on $[0,\mathcal{T}]$. Based on $(\mathcal{H}_3)$, Lemmas 2.1 and 2.2, we get

$$
\begin{aligned}
&\mathcal{W}_1(\varpi)=K(\mathcal{U}_1(\varpi))\le K\left(\overline{conv}(\mathcal{F}_1(\mathcal{U}_1)(\varpi)\cup\{0\})\right)\le K(\mathcal{F}_1(\mathcal{U}_1)(\varpi))\\
&\le K\Bigg\{\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&r+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\|\mathfrak{g}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(m,\mathfrak{w}(m),\mathfrak{v}(m),\mathfrak{u}(m))\|_\infty dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\|\mathfrak{h}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\|\mathfrak{f}(\tau,\mathfrak{w}(\tau),\mathfrak{v}(\tau),\mathfrak{u}(\tau))\|_\infty d\tau\right)d\rho\Bigg\}:(\mathfrak{w},\mathfrak{v},\mathfrak{u})\in\mathcal{U}_1\Bigg\}\\
&\le\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{l}_{\mathfrak{g}}^*(\varpi)K(\mathcal{U}_1(\varpi))d\tau\right)d\rho
\end{aligned}
$$

$$
\begin{aligned}
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{l}^*_{\mathfrak{g}}(\varpi) K(\mathcal{U}_1(\varpi)) dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{l}^*_{\mathfrak{f}}(\varpi) K(\mathcal{U}_1(\varpi)) d\tau \right) d\rho \\
&+ \frac{1}{\Delta} \Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5 \Bigg\{ \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{l}^*_{\mathfrak{h}}(\varpi) K(\mathcal{U}_1(\varpi)) d\tau \right) d\rho \\
&+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{l}^*_{\mathfrak{h}}(\varpi) K(\mathcal{U}_1(\varpi)) dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{l}^*_{\mathfrak{g}}(\varpi) K(\mathcal{U}_1(\varpi)) d\tau \right) d\rho \Bigg\} \\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6 \Bigg\{ \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{l}^*_{\mathfrak{g}}(\varpi) K(\mathcal{U}_1(\varpi)) d\tau \right) d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{l}^*_{\mathfrak{g}}(\varpi) K(\mathcal{U}_1(\varpi)) dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{l}^*_{\mathfrak{f}}(\varpi) K(\mathcal{U}_1(\varpi)) d\tau \right) d\rho \Bigg\} \\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4 \Bigg\{ \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{l}^*_{\mathfrak{f}}(\varpi) K(\mathcal{U}_1(\varpi)) d\tau \right) d\rho \\
&+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{l}^*_{\mathfrak{f}}(\varpi) K(\mathcal{U}_1(\varpi)) dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{l}^*_{\mathfrak{h}}(\varpi) K(\mathcal{U}_1(\varpi)) d\tau \right) d\rho \Bigg\} \Bigg) \Bigg] \\
&+ \int_0^{\varpi} e^{-\varphi(\varpi-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{l}^*_{\mathfrak{f}}(\varpi) K(\mathcal{U}_1(\varpi)) d\tau \right) d\rho \\
\leq& (\mathfrak{l}^*_{\mathfrak{f}} \mathcal{P}_1 + \mathfrak{l}^*_{\mathfrak{g}} \mathcal{P}_2 + \mathfrak{l}^*_{\mathfrak{h}} \mathcal{P}_3) \|\mathcal{W}_1\|_\infty.
\end{aligned}
$$

That is,

$$
\|\mathcal{W}_1(\varpi)\| \leq (\mathfrak{l}^*_{\mathfrak{f}} \mathcal{P}_1 + \mathfrak{l}^*_{\mathfrak{g}} \mathcal{P}_2 + \mathfrak{l}^*_{\mathfrak{h}} \mathcal{P}_3) \|\mathcal{W}_1\|_\infty, \tag{3.13}
$$

and it is assumed that

$$
\max\{\mathcal{P}_1 \mathfrak{l}^*_{\mathfrak{f}} + \mathcal{P}_2 \mathfrak{l}^*_{\mathfrak{g}} + \mathcal{P}_3 \mathfrak{l}^*_{\mathfrak{h}}, \mathcal{Q}_1 \mathfrak{l}^*_{\mathfrak{f}} + \mathcal{Q}_2 \mathfrak{l}^*_{\mathfrak{g}} + \mathcal{Q}_3 \mathfrak{l}^*_{\mathfrak{h}}, \mathcal{O}_1 \mathfrak{l}^*_{\mathfrak{f}} + \mathcal{O}_2 \mathfrak{l}^*_{\mathfrak{g}} + \mathcal{O}_3 \mathfrak{l}^*_{\mathfrak{h}}\} < 1, \tag{3.14}
$$

which implies that $\|\mathcal{W}_1\|_\infty = 0$. i.e $\mathcal{W}_1 = 0, \forall\ \varpi \in [0, \mathcal{T}]$. In a like manner, we have $\mathcal{W}_2 = 0, \forall\ \varpi \in [0, \mathcal{T}]$ and $\mathcal{W}_3 = 0, \forall\ \varpi \in [0, \mathcal{T}]$. So $K(\mathcal{U}(\varpi)) \leq K(\mathcal{U}_1(\varpi)) = 0$, $K(\mathcal{U}(\varpi)) \leq K(\mathcal{U}_2(\varpi)) = 0$, and $K(\mathcal{U}(\varpi)) \leq K(\mathcal{U}_3(\varpi)) = 0$, which implies that $\mathcal{U}(\varpi)$ is relatively compact in $\hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}$. Now, Arzela-Ascoli is applicable, which means that $\mathcal{U}$ is relatively compact in $\mathcal{A}_\Psi$, and therefore, using Theorem 3.1, we deduce that the operator $\mathcal{F}$ has a fixed point $(\mathfrak{f}, \mathfrak{g}, \mathfrak{h})$ solution of the problem on $\mathcal{A}_\Psi$. And that ends the proof.

4. Hyers-Ulam stability

This section discusses the stability of boundary value problem solutions for Hyers-Ulam (1.3) by integral representing its solution provided by

$$\mathfrak{w}(\varpi) = \mathcal{F}_1(\mathfrak{w}, \mathfrak{u}, \mathfrak{v})(\varpi),$$
$$\mathfrak{u}(\varpi) = \mathcal{F}_2(\mathfrak{w}, \mathfrak{u}, \mathfrak{v})(\varpi),$$
$$\mathfrak{v}(\varpi) = \mathcal{F}_3(\mathfrak{w}, \mathfrak{u}, \mathfrak{v})(\varpi),$$

where $\mathfrak{F}_1, \mathfrak{F}_2$, and $\mathfrak{F}_3$ are defined by $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3 \in C([0,\mathcal{T}],\mathcal{R}_e) \times C([0,\mathcal{T}],\mathcal{R}_e) \times C([0,\mathcal{T}],\mathcal{R}_e) \to C([0,\mathcal{T}],\mathcal{R}_e)$;

$$\begin{cases} (\mathcal{D}^{\psi} + \varphi\mathcal{D}^{\psi-1})\mathfrak{w}(\varpi) - \mathfrak{f}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)) = \mathcal{Z}_1(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})(\varpi), \\ (\mathcal{D}^{\xi} + \varphi\mathcal{D}^{\xi-1})\mathfrak{v}(\varpi) - \mathfrak{g}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)) = \mathcal{Z}_2(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})(\varpi), \\ (\mathcal{D}^{\zeta} + \varphi\mathcal{D}^{\zeta-1})\mathfrak{u}(\varpi) - \mathfrak{h}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)) = \mathcal{Z}_3(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})(\varpi), \end{cases} \tag{4.1}$$

for $\varpi \in [0,\mathcal{T}]$. For some $\pi_1, \pi_2, \pi_3 > 0$, the following inequalities are considered:

$$\left\|\mathcal{Z}_1(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})\right\| \leq \pi_1, \qquad \left\|\mathcal{Z}_2(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})\right\| \leq \pi_2 \qquad \left\|\mathcal{Z}_3(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})\right\| \leq \pi_3. \tag{4.2}$$

Definition 4.1. *The tripled system (1.3) is said to be Hyers-Ulam stable, if there exists $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 > 0$, such that for every solution $(\widehat{\mathfrak{w}}, \widehat{\mathfrak{v}}, \widehat{\mathfrak{u}}) \in C([0,\mathcal{T}],\mathcal{R}_e) \times C([0,\mathcal{T}],\mathcal{R}_e) \times C([0,\mathcal{T}],\mathcal{R}_e)$ of the Inequality (4.2), there exists a unique solutions $(\mathfrak{w}, \mathfrak{v}, \mathfrak{u}) \in C([0,\mathcal{T}],\mathcal{R}_e) \times C([0,\mathcal{T}],\mathcal{R}_e) \times C([0,\mathcal{T}],\mathcal{R}_e)$ of problem (1.3) with*

$$\left\|(\mathfrak{w}, \mathfrak{v}, \mathfrak{u}) - (\widehat{\mathfrak{w}}, \widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\right\| \leq \mathcal{K}_1\pi_1 + \mathcal{K}_2\pi_2 + \mathcal{K}_3\pi_3.$$

$$\mathcal{K}_1 = \frac{(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)}{1 - ((\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)(\mu_1 + \sigma_1 + \kappa_1) + (Q_1 + Q_2 + Q_3)(\mu_2 + \sigma_2 + \kappa_2) + (O_1 + O_2 + O_3)(\mu_3 + \phi_3 + \kappa_3))},$$
$$\mathcal{K}_2 = \frac{(Q_1 + Q_2 + Q_3)}{1 - ((\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)(\mu_1 + \sigma_1 + \kappa_1) + (Q_1 + Q_2 + Q_3)(\mu_2 + \sigma_2 + \kappa_2) + (O_1 + O_2 + O_3)(\mu_3 + \phi_3 + \kappa_3))},$$
$$\mathcal{K}_3 = \frac{(O_1 + O_2 + O_3)}{1 - ((\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)(\mu_1 + \sigma_1 + \kappa_1) + (Q_1 + Q_2 + Q_3)(\mu_2 + \sigma_2 + \kappa_2) + (O_1 + O_2 + O_3)(\mu_3 + \phi_3 + \kappa_3))}.$$

Theorem 4.1. *Assume that Theorem 3.1 assumptions hold. Then the boundary value problem (1.3) is Hyers-Ulam stable.*

Proof. Let $(\mathfrak{w}, \mathfrak{v}, \mathfrak{u}) \in C([0,\mathcal{T}],\mathcal{R}_e) \times C([0,\mathcal{T}],\mathcal{R}_e) \times C([0,\mathcal{T}],\mathcal{R}_e)$ be the solution of (1.3) the problem that satisfying (3.1)–(3.3). Let $(\widehat{\mathfrak{w}}, \widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})$ be any solution satisfying (4.2):

$$\begin{cases} (\mathcal{D}^{\psi} + \varphi\mathcal{D}^{\psi-1})\mathfrak{w}(\varpi) = \mathfrak{f}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)) + \mathcal{Z}_1(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})(\varpi), \\ (\mathcal{D}^{\xi} + \varphi\mathcal{D}^{\xi-1})\mathfrak{v}(\varpi) = \mathfrak{g}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)) + \mathcal{Z}_2(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})(\varpi), \\ (\mathcal{D}^{\zeta} + \varphi\mathcal{D}^{\zeta-1})\mathfrak{u}(\varpi) = \mathfrak{h}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)) + \mathcal{Z}_3(\mathfrak{w}, \mathfrak{v}, \mathfrak{u})(\varpi), \end{cases} \tag{4.3}$$

for $\varpi \in [0, \mathcal{T}]$. So,

$$
\begin{aligned}
\widehat{\mathfrak{w}}(\varpi) =& \mathfrak{F}_1(\widehat{\mathfrak{w}}, \widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})(\varpi) \\
&+ \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho \\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho \\
&+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\Bigg\} \\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\Bigg\} \\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho \\
&+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\Bigg\}\Bigg)\Bigg] \\
&+ \int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
&|\widehat{\mathfrak{w}}(\varpi) - \mathfrak{F}_1(\widehat{\mathfrak{w}}, \widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})(\varpi)| = \\
&\times \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Delta}\left(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\left\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\right.\right.\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\right)d\tau\right)d\rho\\
&\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\right\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\left\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\right.\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\right)d\tau\right)d\rho\\
&\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\right\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\left\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\right.\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\right)d\tau\right)d\rho\\
&\left.\left.\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho\right\}\right)\right]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\right)d\rho,\\
\leq&\frac{(\varphi-1+e^{-\varphi})}{\varphi^2\mathcal{E}_1}\left\{\left[\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\left\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)\right.\right.\right.\\
&\left.\left.+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\right\}\right]\pi_1\\
&+\left[\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}\right.\\
&+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}\\
&\left.+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\right]\pi_2+\left[\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\left\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\right\}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\left\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\right.\right.\\
&\left.\left.\left.+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\right\}\right]\right\}\pi_3+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\psi)}\\
\leq&\mathcal{P}_1\pi_1+\mathcal{Q}_1\pi_2+\mathcal{O}_1\pi_3.
\end{aligned}
$$

Similarly,

$$\widehat{\mathfrak{v}}(\varpi)=\mathfrak{F}_2(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)$$

$$
\begin{aligned}
&+\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\Bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\Bigg)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\Bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\Bigg)d\tau\Bigg)d\rho\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\Bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\Bigg)d\rho\Bigg\}\\
&+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\Bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\Bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\Bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\Bigg)d\tau\Bigg)d\rho\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\Bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\Bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\Bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\Bigg)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\Bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\Bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{Z}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})dm\Bigg)d\tau\Bigg)d\rho\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\Bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{Z}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\Bigg)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\Bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{Z}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})d\tau\Bigg)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
&|\widehat{\mathfrak{v}}(\varpi)-\mathfrak{F}_2(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)|\le\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\\
&\times\Bigg\{\Bigg[\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\Bigg(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\Bigg)\\
&+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\Bigg\}\Bigg]\pi_1\\
&+\Bigg[\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\Bigg(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\Bigg)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\Bigg\}\\
&+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\Bigg(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\Bigg)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\Bigg\}\\
&+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\Bigg]\pi_2+\Bigg[\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\Bigg\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\Bigg\}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\Bigg(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\Bigg)\\
&+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\Bigg\}\Bigg]\pi_3\Bigg\}+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\phi)}\\
&\le\mathcal{P}_2\pi_1+\mathcal{Q}_2\pi_2+\mathcal{O}_2\pi_3.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|\widehat{\mathfrak{u}}(\varpi)-\mathcal{F}_3(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)| \leq \frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{\varphi\varpi})}{\varphi^3\Delta}\\
&\times\Bigg\{\Bigg[\mathcal{E}_3\mathcal{E}_6\left\{\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}\right\}+\mathcal{E}_1\mathcal{E}_3\left\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)\right.\\
&\left.+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\right\}\Bigg]\pi_1+\Bigg[\mathcal{E}_3\mathcal{E}_6\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)\right.\\
&\left.+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}+\mathcal{E}_2\mathcal{E}_6\left\{\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\right\}\Bigg]\pi_2\\
&+\Bigg[\mathcal{E}_1\mathcal{E}_3\left\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\right\}+\mathcal{E}_2\mathcal{E}_6\left\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\right.\\
&\left.+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\right\}\Bigg]\pi_3\Bigg\}+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\omega)}\\
&\leq\mathcal{P}_3\pi_1+\mathcal{Q}_3\pi_2+\mathcal{O}_3\pi_3.
\end{aligned}
$$

Thus, the operator $\mathcal{F}$, which is given by (3.1)–(3.3), can be extracted from the fixed-point property, as follows:

$$
\begin{aligned}
\left|\mathfrak{w}(\varpi)-\widehat{\mathfrak{w}}(\varpi)\right| &= \left|\mathfrak{w}(\varpi)-\mathcal{F}_1(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)+\mathcal{F}_1(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)-\widehat{\mathfrak{w}}(\varpi)\right|\\
&\leq \left|\mathcal{F}_1(\mathfrak{w},\mathfrak{v},\mathfrak{u})(\varpi)-\mathcal{F}_1(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)\right|+\left|\mathcal{F}_1(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)-\widehat{\mathfrak{w}}(\varpi)\right|\\
&\leq (\mathcal{P}_1\mu_1+\mathcal{Q}_1\sigma_1+\mathcal{O}_1\kappa_1)+(\mathcal{P}_1\mu_2+\mathcal{Q}_1\sigma_2+\mathcal{O}_1\kappa_2)+(\mathcal{P}_1\mu_3+\mathcal{Q}_1\sigma_3+\mathcal{O}_1\kappa_3)\\
&\quad\left\|(\mathfrak{w},\mathfrak{v},\mathfrak{u})-(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})\right\|+(\mathcal{P}_1\pi_1+\mathcal{Q}_1\pi_2+\mathcal{O}_1\pi_3),
\end{aligned}
\tag{4.4}
$$

similarly,

$$
\begin{aligned}
\left|\mathfrak{v}(\varpi)-\widehat{\mathfrak{v}}(\varpi)\right| &= \left|\mathfrak{v}(\varpi)-\mathcal{F}_2(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)+\mathcal{F}_2(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)-\widehat{\mathfrak{v}}(\varpi)\right|\\
&\leq \left|\mathcal{F}_2(\mathfrak{w},\mathfrak{v},\mathfrak{u})(\varpi)-\mathcal{F}_2(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)\right|+\left|\mathcal{F}_2(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)-\widehat{\mathfrak{v}}(\varpi)\right|\\
&\leq (\mathcal{P}_2\mu_1+\mathcal{Q}_2\sigma_1+\mathfrak{R}_2\kappa_1)+(\mathcal{P}_2\mu_2+\mathcal{Q}_2\sigma_2+\mathcal{O}_2\kappa_2)+(\mathcal{P}_2\mu_3+\mathcal{Q}_2\sigma_3+\mathcal{O}_2\kappa_3)\\
&\quad\left\|(\mathfrak{w},\mathfrak{v},\mathfrak{u})-(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})\right\|+(\mathcal{P}_2\pi_1+\mathcal{Q}_2\pi_2+\mathcal{O}_2\pi_3),
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
\left|\mathfrak{u}(\varpi)-\widehat{\mathfrak{u}}(\varpi)\right| &= \left|\mathfrak{u}(\varpi)-\mathcal{F}_3(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)+\mathcal{F}_3(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)-\widehat{\mathfrak{u}}(\varpi)\right|\\
&\leq \left|\mathcal{F}_3(\mathfrak{w},\mathfrak{v},\mathfrak{u})(\varpi)-\mathcal{F}_3(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)\right|+\left|\mathcal{F}_3(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})(\varpi)-\widehat{\mathfrak{u}}(\varpi)\right|\\
&\leq (\mathcal{P}_3\mu_1+\mathcal{Q}_3\sigma_1+\mathcal{O}_3\kappa_1)+(\mathcal{P}_3\mu_2+\mathcal{Q}_3\sigma_2+\mathcal{O}_3\kappa_2)+(\mathcal{P}_3\mu_3+\mathcal{Q}_3\sigma_3+\mathcal{O}_3\kappa_3)\\
&\quad\left\|(\mathfrak{w},\mathfrak{v},\mathfrak{u})-(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})\right\|+(\mathcal{P}_3\pi_1+\mathcal{Q}_3\pi_2+\mathcal{O}_3\pi_3).
\end{aligned}
\tag{4.6}
$$

From(4.4)–(4.6) it follows that

$$
\left\|(\mathfrak{w},\mathfrak{v},\mathfrak{u})-(\widehat{\mathfrak{w}},\widehat{\mathfrak{v}},\widehat{\mathfrak{u}})\right\| \leq (\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)\pi_1+(\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3)\pi_2+(\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3)\pi_3
$$

$$
\begin{aligned}
&+ (\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)(\mu_1 + \sigma_1 + \kappa_1) \\
&+ (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3)(\mu_2 + \sigma_2 + \kappa_2) \\
&+ (\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3)(\mu_3 + \sigma_3 + \kappa_3)\left\|(\mathfrak{w}, \mathfrak{v}, \mathfrak{u}) - (\widehat{\mathfrak{w}}, \widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\right\|,
\end{aligned}
$$

then

$$
\begin{aligned}
&\left\|(\mathfrak{w}, \mathfrak{v}, \mathfrak{u}) - (\widehat{\mathfrak{w}}, \widehat{\mathfrak{v}}, \widehat{\mathfrak{u}})\right\| \\
&\leq \frac{(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)\pi_1 + (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3)\pi_2 + (\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3)\pi_3}{1 - ((\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)(\mu_1 + \sigma_1 + \kappa_1) + (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3)(\mu_2 + \sigma_2 + \kappa_2) + (\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3)(\mu_3 + \phi_3 + \kappa_3))} \\
&\leq \mathcal{K}_1\pi_1 + \mathcal{K}_2\pi_2 + \mathcal{K}_3\pi_3,
\end{aligned}
$$

Thus, the boundary value problem (1.3) is Hyers-Ulam stable.

5. Example

Example 5.1. *Consider the following tripled fractional differential system of sequential type.*

$$
\begin{cases}
({}^c\mathfrak{D}^{\psi} + \varphi {}^c\mathfrak{D}^{\psi-1})\mathfrak{w}(\varpi) = \mathfrak{f}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)), \quad 2 < \psi \leq 3, \\
({}^c\mathfrak{D}^{\phi} + \varphi {}^c\mathfrak{D}^{\phi-1})\mathfrak{v}(\varpi) = \mathfrak{g}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)), \quad 2 < \phi \leq 3, \\
({}^c\mathfrak{D}^{\omega} + \varphi {}^c\mathfrak{D}^{\omega-1})\mathfrak{u}(\varpi) = \mathfrak{h}(\varpi, \mathfrak{w}(\varpi), \mathfrak{v}(\varpi), \mathfrak{u}(\varpi)), \quad 3 < \omega \leq 4, \\
\mathfrak{w}(0) = 0, \qquad \mathfrak{w}'(0) = 0, \qquad \mathfrak{w}(\mathcal{T}) = \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \mathfrak{v}(\zeta_j) + \Pi_1 \mathcal{I}^{\varsigma}\mathfrak{v}(\vartheta), \\
\mathfrak{v}(0) = 0, \qquad \mathfrak{v}'(0) = 0, \qquad \mathfrak{v}(\mathcal{T}) = \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \mathfrak{u}(\zeta_j) + \Pi_2 \mathcal{I}^{\varrho}\mathfrak{u}(\vartheta), \\
\mathfrak{u}(0) = 0, \qquad \mathfrak{u}'(0) = 0, \qquad \mathfrak{u}''(0) = 0, \qquad \mathfrak{u}(\mathcal{T}) = \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \mathfrak{w}(\zeta_j) + \Pi_3 \mathcal{I}^{\delta}\mathfrak{w}(\vartheta),
\end{cases}
\tag{5.1}
$$

Here $\psi = \frac{7}{3}, \phi = \frac{5}{2}, \omega = \frac{10}{3}; \varsigma = 9/20; \varrho = 11/20; \delta = 13/20; \vartheta = 93/50; \zeta_j = 36/25; \Upsilon_1 = 17/400; \Upsilon_2 = 15/300; \Upsilon_3 = 13/200; \Pi_1 = 17/200; \Pi_2 = 8/125; \Pi_3 = 6/68; \mathcal{T} = 1; \zeta_1 = 1/20; \zeta_2 = 2/20; \nu_1 = 1/100; \nu_2 = 1/50; \sigma_1 = 1/1000; \sigma_2 = 1/500; k = 4; \Delta = 0.067506818609056$ *with the given data, it is found that*

$$
\begin{aligned}
&\mathcal{P}_1 = 1.79143780787545, \mathcal{P}_2 = 0.83611149394660, \mathcal{P}_3 = 0.522203861964576, \\
&\mathcal{Q}_1 = 0.400530445936702, \mathcal{Q}_2 = 1.56029202354379, \mathcal{Q}_3 = 0.109316638513044, \\
&\mathcal{O}_1 = 0.190264056004677, \mathcal{O}_2 = 0.748929077567517, \mathcal{O}_3 = 0.745842301275926.
\end{aligned}
$$

To demonstrate the Theorem 3.1, we will use the following

$$
\begin{aligned}
\mathfrak{f}(\varpi, \mathfrak{w}, \mathfrak{v}, \mathfrak{u}) &= \frac{e^{-1}}{\sqrt{8+\varpi^2}} \cos \mathfrak{w} + \cos \varpi, \\
\mathfrak{g}(\varpi, \mathfrak{w}, \mathfrak{v}, \mathfrak{u}) &= \frac{1}{25+\varpi^2} (\sin \mathfrak{w} + |\mathfrak{v}|) + e^{-\varpi}, \\
\mathfrak{h}(\varpi, \mathfrak{w}, \mathfrak{v}, \mathfrak{u}) &= \frac{e^{-\varpi}}{9} \sin \mathfrak{u} + \tan^{-1} \varpi,
\end{aligned} \tag{5.2}
$$

which clearly satisfies condition ($\mathcal{H}_2$) *with* $\mathfrak{l}_{\mathfrak{f}}^* = \frac{1}{9e}, \mathfrak{l}_{\mathfrak{g}}^* = \frac{1}{26}$, *and* $\mathfrak{l}_{\mathfrak{h}}^* = \frac{1}{9e}$. *Moreover,*

$$
\max\{\mathcal{P}_1 \mathfrak{l}_{\mathfrak{f}}^* + \mathcal{P}_2 \mathfrak{l}_{\mathfrak{g}}^* + \mathcal{P}_3 \mathfrak{l}_{\mathfrak{h}}^*, \mathcal{Q}_1 \mathfrak{l}_{\mathfrak{f}}^* + \mathcal{Q}_2 \mathfrak{l}_{\mathfrak{g}}^* + \mathcal{Q}_3 \mathfrak{l}_{\mathfrak{h}}^*, \mathcal{O}_1 \mathfrak{l}_{\mathfrak{f}}^* + \mathcal{O}_2 \mathfrak{l}_{\mathfrak{g}}^* + \mathcal{O}_3 \mathfrak{l}_{\mathfrak{h}}^*\} < 1,
$$

$$
\max\{0.7309375523, 0.2139832533, 0.3115010395\} = 0.7309375523 < 1. \tag{5.3}
$$

So all conditions of Theorem 3.1 are satisfied, that is the problem (5.1) has at least one solution.

Figures 1–3 show the impact of fractional orders (ψ, ϕ, and ω) on the condition $\mathcal{P}_i, \mathcal{Q}_i, \mathcal{O}_i, i = 1, 2, 3$, given by (3.4) is represented graphically. Based on the $\mathcal{P}, \mathcal{Q}$, and $\mathcal{O}$ values given by (3.4) and the conditions $\mathcal{H}_1$ and $\mathcal{H}_2$, the graphs shown above describe the behavior of the solution of the problem (5.1). For different values of ψ, ϕ, and ω. An important observation to be taken in consideration is that: when the orders (ψ, ϕ, and ω) are small, the values of $\mathcal{P}, \mathcal{Q}$ and $\mathcal{O}$ decrease with increasing in time. As the to orders (ψ, ϕ, and ω) increase, the values $\mathcal{P}, \mathcal{Q}$ and $\mathcal{O}$ increase as well.

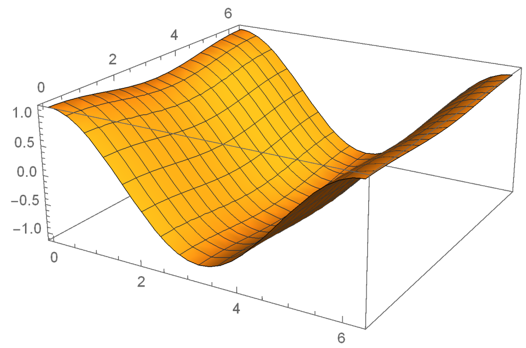

Figure 1. Graph of the approximate solution $\mathfrak{w}(\varpi)$ for various values of ψ.

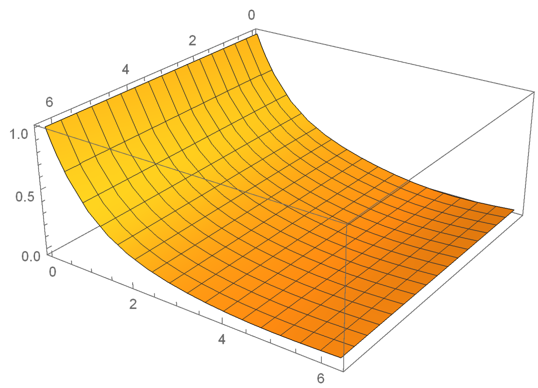

Figure 2. Graph of the approximate solution $\mathfrak{v}(\varpi)$ for various values of ϕ.

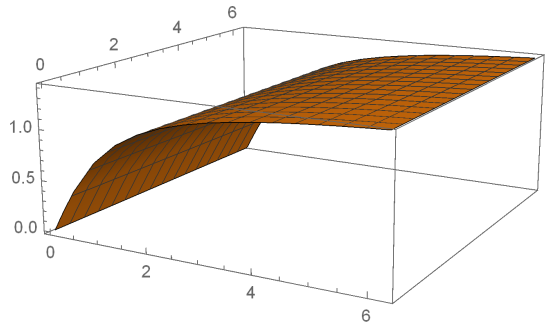

Figure 3. Graph of the approximate solution $\mathfrak{u}(\varpi)$ for various values of ω.

6. Conclusions

In this article, we study a tripled system of sequential fractional differential equations of order χ. Within the proposed system, tripled boundary conditions determine the existence and uniqueness of solutions. The existence result on the basis of mönch's fixed point theorem. The stability of the Hyers-Ulam solutions was investigated. We provide examples to demonstrate the study's generalization. The work described in this article is unique, and it adds a lot to the existing body of knowledge on the subject. When the parameters in systems $(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Pi_1, \Pi_2, \Pi_3)$ were specified, our results conformed to a few special cases. Assume we formulate the system (1.3) as follows: in the presented findings, take Π_1, Π_2, and Π_3.

$$\begin{cases} \mathfrak{w}(0) = 0, \quad \mathfrak{w}'(0) = 0, \quad \mathfrak{w}(1) = \Pi_1 \mathcal{I}^{\varsigma} \mathfrak{v}(\vartheta), \\ \mathfrak{v}(0) = 0, \quad \mathfrak{v}'(0) = 0, \quad \mathfrak{v}(1) = \Pi_2 \mathcal{I}^{\varrho} \mathfrak{u}(\vartheta), \\ \mathfrak{u}(0) = 0, \quad \mathfrak{u}'(0) = 0, \quad \mathfrak{u}''(0) = 0, \quad \mathfrak{u}(1) = \Pi_3 \mathcal{I}^{\delta} \mathfrak{w}(\vartheta). \end{cases} \tag{6.1}$$

The boundary value problems given in (6.1) can be solved by the methodology employed in the previous sections.

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Competing interests

The authors declare that there is no conflict of interest.

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