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Research article

On pairs of equations in eight prime cubes and powers of 2

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Abstract: In this paper, it is proved that every pair of large positive even integers satisfying some necessary conditions can be represented in the form of a pair of eight cubes of primes and 287 powers of 2. This improves the previous result.

Keywords: circle method; linnik problem; powers of 2
Mathematics Subject Classification: 11P32, 11P05, 11P55

1. Introduction

In 1951 and 1953, Linnik [4, 5] considered a problem related to Goldbach's problem. He proved that each sufficiently large positive even integer N can be written as a sum of two primes and k powers of 2, namely

$$N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \tag{1.1}$$

Later in 2002, Heath-Brown and Puchta [1] showed that $k = 13$ and $k = 7$ under the assumption of Generalized Riemann Hypothesis. In 2003, Pintz and Ruzsa [12] obtained that $k = 8$ unconditionally. Recently, Elsholtz showed that $k = 12$ in an unpublished manuscript. This was also proved by Liu and Lü [11] independently.

In 2001, Liu and Liu [6] showed that each large positive even integer N was a sum of eight prime cubes and k powers of 2, namely

$$N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \tag{1.2}$$

The acceptable value was improved by Liu and Lü [8], Platt and Trudgian [13] and Zhao and Ge [16].

As an extension, recently, Liu [10] considered that every pair of large positive even integers satisfying $N_2 \gg N_1 > N_2$ can be written as

$$\begin{cases} N_1 = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{\nu_1} + \cdots + 2^{\nu_k}, \\ N_2 = p_9^3 + p_{10}^3 + \cdots + p_{16}^3 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \end{cases} \tag{1.3}$$

He proved that (1.3) was solvable when $k = 1432$. Later Platt and Trudgian [13], Zhao [15] and Liu [7] improved it to 1319, 648 and 609, respectively.

In this paper, we sharpened the above result and obtained the following theorem.

Theorem 1.1. *For $k = 287$, the concurrent equations of* (1.3) *are solvable for every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$.*

We can establish Theorem 1.1 by using the Hardy-Littlewood circle method in combination with some new technologies of Hu et al. [2] and Hu and Yang [3].

2. Proof of Theorem 1.1

Now we can give an outline for the proof of Theorem 1.1.

Let N_i with $i = 1, 2$ be sufficiently large positive even integers. As in [8], in order to use the circle method, we set

$$P_i = N_i^{1/9-2\epsilon}, \quad Q_i = N_i^{8/9+\epsilon}, \quad L = \log_2 N_1$$

for $i = 1, 2$.

For any integers a_1, a_2, q_1, q_2 satisfying

$$1 \leqslant a_1 \leqslant q_1 \leqslant P_1, (a_1, q_1) = 1,$$

$$1 \leqslant a_2 \leqslant q_2 \leqslant P_2, (a_2, q_2) = 1,$$

we can define the major arcs $\mathfrak{M}_g$, $\mathfrak{M}_y$ and minor arcs $\mathfrak{m}_g$, $\mathfrak{m}_y$ as usual, namely

$$\mathfrak{M}_{\mathrm{i}} = \bigcup_{q \leqslant P_i} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}_{\mathrm{i}}(a, q), \quad \mathfrak{m}_{\mathrm{i}} = [1/Q_i, 1 + 1/Q_i] \setminus \mathfrak{M}_{\mathrm{i}},$$

where $i = 1, 2$ and

$$\mathfrak{M}_{\mathrm{i}}(a, q) = \{\alpha_i \in [0, 1] : |\alpha_i - a/q| \leqslant 1/(qQ_i)\}.$$

By the definitions of P_i and Q_i, we know that the arcs $\mathfrak{M}_i(a, q)$ are disjoint. We also let

$$\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{M}_2 = \{(\alpha_1, \alpha_2) \in [0, 1]^2 : \alpha_1 \in \mathfrak{M}_1, \alpha_2 \in \mathfrak{M}_2\},$$

$$\mathfrak{m} = [1/Q_i, 1 + 1/Q_i]^2 \setminus \mathfrak{M}.$$

As in [3], for convenience, let $\delta = 10^{-4}$ and

$$U_i = \left(\frac{N_i}{16(1 + \delta)}\right)^{1/3}, \quad V_i = U_i^{5/6}$$

for $i = 1, 2$. Let

$$S(\alpha_i, U_i) = \sum_{p \sim U_i} (\log p) e(p^3 \alpha_i), \quad T(\alpha_i, V_i) = \sum_{p \sim V_i} (\log p) e(p^3 \alpha_i),$$

$$G(\alpha_i) = \sum_{v \leqslant L} e(2^v \alpha_i),$$

$$\mathscr{E}_\lambda = \{\alpha_i \in [0,1] : |G(\alpha_i)| \geqslant \lambda L\},$$

where $i = 1, 2$.

Let

$$r(N_1, N_2) = \sum \log p_1 \log p_2 \cdots \log p_{16}$$

denote the weighted number of solutions of (1.3) in $(p_1, ..., p_{16}, v_1, ..., v_k)$ with

$$p_1, ..., p_4 \sim U_1, \quad p_5, ..., p_8 \sim V_1,$$

$$p_9, ..., p_{12} \sim U_2, \quad p_{13}, ...p_{16} \sim V_2, \quad v_j \leqslant L,$$

where $j = 1, 2, ..., k$. Then we have

$$\begin{aligned} & r(N_1, N_2) \\ = & \left(\iint\limits_{\mathfrak{M}} + \iint\limits_{\mathfrak{m} \bigcap \mathfrak{E}_\lambda} + \iint\limits_{\mathfrak{m} \backslash \mathfrak{E}_\lambda} \right) S^4(\alpha_1, U_1) T^4(\alpha_1, V_1) S^4(\alpha_2, U_2) T^4(\alpha_2, V_2) \\ & \times G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\ := & \ r_1(N_1, N_2) + r_2(N_1, N_2) + r_3(N_1, N_2). \end{aligned}$$

We can prove Theorem 1.1 by estimating $r_1(N_1, N_2)$, $r_2(N_1, N_2)$ and $r_3(N_1, N_2)$. We want to show that $r(N_1, N_2) > 0$ for $N_2 \gg N_1 > N_2$.

For a Dirichlet character χ mod q, let

$$C(\chi, a) = \sum_{h=1}^{q} \overline{\chi}(h) e\left(\frac{ah^3}{q}\right), \quad C(q, a) = C(\chi^0, a).$$

If $\chi_1, ..., \chi_8$ are characters mod q, then we write

$$B(n, q; \chi_1, ..., \chi_8) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C(\chi_1, a) C(\chi_2, a) \cdots C(\chi_8, a) e\left(-\frac{an}{q}\right),$$

$$B(n, q) = B(n, q; \chi^0, ..., \chi^0),$$

$$A(n, q) = \frac{B(n, q)}{\varphi^4(q)}, \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q).$$

Lemma 2.1. *Let* $N_1 \equiv N_2 \equiv 0 (\bmod\ 2)$, $\mathscr{A}(N_i, k) = \{n_i \geqslant 2 : n_i = N_i - 2^{v_1} - \cdots - 2^{v_k}\}$ *and* $k \geqslant 35$. *Then we have*

$$\sum_{\substack{n_1 \in \mathscr{A}(N_1, k) \\ n_2 \in \mathscr{A}(N_2, k) \\ n_1 \equiv n_2 \equiv 0 (\bmod\ 2)}} \mathfrak{S}(n_1) \mathfrak{S}(n_2) \geqslant 0.89094 L^k.$$

Proof. For $k \geqslant 35, A(n_i, p^k) = 0$. Now since $A(n_i, p)$ is multiplicative, we can get

$$\mathfrak{S}(n_i) = \prod_{p=2}^{\infty}(1 + A(n_i, p)).$$

With a similar argument of Lemma 2.3 in the paper by Zhao [15], we have

$$\mathfrak{S}(n_i) = 2\left(1 - \frac{1}{2^8}\right)\prod_{p>3}(1 + A(n_i, p)),$$

$$\prod_{p\geqslant 17}(1 + A(n, p)) \geqslant C_0 := 0.82067.$$

Let $m_0 = 14$. Now we can get

$$\begin{aligned}
&\sum_{\substack{n_1 \in \mathscr{B}(N_1,k)\\ n_2 \in \mathscr{B}(N_2,k)\\ n_1 \equiv n_2 \equiv 0 (\bmod 2)}} \mathfrak{S}(n_1)\mathfrak{S}(n_2)\\
\geqslant\ & (1.9921875C_0)^2 \sum_{\substack{n_1 \in \mathscr{B}(N_1,k)\\ n_2 \in \mathscr{B}(N_2,k)\\ n_1 \equiv n_2 \equiv 0 (\bmod 2)}} \prod_{3<p<m_0}(1 + A(n_1, p)) \prod_{3<p<m_0}(1 + A(n_2, p))\\
\geqslant\ & (1.9921875C_0)^2 \sum_{1\leqslant j\leqslant q} \sum_{\substack{n_1 \in \mathscr{B}(N_1,k)\\ n_2 \in \mathscr{B}(N_2,k)\\ n_1 \equiv n_2 \equiv 0 (\bmod 2)\\ n_1 \equiv n_2 \equiv j (\bmod q)}} \prod_{3<p<m_0}(1 + A(n_1, p)) \prod_{3<p<m_0}(1 + A(n_2, p))\\
\geqslant\ & (1.9921875C_0)^2 \sum_{1\leqslant j\leqslant q} \prod_{3<p<m_0}(1 + A(j, p)) \prod_{3<p<m_0}(1 + A(j, p)) \sum_{\substack{n_1 \in \mathscr{B}(N_1,k)\\ n_2 \in \mathscr{B}(N_2,k)\\ n_1 \equiv n_2 \equiv 0 (\bmod 2)\\ n_1 \equiv n_2 \equiv j (\bmod q)}} 1,\\
\geqslant\ & (1.9921875C_0)^2 \sum_{1\leqslant j\leqslant q} \prod_{3<p<m_0}(1 + A(j, p))^2 \sum_{\substack{n_1 \in \mathscr{B}(N_1,k)\\ n_1 \equiv 0 (\bmod 2)\\ n_1 \equiv j (\bmod q)}} 1,
\end{aligned}$$

where $q = \prod_{3<p<m_0} p$. By the result obtained by Zhao and Ge [16, Lemma 2.3], we have

$$\sum_{\substack{n_1 \in \mathscr{B}(N_1,k)\\ n_1 \equiv 0 (\bmod 2)\\ n_1 \equiv j (\bmod q)}} 1 \geqslant \frac{(1 - 0.000064)L^k}{3q} + O(L^{k-1}).$$

Noting that

$$\begin{aligned}
\sum_{j=1}^{p}(1 + A(j, p))^2 &= p + 2\sum_{j=1}^{p} A(j, p) + \sum_{j=1}^{p}(A(j, p))^2 = p + \sum_{j=1}^{p}(A(j, p))^2\\
&\geqslant p,
\end{aligned}$$

therefore

$$
\begin{aligned}
&\sum_{\substack{n_1 \in \mathscr{B}(N_1,k)\\ n_2 \in \mathscr{B}(N_2,k)\\ n_1 \equiv n_2 \equiv 0 (\bmod 2)}} \mathfrak{S}(n_1)\mathfrak{S}(n_2)\\
\geqslant\ & (1.9921875C_0)^2 \sum_{j=1}^{p} \prod_{3<p<m_0} (1+A(j,p))^2 \frac{(1-0.000064)L^k}{3q} + O(L^{k-1})\\
\geqslant\ & \frac{1}{3}(1.9921875C_0)^2 \prod_{3<p<m_0} \sum_{j=1}^{p} (1+A(j,p))^2 \frac{(1-0.000064)L^k}{q} + O(L^{k-1})\\
\geqslant\ & \frac{1}{3}(1.9921875C_0)^2 (1-0.000064)L^k + O(L^{k-1}).
\end{aligned}
$$

Then the lemma follows since L is sufficiently large. □

Lemma 2.2. *Let N_1 and N_2 are sufficiently large positive even integers satisfying $N_2 \gg N_1 > N_2$,*

$$r_1(N_1, N_2) \geqslant 1.26 \times 10^{-4} U_1 V_1^4 U_2 V_2^4 L^k.$$

Proof. By Lemma 2.1 in Liu and Lü [8], we note that

$$
\begin{aligned}
& r_1(N_1, N_2)\\
=\ & \iint\limits_{\mathfrak{M}} S^4(\alpha_1, U_1) T^4(\alpha_1, V_1) S^4(\alpha_2, U_2) T^4(\alpha_2, V_2)\\
& \times G^k(\alpha_1+\alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2\\
\geqslant\ & \left(\frac{1}{3^8}\right)^2 \sum_{\substack{n_1 \in \mathscr{A}(N_1,k)\\ n_2 \in \mathscr{A}(N_2,k)}} \mathfrak{S}(n_1)\mathfrak{S}(n_2) J(n_1) J(n_2).
\end{aligned}
$$

We also note that $J(n_i) > 78.15468 U_i V_i^4$ by Liu and Lü [8, Lemma 3.3]. Then the lemma follows from Lemma 2.1. □

Lemma 2.3. *Let $\alpha = a/q + \lambda$ be subject to $1 \leqslant a \leqslant q, (a,q) = 1$ and $|\lambda| \leqslant 1/qQ$, with $Q = U^{12/7}$; then, we have*

$$\sum_{p \sim U} (\log p) e(p^3 \alpha) \ll U^{1-1/12+\epsilon} + \frac{q^{-1/6} U^{1+\epsilon}}{(1+|\lambda| U^3)^{1/2}}.$$

Proof. This is Lemma 8.5 in Zhao [14]. □

Lemma 2.4. *Let $\mathfrak{m}$ and $S(\alpha_i, U_i)$ be defined as before; then,*

$$\max_{\alpha \in C(\mathscr{M})} |S(\alpha_i, U_i)| \ll U_i^{1-1/12+\epsilon}.$$

Proof. We can find that the proof of this lemma is similar to that of Lemma 3.4 in Liu and Lü [8]. We only need to change 1/14 to 1/12 for Lemma 2.4 in the proof of Liu and Lü [8, Lemma 3.4]. □

Lemma 2.5. *Let* $\operatorname{meas}(\mathscr{E}_\lambda)$ *denotes the measure of* $\mathscr{E}_\lambda$. *We have*

$$\operatorname{meas}(\mathscr{E}_\lambda) \ll N_1^{-E(\lambda)},$$

with $E(0.9532) > 8/9 + 10^{-10}$.

Proof. Similar to the proof of Liu and Lü [8, Lemma 3.5], we can calculate by computer to prove this lemma. □

Lemma 2.6. *Let* N_1 *and* N_2 *are sufficiently large positive even integers satisfying* $N_2 \gg N_1 > N_2$,

$$r_2(N_1, N_2) \ll U_1 V_1^4 U_2 V_2^4 L^{k-1},$$

with $\lambda = 0.9532$.

Proof. According to the definition of $\mathfrak{m}$, we have

$$\mathfrak{m} \subset \{(\alpha_1, \alpha_2) : \alpha_1 \in \mathfrak{m}_1, \alpha_2 \in [0, 1]\} \cup \{(\alpha_1, \alpha_2) : \alpha_1 \in [0, 1], \alpha_2 \in \mathfrak{m}_2\}.$$

Then

$$\begin{aligned}
& r_2(N_1, N_2) \\
= & \iint_{\mathfrak{m}\bigcap\mathscr{E}_\lambda} S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2) \\
& \times G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\
\ll & L^k\Big(\iint_{\substack{(\alpha_1,\alpha_2)\in\mathfrak{m}_1\times[0,1] \\ |G(\alpha_1+\alpha_2)|\geqslant\lambda L}} |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)|\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\
& + \iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]\times\mathfrak{m}_2 \\ |G(\alpha_1+\alpha_2)|\geqslant\lambda L}} |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)|\mathrm{d}\alpha_1\mathrm{d}\alpha_2\Big) \\
:= & L^k(I_1 + I_2).
\end{aligned}$$

Then we have

$$\begin{aligned}
I_1 &= \iint_{\substack{(\alpha_1,\alpha_2)\in\mathfrak{m}_1\times[0,1] \\ |G(\alpha_1+\alpha_2)|\geqslant\lambda L}} |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)|\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \\
&\ll U_1^{11/3+\epsilon}V_1^4 \iint_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2 \\ |G(\alpha_1+\alpha_2)|\geqslant\lambda L}} |S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)|\mathrm{d}\alpha_1\mathrm{d}\alpha_2,
\end{aligned}$$

where we use Lemma 2.5 and the trivial bound of $T(\alpha_1, V_1)$.

Now we use the variable substitution $\beta = \alpha_1 + \alpha_2$ and get

$$\iint\limits_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |G(\alpha_1+\alpha_2)|\geqslant\lambda L}} |S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)|\mathrm{d}\alpha_1\mathrm{d}\alpha_2 = \int_0^1 |S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)|\Big(\int\limits_{\substack{\beta\in[\alpha_2,1+\alpha_2]\\ |G(\beta)|\geqslant\lambda L}} \mathrm{d}\beta\Big)\mathrm{d}\alpha_2.$$

By Lemma 2.6 in the paper by Hu and Yang [3], we have

$$\int_0^1 |S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)|\mathrm{d}\alpha_2 \ll U_2V_2^4.$$

From Lemma 2.5 we have

$$\iint\limits_{\substack{(\alpha_1,\alpha_2)\in[0,1]^2\\ |G(\alpha_1+\alpha_2)|\geqslant\lambda L}} |S^4(\alpha_2,U_2)T^4(\alpha_2,V_2)|\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \ll U_2V_2^4N_1^{-E(\lambda)}.$$

We choose $\lambda = 0.9532$ and get

$$I_1 \ll U_1^{11/3-8/3-\epsilon}V_1^4U_2V_2^4 \ll U_1^{1-\epsilon}V_1^4U_2V_2^4,$$

since $N_2 \gg N_1 > N_2$. Similarly,

$$I_2 \ll U_2^{11/3-8/3-\epsilon}V_2^4U_1V_1^4 \ll U_2^{1-\epsilon}V_2^4U_1V_1^4,$$

Then

$$r_2(N_1,N_2) \ll (U_1^{1-\epsilon}V_1^4U_2V_2^4 + U_2^{1-\epsilon}V_2^4U_1V_1^4)L^k \ll U_1V_1^4U_2V_2^4L^{k-1}.$$

□

To estimate $r_3(N_1,N_2)$, first we need to consider the upper bound for the number of solutions of the equation

$$n = p_1^3+\cdots+p_4^3-p_5^3-\cdots-p_8^3, \quad 0\leqslant |n| \leqslant N_i. \tag{2.1}$$

Lemma 2.7. *Let* $n \equiv 0(\mathrm{mod}\ 2)$ *be an integer and* $\varrho_i(n)$ *be the number of representations of* n *in the form of (2.1) that are subject to*

$$p_1,p_2,p_5,p_6 \sim U_i, \quad p_3,p_4,p_7,p_8 \sim V_i, \quad i=1,2.$$

Then for all $0\leqslant |n| \leqslant N_i$,

$$\varrho_i(n) \leqslant bU_iV_i^4L^{-8}$$

with $b = 147185.22$.

Proof. This lemma is Lemma 2.1 in the paper by Liu [9]. □

Lemma 2.8. *Let N_1 and N_2 be sufficiently large positive even integers satisfying $N_2 \gg N_1 > N_2$,*

$$r_3(N_1, N_2) \leqslant 117.04\lambda^k U_1 V_1^4 U_2 V_2^4 L^k.$$

Proof. According to the definitions of $\mathfrak{m}$ and $\mathscr{E}_\lambda$, by Lemma 2.7 and the definition of $\varrho(n)$ we have

$$\begin{aligned} & r_3(N_1, N_2) \\ \leqslant\ & (\lambda L)^k \iint\limits_{(\alpha_1,\alpha_2)\in[0,1]^2} |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)| d\alpha_1 d\alpha_2 \\ \leqslant\ & (\lambda L)^k \int_0^1 |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)| d\alpha_1 \int_0^1 |S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)| d\alpha_2 \\ \leqslant\ & (\lambda L)^k (\log(2U_1))^4 (\log(2V_1))^4 (\log(2U_2))^4 (\log(2V_2))^4 \varrho_1(0)\varrho_2(0) \\ \leqslant\ & 117.04\lambda^k U_1 V_1^4 U_2 V_2^4 L^k. \end{aligned}$$

□

Combining Lemmas 2.2, 2.6 and 2.8, we can obtain

$$r(N_1, N_2) > 1.26 \times 10^{-4} U_1 V_1^4 U_2 V_2^4 L^k - 117.04\lambda^k U_1 V_1^4 U_2 V_2^4 L^k.$$

Therefore we solve the inequality

$$r(N_1, N_2) > 0$$

and obtain $k \geqslant 287$. Now the proof of Theorem 1.1 is complete.

3. Conclusions

To sum up, we deduce that every pair of sufficiently large even integers N_1, N_2 satisfying $N_2 \gg N_1 > N_2$ can be represented in the form of a pair of eight cubes of primes and 287 powers of 2.

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Conflict of interest

The authors declare that they have no competing interests.

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