Research article

On pairs of equations in eight prime cubes and powers of 2

Gen Li\textsuperscript{1}, Liqun Hu\textsuperscript{2,*} and Xianjiu Huang\textsuperscript{1,2}

\textsuperscript{1} Jiu luan Academy, Nanchang University, Nanchang, Jiangxi 330031, China
\textsuperscript{2} Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, China

* Correspondence: Email: huliqun@ncu.edu.cn.

Abstract: In this paper, it is proved that every pair of large positive even integers satisfying some necessary conditions can be represented in the form of a pair of eight cubes of primes and 287 powers of 2. This improves the previous result.

Keywords: circle method; linnik problem; powers of 2

Mathematics Subject Classification: 11P32, 11P05, 11P55

1. Introduction

In 1951 and 1953, Linnik [4, 5] considered a problem related to Goldbach’s problem. He proved that each sufficiently large positive even integer $N$ can be written as a sum of two primes and $k$ powers of 2, namely

$$N = p_1 + p_2 + 2^{v_1} + \cdots + 2^{v_k}. \quad (1.1)$$

Later in 2002, Heath-Brown and Puchta [1] showed that $k = 13$ and $k = 7$ under the assumption of Generalized Riemann Hypothesis. In 2003, Pintz and Ruzsa [12] obtained that $k = 8$ unconditionally. Recently, Elsholtz showed that $k = 12$ in an unpublished manuscript. This was also proved by Liu and Lü [11] independently.

In 2001, Liu and Liu [6] showed that each large positive even integer $N$ was a sum of eight prime cubes and $k$ powers of 2, namely

$$N = p_3^1 + p_3^2 + \cdots + p_3^8 + 2^{v_1} + \cdots + 2^{v_k}. \quad (1.2)$$

The acceptable value was improved by Liu and Lü [8], Platt and Trudgian [13] and Zhao and Ge [16].

As an extension, recently, Liu [10] considered that every pair of large positive even integers satisfying $N_2 \gg N_1 > N_2$ can be written as

$$\begin{cases} N_1 = p_3^1 + p_3^2 + \cdots + p_3^8 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_3^9 + p_3^{10} + \cdots + p_3^{16} + 2^{v_1} + \cdots + 2^{v_k}. \end{cases} \quad (1.3)$$
He proved that (1.3) was solvable when \( k = 1432 \). Later Platt and Trudgian [13], Zhao [15] and Liu [7] improved it to \( 1319, 648 \) and \( 609 \), respectively.

In this paper, we sharpened the above result and obtained the following theorem.

**Theorem 1.1.** For \( k = 287 \), the concurrent equations of (1.3) are solvable for every pair of sufficiently large positive even integers \( N_1 \) and \( N_2 \) satisfying \( N_2 \gg N_1 > N_2 \).

We can establish Theorem 1.1 by using the Hardy-Littlewood circle method in combination with some new technologies of Hu et al. [2] and Hu and Yang [3].

2. Proof of Theorem 1.1

Now we can give an outline for the proof of Theorem 1.1.

Let \( N_i \) with \( i = 1, 2 \) be sufficiently large positive even integers. As in [8], in order to use the circle method, we set

\[
P_i = N_i^{1/9-2\epsilon}, \quad Q_i = N_i^{9/9+\epsilon}, \quad L = \log_2 N_i
\]

for \( i = 1, 2 \).

For any integers \( a_1, a_2, q_1, q_2 \) satisfying

\[
1 \leq a_1 \leq q_1 \leq P_1, (a_1, q_1) = 1, \\
1 \leq a_2 \leq q_2 \leq P_2, (a_2, q_2) = 1,
\]

we can define the major arcs \( \mathcal{M}_i, \mathcal{M}_y \) and minor arcs \( m_y, m_y \) as usual, namely

\[
\mathcal{M}_i = \bigcup_{q \leq P_i} \bigcup_{1 \leq a \leq q \atop (a,q)=1} \mathcal{M}_i(a, q), \quad m_i = [1/Q_i, 1 + 1/Q_i] \setminus \mathcal{M}_i,
\]

where \( i = 1, 2 \) and

\[
\mathcal{M}_i(a, q) = \{\alpha_i \in [0, 1] : |\alpha_i - a/q| \leq 1/(qQ_i)\}.
\]

By the definitions of \( P_i \) and \( Q_i \), we know that the arcs \( \mathcal{M}_i(a, q) \) are disjoint. We also let

\[
\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 = \{(\alpha_1, \alpha_2) \in [0, 1]^2 : \alpha_1 \in \mathcal{M}_1, \alpha_2 \in \mathcal{M}_2\},
\]

\[
m = [1/Q_i, 1 + 1/Q_i]^2 \setminus \mathcal{M}.
\]

As in [3], for convenience, let \( \delta = 10^{-4} \) and

\[
U_i = \left( \frac{N_i}{16(1 + \delta)} \right)^{1/3}, \quad V_i = U_i^{5/6}
\]

for \( i = 1, 2 \). Let

\[
S(\alpha_i, U_i) = \sum_{p - \mathcal{U}_i} (\log p) \exp(p^3 \alpha_i), \quad T(\alpha_i, V_i) = \sum_{p - \mathcal{V}_i} (\log p) \exp(p^3 \alpha_i),
\]

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We can prove Theorem 1.1 by estimating $r = \{x \in [0, 1] : |G(\chi)| > \lambda L\}$, where $i = 1, 2$.

Let

$$r(N_1, N_2) = \sum \log p_1 \log p_2 \cdots \log p_{16}$$

denote the weighted number of solutions of (1.3) in $(p_1, \ldots, p_{16}, v_1, \ldots, v_k)$ with

$$p_1, \ldots, p_4 \sim U_1, \quad p_5, \ldots, p_8 \sim V_1,$$

$$p_9, \ldots, p_{12} \sim U_2, \quad p_{13}, \ldots, p_{16} \sim V_2, \quad v_j \leq L,$$

where $j = 1, 2, \ldots, k$. Then we have

$$r(N_1, N_2) = \left(S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)\right.$$

$$\times G(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)\alpha_1 d\alpha_1 \alpha_2$$

$$:= r_1(N_1, N_2) + r_2(N_1, N_2) + r_3(N_1, N_2).$$

We can prove Theorem 1.1 by estimating $r_1(N_1, N_2), r_2(N_1, N_2)$ and $r_3(N_1, N_2)$. We want to show that $r(N_1, N_2) > 0$ for $N_2 \gg N_1 > N_2$.

For a Dirichlet character $\chi \mod q$, let

$$C(\chi, a) = \sum_{h=1}^{q} \overline{X}(h)e\left(\frac{ah^2}{q}\right), \quad C(q, a) = C(\chi^0, a).$$

If $\chi_1, \ldots, \chi_8$ are characters mod $q$, then we write

$$B(n, q; \chi_1, \ldots, \chi_8) = \sum_{\chi_1, \ldots, \chi_8} C(\chi_1, a)C(\chi_2, a) \cdots C(\chi_8, a)e\left(-\frac{an}{q}\right),$$

$$B(n, q) = B(n, q; \chi^0, \ldots, \chi^0),$$

$$A(n, q) = \frac{B(n, q)}{\varphi^4(q)}, \quad \Xi(n) = \sum_{q=1}^{\infty} A(n, q).$$

**Lemma 2.1.** Let $N_1 \equiv N_2 \equiv 0 (\mod 2), \mathcal{O}(N, k) = \{n_i \geq 2 : n_i = N_i - 2^{n_1} - \cdots - 2^{n_k}\}$ and $k \geq 35$. Then we have

$$\sum_{n_1 \in \mathcal{O}(N_1, k)} \Xi(n_1) \Xi(n_2) \geq 0.89094L^k.$$
Proof. For $k \geq 35$, $A(n_i, p^k) = 0$. Now since $A(n_i, p)$ is multiplicative, we can get

$$\mathcal{E}(n_i) = \prod_{p=2}^{\infty} (1 + A(n_i, p)).$$

With a similar argument of Lemma 2.3 in the paper by Zhao [15], we have

$$\mathcal{E}(n_i) = 2 \left(1 - \frac{1}{2^8}\right) \prod_{p>3} (1 + A(n_i, p)),$$

$$\prod_{p>17} (1 + A(n, p)) \geq C_0 := 0.82067.$$

Let $m_0 = 14$. Now we can get

$$\sum_{n_1 \in \mathbb{N}, n_2 \in \mathbb{N}} \mathcal{E}(n_1) \mathcal{E}(n_2) \geq (1.9921875C_0)^2 \sum_{n_1 \in \mathbb{N}, n_2 \in \mathbb{N}} \prod_{3<p<m_0} (1 + A(n_1, p)) \prod_{3<p<m_0} (1 + A(n_2, p)) \prod_{1 \in \mathbb{N}, 3<p<m_0} (1 + A(j, p)) \prod_{3<p<m_0} (1 + A(j, p)) \sum_{n_1 \in \mathbb{N}, n_2 \in \mathbb{N}} 1,$$

where $q = \prod_{3<p<m_0} p$. By the result obtained by Zhao and Ge [16, Lemma 2.3], we have

$$\sum_{n_1 \in \mathbb{N}, n_1 \equiv 0 \pmod{2}, n_1 \equiv j \pmod{q}} 1 \geq \frac{(1 - 0.000064)L^k}{3q} + O(L^{k-1}).$$

Noting that

$$\sum_{j=1}^{p} (1 + A(j, p))^2 = p + 2 \sum_{j=1}^{p} A(j, p) + \sum_{j=1}^{p} (A(j, p))^2 = p + \sum_{j=1}^{p} (A(j, p))^2 \geq p,$$

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Therefore
\[
\sum_{n_1 \in \mathbb{Z}[N_1,k], n_2 \in \mathbb{Z}[N_2,k]} \Xi(n_1) \Xi(n_2)
\]
\[
\geq (1.9921875C_0)^2 \sum_{j=1}^{p} \prod_{3<p<m_0} (1 + A(j, p))^2 \frac{(1 - 0.000064)L^k}{3q} + O(L^{k-1})
\]
\[
\geq \frac{1}{3} (1.9921875C_0)^2 \sum_{3<p<m_0} \prod_{j=1}^{p} (1 + A(j, p))^2 \frac{(1 - 0.000064)L^k}{q} + O(L^{k-1})
\]
\[
\geq \frac{1}{3} (1.9921875C_0)^2 (1 - 0.000064)L^k + O(L^{k-1}).
\]

Then the lemma follows since \( L \) is sufficiently large. □

**Lemma 2.2.** Let \( N_1 \) and \( N_2 \) are sufficiently large positive even integers satisfying \( N_2 \gg N_1 > N_2 \),

\[
r_1(N_1, N_2) \geq 1.26 \times 10^{-4} U_1^4 U_2 V_2^4 L^k.
\]

**Proof.** By Lemma 2.1 in Liu and Lü [8], we note that

\[
r_1(N_1, N_2) = \int \int S^4(\alpha_1, U_1) T^4(\alpha_1, V_1) S^4(\alpha_2, U_2) T^4(\alpha_2, V_2)
\]
\[
\times G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2
\]
\[
\geq \left( \frac{1}{3L} \right)^2 \sum_{n_1 \in \mathbb{Z}[N_1,k], n_2 \in \mathbb{Z}[N_2,k]} \Xi(n_1) \Xi(n_2) J(n_1) J(n_2).
\]

We also note that \( J(n_i) > 78.15468U_i V_i^4 \) by Liu and Lü [8, Lemma 3.3]. Then the lemma follows from Lemma 2.1. □

**Lemma 2.3.** Let \( \alpha = a/q + \lambda \) be subject to \( 1 \leq a \leq q, (a, q) = 1 \) and \( |\lambda| \leq 1/qQ \), with \( Q = U^{12/7} \); then, we have

\[
\sum_{p-U} (\log p) e(p^3 \alpha) \ll U^{1-1/12+\epsilon} + \frac{q^{-1/6} U^{1+\epsilon}}{(1 + |\lambda| U^3)^{1/2}}.
\]

**Proof.** This is Lemma 8.5 in Zhao [14]. □

**Lemma 2.4.** Let \( m \) and \( S(\alpha, U_i) \) be defined as before; then,

\[
\max_{\alpha \in \mathbb{C}(\#)} |S(\alpha, U_i)| \ll U_i^{1-1/12+\epsilon}.
\]

**Proof.** We can find that the proof of this lemma is similar to that of Lemma 3.4 in Liu and Lü [8]. We only need to change 1/14 to 1/12 for Lemma 2.4 in the proof of Liu and Lü [8, Lemma 3.4]. □
Lemma 2.5. Let $\text{meas}(E_i)$ denote the measure of $E_i$. We have

$$\text{meas}(E_i) \ll N_1^{-E_i},$$

with $E(0.9532) > 8/9 + 10^{-10}$.

Proof. Similar to the proof of Liu and Lü [8, Lemma 3.5], we can calculate by computer to prove this lemma. □

Lemma 2.6. Let $N_1$ and $N_2$ are sufficiently large positive even integers satisfying $N_2 \gg N_1 > N_2$,

$$r_2(N_1, N_2) \ll U_1V_1^4 U_2 V_2^{2L-1},$$

with $\lambda = 0.9532$.

Proof. According to the definition of $m$, we have

$$m \subset \{(\alpha_1, \alpha_2) : \alpha_1 \in m_1, \alpha_2 \in [0, 1] \} \cup \{(\alpha_1, \alpha_2) : \alpha_1 \in [0, 1], \alpha_2 \in m_2 \}.$$

Then

$$r_2(N_1, N_2) = \int_{m \cap E_i} S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)$$

$$\times G(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2$$

$$\ll L^k \left( \int_{(\alpha_1, \alpha_2) \in m \times [0, 1]} |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)| d\alpha_1 d\alpha_2 ight)$$

$$+ \int_{(\alpha_1, \alpha_2) \in [0, 1] \times m} |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)| d\alpha_1 d\alpha_2$$

$$:= L^k (I_1 + I_2).$$

Then we have

$$I_1 = \int_{(\alpha_1, \alpha_2) \in m \times [0, 1]} |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)| d\alpha_1 d\alpha_2$$

$$\ll U_1^{11/3+e} V_1^4 \int_{(\alpha_1, \alpha_2) \in [0, 1]^2} |S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)| d\alpha_1 d\alpha_2,$$

where we use Lemma 2.5 and the trivial bound of $T(\alpha, V_1)$.

Now we use the variable substitution $\beta = \alpha_1 + \alpha_2$ and get
\[
\int\int_{(\alpha_1, \alpha_2) \in [0, 1]^2 \setminus G(\alpha_1 + \alpha_2) \geq \lambda L} |S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)| \, d\alpha_1 \, d\alpha_2 \\
= \int_0^1 |S^4(\alpha_2, U_2)T^4(\alpha_2, V_2)| \left( \int_{\beta \in \alpha_1 + \alpha_2} \, d\beta \right) \, d\alpha_2.
\]

By Lemma 2.6 in the paper by Hu and Yang [3], we have
\[
Z_{10} \mid S^4(\alpha_2, U_2)T^4(\alpha_2, V_2) \mid d\alpha_1 \ll U_2 V_2^4.
\]

From Lemma 2.5 we have
\[
Z_{10} \mid S^4(\alpha_2, U_2)T^4(\alpha_2, V_2) \mid d\alpha_1 \ll U_2 V_2^4 N_1^{-E(\lambda)}.
\]

We choose \( \lambda = 0.9532 \) and get
\[
I_1 \ll U_1^{11/3 - 8/3 - \epsilon} V_1^4 U_2 V_2^4 \ll U_1^{1-\epsilon} V_1^4 U_1 V_1^4,
\]
since \( N_2 \gg N_1 > N_2 \). Similarly,
\[
I_2 \ll U_2^{11/3 - 8/3 - \epsilon} V_2^4 U_1 V_1^4 \ll U_2^{1-\epsilon} V_2^4 U_1 V_1^4.
\]

Then
\[
I_2(N_1, N_2) \ll (U_1^{1-\epsilon} V_1^4 U_2 V_2^4 + U_2^{1-\epsilon} V_2^4 U_1 V_1^4)L^k \ll U_1 V_1 U_2 V_2 L^{k-1}.
\]

\( \square \)

To estimate \( r_3(N_1, N_2) \), first we need to consider the upper bound for the number of solutions of the equation
\[
n = p_1^3 + \cdots + p_5^3 - p_5^3 - \cdots - p_8^3, \quad 0 \leq |n| \leq N_i. \tag{2.1}
\]

**Lemma 2.7.** Let \( n \equiv 0 \pmod{2} \) be an integer and \( \varrho_i(n) \) be the number of representations of \( n \) in the form (2.1) that are subject to
\[
p_1, p_2, p_5, p_6 \sim U_i, \quad p_3, p_4, p_7, p_8 \sim V_i, \quad i = 1, 2.
\]

Then for all \( 0 \leq |n| \leq N_i \),
\[
\varrho_i(n) \leq b U_i V_i L^{-8}
\]
with \( b = 147185.22 \).

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**Proof.** This lemma is Lemma 2.1 in the paper by Liu [9].

**Lemma 2.8.** Let $N_1$ and $N_2$ be sufficiently large positive even integers satisfying $N_2 \gg N_1 > N_2$,

\[ r_3(N_1, N_2) \leq 117.04 \lambda^k U_1^4 V_1^4 U_2^4 V_2^4 L^k. \]

**Proof.** According to the definitions of $\omega$ and $\varphi$, by Lemma 2.7 and the definition of $\varphi(n)$ we have

\[
\begin{align*}
\quad & r_3(N_1, N_2) \\
\leq & (\lambda L)^k \int_0^1 |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)| \mathrm{d}\alpha_1 \\
\leq & (\lambda L)^k \int_0^1 |S^4(\alpha_1, U_1)T^4(\alpha_1, V_1)| \mathrm{d}\alpha_1 \\
\leq & (\lambda L)^k (\log(2U_1))^4 (\log(2V_1))^4 (\log(2U_2))^4 (\log(2V_2))^4 \varphi_1(0) \varphi_2(0) \\
\leq & 117.04 \lambda^k U_1^4 V_1^4 U_2^4 V_2^4 L^k.
\end{align*}
\]

\[ \square \]

Combining Lemmas 2.2, 2.6 and 2.8, we can obtain

\[ r(N_1, N_2) > 1.26 \times 10^{-4} U_1^4 V_1^4 U_2^4 V_2^4 L^k - 117.04 \lambda^k U_1^4 V_1^4 U_2^4 V_2^4 L^k. \]

Therefore we solve the inequality

\[ r(N_1, N_2) > 0 \]

and obtain $k \geq 287$. Now the proof of Theorem 1.1 is complete.

3. Conclusions

To sum up, we deduce that every pair of sufficiently large even integers $N_1, N_2$ satisfying $N_2 \gg N_1 > N_2$ can be represented in the form of a pair of eight cubes of primes and 287 powers of 2.

Acknowledgments

This work was supported by the Natural Science Foundation of Jiangxi Province for Distinguished Young Scholars (Grant No. 20212ACB211007), Natural Science Foundation of China (Grant No. 11761048) and Natural Science Foundation of Tianjin City (Grant No. 19JCQNJC14200). The authors would like to express their sincere thanks to the referee for many useful suggestions and comments on the manuscript.

Conflict of interest

The authors declare that they have no competing interests.
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