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Common fixed points of locally contractive mappings in bicomplex valued metric spaces with application to Urysohn integral equation

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Abstract: The aim of this article is to obtain common fixed points of locally contractive mappings in the setting of bicomplex valued metric spaces. Our investigations generalize some conventional theorems of literature. Furthermore, we supply a significant example to manifest the authenticity of the proved results. As an application, we solve the solution of the integral equation by using our main result.

Keywords: common fixed point; bicomplex valued metric space; closed ball; generalized contractions; Urysohn integral equation
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1. Introduction

The emergence of complex numbers was established in the 17^{th} century by Sir Carl Fredrich Gauss but his work was not on record; then, in the year 1840 Augustin Louis Cauchy started doing analysis of complex numbers, and he is known to be an effective founder of complex analysis. The theory of complex numbers has its source in that the solution of $ax^2 + bx + c = 0$ was not worthwhile for $b^2 - 4ac < 0$, in the set of real numbers. Under this backdrop, Euler was the first mathematician who presented the symbol i, for $\sqrt{-1}$ with the property $i^2 = -1$.

On the other hand, the beginning of bicomplex numbers was set up by Segre [1] who provided a commutative substitute to the skew field of quaternions. These numbers generalize complex numbers more firmly and precisely to quaternions. For a comprehensive review of investigations into bicomplex numbers, we refer the researchers to [2]. In 2011, Azam et al. [3] gave the concept of a complex valued metric space (CVMS) as a generalization of a classical metric space. In 2017, Choi et al. [4] combined the concepts of bicomplex numbers and CVMSs and introduced the notion of bicomplex valued metric spaces (bi-CVMSs); they established common fixed point results for weakly compatible mappings. Jebril et al. [5], utilized this notion of newly introduced space and obtained common fixed point results under rational contractions for a pair of mappings in the environment of bi-CVMSs. Subsequently, Beg et al. [6] strengthened the concept of bi-CVMS and proved generalized fixed point theorems. Later on, Gnanaprakasam et al. [7] established some common fixed point results for rational contraction in bi-CVMSs and solved a system of linear equations as application of their main result. For more details in the direction of CVMSs and bi-CVMSs, we refer the researchers to [8–29].

In this article, we obtain common fixed points of locally contractive mappings of rational expressions in bi-CVMSs. We also provide a significant example to show the originality of obtained results. As an application, we explore the solutions of integral equations.

2. Preliminaries

We represent $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ as the set of real, complex and bicomplex numbers respectively. Segre [1] defined the notion of a bicomplex number as follows:

$$\ell = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units i_1 and i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we represent the set of bicomplex numbers by $\mathbb{C}_2$ and it is defined as

$$\mathbb{C}_2 = \{\ell : \ell = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 : a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$$

that is

$$\mathbb{C}_2 = \{\ell : \ell = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\}$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$. If $\ell = z_1 + i_2 z_2$ and $\hbar = \omega_1 + i_2 \omega_2$ are any two bicomplex numbers then the sum is

$$\ell \pm \hbar = (z_1 + i_2 z_2) \pm (\omega_1 + i_2 \omega_2) = (z_1 \pm \omega_1) + i_2 (z_2 \pm \omega_2)$$

and the product is

$$\ell \cdot \hbar = (z_1 + i_2 z_2) \cdot (\omega_1 + i_2 \omega_2) = (z_1 \omega_1 - z_2 \omega_2) + i_2 (z_1 \omega_2 + z_2 \omega_1).$$

There are four idempotent elements in $\mathbb{C}_2$, which are, $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ out of which e_1 and e_2 are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be given as the combination of $e_1$and e_2, namely

$$\ell = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This characterization of ℓ is studied as the idempotent characterization of $\mathbb{C}_2$ and the complex coefficients $\ell_1 = (z_1 - i_1 z_2)$ and $\ell_2 = (z_1 + i_1 z_2)$ are familar as idempotent components of ℓ.

A member $\ell = z_1 + i_2 z_2 \in \mathbb{C}_2$ is called invertible if there exists one more member $\hbar \in \mathbb{C}_2$ such that $\ell\hbar = 1$ and $\hbar$ is called the multiplicative inverse of ℓ. Accordingly ℓ is called the multiplicative inverse of $\hbar$. A member which has an inverse in $\mathbb{C}_2$ is called the nonsingular element of $\mathbb{C}_2$ and a member which does not have an inverse in $\mathbb{C}_2$ is called the singular element of $\mathbb{C}_2$.

A member $\ell = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular iff $\left|z_1^2 + z_2^2\right| \neq 0$ and singular iff $\left|z_1^2 + z_2^2\right| = 0$. The inverse of ℓ is defined as

$$\ell^{-1} = \hbar = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Note that 0 in $\mathbb{C}_0$ and $0 = 0 + i0$ in $\mathbb{C}_1$ are the only members which do not have a multiplicative inverse. We represent the set of a singular elements of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\aleph_0$ and $\aleph_1$ respectively. But in $\mathbb{C}_2$, there are more than one members which do not have multiplicative inverse. We represent the set of singular member of $\mathbb{C}_2$ by $\aleph_2$. Evidently $\aleph_0 = \aleph_1 \subset \aleph_2$.

A bicomplex number $\ell = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_{2\times 2}$$

is degenerated. In that case ℓ^{-1} exists and this is also degenerated.

The norm $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\ell\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}}, \end{aligned}$$

where $\ell = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with reference to defined norm is a norm linear space, also $\mathbb{C}_2$ is complete, hence $\mathbb{C}_2$ is the Banach space. If $\ell, \hbar \in \mathbb{C}_2$, then

$$\|\ell\hbar\| \leq \sqrt{2}\,\|\ell\|\,\|\hbar\|$$

holds instead of

$$\|\ell\hbar\| \leq \|\ell\|\,\|\hbar\|$$

therefore $\mathbb{C}_2$ is not the Banach algebra. The partial order relation $\preceq_{i_2}$ on $\mathbb{C}_2$ is defined as follows:

$$\ell \preceq_{i_2} \hbar \Leftrightarrow \operatorname{Re}(z_1) \leq \operatorname{Re}(\omega_1) \text{ and } \operatorname{Im}(z_2) \leq \operatorname{Im}(\omega_2)$$

where $\ell = z_1 + i_2 z_2$ and $\hbar = \omega_1 + i_2 \omega_2 \in \mathbb{C}_2$.

It follows that

$$\ell \preceq_{i_2} \hbar$$

if one of these assertions is satisfied:

$$\begin{aligned} &\text{(i)} && (z_1) = \omega_1,\ z_2 \prec \omega_2, \\ &\text{(ii)} && z_1 \prec \omega_1,\ z_2 = \omega_2, \\ &\text{(iii)} && z_1 \prec \omega_1,\ z_2 \prec \omega_2, \end{aligned}$$

$$\text{(iv)} \qquad z_1 = \omega_1,\ z_2 = \omega_2.$$

Specially, we can write $\ell \precnsim_{i_2} \hbar$ if $\ell \preceq_{i_2} \hbar$ and $\ell \neq \hbar$; that is, one of the assertions (i)–(iii) is satisfied and we will write $\ell \prec_{i_2} \hbar$ if only (iii) is satisfied. For $\ell, \hbar \in \mathbb{C}_2$, we have

(i) $\ell \preceq_{i_2} \hbar \Longrightarrow \|\ell\| \leq \|\hbar\|$,
(ii) $\|\ell + \hbar\| \leq \|\ell\| + \|\hbar\|$,
(iii) $\|a\ell\| \leq a\,\|\hbar\|$, where a is a non negative real number,
(iv) $\|\ell\hbar\| \leq \sqrt{2}\,\|\ell\|\,\|\hbar\|$,
(v) $\left\|\ell^{-1}\right\| = \|\ell\|^{-1}$,
(vi) $\left\|\frac{\ell}{\hbar}\right\| = \frac{\|\ell\|}{\|\hbar\|}$, if $\hbar$ is a degenerated bicomplex number.

3. Bicomplex valued metric space

Choi et al. [4] defined the bi-CVMS as follows:

Definition 1. ([4]) Let $\mathcal{W} \neq \emptyset$ and $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}_2$ be a mapping satisfying

(i) $0 \preceq_{i_2} d(\ell, \hbar)$ and $d(\ell, \hbar) = 0 \Longleftrightarrow \ell = \hbar$,
(ii) $d(\ell, \hbar) = d(\hbar, \ell)$,
(iii) $d(\ell, \hbar) \preceq_{i_2} d(\ell, \nu) + d(\nu, \hbar)$

for all $\ell, \hbar, \nu \in \mathcal{W}$; then, $(\mathcal{W}, d)$ is a bi-CVMS.

Example 1. ([6]) Let $\mathcal{W} = \mathbb{C}_2$ and $\ell, \hbar \in \mathcal{W}$. Define $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}_2$ by

$$d(\ell, \hbar) = |z_1 - \omega_1| + i_2\,|z_2 - \omega_2|$$

where $\ell = z_1 + i_2 z_2$, $\hbar = \omega_1 + i_2\omega_2 \in \mathbb{C}_2$. Then, $(\mathcal{W}, d)$ is a bi-CVMS.

Lemma 1. *([6]) Let $(\mathcal{W}, d)$ be a bi-CVMS and let $\{\ell_n\} \subseteq \mathcal{W}$. Then $\{\ell_n\}$ converges to ℓ if and only if $\|d(\ell_n, \ell)\| \to 0$ as $n \to \infty$.*

Lemma 2. *([6]) Let $(\mathcal{W}, d)$ be a bi-CVMS and let $\{\ell_n\} \subseteq \mathcal{W}$. Then $\{\ell_n\}$ is a Cauchy sequence if and only if $\|d(\ell_n, \ell_{n+m})\| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

4. Main results

Now we present our main result in this way.

Theorem 1. *Let $(\mathcal{W}, d)$ be a complete bi-CVMS and $\daleth_1, \daleth_2 : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5 \in [0, 1)$ with $\aleph_1 + \sqrt{2}\,(\aleph_2 + \aleph_3 + 2\aleph_4 + 2\aleph_5) < 1$ such that*

$$\begin{aligned} d(\daleth_1 u, \daleth_2 \varrho) \preceq_{i_2} \aleph_1 d(u, \varrho) &+ \aleph_2 \frac{d(u, \daleth_1 u)\, d(\varrho, \daleth_2 \varrho)}{1 + d(u, \varrho)} + \aleph_3 \frac{d(\varrho, \daleth_1 u)\, d(u, \daleth_2 \varrho)}{1 + d(u, \varrho)} \\ &+ \aleph_4 \frac{d(u, \daleth_1 u)\, d(u, \daleth_2 \varrho)}{1 + d(u, \varrho)} + \aleph_5 \frac{d(\varrho, \daleth_1 u)\, d(\varrho, \daleth_2 \varrho)}{1 + d(u, \varrho)} \end{aligned} \tag{4.1}$$

for all $u_0, u, \varrho \in \overline{B(u_0,\rho)}$, $\rho \in \mathbb{C}_2$ *and*

$$\|d(u_0, \beth_1 u_0)\|\,| \le (1-\lambda)\rho \tag{4.2}$$

where

$$\lambda = \max\{(\frac{\aleph_1 + \sqrt{2}\aleph_4}{1-\sqrt{2}\aleph_2 - \sqrt{2}\aleph_4}), (\frac{\aleph_1 + \sqrt{2}\aleph_5}{1-\sqrt{2}\aleph_2 - \sqrt{2}\aleph_5})\},$$

then there exists a unique point $u^* \in \overline{B(u_0,\rho)}$ *such that* $u^* = \beth_1 u^* = \beth_2 u^*$.

Proof. Let $u_0 \in \mathcal{W}$ and define

$$u_{2n+1} = \beth_1 u_{2n} \text{ and } u_{2n+2} = \beth_2 u_{2n+1},$$

for $n = 0, 1, 2, \ldots$. Now we show that $u_n \in \overline{B(u_0,\rho)}$, for all $n \in \mathbb{N}$. By the fact that

$$\lambda = \max\{(\frac{\aleph_1 + \sqrt{2}\aleph_4}{1-\sqrt{2}\aleph_2 - \sqrt{2}\aleph_4}), (\frac{\aleph_1 + \sqrt{2}\aleph_5}{1-\sqrt{2}\aleph_2 - \sqrt{2}\aleph_5})\} < 1$$

and given the inequality (4.2), we have

$$\|d(u_0, \beth_1 u_0)\|\,| \le \rho.$$

It implies that $u_1 \in \overline{B(u_0,\rho)}$. Let $u_2, ..., u_j \in \overline{B(u_0,\rho)}$ for some $j \in N$. If $j = 2n+1$, where $n = 0, 1, 2, \ldots \frac{j-1}{2}$ or $j = 2n+2$, where $n = 0, 1, 2, \ldots, \frac{j-2}{2}$; then, by (4.1), we have

$$\begin{aligned}
d(u_{2n+1}, u_{2n+2}) &= d(\beth_1 u_{2n}, \beth_2 u_{2n+1}) \\
&\preceq_{i_2} \aleph_1 d(u_{2n}, u_{2n+1}) + \aleph_2 \frac{d(u_{2n+1}, \beth_2 u_{2n+1})\, d(u_{2n}, \beth_1 u_{2n})}{1 + d(u_{2n}, u_{2n+1})} \\
&+ \aleph_3 \frac{d(u_{2n}, \beth_2 u_{2n+1})\, d(u_{2n+1}, \beth_1 u_{2n})}{1 + d(u_{2n}, u_{2n+1})} \\
&+ \aleph_4 \frac{d(u_{2n}, \beth_2 u_{2n+1})\, d(u_{2n}, \beth_1 u_{2n})}{1 + d(u_{2n}, u_{2n+1})} \\
&+ \aleph_5 \frac{d(u_{2n+1}, \beth_2 u_{2n+1})\, d(u_{2n+1}, \beth_1 u_{2n})}{1 + d(u_{2n}, u_{2n+1})}.
\end{aligned}$$

Now $u_{2n+1} = \beth_1 u_{2n}$ implies that $d(u_{2n+1}, \beth_1 u_{2n}) = 0$, so we have

$$d(u_{2n+1}, u_{2n+2}) \preceq_{i_2} \aleph_1 d(u_{2n}, u_{2n+1}) + \aleph_2 \frac{d(u_{2n+1}, u_{2n+2})\, d(u_{2n}, u_{2n+1})}{1 + d(u_{2n}, u_{2n+1})}$$

$$+\aleph_4 \frac{d(u_{2n}, u_{2n+2})\, d(u_{2n}, u_{2n+1})}{1 + d(u_{2n}, u_{2n+1})}.$$

This implies that

$$|d(u_{2n+1}, u_{2n+2})| \;\; \le \;\; \aleph_1 |d(u_{2n}, u_{2n+1})| + \sqrt{2}\aleph_2 \frac{\|d(u_{2n+1}, u_{2n+2})\|\, \|d(u_{2n}, u_{2n+1})\|}{\|1 + d(u_{2n}, u_{2n+1})\|}$$

$$+\sqrt{2}\aleph_4 \frac{\|d(u_{2n},u_{2n+2})\|\,\|d(u_{2n},u_{2n+1})\|}{\|1+d(u_{2n},u_{2n+1})\|}.$$

Since $\|1+d(u_{2n},u_{2n+1})\| > \|d(u_{2n},u_{2n+1})\|$, we have

$$\|d(u_{2n+1},u_{2n+2})\|\,| \leq \aleph_1 \|d(u_{2n},u_{2n+1})\| + \sqrt{2}\aleph_2 \|d(u_{2n+1},u_{2n+2})\| + \sqrt{2}\aleph_4 \|d(u_{2n},u_{2n+2})\|,$$

which implies that by triangular inequality

$$|d(u_{2n+1},u_{2n+2})| \leq \frac{(\aleph_1+\sqrt{2}\aleph_4)}{(1-\sqrt{2}\aleph_2-\sqrt{2}\aleph_4)}|d(u_{2n},u_{2n+1})|. \tag{4.3}$$

Similarly, we get

$$\begin{aligned} d(u_{2n+2},u_{2n+3}) &= d(\beth_1 u_{2n+2},\beth_2 u_{2n+1}) \\ &\preceq_{i_2} \aleph_1 d(u_{2n+2},u_{2n+1}) + \aleph_2 \frac{d(u_{2n+1},\beth_2 u_{2n+1})\, d(u_{2n+2},\beth_1 u_{2n+2})}{1+d(u_{2n+2},u_{2n+1})} \\ &+ \aleph_3 \frac{d(u_{2n+2},\beth_2 u_{2n+1})\, d(u_{2n+1},\beth_1 u_{2n+2})}{1+d(u_{2n+2},u_{2n+1})} \\ &+ \aleph_4 \frac{d(u_{2n+2},\beth_2 u_{2n+1})\, d(u_{2n+2},\beth_1 u_{2n+2})}{1+d(u_{2n+2},u_{2n+1})} \\ &+ \aleph_5 \frac{d(u_{2n+1},\beth_2 u_{2n+1})\, d(u_{2n+1},\beth_1 u_{2n+2})}{1+d(u_{2n+2},u_{2n+1})}. \end{aligned}$$

Now $u_{2n+2} = \beth_2 u_{2n+1}$ implies that $d(u_{2n+2},\beth_2 u_{2n+1}) = 0$, we have

$$d(u_{2n+2},u_{2n+3}) \preceq_{i_2} \aleph_1 d(u_{2n+2},u_{2n+1}) + \aleph_2 \frac{d(u_{2n+1},u_{2n+2})\, d(u_{2n+2},u_{2n+3})}{1+d(u_{2n+2},u_{2n+1})}$$

$$+\aleph_5 \frac{d(u_{2n+1},u_{2n+2})\, d(u_{2n+1},u_{2n+3})}{1+d(u_{2n+2},u_{2n+1})}.$$

This implies that

$$\begin{aligned} \|d(u_{2n+2},u_{2n+3})\| \leq{}& \aleph_1 \|d(u_{2n+2},u_{2n+1})\| \\ &+ \sqrt{2}\aleph_2 \frac{\|d(u_{2n+1},u_{2n+2})\|\,\|d(u_{2n+2},u_{2n+3})\|}{\|1+d(u_{2n+1},u_{2n+2})\|} \\ &+ \sqrt{2}\aleph_5 \frac{\|d(u_{2n+1},u_{2n+2})\|\,\|d(u_{2n+1},u_{2n+3})\|}{\|1+d(u_{2n+2},u_{2n+1})\|}. \end{aligned}$$

Since $\|1+d(u_{2n+2},u_{2n+1})\| > \|d(u_{2n+2},u_{2n+1})\|$, we have

$$\begin{aligned} \|d(u_{2n+2},u_{2n+3})\| \leq{}& \aleph_1 \|d(u_{2n+2},u_{2n+1})\| \\ &+ \sqrt{2}\aleph_2 \|d(u_{2n+2},u_{2n+3})\| + \sqrt{2}\aleph_5 \|d(u_{2n+1},u_{2n+3})\|. \end{aligned}$$

This implies the following given by triangular inequality

$$|d(u_{2n+2},u_{2n+3})| \leq \frac{(\aleph_1+\sqrt{2}\aleph_5)}{(1-\sqrt{2}\aleph_2-\sqrt{2}\aleph_5)}|d(u_{2n+2},u_{2n+1})|. \tag{4.4}$$

Putting

$$\lambda = \max\{(\frac{\aleph_1 + \sqrt{2}\aleph_4}{1 - \sqrt{2}\aleph_2 - \sqrt{2}\aleph_4}), (\frac{\aleph_1 + \sqrt{2}\aleph_5}{1 - \sqrt{2}\aleph_2 - \sqrt{2}\aleph_5})\} < 1,$$

we obtain that

$$\left\|d(u_j, u_{j+1})\right\| \leq \lambda^j \|d(u_0, u_1)\| \text{ for some } j \in N. \tag{4.5}$$

Now

$$\begin{aligned}
\left\|d(u_0, u_{j+1})\right\| &\leq \|d(u_0, u_1)\| + ... + \left\|d(u_j, u_{j+1})\right\| \\
&\leq \|d(u_0, u_1)\| + ... + \lambda^j \|d(u_0, u_1)\| \\
&= \|d(u_0, u_1)\| [1 + ... + \lambda^{j-1} + \lambda^j] \\
&\leq (1 - \lambda)(\rho) \frac{(1 - \lambda^{j+1})}{1 - \lambda} \\
&\leq \rho.
\end{aligned}$$

This gives $u_{j+1} \in \overline{B(u_0, \rho)}$. Hence $u_n \in \overline{B(u_0, \rho)}$ for all $n \in \mathbb{N}$. One can easily prove that

$$\|d(u_n,\ u_{n+1}\| \leq \lambda^n \|d(u_0,\ u_1)\|$$

for all n. Now for $m > n$ and by the triangular inequality, we have

$$\begin{aligned}
\|d(u_n, u_m)\| \leq\ & \lambda^n \|d(u_0, u_1)\| \\
& + \lambda^{n+1} \|d(u_0, u_1)\| \\
& + \cdots \\
& + \lambda^{m-1} \|d(u_0, u_1)\| \\
\leq\ & \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right] \|d(u_0, u_1)\|.
\end{aligned}$$

Now, by taking $n \to \infty$, we get

$$\|d(u_n, u_m)\| \to 0.$$

By Lemma (2), we conclude that the sequence $\{u_n\}$ is a Cauchy sequence in $\overline{B(u_0, \rho)}$. Consequently there exists $u^* \in \overline{B(u_0, \rho)}$ such that $\lim_{n\to\infty} u_n = u^*$. It follows that $u^* = \beth_1 u^*$; otherwise, $d(u^*, \beth_1 u^*) = \upsilon \succ 0$ and we would then have

$$\begin{aligned}
\upsilon &\preceq_{i_2} (d(u^*, u_{2n+2}) + d(u_{2n+2}, \beth_1 u^*)) \\
&= d(u^*, u_{2n+2}) + d(\beth_2 u_{2n+1}, \beth_1 u^*) \\
&\preceq_{i_2} \begin{pmatrix} d(u^*, u_{2n+2}) + \aleph_1 d(u_{2n+1}, u^*) + \aleph_2 \frac{d(u_{2n+1}, \beth_2 u_{2n+1}) d(u^*, \beth_1 u^*)}{1 + d(u^*, u_{2n+1})} \\ + \aleph_3 \frac{d(u_{2n+1}, \beth_1 u^*) d(u^*, \beth_2 u_{2n+1})}{1 + d(u^*, u_{2n+1})} \\ + \aleph_4 \frac{d(u^*, \beth_2 u_{2n+1}) d(u^*, \beth_1 u^*)}{1 + d(u^*, u_{2n+1})} \\ + \aleph_5 \frac{d(u_{2n+1}, \beth_2 u_{2n+1}) d(u_{2n+1}, \beth_1 u^*)}{1 + d(u^*, u_{2n+1})} \end{pmatrix}
\end{aligned}$$

which implies that

$$\|\upsilon\| \leqslant \begin{pmatrix} \|d(u^*, u_{2n+2})\| + \aleph_1 \|d(u_{2n+1}, u^*)\| + \sqrt{2}\aleph_2 \frac{\|d(u_{2n+1}, u_{2n+2})\| \|\upsilon\|}{\|1 + d(u^*, u_{2n+1})\|} \\ + \sqrt{2}\aleph_3 \frac{\|d(u_{2n+1}, \beth_1 u^*)\| \|d(u^*, u_{2n+2})\|}{\|1 + d(u^*, u_{2n+1})\|} \\ + \sqrt{2}\aleph_4 \frac{\|d(u^*, u_{2n+2})\| \|\upsilon\|}{\|1 + d(u^*, u_{2n+1})\|} \\ + \sqrt{2}\aleph_5 \frac{\|d(u_{2n+1}, u_{2n+2})\| \|d(u_{2n+1}, \beth_1 u^*)\|}{\|1 + d(u^*, u_{2n+1})\|} \end{pmatrix}.$$

That is $\|\upsilon\| = 0$, which is a contradiction. Thus $u^* = \beth_1 u^*$. Similarly, we can prove that $u^* = \beth_2 u^*$. $\square$

Now we show the uniqueness of the common fixed point. We suppose $u^/$ in $\mathcal{W}$ is another common fixed point of $\beth_1$ and $\beth_2$, that is $u^/ = \beth_1 u^/ = \beth_2 u^/$ which is distinct from u^* that is $u^* \neq u^/$. Now by (4.1), we have

$$\begin{aligned} d(u^*, u^/) &= d(\beth_1 u^*, \beth_2 u^/) \\ &\preceq_{i_2} \aleph_1 d(u^*, u^/) + \aleph_2 \frac{d(u^*, \beth_1 u^*)\, d\left(u^/, \beth_2 u^/\right)}{1 + d(u^*, u^/)} \\ &+ \aleph_3 \frac{d\left(u^/, \beth_1 u^*\right) d\left(u^*, \beth_2 u^/\right)}{1 + d(u^*, u^/)} \\ &+ \aleph_4 \frac{d(u^*, \beth_1 u^*)\, d\left(u^*, \beth_2 u^/\right)}{1 + d(u^*, u^/)} \\ &+ \aleph_5 \frac{d\left(u^/, \beth_1 u^*\right) d\left(u^/, \beth_2 u^/\right)}{1 + d(u^*, u^/)}, \end{aligned}$$

so that

$$\begin{aligned} \left\|d(u^*, u^/)\right\| \quad \leq \quad & \aleph_1 \left\|d(u^*, u^/)\right\| + \sqrt{2}\aleph_2 \frac{\|d(u^*, \beth_1 u^*)\| \left\|d\left(u^/, \beth_2 u^/\right)\right\|}{\left\|1 + d(u^*, u^/)\right\|} \\ &+ \sqrt{2}\aleph_3 \frac{\left\|d\left(u^/, \beth_1 u^*\right)\right\| \left\|d\left(u^*, \beth_2 u^/\right)\right\|}{\left\|1 + d(u^*, u^/)\right\|} \\ &+ \sqrt{2}\aleph_4 \frac{\|d(u^*, \beth_1 u^*)\| \left\|d\left(u^*, \beth_2 u^/\right)\right\|}{\left\|1 + d(u^*, u^/)\right\|} \\ &+ \sqrt{2}\aleph_5 \frac{\left\|d\left(u^/, \beth_1 u^*\right)\right\| \left\|d\left(u^/, \beth_2 u^/\right)\right\|}{\left\|1 + d(u^*, u^/)\right\|}. \end{aligned}$$

Since $\left\|1 + d(u^*, u^/)\right\| > \left\|d(u^*, u^/)\right\|$, we have

$$\left\|d(u^*, u^/)\right\| \leq (\aleph_1 + \sqrt{2}\aleph_3)|d(u^*, u^/)|.$$

This is contradiction to $\aleph_1 + \sqrt{2}\aleph_3 < 1$. Hence, $u^/ = u^*$. Therefore u^* is a unique common fixed point of $\beth_1$ and $\beth_2$.

Corollary 1. *Let $(\mathcal{W}, d)$ be a complete bi-CVMS and $\beth : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5 \in [0, 1)$ with $\aleph_1 + \sqrt{2}(\aleph_2 + \aleph_3 + 2\aleph_4 + 2\aleph_5) < 1$ and $\beth$ satisfies*

$$d(\beth u, \beth \varrho) \preceq_{i_2} \aleph_1 d(u, \varrho) + \aleph_2 \frac{d(u, \beth u)\, d(\varrho, \beth \varrho)}{1 + d(u, \varrho)} + \aleph_3 \frac{d(\varrho, \beth u)\, d(u, \beth \varrho)}{1 + d(u, \varrho)}$$

$$+\aleph_4 \frac{d(u, \beth u)\, d(u, \beth \varrho)}{1 + d(u, \varrho)} + \aleph_5 \frac{d(\varrho, \beth u)\, d(\varrho, \beth \varrho)}{1 + d(u, \varrho)}$$

for all $u_0, u, \varrho \in \overline{B(u_0,\rho)}$, $\rho \in \mathbb{C}_2$ *and*

$$|d(u_0, \beth_1 u_0)| \leq (1-\lambda)|\rho|$$

where

$$\lambda = \max\{(\frac{\aleph_1 + \sqrt{2}\aleph_4}{1-\sqrt{2}\aleph_2 - \sqrt{2}\aleph_4}), (\frac{\aleph_1 + \sqrt{2}\aleph_5}{1-\sqrt{2}\aleph_2 - \sqrt{2}\aleph_5})\},$$

then there exists a unique point $u^* \in \overline{B(u_0,\rho)}$ *such that* $u^* = \beth u^*$.

Proof. Take $\beth_1 = \beth_2 = \beth$ in Theorem 1. □

We provide the following example in order to show the validity of our main result.

Example 2. Let $\mathcal{W} = [0,\infty)$ and define $d : \mathcal{W}\times\mathcal{W} \longrightarrow \mathbb{C}_2$ as follows:

$$d(u_{1,}u_2) = (1+i_2)\left|u_{1,} - u_2\right|.$$

Then, $(\mathcal{W}, d)$ is a complete bi-CVMS. Take $u_0 = \frac{1}{2}$ and $\rho = \frac{1}{2} + \frac{1}{2}i_2$. Then $\overline{B(u_0,\rho)} = [0,1]$. Define $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$ as

$$\beth_1 u = \frac{u}{4}$$

and

$$\beth_2 u = \frac{u}{5}.$$

Then with $\aleph_1 = \frac{1}{6}, \aleph_2 = \frac{1}{24}, \aleph_3 = \frac{1}{2}, \aleph_4 = \frac{1}{25}$ and $\aleph_5 = \frac{1}{26}$, all the assumptions of Theorem 1 are satisfied; hence, $0 \in \overline{B(u_0,\rho)}$ is a unique common fixed point of $\beth_1$ and $\beth_2$.

Now we derive some results from our main Theorem 1 by setting some constants equal to zero.

Corollary 2. *Let* $(\mathcal{W}, d)$ *be a complete bi-CVMS and* $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3, \aleph_4 \in [0,1)$ *with* $\aleph_1 + \sqrt{2}(\aleph_2 + \aleph_3 + 2\aleph_4) < 1$ *and* $\beth_1, \beth_2$ *satisfy*

$$\begin{aligned} d(\beth_1 u, \beth_2 \varrho) \preceq_{i_2} & \aleph_1 d(u,\varrho) + \aleph_2 \frac{d(u,\beth_1 u)\, d(\varrho, \beth_2\varrho)}{1+d(u,\varrho)} \\ & + \aleph_3 \frac{d(\varrho,\beth_1 u)\, d(u, \beth_2\varrho)}{1+d(u,\varrho)} + \aleph_4 \frac{d(u,\beth_1 u)\, d(u, \beth_2\varrho)}{1+d(u,\varrho)} \end{aligned}$$

for all $u_0, u, \varrho \in \overline{B(u_0,\rho)}$, $0 \prec \rho \in \mathbb{C}_2$ *and*

$$|d(u_0, \beth_1 u_0)| \leq (1-\lambda)|\rho|$$

where $\lambda = \max\{(\frac{\aleph_1+\sqrt{2}\aleph_4}{1-\sqrt{2}\aleph_2-\sqrt{2}\aleph_4}), (\frac{\aleph_1}{1-\sqrt{2}\aleph_2})\}$; *then, there exists a unique point* $u^* \in \overline{B(u_0,\rho)}$ *such that* $u^* = \beth_1 u^* = \beth_2 u^*$.

Proof. Take $\aleph_5 = 0$ in Theorem 1. □

Corollary 3. *Let* $(\mathcal{W}, d)$ *be a complete bi-CVMS and* $\beth : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3, \aleph_4 \in [0, 1)$ *with* $\aleph_1 + \sqrt{2}\,(\aleph_2 + \aleph_3 + 2\aleph_4) < 1$ *and* $\beth$ *satisfies*

$$\begin{aligned} d(\beth u, \beth \varrho) \precsim{} & \aleph_1 d(u, \varrho) + \aleph_2 \frac{d\,(u, \beth u)\,d\,(\varrho, \beth \varrho)}{1 + d\,(u, \varrho)} \\ & + \aleph_3 \frac{d\,(\varrho, \beth u)\,d\,(u, \beth \varrho)}{1 + d\,(u, \varrho)} + \aleph_4 \frac{d\,(u, \beth u)\,d\,(u, \beth \varrho)}{1 + d\,(u, \varrho)} \end{aligned}$$

for all $u_0, u, \varrho \in \overline{B(u_0, \rho)}$, $0 \prec \rho \in \mathbb{C}_2$ *and*

$$|d(u_0, \beth u_0)| \leq (1 - \lambda)|\rho|$$

where

$$\lambda = \max\{(\frac{\aleph_1 + \sqrt{2}\aleph_4}{1 - \sqrt{2}\aleph_2 - \sqrt{2}\aleph_4}), (\frac{\aleph_1}{1 - \sqrt{2}\aleph_2})\};$$

then, there exists a unique point $u^* \in \overline{B(u_0, \rho)}$ *such that* $u^* = \beth u^*$.

Proof. Take $\beth_1 = \beth_2 = \beth$ in Corollary 2. □

Corollary 4. *Let* $(\mathcal{W}, d)$ *be a complete bi-CVMS and* $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3, \aleph_5 \in [0, 1)$ *with* $\aleph_1 + \sqrt{2}\,(\aleph_2 + \aleph_3 + 2\aleph_5) < 1$ *and* $\beth_1, \beth_2$ *satisfy*

$$\begin{aligned} d(\beth_1 u, \beth_2 \varrho) \preceq_{i_2}{} & \aleph_1 d(u, \varrho) + \aleph_2 \frac{d\,(u, \beth_1 u)\,d\,(\varrho, \beth_2 \varrho)}{1 + d\,(u, \varrho)} \\ & + \aleph_3 \frac{d\,(\varrho, \beth_1 u)\,d\,(u, \beth_2 \varrho)}{1 + d\,(u, \varrho)} + \aleph_5 \frac{d\,(\varrho, \beth_1 u)\,d\,(\varrho, \beth_2 \varrho)}{1 + d\,(u, \varrho)} \end{aligned}$$

for all $u_0, u, \varrho \in \overline{B(u_0, \rho)}$, $0 \prec \rho \in \mathbb{C}_2$ *and*

$$|d(u_0, \beth_1 u_0)| \leq (1 - \lambda)|\rho|$$

where

$$\lambda = \max\{(\frac{\aleph_1}{1 - \sqrt{2}\aleph_2}), (\frac{\aleph_1 + \sqrt{2}\aleph_5}{1 - \sqrt{2}\aleph_2 - \sqrt{2}\aleph_5})\};$$

then, there exists a unique point $u^* \in \overline{B(u_0, \rho)}$ *such that* $u^* = \beth_1 u^* = \beth_2 u^*$.

Proof. Take $\aleph_4 = 0$ in Theorem 1. □

Corollary 5. *Let* $(\mathcal{W}, d)$ *be a complete bi-CVMS and* $\beth : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3, \aleph_5 \in [0, 1)$ *with* $\aleph_1 + \sqrt{2}\,(\aleph_2 + \aleph_3 + 2\aleph_5) < 1$ *and* $\beth$ *satisfies*

$$\begin{aligned} d(\beth u, \beth \varrho) \preceq_{i_2}{} & \aleph_1 d(u, \varrho) + \aleph_2 \frac{d\,(u, \beth u)\,d\,(\varrho, \beth \varrho)}{1 + d\,(u, \varrho)} \\ & + \aleph_3 \frac{d\,(\varrho, \beth u)\,d\,(u, \beth \varrho)}{1 + d\,(u, \varrho)} + \aleph_5 \frac{d\,(\varrho, \beth u)\,d\,(\varrho, \beth \varrho)}{1 + d\,(u, \varrho)} \end{aligned}$$

for all $u_0, u, \varrho \in \overline{B(u_0,\rho)}$, $0 \prec \rho \in \mathbb{C}_2$ *and*

$$|d(u_0, \beth u_0)| \leq (1-\lambda)|\rho|$$

where

$$\lambda = \max\{(\frac{\aleph_1}{1-\sqrt{2}\aleph_2}), (\frac{\aleph_1+\sqrt{2}\aleph_5}{1-\sqrt{2}\aleph_2-\sqrt{2}\aleph_5})\};$$

then, there exists a unique point $u^* \in \overline{B(u_0,\rho)}$ *such that* $u^* = \beth u^*$.

Proof. Set $\beth_1 = \beth_2 = \beth$ in Corollary 4. □

Corollary 6. *Let* $(\mathcal{W}, d)$ *be a complete bi-CVMS and* $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3 \in [0,1)$ *with* $\aleph_1 + \sqrt{2}(\aleph_2+\aleph_3) < 1$ *and* $\beth_1,\beth_2$ *satisfy*

$$d(\beth_1 u, \beth_2 \varrho) \preceq_{i_2} \aleph_1 d(u,\varrho) + \aleph_2 \frac{d(u,\beth_1 u)\, d(\varrho,\beth_2\varrho)}{1+d(u,\varrho)} + \aleph_3 \frac{d(\varrho,\beth_1 u)\, d(u,\beth_2\varrho)}{1+d(u,\varrho)}$$

for all $u_0, u, \varrho \in \overline{B(u_0,\rho)}$, $0 \prec \rho \in \mathbb{C}_2$ *and*

$$|d(u_0, \beth_1 u_0)| \leq (1-\lambda)|\rho|$$

where $\lambda = \frac{\aleph_1}{1-\sqrt{2}\aleph_2}$; *then, there exists a unique point* $u^* \in \overline{B(u_0,\rho)}$ *such that* $u^* = \beth_1 u^* = \beth_2 u^*$.

Proof. Choose $\aleph_4 = \aleph_5 = 0$ in Theorem 1. □

Corollary 7. *Let* $(\mathcal{W}, d)$ *be a complete bi-CVMS and* $\beth : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2, \aleph_3 \in [0,1)$ *with* $\aleph_1 + \sqrt{2}(\aleph_2+\aleph_3) < 1$ *and* $\beth$ *satisfies*

$$d(\beth u, \beth \varrho) \preceq_{i_2} \aleph_1 d(u,\varrho) + \aleph_2 \frac{d(u,\beth u)\, d(\varrho,\beth\varrho)}{1+d(u,\varrho)} + \aleph_3 \frac{d(\varrho,\beth u)\, d(u,\beth\varrho)}{1+d(u,\varrho)}$$

for all $u_0, u, \varrho \in \overline{B(u_0,\rho)}$, $0 \prec \rho \in \mathbb{C}_2$ *and*

$$|d(u_0, \beth u_0)| \leq (1-\lambda)|\rho|$$

where $\lambda = \frac{\aleph_1}{1-\sqrt{2}\aleph_2}$; *then, there exists a unique point* $u^* \in \overline{B(u_0,\rho)}$ *such that* $u^* = \beth u^*$.

Proof. Take $\beth_1 = \beth_2 = \beth$ in Corollary 6. □

Corollary 8. *Let* $(\mathcal{W}, d)$ *be a complete bi-CVMS and* $\beth_1, \beth_2 : \mathcal{W} \to \mathcal{W}$. *Suppose that there exist* $\aleph_1, \aleph_2 \in [0,1)$ *with* $\aleph_1 + \sqrt{2}\aleph_2 < 1$ *and* $\beth_1$, $\beth_2$ *satisfy*

$$d(\beth_1 u, \beth_2 \varrho) \preceq_{i_2} \aleph_1 d(u,\varrho) + \aleph_2 \frac{d(u,\beth_1 u)\, d(\varrho,\beth_2\varrho)}{1+d(u,\varrho)}$$

for all $u_0, u, \varrho \in \overline{B(u_0,\rho)}$, $0 \prec \rho \in \mathbb{C}_2$ *and*

$$|d(u_0, \beth_1 u_0)| \leq (1-\lambda)|\rho|$$

where $\lambda = \frac{\aleph_1}{1-\sqrt{2}\aleph_2}$; *then, there exists a unique point* $u^* \in \overline{B(u_0,\rho)}$ *such that* $u^* = \beth_1 u^* = \beth_2 u^*$.

Proof. Take $\aleph_3 = \aleph_4 = \aleph_5 = 0$ in Theorem 1. $\square$

Corollary 9. *Let $(\mathcal{W}, d)$ be a complete bi-CVMS and $\beth : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2 \in [0, 1)$ with $\aleph_1 + \sqrt{2}\aleph_2 < 1$ and $\beth$ satisfies*

$$d(\beth u, \beth \varrho) \preceq_{i_2} \aleph_1 d(u, \varrho) + \aleph_2 \frac{d(u, \beth u)\, d(\varrho, \beth \varrho)}{1 + d(u, \varrho)}$$

for all $u_0, u, \varrho \in \overline{B(u_0, \rho)}$, $0 \prec \rho \in \mathbb{C}_2$ and

$$|d(u_0, \beth u_0)| \leq (1 - \lambda)|\rho|$$

where $\lambda = \frac{\aleph_1}{1-\sqrt{2}\aleph_2}$; then, there exists a unique point $u^* \in \overline{B(u_0, \rho)}$ such that $u^* = \beth u^*$.

Proof. Take $\beth_1 = \beth_2 = \beth$ in Corollary 8. $\square$

Now we establish the following result for two finite families of mappings as an application of Theorem 1.

Theorem 2. *If $\{\aleph_i\}_1^m$ and $\{\Re_i\}_1^n$ are two finite pairwise commuting finite families with a self-mapping defined on a complex valued extended b-metric space with $\varphi : \mathcal{W} \times \mathcal{W} \to [1, \infty)$ such that the mappings $\Re$ and $\Im$ (with $\Im = \aleph_1 \aleph_2 \cdots \aleph_m$ and $\Re = \Re_1 \Re_2 \cdots \Re_n$) satisfy (4.1) and (4.2); then, the component mappings of these $\{\aleph_i\}_1^m$ and $\{\Re_i\}_1^n$ have a unique common fixed point.*

Proof. By Theorem 1, one can get $\Im u^* = \Re u^* = u^*$, which is unique. Now by pairwise commutativity of $\{\aleph_i\}_1^m$ and $\{\Re_i\}_1^n$,(for every $1 \leq k \leq m$) one can write $\aleph_k u^* = \aleph_k \aleph u^* = \aleph \aleph_k u^*$ and $\aleph_k u^* = \aleph_k \Re u^* = \Re \aleph_k u^*$ which manifest that $\aleph_k u^*$, $\forall k$ is also a common fixed point of $\Im$ and $\Re$. Now utilizing the uniqueness, one can write $\Im_k u^* = u^*$ (for every k) which shows that u^* is a common fixed point of $\{\Im_i\}_1^m$. By doing the same strategy, we can prove that $\Re_k u^* = u^*$ $(1 \leq k \leq n)$. Hence $\{\aleph_i\}_1^m$ and $\{\Re_i\}_1^n$ have a unique common fixed point. $\square$

Corollary 10. *Let $(\mathcal{W}, d)$ be a complete bi-CVMS and $F, G : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5 \in [0, 1)$ with $\aleph_1 + \sqrt{2}(\aleph_2 + \aleph_3 + 2\aleph_4 + 2\aleph_5) < 1$ and F, G satisfy*

$$d(F^m u, G^n \varrho) \preceq_{i_2} \aleph_1 d(u, \varrho) + \aleph_2 \frac{d(u, F^m u)\, d(\varrho, G^n \varrho)}{1 + d(u, \varrho)} + \aleph_3 \frac{d(\varrho, F^m u)\, d(u, G^n \varrho)}{1 + d(u, \varrho)}$$

$$+ \aleph_4 \frac{d(u, F^m u)\, d(u, G^n \varrho)}{1 + d(u, \varrho)} + \aleph_5 \frac{d(\varrho, F^m u)\, d(\varrho, G^n \varrho)}{1 + d(u, \varrho)}$$

for all $u_0, u, \varrho \in \overline{B(u_0, \rho)}$, $0 \prec \rho \in \mathbb{C}_2$ and

$$|d(u_0, G^n u_0)| \leq (1 - \lambda)|\rho|$$

where

$$\lambda = \max\{(\frac{\aleph_1 + \sqrt{2}\aleph_4}{1 - \sqrt{2}\aleph_2 - \sqrt{2}\aleph_4}), (\frac{\aleph_1 + \sqrt{2}\aleph_5}{1 - \sqrt{2}\aleph_2 - \sqrt{2}\aleph_5})\};$$

then, there exists a unique point $u^* \in \overline{B(u_0, \rho)}$ such that $u^* = F u^* = G u^*$.

Proof. Take $\aleph_1 = \aleph_2 = \cdots = \aleph_m = F$ and $\Re_1 = \Re_2 = \cdots = \Re_n = G$, in Theorem 2. □

Corollary 11. *Let $(\mathcal{W}, d)$ be a complete bi-CVMS and $\beth : \mathcal{W} \to \mathcal{W}$. Suppose that there exist $\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5 \in [0, 1)$ with $\aleph_1 + \sqrt{2}\,(\aleph_2 + \aleph_3 + 2\aleph_4 + 2\aleph_5) < 1$ and $\beth$ satisfies*

$$d(\beth^n u, \beth^n \varrho) \preceq_{i_2} \aleph_1 d(u, \varrho) + \aleph_2 \frac{d\,(u\beth^n u)\, d\,(\varrho, \beth^n \varrho)}{1 + d\,(u, \varrho)} + \aleph_3 \frac{d\,(\varrho, \beth^n u)\, d\,(u, \beth^n \varrho)}{1 + d\,(u, \varrho)}$$

$$+\aleph_4 \frac{d\,(u, \beth^n u)\, d\,(u, \beth^n \varrho)}{1 + d\,(u, \varrho)} + \aleph_5 \frac{d\,(\varrho, \beth^n u)\, d\,(\varrho, \beth^n \varrho)}{1 + d\,(u, \varrho)}$$

for all $u_0, u, \varrho \in \overline{B(u_0, \rho)}$, $0 \prec \rho \in \mathbb{C}_2$ and

$$|d(u_0, \beth^n u_0)| \leq (1 - \lambda)|\rho|$$

where

$$\lambda = \max\{(\frac{\aleph_1 + \sqrt{2}\aleph_4}{1 - \sqrt{2}\aleph_2 - \sqrt{2}\aleph_4}), (\frac{\aleph_1 + \sqrt{2}\aleph_5}{1 - \sqrt{2}\aleph_2 - \sqrt{2}\aleph_5})\};$$

then, there exists a unique point $u^* \in \overline{B(u_0, \rho)}$ such that $u^* = \beth u^*$.

Take $m = n$ and $F = G = \beth$ in Corollary 10.

5. Applications

In this section, we show the importance and applicability of the established results.

Let $\mathcal{W} = C([a, b], \mathbb{R})$, $a > 0$ where $C[a, b]$ denotes the set of all real continuous functions defined on the closed interval $[a, b]$ and $d : \mathcal{W} \times \mathcal{W} \to \mathbb{C}_2$ is defined as follows:

$$d(u, \varrho) = (1 + i)\,(|u\,(t) - \varrho\,(t)|)$$

for all $u, \varrho \in \mathcal{W}$ and $t \in [a, b]$, where $|\cdot|$ is the usual real modulus. Then, $(\mathcal{W}, d)$ is a complete bi-CVMS. Consider the Urysohn integral equations

$$u(t) = \frac{1}{b - a} \int_a^b K_1(t, s, u(s))ds + g(t), \tag{5.1}$$

$$u(t) = \frac{1}{b - a} \int_a^b K_2(t, s, u(s))ds + g(t), \tag{5.2}$$

where $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and $t \in [a, b]$. We define partial order $\preceq_{i_2}$ in $\mathbb{C}_2$ as $u\,(t) \preceq_{i_2} \varrho\,(t)$ if and only if $u \leq \varrho$.

Theorem 3. *Define the continuous mappings $\beth_1, \beth_2$: $\mathcal{W} \to \mathcal{W}$ by*

$$\beth_1 u(t) = \frac{1}{b - a} \int_a^b K_1(t, s, u(s))ds + g(t),$$

$$\beth_2 u(t) = \frac{1}{b - a} \int_a^b K_2(t, s, u(s))ds + g(t),$$

for all $t \in [a, b]$. *Suppose the following inequality*

$$|K_1(t, s, u(s)) - K_2(t, s, \varrho(s))| \leq \aleph_1 |u(s) - \varrho(s)|$$

holds, for all $u, \varrho \in \mathcal{W}$ *with* $u \neq \varrho$ *and* $\aleph_1 < 1$; *then, the integral operators defined by (5.1) and (5.2) have a unique common solution.*

Proof. Consider

$$\begin{aligned}
(1+i_2)\left|\beth_1 u(t) - \beth_2 h(t)\right| &= (1+i_2)\left(\frac{1}{b-a}\left|\int_a^b K_1(t,s,u(s))ds - \int_a^b K_2(t,s,h(s))ds\right|\right) \\
&\leq (1+i_2)\left(\frac{1}{b-a}\int_a^b |K_1(t,s,u(s)) - K_2(t,s,h(s))|\, ds\right) \\
&\leq (1+i_2)\left(\frac{\aleph_1}{b-a}\int_a^b |u(s)-\varrho(s)|\, ds\right) \\
&\leq \rho\,(1+i_2)\,|u(s)-\varrho(s)|\,.
\end{aligned}$$

Thus,

$$d\,(\beth_1 u, \beth_2 \varrho) \preceq_{i_2} \aleph_1 d(u, \varrho).$$

Now with $\aleph_2 = \aleph_3 = \aleph_4 = \aleph_5 = 0$, all the assumptions of Theorem (1) are satisfied and the integral equations (5.1) and (5.2) have a unique common solution. □

6. Conclusions

In this article, we have utilized the notion of bi-CVMS and secured common fixed point results for rational contractions on a closed ball. We have derived common fixed points and the fixed points of single valued mappings for contractions on a closed ball. We expect that the obtained consequences in this article will form up to date relations for researchers who are employing in bi-CVMS.

The future work in this area will focus on studying the common fixed points of single valued and multivalued mappings in the setting of bi-CVMS. Differential and integral equations can be solved as applications of these results.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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