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Research article

Applying fixed point techniques for obtaining a positive definite solution to nonlinear matrix equations

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Abstract: In this manuscript, the concept of rational-type multivalued $F-$contraction mappings is investigated. In addition, some nice fixed point results are obtained using this concept in the setting of $MM-$spaces and ordered $MM-$spaces. Our findings extend, unify, and generalize a large body of work along the same lines. Moreover, to support and strengthen our results, non-trivial and extensive examples are presented. Ultimately, the theoretical results are involved in obtaining a positive, definite solution to nonlinear matrix equations as an application.

Keywords: fixed point technique; multivalued $F-$contraction; $m-$metric space; multivalued mapping; nonlinear matrix equations
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1. Introduction and preliminaries

The study of nonlinear matrix equations has long attracted the attention of nonlinear analysis, which includes control theory, dynamical programming, stochastic filtering, queuing theory, statistics, and a number of other mathematical and practical disciplines.

Run and Reuring [1] extended the Banach contraction principle to the configuration of ordered metric space, where some applications are examined. The problem of the existence and uniqueness of a fixed point for the contraction type operator on a partially ordered set has since been studied by a number of scholars.

In the last few decades, Nadler [2] explored the idea of Banach contraction [3] in the case of multivalued mapping. Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping mapping so that

$$H_d(Tx, Ty) \leq \gamma d(x, y), \text{ for all } x, y \in x, \ \gamma \in [0, 1),$$

where H_d is a Hausdorff with respect to metric d and $CB(X)$ is a non-empty closed and bounded subset of X. Then T admits a fixed point.

Recently, well known articles have been published in this direction which greatly assist readers in finishing their works in the field of fixed point for multivalued mappings, see [4–7].

In recent research, many writers have developed new fixed point theorems by taking into consideration some sophisticated contractive conditions on diffident spaces in order to close gaps in the literature. As a result, we hope to fill one of the gaps in the literature on metric fixed point theory with this study. Consequently, we present some fixed point theorems based on a recent contractive approach known as "rational type multivalued $F-$contraction mapping on complete $M-$metric space".

Now, we recall some basic concepts of $M-$metric space. Throughout this paper, the symbols, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ represent respectively set of all natural numbers, real numbers and positive real numbers.

Consider a nonempty set X and a mapping $m : X \times X \to \mathbb{R}^+$. Setting

$$\begin{aligned} m_{xy} &= \min\{m(x, x), m(y, y)\}, \\ M_{xy} &= \max\{m(x, x), m(y, y)\}. \end{aligned}$$

Asadi et al. [8] presented the notion of an $M-$metric space as a real generalization of partial and ordinary metric space as follows:

Definition 1.1. [8] Let X be a non-empty set. Then an $M-$metric is a mapping $m : X \times X \to \mathbb{R}^+$ fulfilling the assertions below, for all $x, y, z \in X$,

(i) $m(x, y) = m(y, x)$;
(ii) $m_{xy} \leq m(x, y)$;
(iii) $m(x, x) = m(y, y) = m(x, y)$ iff $x = y$;
(iv) $m(x, y) - m_{xy} \leq (m(x, z) - m_{xz}) + (m(x, y) - m_{xy})$.

Then, the pair (X, m) is called $M-$metric space.

Remark 1.2. [8] For any $x, y, z \in X$, the observations below hold:

(i) $0 \leq M_{xy} + m_{xy} = m(x, x) + m(y, y)$;
(ii) $0 \leq M_{xy} - m_{xy} = |m(x, x) - m(y, y)|$;
(iii) $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + \left(M_{yz} - m_{yz}\right)$.

Example 1.3. [8] Let X be an $M-$metric space. Define m^w, $m^s : X \times X \to \mathbb{R}^+$ by

$$m^w(x, y) = m(x, y) - 2m_{xy} + M_{xy},$$

and

$$m^s = \begin{cases} m(x,y) - m_{xy}, \text{ if } x \neq y, \\ 0, \text{ if } x = y. \end{cases}$$

Then m^w and m^s define ordinary metrics.

In the context of $M-$metric space X, let $B_m(x,\epsilon) = \{x \in X : m(x,y) < m_{xy} + \epsilon, \forall x \in X, \varepsilon > 0\}$ be an open ball with center x and radius ϵ. The collection $\{B_m(x,\epsilon) : x \in X,\ \epsilon > 0\}$ represents a basis for the T_0 topology τ_m. For more details on the contributions of fixed points in the metric space, we present a series of papers [9–11].

Definition 1.4. [8] Let $\{x_n\}$ be a sequence in $M-$metric space X

(i) $\{x_n\}$ is called $m-$convergent to x in X iff

$$\lim_{n\to\infty} \left(m(x_n, x) - m_{x_n x}\right) = 0.$$

(ii) If $\lim_{n,m\to\infty}\left(m(x_n,x_m) - m_{x_n x_m}\right)$ and $\lim_{n,m\to\infty}\left(M_{x_n,x_m} - m_{x_n x_m}\right)$ exist and are finite. Then, the sequence $\{x_n\}$ is called $M-$Cauchy.

(iii) If every $M-$Cauchy $\{x_n\}$ is $m-$convergent with respect to τ_m to x in X such that

$$\lim_{n\to\infty} m(x_n, x) - m_{x_n x} = 0, \text{ and } \lim_{n\to\infty}\left(M_{x_n,x} - m_{x_n x}\right) = 0.$$

Then, X is called a complete $M-$metric space.

Lemma 1.5. *[8] Let X be an $M-$metric space. Then*

(i) *$\{x_n\}$ is an $M-$Cauchy sequence in X iff $\{\wp_n\}$ is a Cauchy sequence in a metric space (X, m^w),*

(ii) *an $M-$metric space X is complete iff the metric space (X, m^w) is complete. Moreover,*

$$\lim_{n\to\infty} m^w(x_n, x) = 0 \textit{ iff } \left(\lim_{n\to\infty}\left(m(x_n,x) - m_{x_n x}\right) = 0,\ \lim_{n\to\infty}\left(M_{x_n x} - m_{x_n x}\right) = 0\right).$$

Lemma 1.6. *[8] Suppose that $\{x_n\} \to x$ and $\{y_n\} \to y$ as $n \to \infty$ in $M-$metric space X. Then, we have $\left(m(x_n, y_n) - m_{x_n y_n}\right) \to \left(m(x,y) - m_{x,y}\right)$ as $n \to \infty$.*

Lemma 1.7. *[8] Assume that $\{x_n\} \to x$ as $n \to \infty$ in $M-$metric space X. Then, for all $x \in X$, we get $\left(m(x_n, y) - m_{x_n y}\right) \to \left(m(x,y) - m_{x,y}\right)$ as $n \to \infty$.*

Lemma 1.8. *[8] Suppose that $\{x_n\} \to x$ and $\{x_n\} \to y$ as $n \to \infty$ in $M-$metric space X Then $m(x,y) = m_{x,y}$. Further, if $m(x,x) = m(x,y)$ then $x = y$.*

Definition 1.9. [12] Define $H_m : CB^m(X) \times CB^m(X) \to [0,\infty)$ by

$$H_m(K, X) = \max\{\nabla_m(K,X), \nabla_m(X,K)\},$$

where

$$\begin{aligned} m(x,X) &= \inf\{m(x,y) : y \in X\}, \\ \nabla_m(K,X) &= \sup\{m(x,X) : x \in K\}. \end{aligned}$$

Lemma 1.10. *[12] Let K be any non-empty set in $M-$metric space X, then*

$$x \in \overline{K} \text{ iff } m(x,K) = \sup_{x\in K}\{m_{xy}\}.$$

Proposition 1.11. *[12] If $A, B, C \in CB^m(X)$, then*

(*i*) $\nabla_m(A,A) = \sup_{x\in A}\left\{\sup_{y\in A} m_{xy}\right\}$,
(*ii*) $\left(\nabla_m(A,B) - \sup_{x\in A}\sup_{y\in B} m_{xy}\right) \leq (\nabla_m(A,C) - \inf_{x\in A}\inf_{z\in C} m_{xz}) + \left(\nabla_m(C,B) - \inf_{z\in C}\inf_{y\in B} m_{zy}\right)$.

Proposition 1.12. *[12] If $A, B, C \in CB^m(X)$, then*

(*i*) $H_m(A,A) = \nabla_m(A,A) = \sup_{x\in A}\left\{\sup_{y\in A} m_{xy}\right\}$,
(*ii*) $H_m(A,B) = H_m(B,A)$,
(*iii*) $H_m(A,B) - \sup_{x\in A}\sup_{y\in A} m_{xy}) \leq H_m(A,C) + H_m(B,C) - \inf_{x\in A}\inf_{z\in C} m_{xz} - \inf_{z\in C}\inf_{y\in B} m_{yz}$.

Lemma 1.13. *[12] Consider $A, B \in CB^m(X)$ and $h > 1$. Then, for each $x \in A$, there exists at the least one $y \in B$ such that*

$$m(x,y) \leq hH_m(A,B).$$

Lemma 1.14. *[12] Consider $A, B \in CB^m(X)$ and $l > 0$. Then, for each $x \in A$, there exist at least one $y \in B$ such that*

$$m(x,y) \leq H_m(A,B) + l.$$

Theorem 1.15. *[12] Let X be a complete $M-$metric space and $T : X \to CB^m(X)$ be a set-valued mapping. If there is $h \in (0,1)$ so that*

$$H_m(T(x), T(y)) \leq hm(x,y), \tag{1.1}$$

for all $x, y \in X$. Then, x^* is a fixed point T.

Recently, Wardowski [13] consider the following family of function to give more general contractive condition for fixed point theory on metric space:

$$\nabla_F = \{F : R^+ \to R : \ F \text{ satisfied } (F_1),\ (F_2) \text{ and } F_3\}$$

and

$$\chi_* = \{F \in \nabla_F : \ F \text{ satisfied } F_4\}$$

where the conditions presented as follows:

(F_1) F is strictly increasing,
(F_2) For all a sequence $\{j_n\} \subseteq R^+$, $\lim_{n\to\infty} j_n = 0$ iff $\lim_{n\to\infty} F(j_n) = -\infty$,
(F_3) There exist $0 < k < 1$ such that $\lim_{j\to 0^+} j^k F(j) = 0$, and
(F_4) $F(\inf A) = \inf F(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

In the context of various spaces, a number of articles on $F-$contraction and associated fixed point theorems were published. For more details, see [14–18].

In [1], two class of matrix equations have been investigated by Ran and Reurings as follows:

$$D = \alpha \pm \sum_{j=1}^{m} x_j^* D x_j$$

where α is an $n \times n$ positive definite matrix and the x_j are $n \times n$ arbitrary matrices. Under some hypotheses, they established the existence and uniqueness of positive definite solutions to the above equation. Duan et al. [19] generalized the above system by making a small change as follows:

$$D = \alpha \pm \sum_{j=1}^{m} x_j^* D^{\rho_j} x_j,$$

where $0 < \left|\rho_j\right| < 1$. They investigated the existence and uniqueness of a positive definite solution to such an equation on the basis of a fixed point theorem for mixed monotone mappings. This form of matrix equation frequently occurs in a variety of fields, including ladder networks [20, 21], dynamic programming [22, 23], control theory [24, 25], zeroing neural network methods for solving the Yang-Baxter-like matrix equation and more results in this direction see [26–33] etc.

Motivated by the above works, this paper is devoted to discussing a positive definite solution of a nonlinear matrix equation in the form of

$$D = \alpha + \sum_{i=1}^{p} E_i^* \Psi(D) E_i,$$

where α is a positive definite matrix which that belongs to $(J(n) \subseteq Q(n))$, and the mapping $\Psi : J(n) \to J(n)$ is continuous in the trace norm. To implement this strategy, we use the fixed point-method under rational type multivalued $F-$contraction mapping in the context of complete $M-$metric space and ordered $M-$metric space. Our results generalize the results of Altun et al. [34] and Kumar et al. [35]. Finally, we give non-trivial extensive examples to show that our concepts are meaningful and to support our results.

2. Main results

We begin with the following definition.

Definition 2.1. Let X be an $M-$metric space and $T : X \to CB^m(X)$ be a mapping. Then, T is said to be multivalued $F-$contraction if $F \in \nabla_F$ and there exists $\tau > 0$ such that

$$H_m(Tx, Ty) > 0 \Rightarrow \tau + F(m(Tx, Ty)) \leq F(m(x, y)). \tag{2.1}$$

Example 2.2. Consider $X = [0, 1]$ endowed with $m(x, y) = \frac{1}{2}|x + y| + \frac{1}{2}\min\{x, y\}$ for all $x, y \in X$. Clearly (X, m) be a complete $M-$metric space. Describe a mapping $T : X \to CB^m(X)$ as

$$T(x) = \begin{cases} \left\{\frac{1}{2}, \frac{1}{4}\right\}, & \text{if } x = 1 \\ \left\{\frac{1}{2}\right\}, & \text{if } x \neq 1. \end{cases}$$

Clearly, T is a multivalued mapping. Define a function $F: R^{+} \to R$ by $F(x) = \ln(x)$ for all $x \in R^{+}$ and $\tau = \ln\left(\frac{4}{3}\right)$. Now, we claim that T fulfills (2.1). First, for $x \neq 1$, we have

$$\begin{aligned}
\ln\left(\frac{4}{3}\right) + \ln\left(H_m\left(T(x), T(1)\right)\right) &= \ln\left(\frac{4}{3}\right) + \ln\left(\max\left\{\begin{array}{c} \nabla_m\left\{\left\{\frac{1}{2}\right\}, \left\{\frac{1}{4}, \frac{1}{2}\right\}\right\}, \\ \nabla_m\left\{\left\{\frac{1}{4}, \frac{1}{2}\right\}, \left\{\frac{1}{2}\right\}\right\} \end{array}\right\}\right) \\
&= \ln\left(\frac{4}{3}\right) + \ln\left(\max\left\{\inf\left\{\frac{5}{8}, \frac{3}{4}\right\}\right\}, \sup\left\{\frac{3}{4}, \frac{1}{2}\right\}\right) \\
&= \ln\left(\frac{4}{3}\right) + \ln\left(\max\left\{\frac{5}{8}\right\}, \left\{\frac{3}{4}\right\}\right) \\
&\leq \ln\left(\frac{4}{6}\right) + \ln\left(\frac{3}{4}\right) \leq \ln\left(m(x, 1)\right) \\
&\leq \ln\left(\frac{1}{2}\right) \leq \ln\left(\frac{1}{2}|x + y| + \frac{1}{2}\min\{x, y\}\right).
\end{aligned}$$

Hence, the condition (2.1) is true.

Definition 2.3. Let X be an $M-$metric space and $T: X \to CB^{m}(X)$ be a given mapping. We say that T is rational type multivalued $F-$contraction mapping if $F \in \nabla_F$ and there exist $\tau > 0$ such that

$$H_m\left(T(x), T(y)\right) > 0 \Rightarrow \tau + F\left(H_m\left(T(x), T(y)\right)\right) \leq F\left(A_M(x, y)\right), \tag{2.2}$$

where

$$A_M(x, y) = \max\left\{m(x, y), M(x, T(x)), M(y, T(y)), \frac{M(y, T(y))\left[1 + M(x, T(x))\right]}{1 + m(x, y)}\right\}.$$

Lemma 2.4. *Let X be an $M-$metric space and $K(X)$ be a compact subset of X. Consider $A \subseteq K(X)$ and $f: A \to K(X)$. The, following statements are equivalent:*

(*I*) *f is continuous.*
(*II*) *Since $K(X)$ is compact, then for any convergence subsequence $x_{n_k} \to c$, $f(x_{n_k}) \to f(c)$ for any point $x \in A$.*

Proof. $(I) \Rightarrow (II)$.

Assume that f is continuous. For any $\epsilon > 0$, there exist $\delta > 0$ such that

$$f\left(B_m(c, \delta)\right) \subseteq f\left(B_m(c, \epsilon)\right),$$

$$m(x, c) < m_{xc} + \delta \Rightarrow m\left(f(x), f(c)\right) < m_{f(x)(f(c))} + \epsilon,$$

and

$$m(x, c) - m_{xc} < \delta \Rightarrow m\left(f(x), f(c)\right) - m_{f(x)(f(c))} < \epsilon.$$

Now, we want to prove that $f(x_{n_k}) \to f(c)$. Suppose that $x_{n_k} \to c$ in t_m, so $\lim\left(m(x_{n_k}, c) - m_{x_{n_k}c}\right) = 0$. Then

$$m(x_{n_k}, c) - m_{x_{n_k}c} < \delta \Rightarrow m\left(f(x_{n_k}), f(c)\right) - m_{f\left(x_{n_k}\right)(f(c))} < \epsilon,$$

which implies that $f(x_{n_k}) \to f(c)$.

$(II) \Rightarrow (I)$.

Using the method of contradiction, we want to prove that f is continuous. So, suppose not, then there exists $\epsilon > 0$ such that

$$m(x,c) - m_{xc} < \delta \Rightarrow m(f(x), f(c)) - m_{f(x)(f(c))} \geq \epsilon.$$

Since $f(x_{n_k}) \to f(c)$, and choosing any $\delta = \frac{1}{\wp_{n_k}}$ for any $n \in N$ such that

$$m(x_{n_k}, c) - m_{x_{n_k}c} < \frac{1}{x_{n_k}} \Rightarrow m(f(x_{n_k}) f(c)) - m_{f(x_{n_k})(f(c))} \geq \epsilon,$$

which implies that $x_{n_k} \to c$ while $f(x_{n_k}) \nrightarrow f(c)$, this contradicts the fact that $f(x_{n_k}) \to f(c)$. Hence f is continuous. □

Theorem 2.5. *Let X be a complete$M-$metric space, $T : X \to K(X)$ be rational type multivalued $F-$contraction mapping and $F \in \nabla_F$. Then T possesses a fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point. Since $T(x)$ is non-empty for all $x \in X$, we can choose $x_1 \in T(x_0)$. If $x_1 \in T(x_1)$ then x_1 is a fixed point of T and the proof is finished. Now, consider $x_1 \notin T(x_1)$. Since $T(x_1)$ is closed and$M(x_1, T(x_1)) > 0$, we have $M(x_1, T(x_1)) \leq H_m(T(x_0), T(x_1))$. Using (F_1), we get $F(M(x_1, T(x_1))) \leq F(H_m(T(x_0), T(x_1)))$.

Applying (2.2) one can write

$$F(M(x_1, T(x_1))) \leq F(H_m(T(x_0), T(x_1))) \leq F(A_M(x_0, x_1)) - \tau \tag{2.3}$$

$$= F\left(\begin{array}{c} \max m(x_0,x_1), M(x_0,T(x_0)), M(x_1,T(x_1)), \\ \frac{M(x_1,T(x_1))[1+M(x_0,T(x_0))]}{1+m(x_0,x_1)} \end{array} \right) - \tau$$

$$= F(\max m(x_0,x_1), M(x_1,T(x_1)) - \tau. \tag{2.4}$$

Since

$$\begin{aligned} A_M(x_0,x_1) &= \max\left\{ \begin{array}{c} m(x_0,x_1), M(x_0,T(x_0)), M(x_1,T(x_1)), \\ \frac{m(x_1,T(x_1))[1+M(x_0,T(x_0))]}{1+m(x_0,x_1)} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} m(x_0,x_1), m(x_1,T(\wp_1)), \\ \frac{m(x_1,T(x_1))[1+m(x_0,T(x_0))]}{1+m(x_0,x_1)} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} m(x_0,x_1), m(x_1,T(x_1)), \\ \frac{m(x_1,T(x_1))[1+m(x_0,x_1)]}{1+m(x_0,x_1)} \end{array} \right\} \\ &= \max\{m(x_0,x_1), M(x_1,T(x_1))\}. \end{aligned}$$

Now, if $m(x_0,x_1) \leq M(x_1,T(x_1))$ then from (2.4), we get

$$F(M(x_1,T(x_1))) \leq F(M(x_1,T(x_1))) - \tau,$$

which is a contradiction. Thus $M(x_1,T(x_1)) < m(x_0,x_1)$ and so from (2.3), one has

$$F(M(x_1,T(x_1))) \leq F(m(x_0,x_1)) - \tau. \tag{2.5}$$

Since $T(x_1)$ is compact, then there exists $x_2 \in T(x_1)$ such that $m(x_1, x_2) = M(x_1, T(x_1))$, using (2.4), we obtain that

$$F(m(x_1, x_2)) \leq F(m(x_0, x_1)) - \tau \text{ for all } n \in \mathbb{N}.$$

Because $x_1 \in T(x_0)$ and $x_2 \in T(x_1)$ with $M(x_1, T(x_1)) \leq H_m(T(x_0), T(x_1))$. Then, we have

$$F(M(x_2, T(x_2))) \leq F(H_m(T(x_1), T(x_2))) \leq F(A_M(x_1, x_2)) - \tau. \tag{2.6}$$

By considering the same way, we deduce that

$$A_M(x_1, x_2) \leq \max\{m(x_1, x_2), m(x_1, T(x_2))\}.$$

Thus, from (2.6) one gets

$$F(m(x_2, T(x_2))) \leq F(m(x_1, x_2)) - \tau. \tag{2.7}$$

As $T(x_2)$ is compact, then there exists $\wp_3 \in T(x_2)$ such that $m(x_2, x_3) = M(x_2, T(x_2))$. Hence, we have

$$F(M(x_2, x_3)) \leq F(m(x_1, x_2)) - \tau.$$

Impermanent this procedure in the same fashion. We get $\{x_n\} \in U$ such that

$$F(m(x_n, x_{n+1})) \leq F(m(x_{n-1}, x_n)) - \tau \text{ for all } n \in \mathbb{N}. \tag{2.8}$$

If there exists $n_0 \in \mathbb{N}$ so that $x_{n_0} \in T(x_{n_0})$, then x_{n_0} is a FP of T and the proof is over. Now, suppose that for every $n \in \mathbb{N}$, $x_n \notin T(x_n)$. Set $\delta_n = m(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then $\delta_n > 0$ for all $n \in \mathbb{N}$, and using (2.7) we deduce that

$$F(\delta_n) \leq F(\delta_{n-1}) - \tau \leq F(\delta_{n-1}) - 2\tau \leq ... \leq F(\delta_0) - n\tau. \tag{2.9}$$

From the above inequality, we get $\lim_{n\to\infty} F(\delta_n) = -\infty$ then by (F_2) we have $\lim_{n\to\infty} \delta_n = 0$. Then by (F_3), there exist $k \in (0,1)$ so that $\lim_{n\to\infty} \delta_n^k F(\delta_n) = 0$. From (2.8) the following is true for all $n \in \mathbb{N}$.

$$\delta_n^k (F(\delta_n) - F(\delta_0)) \leq -\delta_n^k n\tau \leq 0. \tag{2.10}$$

Letting $n \to \infty$ in (2.10), we get

$$\lim_{n\to\infty} n\delta_n^k = 0. \tag{2.11}$$

From (2.11), there exists $n_1 \in \mathbb{N}$ such that $n\delta_n^k \leq 1$ for all $n \geq n_1$, then we

$$\delta_n \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1. \tag{2.12}$$

In order to show that $\{x_n\}$ is a Cauchy sequence in $MM-$space. Consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then from (2.12) and triangle inequality of $MM-$space, one can obtain

$$\begin{aligned} m(x_n, x_m) - m_{x_n, x_m} &\leq m(x_n, x_{n+1}) - m_{x_n, x_{n+1}} + m(x_{n+1}, x_{n+2}) - m_{x_{n+1}, \wp_{n+2}} + \\ &\quad ... + m(x_{m-1}, x_m) - m_{x_{m-1}, x_m} \\ &\leq m(x_n, x_{n+1}) + m(x_{n+1}, x_{n+2}) + ... + +m(x_{m-1}, x_m) \\ &\leq \delta_n + \delta_{n+1} + ... + \delta_{m-1} \end{aligned}$$

$$= \sum_{i=n}^{m-1} \delta_i \leq \sum_{i=n}^{\infty} \delta_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ leads to the limit goes to 0 as $n \to \infty$. Hence $m(x_n, x_m) - m_{x_n, x_m} \to 0$. Thus $\{x_n\}$ is an M-Cauchy sequence in X. Since X is an m-complete, there exists $\xi \in X$ so that $\{x_n\}$ converges to ξ that is $m(x_n, \xi) - m_{x_n,\xi} \to 0$ as $n \to \infty$. Now suppose that F is continuous. In this case, we claim that $\xi \in T(\xi)$. Assume the contrary, that is ξ is not contained in $T(\xi)$. Then there exist $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $M(x_{n_k}, T(\xi)) > 0$ for all $n_k \geq n_0$. On the other hand, there exist $n \in N$ such that $\wp_n \in T(\xi)$ for all $n \geq n_1$ which implies that a contradiction, because ξ is not in $T(\xi)$ since $H(T(x_{n_k})), T(y) > 0$ for all $n_k \geq n_0$. Hence, we get

$$\tau + F(M(x_{n_{k+1}}, T(\xi))) \leq F(H_m(x_{n_{k+1}}, T(\xi))) \leq F(A_M(x_{n_k}, \xi)), \tag{2.13}$$

where

$$\begin{aligned} A_M(x_{n_k}, \xi) &= \max\left\{ \begin{array}{c} m(x_{n_k}, \xi), M(x_{n_k}, T(x_{n_k}), M(\xi, T(\xi))), \\ \frac{M(\xi, T(\xi))\left[1 + M\left(x_{n_k}, T\left(x_{n_k}\right)\right)\right]}{1 + m\left(x_{n_k}, \xi\right)} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} m(x_{n_k}, \xi), m(x_{n_k}, \wp_{n_{k+1}}), M(\xi, T(\xi)), \\ \frac{M(\xi, T(\xi))\left[1 + m\left(x_{n_k}, x_{n_{k+1}}\right)\right]}{1 + m\left(x_{n_k}, \xi\right)} \end{array} \right\}. \end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k \to \infty} A_M(x_{n_k}, \xi) = M(\xi, T(\xi)). \tag{2.14}$$

Passing $k \to \infty$ in (2.13) and using (2.14) and from the continuity of F, we get

$$\tau + F(M(\xi, T(\xi))) \leq F(M(\xi, T(\xi))),$$

a contradiction. Therefore $\Im \in T(\Im) = T(\Im)$. Hence $\Im$ is a fixed point of T. □

The next results in this part is to obtain fixed point consequences for rational type multivalued $F-$contraction mapping under Wordoski's axioms $(F_1 - F_4)$ by considering $CB^m(X)$ instead of $K(X)$.

Theorem 2.6. *Let X be a complete $M-$metric space and $T : X \to CB^m(X)$ be a rational type multivalued $F-$contraction mapping. If $F \in \chi_*$ so that*

$$F(\inf A) = \inf F(A) \textit{ for all } A \subset (0, \infty) \textit{ with } \inf F(A) > 0. \tag{2.15}$$

Then T owns a fixed point.

Proof. In same of the proof of Theorem (2.5) and using the assertion (F_4), we have

$$\begin{aligned} F(M(x_1, T(x_1))) &= F(\inf\{m(x_1, y) : y \in T(x_1)\}) \\ &= \inf\{F(m(x_1, y) : y \in T(x_1))\}. \end{aligned}$$

Using (2.15), we get

$$\inf\{F(m(x_1, y) : y \in T(x_1))\} < F(m(x_0, x_1)) - \frac{\tau}{2}.$$

Thus, there is $x_2 \in T(x_1)$ so that

$$F(m(x_1, x_2)) \le F(m(x_0, x_1)) - \frac{\tau}{2}.$$

In the same way as for the proof of Theorem (2.5), we can reach the end of the proof. □

Corollary 2.7. *Let X be a complete $M-$metric space and $T : X \to CB^m(X)$ be a rational type multivalued $F-$contraction mapping. If $F \in \chi_*$ so that*

$$F(\inf A) = \inf F(A) \text{ for all } A \subset (0, \infty) \text{ with } \inf F(A) > 0$$

$$\tau + F(H_m(T(x), T(y))) \le F\left(\frac{1}{2}\left[m(x, T(x)) + m(y, T(y))\right]\right). \tag{2.16}$$

Then T has a fixed point.

Corollary 2.8. *Let X be an complete $M-$metric space and let $T : X \to X$ be rational type $F-$contraction, if $F \in \nabla_F$ and there exist $\tau > 0$ such that*

$$\tau + F(H_m(T(x), T(y))) \le F(A_M(x, y))$$

where

$$A_M(x, y) = \max\left\{m(x, y), m(x, T(y)), m(y, T(y)), \frac{m(y, T(y))[1 + m(x, T(x))]}{1 + m(x, y)}\right\}$$

for all$x, y \in X$. Then T has a fixed point.

Now, we shall introduce a multivalued $F-$contraction in ordered $M-$metric spaces.

Let $X \neq \phi$. If $(X, \preceq)$ is a partially ordered set on $M-$metric space X, then $(X, \preceq, m)$ is called ordered $M-$metric space. We say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$ hold. Moreover, a mapping $T : X \to X$ is said to be non-decreasing if $T(x) \preceq T(y)$ whenever $x \preceq y$, for all $x, y \in X$. Further, an ordered $M-$metric space $(X, m, \preceq)$ is regular if for every non-decreasing sequence $\{x_n\}$ in X, convergent to $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Definition 2.9. Let $(X, \preceq)$ be a partial ordered and K, L be two non-empty subset of X. Then we define two relation between K and L as follows:

(i) $K \prec_1 L$ if for each $x \in K$, there exist $y \in L$ such that $x \preceq y$,
(ii) $K \prec_2 L$ if for each $x \in K$ and $y \in L$, we have $x \preceq y$.

Theorem 2.10. *Let $(X, m, \preceq)$ be an ordered complete $M-$metric space and $T : X \to K(X)$ be a rational type multivalued $F-$contraction mapping, if $F \in \nabla_F$ and there is $\tau > 0$ so that*

$$H_m(T(x), T(y)) > 0 \Rightarrow \tau + F(H_m(T(x), T(y))) \le F(A_M(x, y)), \tag{2.17}$$

where

$$A_M(x, y) = \max\left\{m(x, y), M(x, T(x)), M(y, T(y)), \frac{M(y, T(y))[1 + M(x, T(x))]}{1 + m(x, y)}\right\}$$

for all comparable $x, y \in X$. If the assertions below hold:
(i) There exist $x_0 \in X$ such that $\{x_0\} \prec_1 T(x_0)$.
(ii) For $x, y \in X$, $x \preceq y$ implies that $T(x) \prec_2 T(y)$.
(iii) X is regular, then T owns a fixed point in X.

Proof. Based on assertion (i), there exists $x_1 \in T(x_0)$ such that $x_0 \preceq x_1$. According to assertion (ii), we have $T(x_0) \prec_2 T(x_1)$. If $x_1 \in T(x_1)$ then x_1 is a FP of T and the proof is finished. So, suppose that $x_1 \notin T(x_1)$. Since $T(x_1)$ is closed and $M(x_1, T(x_1)) > 0$. Then, we get $M(x_1, T(x_1)) \leq H_m(T(x_0), T(x_1))$. Again by (F_1), we obtain that $F(M(x_1, T(x_1))) \leq F(H_m(T(x_0), T(x_1)))$.

Now, using (2.17), we have

$$\begin{aligned} F(M(x_1,T(x_1))) \leq F(H_m(T(x_0),T(x_1))) &\leq F(A_M(x_0,x_1)) - \tau \\ &= F\left(\max\left\{\begin{array}{c} m(x_0,x_1), M(x_0,T(x_0)), M(x_1,T(x_1)), \\ \frac{M(x_1,T(x_1))[1+M(x_0,T(x_0))]}{1+m(x_0,x_1)} \end{array}\right\} - \tau\right) \\ &= F(\max\{m(x_0,x_1), M(x_1,T(x_1))\}) - \tau, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} A_M(x_0,x_1) &= \max\left\{\begin{array}{c} m(x_0,x_1), M(x_0,T(x_0)), M(x_1,T(x_1)), \\ \frac{m(x_1,T(x_1))[1+M(x_0,T(x_0))]}{1+m(x_0,x_1)} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} m(x_0,x_1), m(x_1,T(x_1)), \\ \frac{m(x_1,T(x_1))[1+m(x_0,T(x_0))]}{1+m(x_0,x_1)} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} m(x_0,x_1), m(x_1,T(x_1)), \\ \frac{m(x_1,T(x_1))[1+m(x_0,x_1)]}{1+m(x_0,x_1)} \end{array}\right\} \\ &= \max\{m(x_0,x_1), M(x_1,T(x_1))\}. \end{aligned}$$

Now, if $m(x_0, x_1) \leq M(x_1, T(x_1))$ then from (2.18), we obtain that

$$F(M(x_1,T(x_1))) \leq F(M(x_1,T(x_1))) - \tau$$

which is a contradiction. Thus $M(x_1, T(x_1)) < m(x_0, x_1)$ and so from (2.3), we have

$$F(M(x_1,T(x_1))) \leq F(m(x_0,x_1)) - \tau. \tag{2.19}$$

Because $T(x_1)$ is compact, then there exists $x_2 \in T(x_1)$ with $x_1 \preceq \wp_2$ such that $m(x_1, x_2) = M(x_1, T(x_1))$, from (2.19), we get

$$F(m(x_1,x_2)) \leq F(m(x_0,x_1)) - \tau \text{ for all } n \in \mathbb{N}.$$

Since $x_1 \in T(x_0)$ and $x_2 \in T(x_1)$ with $M(x_1, T(x_1)) \leq H_m(T(x_0), T(x_1))$. Then, we get

$$F(M(x_2,T(x_2))) \leq F(H_m(T(x_1),T(x_2))) \leq F(A_M(x_1,x_2)) - \tau. \tag{2.20}$$

Considering the same way, we deduce that

$$A_M(x_1,x_2) \leq \max\{m(x_1,x_2), m(x_1,T(x_2))\}.$$

It follows from (2.20) that

$$F(m(x_2,T(x_2))) \leq F(m(x_1,x_2)) - \tau. \tag{2.21}$$

The compactness of $T(x_2)$ leads to with the assumption (*ii*), there exists $x_3 \in T(x_2)$ with $x_1 \preceq x_2$ such that $m(x_2, \wp_3) = M(x_2, T(x_2))$. Therefore, we have

$$F(M(x_2, x_3)) \leq F(m(x_1, x_2)) - \tau.$$

continuing this process in the same manner in the proof of Theorem (2.5), in the view of condition (*ii*), we can construct a monotone non-decreasing sequence in X such that $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$, and

$$x_0 \preceq x_1 \preceq \ldots \preceq \; x_n \ldots,$$

which implies that x_n and x_{n+1} are comparable and hence $T(x_n) \prec_2 T(x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Next, proceeding as in the proof of Theorem (2.5), we get $\{x_n\}$ is m-Cauchy sequence. So, there exist $\xi \in X$ such that $\{x_n\}$ converges to ξ that is $m(x_n, \xi) - m_{x_n,\xi} \to 0$ as $n \to \infty$. From condition (*iii*) we deduce that $x_n \preceq \xi$ for all $n \in \mathbb{N} \cup \{0\}$. From now, the rest of the proof can be completed as in the proof of Theorem (2.5). □

Example 2.11. Let $X = [0, \infty)$ endowed with $m(x, y) = \frac{x+y}{2}$ for all $x, y \in X$. Clearly (X, m) be a complete $M-$metric space. Define the mapping $T : X \to CB^m(X)$ by

$$T(x) = \begin{cases} \left[\frac{\wp}{3}, \frac{\wp}{2}\right], & \text{if } x \in [0, 3] \cap [0, \infty), \\ \left[\frac{2\wp}{3}, \frac{4\wp}{9}\right], & \text{if } x \in (0, 3) \cap [0, \infty). \end{cases}$$

Clearly, T is a multivalued mapping. Define a function $F : R^+ \to R$ by $F(x) = \ln(x)$ for all $x \in R^+$ and $\tau = \left(0, \frac{8}{90}\right]$. Now, we want to prove that T satisfies the condition (2.2). First, from main contraction, we can write

$$\begin{aligned}
\tau + \ln(H_m(T(x), T(y))) &= \tau + \ln\left(\max\left\{\sup_{p \in T(x)} m(p, T(y)), \sup_{q \in T(y)} m(T(x), q)\right\}\right) \\
&= \tau + \ln\left(\max\left\{\begin{array}{c} \sup_{p \in T(x)} m\left(p, \left[\frac{2\mathfrak{I}}{3}, \frac{4\mathfrak{I}}{9}\right]\right), \\ \sup_{q \in T(y)} m\left(\left[\frac{x}{3}, \frac{x}{2}\right], q\right) \end{array}\right\}\right) \\
&= \tau + \ln\left(\max\left\{m\left(\frac{x}{3}, \frac{2y}{3}\right), m\left(\frac{2y}{3}, \frac{x}{2}\right)\right\}\right) \\
&= \tau + \ln\left(\max\left\{\left(\frac{\frac{x}{3} + \frac{2y}{3}}{2}\right), \left(\frac{\frac{2y}{3} + \frac{x}{2}}{2}\right)\right\}\right) \\
&= \tau + \ln\left(\max\left\{\left(\frac{x + 2y}{6}\right), \left(\frac{4y + 3x}{12}\right)\right\}\right) \\
&\leq \ln\left(\max\left\{\begin{array}{c} m(x, y), M(x, T(x)), M(y, T(y)), \\ \frac{M(y, T(y))[1 + M(x, T(x))]}{1 + m(x, y)} \end{array}\right\}\right).
\end{aligned}$$

Thus, all the axioms of Theorems (2.5) and (2.6) are satisfied and 0 is a fixed point of T.

3. Application to nonlinear matrix equations

In this part, we use the theoretical results obtained in the above part to find the existence of the positive definite solution to the following nonlinear matrix equation in equation:

$$D = \alpha + \sum_{i=1}^{p} E_i^* \Psi(D) E_i \tag{3.1}$$

where α is a positive definite matrix , $E_1, E_2, ...E_m$ are $n \times n$ matrices, Ψ is a self map on the set of all $n \times n$ Herniation matrices, which maps set of all $n \times n$ Herniation positive definite into itself matrices. Set

$$J(n) = \{D : D \text{ is } n \times n \text{ Herniation matrix}\}.$$

The spectral norm is denoted by $\|.\|_1$, i.e.,

$$\|E\|_1 = (\mu^+ (E^* E))^{\frac{1}{2}},$$

where $\mu^+ (E^* E)$ is the greatest eigenvalue of the matrix $E^* E$. The Ky Fan norm is given as

$$\|E\|_1 = \sum_{i=1}^{n} S_i(E),$$

where $\{S_1(E), S_2(E), S_3(E), ...S_n(E)\}$ is the set of the singular value of E. Further more,

$$\|E\| = tr\left((E^* E)^{\frac{1}{2}}\right),$$

which is $tr(E)$ for (Hermitian) nonnegative matrices and

$$Q(n) = \{D \in Q(n) : D \text{ is positive definite}\}.$$

Then, $F \in \nabla_f$ Assume that $(J(n), m)$ is a complete $M-$metric space, where

$$m(D, P) = \frac{1}{2}\|D + P\|_1 = \frac{1}{2}(tr(D + P)). \tag{3.2}$$

Theorem 3.1. *Let $\alpha \in Q(n)$ and $\Psi : J(n) \to J(n)$ be a mapping which maps $Q(n)$ to $Q(n)$. Suppose the following two axioms hold*

(i) *there exist a positive number N so that $\sum_{i=1}^{m} E_i E_i^* < N I_n$ and $\sum_{i=1}^{m} E_i^* \Psi(\alpha) E_i > 0$,*

(ii) *for all $D, P \in Q(n)$, we have*

$$m(\Psi(D), \Psi(P)) \leq \frac{1}{N} A_M(D, P) e^{-\left(\frac{2 + \frac{1}{2}\|D+P\|}{2 \cdot \frac{1}{2}\|D+P\|}\right)}$$

where

$$A_M(D, P) = \max\left\{m(D, P), m(D, \Psi(D)), m(P, \Psi(P)), \frac{m(P, \Psi(P))[1 + m(D, \Psi(D))]}{1 + m(D, P)}\right\}.$$

Then Eq (3.1) *has a solution in $Q(n)$.*

Proof. Define $\sigma : Q(n) \to Q(n)$ and $F : R^+ \to R^+$ by

$$\sigma(D) = \alpha + \sum_{i=1}^{m} E_i^* \Psi(D) E_i,$$

and $F(r) = \ln r$, respectively. Then a FP of σ is a solution of (3.1). Let $D, P \in Q(n)$ with $D \neq P$. Then, for $m(D, P) > 0$, we deduce that $\tau(t) = \frac{1}{\frac{t}{2}} + \frac{1}{2}$, and

$$\begin{aligned}
m(\sigma(D), \sigma(P)) &= \frac{1}{2}\|\sigma(D) + \sigma(P)\|_1 \\
&= \frac{1}{2}(tr(\sigma(D) + \sigma(P))) \\
&= \sum_{i=1}^{m} \frac{1}{2}(tr(E_i E_i^*(\sigma(D) + \sigma(P)))) \\
&= \frac{1}{2} tr\left(\left(\sum_{i=1}^{m} E_i E_i^*\right)\sigma(D) + \sigma(P)\right) \\
&\leq \left\|\sum_{i=1}^{m} E_i E_i^*\right\| \frac{1}{2}\|\sigma(D) + \sigma(P)\|_1 \\
&\leq \frac{\left\|\sum_{i=1}^{m} E_i E_i^*\right\|}{N} A_M(D, P) e^{-\frac{2+\frac{1}{2}\|D+P\|}{2.\frac{1}{2}\|D+P\|}} \\
&< A_M(D, P) e^{-\frac{2+\frac{1}{2}\|D+P\|}{2.\frac{1}{2}\|D+P\|}}.
\end{aligned}$$

Hence

$$\ln\left(\frac{1}{2}\|\sigma(D) + \sigma(P)\|\right) < \ln\left(A_M(D, P) e^{-\frac{2+\frac{1}{2}\|D+P\|}{2.\frac{1}{2}\|D+P\|}}\right) = \ln\left(A_M(D, P) e^{-\frac{2+\frac{1}{2}\|D+P\|}{2.\frac{1}{2}\|D+P\|}}\right),$$

which implies that

$$\frac{1}{\frac{1}{2}\|D + P\|} + \frac{1}{2} + \ln\left(\frac{1}{2}\|\sigma(D) + \sigma(P)\|\right) < \ln(A_M(D, P)).$$

Consequently,

$$\tau(m(D, P)) + F(m(\sigma(D), \sigma(P))) < F(A_M(D, P)).$$

Hence, all requirements of Corollary (2.9) immediately hold. Thus, σ have a fixed point which is a solution to the system (3.1) in $Q(n)$. ◻

Example 3.2. Consider the following matrix equation:

$$D = \alpha + \sum_{i=1}^{2} E_i^* \Psi(D) E_i, \tag{3.2}$$

where

$$\alpha = \begin{pmatrix} 0.1 & 0.01 & 0.01 \\ 0.01 & 0.1 & 0.01 \\ 0.01 & 0.01 & 0.1 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 0.4 & 0.01 & 0.01 \\ 0.01 & 0.4 & 0.01 \\ 0.01 & 0.01 & 0.4 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0.6 & 0.01 & 0.01 \\ 0.01 & 0.6 & 0.01 \\ 0.01 & 0.01 & 0.6 \end{pmatrix},$$

Define $\Psi : \sigma(3) \to \sigma(3)$, by

$$\Psi(D) = \frac{D}{3}.$$

Define a mapping $\sigma : \sigma(3) \to \sigma(3)$ by

$$\sigma(D) = \alpha + \sum_{i=1}^{m} E_i^* \Psi(D) E_i.$$

Hence, all conditions of Theorem (3.1) are satisfied with $N = \frac{6}{10}$. Therefore, the problem (3.2) has a solution.

4. Conclusions

In this article, we have achieved fixed point results for modified multivalued rational type $F-$contraction in complete $M-$metric spaces and ordered $M-$metric spaces. Our generalized results are based on two fixed point results for multivalued $M-$metric metric space of Altun et al. [34] and Kumar et al. [35]. Finally, we present an application dealing with the existence of positive definite solutions for non-linear matrix equations.

Availability of data and material

The data used to support the findings of this study are available from the corresponding author upon request.

Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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