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Research article

Some topological aspects of interval spaces

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Abstract: In previous papers, several T_0, T_2 objects, D-connectedness and zero-dimensionality in topological categories have been introduced and compared. In this paper, we characterize separated objects, T_0, $\mathbf{T_0}$, T_1, Pre-T_2 and several versions of Hausdorff objects in the category of interval spaces and interval-preserving mappings and examine their mutual relationship. Further, we give the characterization of the notion of closedness and D-connectedness in interval spaces and study some of their properties. Finally, we introduce zero-dimensionality in this category and show its relation to D-connectedness.

1. Introduction

Convexity is a fundamental feature in many fields of mathematics. However, in vector spaces, it is not the best environment for understanding the basic characteristic of convex sets. As a remedy, abstract convex structures [40] came into existence and have many applications in different areas of mathematics, including topology, graph theory and lattice theory (see [39], [35] and [32]). Convex structures can be determined in several different ways, including through the use of the algebraic closure operator and hull operators. In 1971, Calder [17] introduced the concept of Interval operators which is a natural generalization of intervals and it also provides a natural and frequent method of constructing convex structures. Interval operators have many applications in planer geometry such as Pasch-Peano (PP) spaces.

In 1921, Sierpinski [20] introduced zero-dimensional topological space consisting of a basis that is clopen and it has been utilized to construct several well-known classes of topological spaces such as

Lusin spaces [16], non-Archimedean spaces [19] and stone spaces [21]. Recently, Stine put forward this notion to an arbitrary topological category [36, 37].

Classical separation axioms of topology have been put forward for topological categories by numerous authors [2, 18] using different approaches. In 1991, Baran [2, 18] introduced T_0, T_1 and T_2 objects and (strongly) closed objects in a set-based topological category by using initial, final lifts and (in) discrete objects. Further, he introduced the concept of pre-T_2 in topological space and later on, extended it to a set-based topological category [2, 9]. T_0 objects and the notion of closedness are widely used to define and characterize various forms of Hausdorff objects [5], connectedness [8] and sobriety [11] in some topological categories [11, 23, 33].

In 1994, Mielke [30] showed the important role of pre-T_2 objects in the general theory of geometric realization, their associated intervals and corresponding homotopic structures. Also, in 1999, Mielke [31] used pre-T_2 objects of topological categories to characterize decidable objects in topos theory, where $X \in Obj(\mathcal{E})$ with $\mathcal{E}$ as a topos [21], is called decidable if the diagonal $\Delta \subset X^2$ is a complemented subobject.

Other uses of pre-T_2 objects include defining various forms of Hausdorff objects [5], T_3 and T_4 objects [7] in some well-known topological categories [14, 25]. There is also, a relationship between pre-T_2 objects and partitions, as well as equivalence relations in case of **Top** see [36] in the some other categories see [10, 12, 13, 15].

The salient objectives of the paper are stated as follows:

(1) To characterize separated, T_0, $\mathbf{T_0}$, T_1, pre-T_2, T_2, ST_2 and NT_2 interval spaces, and examine their mutual relationship;

(2) To give the characterization of closedness of singleton sets and D-connectedness in the category **IS** (i.e., the category of interval spaces and interval preserving mappings);

(3) To examine the zero-dimensionality and study its relation to D-connectedness in the category of interval spaces and interval preserving mappings.

2. Preliminaries

Let X be a non-empty set and $\{B_i\}_{i \in I} \overset{dir}{\subseteq} P(X)$ denotes the directed subset of X, which means that, for any $E, F \in \{B_i\}_{i \in I}$, there exists $G \in \{B_i\}_{i \in I}$ such that $E \subseteq G$ and $F \subseteq G$. For any non-empty sets X and Y, and $f : X \longrightarrow Y$ be any mapping. Define forward mapping $f^{\rightarrow} : P(X) \longrightarrow P(Y)$ and backward mapping $f^{\leftarrow} : P(Y) \longrightarrow P(X)$ by $f^{\rightarrow}(E) = \{f(x) \mid x \in E\}$ and $f^{\leftarrow}(G) = \{x \mid f(x) \in G\}$ for any $E \in P(X)$ and $G \in P(Y)$, respectively.

Definition 2.1. *(cf. [40, 41]) A convex structure $\mathfrak{C}$ on the set X is a subset of $P(X)$ satisfying the following:*

(1) $\emptyset, X \in \mathfrak{C}$;

(2) $\{B_i\}_{i \in I} \subseteq \mathfrak{C}$ implies $\bigcap_{i \in I} B_i \in \mathfrak{C}$;

(3) $\{B_i\}_{i \in I} \overset{dir}{\subseteq} \mathfrak{C}$ implies $\bigcup_{i \in I} B_i \in \mathfrak{C}$.

The pair $(X, \mathfrak{C})$ *is called convexity space. The members of* $\mathfrak{C}$ *are called convex sets and their complements are called concave sets.*

A mapping $g : (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ is called convexity preserving mapping provided that $E \in \mathfrak{C}_Y$ implies $g^{\leftarrow}(E) \in \mathfrak{C}_X$. Let **CS** denotes the category of convexity spaces $(X, \mathfrak{C})$ and convexity preserving mappings.

The smallest convex set including a set E is defined as $co(E) = \bigcap \{F : E \subseteq F \in \mathfrak{C}\}$ is called the convex hull of E. A set of type $co(E)$ with E is finite, and it is called polytope [40].

Definition 2.2. *(cf. [40, 41]) A closure operator cl on X is a mapping* $cl : P(X) \longrightarrow P(X)$ *satisfying:*

(1) $cl(\emptyset) = \emptyset$;

(2) $E \subseteq cl(E)$;

(3) $E \subseteq F$ *implies* $cl(E) \subseteq cl(F)$;

(4) $cl(cl(F)) = cl(F)$.

The pair (X, cl) *is called a closure space. Further, the closure space* (X, cl) *is said to be an algebraic closure space if* $cl(E) = \cup\{cl(F) \mid F$ *is a finite subset of* $E\}$ *is satisfied.*

A mapping $g : (X, cl_X) \longrightarrow (Y, cl_Y)$ between two closure spaces is called a closure preserving mapping such that $g^{\rightarrow}(cl_X(E)) \subseteq cl_Y(g^{\rightarrow}(E))$, $\forall E \in P(X)$. Let **CLS** denotes the category of closure spaces and closure preserving mappings, and **ACLS** (the category of algebraic closure spaces and algebraic closure preserving mappings) is the full subcategory of **CLS**. Note that **ACLS** $\cong$ **CS** [40, 41].

Definition 2.3. *(cf. [40, 41]) The mapping* $J : X \times X \to P(X)$ *is called an interval operator satisfying the following:*

(1) For all $x, y \in X$, $x, y \in J(x, y)$ *(Extensive Law)*;

(2) $J(x, y) = J(y, x)$ *(Symmetry Law)*.

The pair (X, J) *is called an interval space, and* $J(x, y)$ *is the interval between x and y.*

The mapping $f : (E, J_E) \longrightarrow (F, J_F)$ is called a interval preserving mapping, if

$$\forall x, y \in X, f^{\rightarrow}(J_E(x, y)) \subseteq J_F((f(x), f(y))).$$

Let **IS** denotes the category of interval spaces and interval preserving mappings. Note that **IS** is the full subcategory of **CS**.

Example 2.1. *(cf. [41]) Let* $\mathbb{R}$ *be the set of real numbers, and define a mapping* $J_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \longrightarrow P(\mathbb{R})$ *by*

$$\forall x, y \in \mathbb{R}, J_{\mathbb{R}}(x, y) = [min\{x, y\}, max\{x, y\}],$$

where $J_{\mathbb{R}}$ *indicates the interval operator on* $\mathbb{R}$.

Example 2.2. *(cf. [40, 41]) Let d be a metric on X, and define a mapping* $J_d : X \times X \longrightarrow P(X)$ *as follows:*

$$\text{for all } x, y \in X, J_d(x, y) = \{k \in X \mid d(x, y) = d(x, k) + d(k, y)\},$$

where J_d indicates the geodesic interval operator on X.

Example 2.3. *(cf. [40]) Let V be a vector space and define a mapping $J_V : V \times V \longrightarrow P(V)$ by $J_V(x, y) = \{xt + (1 - t)y \mid 0 \le t \le 1\}$, where J_V indicates the standard interval operator on the vector space V.*

Example 2.4. *(cf. [40]) Let $(X, \le)$ be a partially ordered set and define a mapping $J_\le : X \times X \longrightarrow P(X)$ as follows:*

$$J_\le(x, y) = \begin{cases} \{x, y\} & \text{if } x, y \text{ are incomparable}; \\ \{z \mid x \le z \le y\} & \text{if } x \le y, \end{cases}$$

where $J_\le$ indicates the ordered interval operator on X.

Example 2.5. *(cf. [40]) Let (M, m) be a median algebra and define a mapping $J_m : M \times M \longrightarrow P(M)$ as follows:*

$$\text{for all } x, y \in M, J_m = \{m(x, y, z) \mid z \in M\} = \{z \in M \mid m(x, y, z) = z\},$$

where J_m indicates the median interval operator on M.

For any interval space (X, J), if for any $x, y, z \in X$ and $w \in J(y, z)$, $t \in J(x, w)$, and then there exists $k \in J(x, y)$ such that $t \in J(z, k)$. This property is known as the Peano Property. Further, if for any $p, x, y \in X$, $z \in J(p, x)$ and $w \in J(p, y)$, then the intervals $J(x, w)$ and $J(z, y)$ intersect. This property is known as the Pasch property [40].

Any interval space (X, J) satisfying the Pasch and Peano properties is called a PP space. Note that every vector space over a totally ordered field is a PP space [40].

Definition 2.4. *(cf. [40, 41]) A convex space $(X, \mathfrak{C})$ is called an arity 2 convex space satisfying the following: for all $B \in P(X)$ and all $x, y \in B$, $co(\{x, y\}) \subseteq B$ implies $B \in \mathfrak{C}$.*

Let **CS(2)** denotes the category of arity 2 convex spaces $(X, \mathfrak{C})$ and convexity preserving mappings. Note that **CS(2)** can be embedded in **IS** as a reflexive subcategory [40, 41].

Proposition 2.1. *(cf. [40, 41]) Suppose $(X, \mathfrak{C})$ is a convex space and define $J^\mathfrak{C} : X \times X \longrightarrow P(X)$ by*

$$\forall x, y \in X, J^\mathfrak{C}(x, y) = co(x, y) = \bigcap_{x, y \in B \in \mathfrak{C}} B.$$

Then $J^\mathfrak{C}$ represents the interval operator on X.

Proposition 2.2. *(cf. [40, 41]) Suppose (X, J) is interval space and define $\mathfrak{C}^J$ by*

$$\mathfrak{C}^J = \{B \in P(X) \mid \forall x, y \in B, J(x, y) \subseteq B\}.$$

Then, $(X, \mathfrak{C}^J)$ is an arity 2 convex space.

A functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathbf{Set}$ (the category of sets and functions) is called topological if (1) $\mathcal{U}$ is concrete (2) $\mathcal{U}$ consists of small fibers and (3) every $\mathcal{U}$-source has a unique initial lift, i.e., if for every source $(f_i : X \to (X_i, \zeta_i))_{i \in I}$ there exists a unique structure ζ on X such that $g : (Y, \eta) \to (X, \zeta)$ is a

morphism iff for each $i \in I$, $f_i \circ g : (Y, \eta) \rightarrow (X_i, \zeta_i)$ is a morphism or equivalently, each $\mathcal{U}$-sink has a unique final lift [1, 38].

Note that a topological functor $\mathcal{U} : \mathcal{E} \longrightarrow \textbf{Set}$ has a left adjoint $\mathcal{D} : \textbf{Set} \longrightarrow \mathcal{E}$, called the discrete functor. An object of the form $X = \mathcal{U}\mathcal{D}(X)$ is called a discrete object in $\mathcal{E}$, i.e., the $\mathcal{E}$-objects X such that every $f : \mathcal{U}X \longrightarrow \mathcal{U}Y$, $Y \in \mathcal{E}$, is an $\mathcal{E}$-morphism.

Also, the functor $\mathcal{U}$ is called a normalized topological functor if the subterminals have a unique structure [1, 38].

Lemma 2.1. *(cf. [41]) Let (X_i, J_i) be the collection of interval space and $(f_i : (X, J_*) \longrightarrow (X_i, J_i))_{i \in I}$ be a source. Then, for any $x, y \in X$,*

$$J_*(x, y) = \bigcap_{i \in I} f_i^{\leftarrow}(J_i(f_i(x), f_i(y)))$$

is the initial interval structure on X.

Lemma 2.2. *(cf. [41]) Let (X, J) be an interval space. Then, we have the following:*

(1) The discrete interval structure on X is defined by $J_{dis}(x, y) = \{x, y\}$ for any distinct $x, y \in X$.

(2) The indiscrete interval structure on X is given by $J_{ind}(x, y) = X$ for any distinct $x, y \in X$.

Remark 2.1. *The topological functor $\mathcal{U} : \textbf{IS} \longrightarrow \textbf{Set}$ is normalized since a unique structure exists on $\emptyset$, the empty set or $X = \{x\}$, i.e., a one-point set for $X \in Obj(\textbf{IS})$ [41].*

3. Separated, pre-Hausdorff and Hausdorff interval spaces

Let X be a set and the wedge $X^2 \bigvee_\Delta X^2$ be two any disjoint copies of X^2 intersecting diagonally. In other words, the pushout of $\Delta : X \longrightarrow X^2$ along itself. A point (x, y) in $X^2 \bigvee_\Delta X^2$ is denoted by $(x, y)_1$ (resp. $(x, y)_2$) if it is in the first (resp. second) component.

Definition 3.1. *(cf. [2]) The mapping $A : X^2 \bigvee_\Delta X^2 \longrightarrow X^3$ is said to be the principal axis mapping provided that*

$$A(x, y)_j = \begin{cases} (x, y, x) & , \ j = 1 \\ (x, x, y) & , \ j = 2. \end{cases}$$

Definition 3.2. *(cf. [2]) The mapping $S : X^2 \bigvee_\Delta X^2 \longrightarrow X^3$ is said to be a skewed axis mapping provided that*

$$S(x, y)_j = \begin{cases} (x, y, y) & , \ j = 1 \\ (x, x, y) & , \ j = 2. \end{cases}$$

Definition 3.3. *(cf. [2]) The mapping $\nabla : X^2 \bigvee_\Delta X^2 \longrightarrow X^2$ is said to be a fold mapping provided that $\nabla(x, y)_j = (x, y)$ for $j = 1, 2$.*

Definition 3.4. *Let $\mathcal{U} : \mathcal{E} \longrightarrow \textbf{Set}$ be a topological functor and $X \in Obj(\mathcal{E})$ with $\mathcal{U}(X) = Y$.*

(1) X is called separated provided that every initial morphism with the domain X is a monomorphism [42].

(2) X is called T_0 provided that the initial lift of the $\mathcal{U}$-source $\{A : Y^2 \bigvee_\Delta Y^2 \to \mathcal{U}(X^3) = Y^3$ and $\nabla : Y^2 \bigvee_\Delta Y^2 \to \mathcal{U}\mathcal{D}(Y^2) = Y^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is the left adjoint of $\mathcal{U}$ [2].

(3) X is called T_1 provided that the initial lift of the $\mathcal{U}$-source $\{S : Y^2 \bigvee_\Delta Y^2 \to \mathcal{U}(X^3) = Y^3$ and $\nabla : Y^2 \bigvee_\Delta Y^2 \to \mathcal{U}\mathcal{D}(Y^2) = Y^2\}$ is discrete [2].

(4) X is called $\mathbf{T_0}$ if X does not contain an indiscrete subspace with at least two points [29].

(5) X is called pre-T_2 iff initial lifts of the $\mathcal{U}$-sources $\{A : Y^2 \bigvee_\Delta Y^2 \to \mathcal{U}(X^3) = Y^3$ and $S : Y^2 \bigvee_\Delta Y^2 \to \mathcal{U}(X^3) = Y^3\}$ coincide [2].

(6) X is called ST_2 provided that X is separated and pre-T_2.

(7) X is called T_2 provided that X is T_0 and pre-T_2 [2].

(8) X is called NT_2 provided that X is $\mathbf{T_0}$ and pre-T_2 [2].

Remark 3.1. *(1) In the category **Top**, separated, T_0 and $\mathbf{T_0}$ (resp. T_1) reduce to the usual T_0 (resp. T_1) of topological spaces. Similarly, ST_2, T_2 and NT_2 reduce to a classical Hausdorff topological space [4, 6, 42].*

(2) If $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is a topological functor, where $\mathcal{B}$ is an elementary topos, then Definition 3.4 is still valid [2].

(3) In any arbitrary topological category, every $\mathbf{T_0}$ object is separated but converse is not in general [42]. Further, the T_0 object and separated object, and T_0 and $\mathbf{T_0}$ objects are independent of each other [4].

(4) In any arbitrary topological category, there is no relation among ST_2, T_2 and NT_2 [5]. However, for any topological functor $\mathcal{U} : pre\text{-}T_2(\mathcal{E}) \longrightarrow \mathbf{Set}$, where pre-$T_2(\mathcal{E})$ is the full subcategory of all pre-T_2 objects in $\mathcal{E}$, all T_0, T_1, ST_2, T_2 and NT_2 objects are equivalent [9].

(5) Let $\mathcal{U} : \mathcal{E} \longrightarrow \mathbf{Set}$ be a topological functor and $X \in Obj(\mathcal{E})$. If X is an indiscrete object, then X is pre-T_2 [9].

Theorem 3.1. *An interval space (X, J) is separated iff X has at most one point.*

Proof. Suppose (X, J) is separated, $X \neq \emptyset$ and $X \neq \{a\}$. Then, there exists $b \in X$ with $a \neq b$. If $X = \{a, b\}$, then $J(a, b) = X$, an indiscrete structure. Let $f : (X, J) \longrightarrow (X, J)$ be a mapping defined by $f(a) = a = f(b)$. Since (X, J) is an indiscrete interval space, f is initial (i.e., $f^\leftarrow(J(f(a), f(b))) = f^\leftarrow(J(a, a)) = X = J(a, b)$) but it is not mono. Hence, (X, J) is not a separated interval space.

Note that every subspace of a separated interval space is separated since the composition of initial lifts is initial and the composition of monomorphisms is a monomorphism. If $\text{Card}X \geq 3$, for any $a, b \in X$ with $a \neq b$ and $M = \{a, b\} \subset X$, then the subinterval structure J_M on M is indiscrete. By Definition 1.1 of [42], a separated interval space can not have an indiscrete subspace with at least two points, which is a contradiction. Hence, X must be the empty set or a one-point set.

Conversely, if $X = \emptyset$ or $X = \{a\}$, then clearly (X, J) is separated.

$\square$

Theorem 3.2. *Every interval space (X, J) is T_0.*

Proof. Let (X, J) be an interval space. We show that (X, J) is T_0. Let $\overline{J}$ be an initial structure on $X^2 \bigvee_\Delta X^2$ induced by $A : X^2 \bigvee_\Delta X^2 \longrightarrow (X^3, J^3)$ and $\nabla : X^2 \bigvee_\Delta X^2 \longrightarrow (X^2, J^2_{dis})$, where J^3 and J^2_{dis} are products and discrete interval structures on X^3 and X^2, respectively. Let $m, n \in X^2 \bigvee_\Delta X^2$.

Case I: If $m = n$, then $\nabla m = \nabla n$ and $pr_k Am = pr_k An$, $k = 1, 2, 3$, where pr_k is the projection mapping $pr_k : X^3 \longrightarrow X$ for $k = 1, 2, 3$.

On the other hand,

$$\nabla^{\leftarrow}(J_{dis}(\nabla m, \nabla n)) = \nabla^{\leftarrow}(J_{dis}(\nabla m, \nabla m)) = \nabla^{\leftarrow}(\{\nabla m\}) = \{m\}$$

and

$$pr_k A^{\leftarrow}(J(pr_k Am, pr_k An)) = pr_k A^{\leftarrow}(J(pr_k Am, pr_k Am)), \quad k = 1, 2, 3.$$

It follows that $m \in pr_k A^{\leftarrow}(J(pr_k Am, pr_k Am))$ for $k = 1, 2, 3$.

By Lemma 2.1, we obtain $\overline{J}(m, m) = \{m\}$, a discrete structure.

Case II: Let $m \neq n$ and $\nabla m = \nabla n$. If $\nabla m = (x, y) = \nabla n$ for some $(x, y) \in X^2$ given that $m \neq n$, consequently, it follows that $m = (x, y)_i$ and $n = (x, y)_j$ with $i \neq j$ and $i, j = 1, 2$.

Suppose $m = (x, y)_1$ and $n = (x, y)_2$. By Lemma 2.2 (1),

$$J_{dis}(\nabla m, \nabla n) = J_{dis}(\nabla(x, y)_1, \nabla(x, y)_2) = J_{dis}((x, y), (x, y)) = \{(x, y)\}$$

and

$$\nabla^{\leftarrow}(J_{dis}(\nabla m, \nabla n)) = \nabla^{\leftarrow}\{(x, y)\} = \{(x, y)_1, (x, y)_2\}.$$

Similarly,

$$pr_1 A^{\leftarrow}(J(pr_1 Am, pr_1 An)) = pr_1 A^{\leftarrow}(J(pr_1 A(x, y)_1, pr_1 A(x, y)_2)) = pr_1 A^{\leftarrow}(J(x, x)),$$

$$pr_2 A^{\leftarrow}(J(pr_2 Am, pr_2 An)) = pr_2 A^{\leftarrow}(J(pr_2 A(x, y)_1, pr_2 A(x, y)_2)) = pr_2 A^{\leftarrow}(J(y, x))$$

and

$$pr_3 A^{\leftarrow}(J(pr_3 Am, pr_3 An)) = pr_3 A^{\leftarrow}(J(pr_3 A(x, y)_1, pr_3 A(x, y)_2)) = pr_3 A^{\leftarrow}(J(x, y)).$$

Since $x = pr_1 A(x, y)_1 = pr_1 A(x, y)_2 \in J(x, x)$, consequently, $(x, y)_1, (x, y)_2 \in pr_1 A^{\leftarrow}(J(x, x))$. Similarly, $x = pr_2 A(x, y)_2 = pr_3 A(x, y)_1 \in J(x, y)$ and $y = pr_2 A(x, y)_1 = pr_3 A(x, y)_2 \in J(x, y)$, and it follows that $(x, y)_1, (x, y)_2 \in pr_k A^{\leftarrow}(J(x, y))$ for $k = 2, 3$.

By Lemma 2.1,

$$\begin{aligned} \overline{J}(m, n) &= pr_k A^{\leftarrow}(J(pr_k Am, pr_k An)) \cap \nabla^{\leftarrow}(J_{dis}(\nabla m, \nabla n)), \quad k = 1, 2, 3 \\ &= pr_k A^{\leftarrow}(J(pr_k Am, pr_k An)) \cap \{(x, y)_1, (x, y)_2\}, \quad k = 1, 2, 3 \\ &= \{(x, y)_1, (x, y)_2\}. \end{aligned}$$

In a similar way, if $m = (x, y)_2$ and $n = (x, y)_1$, then $\overline{J}(m, n) = \{(x, y)_1, (x, y)_2\}$.

Case III: Let $m \neq n$ and $\nabla m \neq \nabla n$. Note that

$$\nabla^{\leftarrow}(J_{dis}(\nabla m, \nabla n)) = \nabla^{\leftarrow}\{\nabla m, \nabla n\} = \{m, n\}$$

and $pr_k Am, pr_k An \in J(pr_k Am, pr_k An)$ for $k = 1, 2, 3$, and consequently, $m, n \in pr_k A^{\leftarrow}(J(pr_k Am, pr_k An))$. By Lemma 2.1, we have

$$\begin{aligned} \overline{J}(m, n) &= pr_k A^{\leftarrow}(J(pr_k Am, pr_k An)) \cap \nabla^{\leftarrow}(J_{dis}(\nabla m, \nabla n)), \quad k = 1, 2, 3 \\ &= \{m, n\}. \end{aligned}$$

Hence $\overline{J}$ is the discrete structure and by Definition 3.4 (ii), (X, J) is T_0. □

Theorem 3.3. *Every interval space* (X, J) *is* T_1.

Proof. The proof is similar to Theorem 3.2. So the proof is omitted. □

Theorem 3.4. *An interval space* (X, J) *is* $\mathbf{T_0}$ *iff* X *has at most one point.*

Proof. Suppose (X, J) is $\mathbf{T_0}$, $X \neq \emptyset$ and $X \neq \{a\}$. Then, there exists $b \in X$ with $a \neq b$. Let $M = \{a, b\}$ and J_M be an interval structure induced by the inclusion mapping $i : M \longrightarrow (X, J)$. By Lemma 2.1, $J_M(a, b) = i^{\leftarrow}(J(i(a), i(b))) = M \cap J(a, b) = M$, i.e., the indiscrete structure on M, which is a contradiction. Thus, X has at most one point.

Conversely, if $X = \emptyset$ or $X = \{a\}$, then clearly (X, J) is $\mathbf{T_0}$. □

Corollary 3.1. *Let* (X, J) *be an interval space. The following statements are equivalent:*

(1) (X, J) *is separated.*

(2) (X, J) *is* $\mathbf{T_0}$.

(3) X *has at most one point.*

Proof. The proof can be deduced from Theorems 3.1 and 3.4. □

Theorem 3.5. *An interval space* (X, J) *is pre-*T_2 *iff* (X, J) *is an indiscrete interval space.*

Proof. Suppose that (X, J) is pre-T_2. If $X = \emptyset$, $X = \{x\}$ or $X = \{x, y\}$, then $J_{dis} = J_{ind} = J$. Now, consider $Card X = 3$, i.e., $X = \{x, y, z\}$. Then, by Definition 2.3, X carries only discrete and indiscrete structures. Assume, on the contrary, that (X, J) is not an indiscrete interval space. It follows that for all $x, y \in X$ with $x \neq y$, $J(x, y) = \{x, y\}$. Let J_A and J_S be initial structures on $X^2 \bigvee_{\Delta} X^2$ induced by $A : X^2 \bigvee_{\Delta} X^2 \longrightarrow (X^3, J^3)$ and $S : X^2 \bigvee_{\Delta} X^2 \longrightarrow (X^3, J^3)$, respectively. Here, J^3 is the product structure on X^3. Also, pr_k is the projection mapping $pr_k : X^3 \longrightarrow X$ for $k = 1, 2, 3$. We show that (X, J) is not pre-T_2, i.e., $J_A(m, n) \neq J_S(m, n)$ for some $m, n \in X^2 \bigvee_{\Delta} X^2$.

Suppose $m = (x, y)_1$ and $n = (z, y)_2 \in X^2 \bigvee_{\Delta} X^2$ for all $x, y, z \in X$ with $x \neq y \neq z$. Note that

$A^{\leftarrow}(\{x, z\} \times X^2) = \{(x, x)_1 = (x, x)_2, (z, z)_1 = (z, z)_2, (x, y)_1, (x, y)_2, (x, z)_1, (x, z)_2, (z, x)_1, (z, x)_2, (z, y)_1, (z, y)_2\},$

$S^{\leftarrow}(\{x, z\} \times X^2) = \{(x, x)_1 = (x, x)_2, (z, z)_1 = (z, z)_2, (x, y)_1, (x, y)_2, (x, z)_1, (x, z)_2, (z, x)_1, (z, x)_2, (z, y)_1, (z, y)_2\},$

$A^{\leftarrow}(X \times \{y, z\} \times X) = \{(y, y)_1 = (y, y)_2, (z, z)_1 = (z, z)_2, (x, y)_1, (y, x)_2, (x, z)_1, (z, x)_2, (y, z)_1, (y, z)_2, (z, y)_1, (z, y)_2\},$

$S^{\leftarrow}(X \times \{y, z\} \times X) = \{(y, y)_1 = (y, y)_2, (z, z)_1 = (z, z)_2, (x, y)_1, (y, x)_2, (x, z)_1, (z, x)_2, (y, z)_1, (y, z)_2, (z, y)_1, (z, y)_2\},$

$A^{\leftarrow}(X^2 \times \{x, y\}) = \{(x, x)_1 = (x, x)_2, (y, y)_1 = (y, y)_2, (x, y)_1, (x, y)_2, (y, x)_1, (y, x)_2, (x, z)_1, (z, x)_2, (y, z)_1, (z, y)_2\}$

and

$$S^{\leftarrow}(X^2 \times \{y\}) = \{(y,y)_1 = (y,y)_2, (x,y)_1, (x,y)_2, (z,y)_1, (z,y)_2\}.$$

By Lemma 2.1,

$$
\begin{aligned}
J_A((x,y)_1, (z,y)_2) &= \bigcap_{k=1}^{3} pr_k A^{\leftarrow}(J(pr_k A(x,y)_1, pr_k A(z,y)_2)) \\
&= pr_1 A^{\leftarrow}(J(x,z)) \cap pr_2 A^{\leftarrow}(J(y,z)) \cap pr_3 A^{\leftarrow}(J(x,y)) \\
&= A^{\leftarrow}(pr_1^{\leftarrow}(J(x,z))) \cap A^{\leftarrow}(pr_2^{\leftarrow}(J(y,z))) \cap A^{\leftarrow}(pr_3^{\leftarrow}(J(x,y))) \\
&= A^{\leftarrow}(\{x,z\} \times X^2) \cap A^{\leftarrow}(X \times \{y,z\} \times X) \cap A^{\leftarrow}(X^2 \times \{x,y\}) \\
&= \{(x,y)_1, (x,z)_1, (z,x)_2, (z,y)_2\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_S((x,y)_1, (z,y)_2) &= \bigcap_{k=1}^{3} pr_k S^{\leftarrow}(J(pr_k S(x,y)_1, pr_k S(z,y)_2)) \\
&= pr_1 S^{\leftarrow}(J(x,z)) \cap pr_2 S^{\leftarrow}(J(y,z)) \cap pr_3 S^{\leftarrow}(J(y,y)) \\
&= S^{\leftarrow}(pr_1^{\leftarrow}(J(x,z))) \cap S^{\leftarrow}(pr_2^{\leftarrow}(J(y,z))) \cap S^{\leftarrow}(pr_3^{\leftarrow}(J(y,y))) \\
&= S^{\leftarrow}(\{x,z\} \times X^2) \cap S^{\leftarrow}(X \times \{y,z\} \times X) \cap S^{\leftarrow}(X^2 \times \{y\}) \\
&= \{(x,y)_1, (z,y)_1, (z,y)_2\}.
\end{aligned}
$$

Therefore, $J_A((x,y)_1, (z,y)_2) \neq J_S((x,y)_1, (z,y)_2)$, and consequently, (X,J) is not pre-T_2.

Now, consider $CardX > 3$. Assume, on the contrary, that (X,J) is not an indiscrete interval space. Then, there exists $M \subset X$ such that $J(x,y) = M$ for all $x,y \in X$ with $\{x,y\} \subset M \neq X$ and $x \neq y$. Then, there exists a point $z \in X$ but $z \neq M$ whenever $J(x,y) = M$ for all $x,y \in X$ with $x \neq y$. Similar to the above, consider $(z,y)_1 \in X^2 \bigvee_\Delta X^2$ for any $z, y \in X$ with $y \neq z$. Since $pr_1 S(z,y)_1 = z \in J(x,z)$, consequently, $(z,y)_1 \in pr_1 S^{\leftarrow}(J(x,z))$. Similarly, $pr_2 S(z,y)_1 = y \in J(y,z)$, and, consequently, $(z,y)_1 \in pr_2 S^{\leftarrow}(J(y,z))$ and $(z,y)_1 \in pr_3 S^{\leftarrow}(J(x,y))$. Thus, by Lemma 2.1, $(z,y)_1 \in J_S((x,y)_1, (z,y)_2)$ for any $(x,y)_1, (z,y)_2 \in X^2 \bigvee_\Delta X^2$. However $(z,y)_1 \notin J_A((x,y)_1, (z,y)_2)$ since $pr_3 A(z,y)_1 = z \notin J(x,y)$ and it follows that $(z,y)_1 \notin pr_3 A^{\leftarrow}(J(x,y))$. Thus, $J_A((x,y)_1, (z,y)_2) \neq J_S((x,y)_1, (z,y)_2)$. Consequently, an interval space (X,J) is not pre-T_2.

Conversely, let (X,J) be an indiscrete interval space. Then, by Remark 3.1 (5), (X,J) is pre-T_2. $\quad\square$

Theorem 3.6. *An interval space (X,J) is T_2 iff (X,J) is an indiscrete interval space.*

Proof. The proof follows from Theorems 3.2 and 3.5. $\quad\square$

Remark 3.2. *(1) In **O-REL** (the category of ordered relative spaces and relative mappings) [27] as well as in **b-UFIL** (the category of b-UFIL spaces and buc mappings) [27, 28], $T_1 \implies T_0 \implies$* **$T_0$** *[22, 34].*

*(2) In $\mathcal{V}$-**Cls** (the category of $\mathcal{V}$-closure spaces and continuous mappings) with $\mathcal{V}$ as an integral quantale [26], $T_2 = T_1 \implies T_0 \implies$* **$T_0$** *[33].*

(3) In **Born** (the category of bornological spaces and bounded mappings), all objects are T_0, T_1 and T_2 [4], and X is separated or $\mathbf{T_0}$ iff X is either empty or a singleton [4]. However, in **Prox** (the category of proximity spaces and proximity mappings), all objects are not T_0, T_1 and T_2 but they are all equal [24].

(4) In **IS**, by Theorems 3.2, 3.3 and 3.6, and Corollary 3.1, we conclude that $\mathbf{T_0} \implies T_0 = T_1$ and $T_2 \implies T_0 = T_1$ but the converse is not true in general.

Corollary 3.2. *An interval space (X, J) is NT_2 iff (X, J) is ST_2 iff X has a cardinality 1.*

Proof. It follows from Theorems 3.1, 3.4 and 3.5. □

4. Notion of closedness and D-connectedness in interval spaces

Let X be any set and $p \in X$. Let the *infinite wedge product of X at p* be the infinitely countable disjoint copies of X identifying at p and denoted by $\bigvee_p^\infty X$.

For a point $x \in \bigvee_p^\infty X$, we write it as x_j if it belongs to the jth component of the infinite wedge product.

Definition 4.1. *(cf. [2]) Let $X^\infty = X \times X \times X \times ......$ be the countable Cartesian product of X.*

(1) *The mapping $A_p^\infty : \bigvee_p^\infty X \longrightarrow X^\infty$ is said to be an infinite principal p-axis mapping provided that*

$$A_p^\infty(x_j) = (p, p, \cdots, p, \underbrace{x}_{j^{th}\ place}, p, \cdots), \quad \forall j \in I.$$

(2) *The mapping $\nabla_p^\infty : \bigvee_p^\infty X \longrightarrow X$ is said to be an infinite fold mapping at p provided that*

$$\nabla_p^\infty(x_j) = x, \quad \forall j \in I.$$

Definition 4.2. *(cf. [2,3]) Let $\mathcal{U} : \mathcal{E} \longrightarrow \mathbf{Set}$ be a topological functor and $X \in Obj(\mathcal{E})$ with $\mathcal{U}(X) = Y$ and $p \in Y$. $\{p\}$ is closed provided that the initial lift of the $\mathcal{U}$-source $\{\bigvee_p^\infty Y \xrightarrow{A_p^\infty} \mathcal{U}X^\infty = Y^\infty$ and $\bigvee_p^\infty Y \xrightarrow{\nabla_p^\infty} \mathcal{U}DY = Y\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is left adjoint of $\mathcal{U}$.*

Remark 4.1. *In **Top**, the closedness of $\{p\}$ reduces to the usual closedness of the singleton set $\{p\}$ [2,3]. Also, for any $X \in obj(\mathbf{Top})$, X is T_1 iff all points of X are closed. However, in an arbitrary topological category, this is not true in general [3].*

Theorem 4.1. *Every singleton set $\{p\}$ in an interval space (X, J) is closed.*

Proof. Let (X, J) be an interval space, $p \in X$. We show that $\{p\}$ is closed. Let $\bar{J}$ be an initial structure on $\bigvee_p^\infty X$ induced by $A_p^\infty : \bigvee_p^\infty X \to (X^\infty, J^\infty)$ and $\nabla_p^\infty : \bigvee_p^\infty X \to (X, J_{dis})$, where J^∞ and J_{dis} are the product interval structures and discrete interval structures on X^∞ and X, respectively. Let $m, n \in \bigvee_p^\infty X$.

If $m = n$, then $\nabla_p^\infty m = \nabla_p^\infty n$ and also $pr_k A_p^\infty m = pr_k A_p^\infty n$ for $k \in I$. Here, pr_k are the projection mappings $pr_k : X^\infty \to X$, where $k \in I$.

By Lemma 2.2 (1),

$$\nabla_p^{\infty\leftarrow}(J_{dis}(\nabla_p^\infty m, \nabla_p^\infty n)) = \nabla_p^\leftarrow(J_{dis}(\nabla_p^\infty m, \nabla_p^\infty m))$$

and it follows that $\nabla_p^{\infty\leftarrow}(\{\nabla_p^{\infty}m\}) = \{m\}$ and

$$pr_kA_p^{\infty\leftarrow}(J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}n)) = pr_kA_p^{\infty\leftarrow}(J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}m)), \quad \forall k \in I.$$

Since $pr_kA_p^{\infty}m \in J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}m)$ for each $k \in I$, consequently, $m \in pr_kA_p^{\infty\leftarrow}(J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}m))$.
By Lemma 2.1,

$$
\begin{aligned}
\bar{J}(m, m) &= pr_kA_p^{\infty\leftarrow}(J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}m)) \cap \nabla_p^{\infty\leftarrow}(J_{dis}(\nabla_p^{\infty}m, \nabla_p^{\infty}m)), \quad k \in I \\
&= pr_kA_p^{\infty\leftarrow}(J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}m)) \cap \{m\}, \quad k \in I \\
&= \{m\}.
\end{aligned}
$$

Let $m \neq n$ and $\nabla_p^{\infty}m = \nabla_p^{\infty}n$. If $\nabla_p^{\infty}m = p = \nabla_p^{\infty}n$, consequently, $m = (p, p, p, ..., p, ...) = p_i = p_j = n$ for all $i, j \in I$, which is a contradiction.

Suppose $\nabla_p^{\infty}m = x = \nabla_p^{\infty}n$, so it follows easily that $m = x_i$ and $n = x_j$ for some $i, j \in I$ with $i \neq j$. Note that

$$J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}n) = J(pr_kA_p^{\infty}x_i, pr_kA_p^{\infty}x_j) = \begin{cases} J(x, p), & if\ k = i \\ J(p, x), & if\ k = j \\ J(p, p), & if\ k \notin \{i, j\}. \end{cases}$$

Since $x = pr_kA_p^{\infty}m \in J(x, p)$, consequently, $m \in pr_kA_p^{\infty\leftarrow}(J(x, p))$ for $k = i$ or $k = j$, and $p = pr_kA_p^{\infty}n \in J(x, p)$ for any $k \in I$; it follows that $n \in pr_kA_p^{\infty\leftarrow}(J(x, p))$. Thus, $m, n \in pr_kA_p^{\infty\leftarrow}(J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}n))$ for any $k \in I$. On the other hand,

$$\nabla_p^{\infty\leftarrow}(J_{dis}(\nabla_p^{\infty}x_i, \nabla_p^{\infty}x_j)) = \nabla_p^{\infty\leftarrow}\{x\} = \{x_i, x_j\} = \{m, n\}.$$

By Lemma 2.1,

$$
\begin{aligned}
\bar{J}(m, n) &= pr_kA_p^{\infty\leftarrow}(J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}n)) \cap \nabla_p^{\infty\leftarrow}(J_{dis}(\nabla_p^{\infty}m, \nabla_p^{\infty}n)), \quad k \in I \\
&= pr_kA_p^{\infty\leftarrow}(J(pr_kAm, pr_kAn)) \cap \{m, n\} = \{m, n\}.
\end{aligned}
$$

Suppose that $m \neq n$ and $\nabla_p^{\infty}m \neq \nabla_p^{\infty}n$.

By Lemma 2.2 (1), $\nabla_p^{\infty\leftarrow}(J_{dis}(\nabla_p^{\infty}m, \nabla_p^{\infty}n)) = \nabla_p^{\leftarrow}(\{\nabla_p^{\infty}m, \nabla_p^{\infty}n\}) = \{m, n\}$ and $m, n \in J(pr_kA_p^{\infty}m, pr_kA_p^{\infty}n)$ for any $k \in I$. By Lemma 2.1, $\bar{J}(m, n) = \{m, n\}$, which is a discrete structure. Thus, by Definition 4.2, $\{p\}$ is closed. $\square$

Definition 4.3. *(cf. [8, 38]) Let $\mathcal{U} : \mathcal{E} \longrightarrow$ **Set** be a topological functor and $X \in Obj(\mathcal{E})$. X is said to be D-connected provided that any morphism from X to any discrete object is constant.*

Remark 4.2. *In **Top**, the D-connectedness reduces to the usual connectedness [8, 38].*

Theorem 4.2. *An interval space (X, J) is D-connected iff there exists a proper subset N of X such that $\{x, y\} \subset J(x, y)$ for some $x \in N$ and $y \in N^c$.*

Proof. Let (X, J) be D-connected and there exists a nonempty subset N of X, $J(x, y) = \{x, y\}$ for all $x \in N$ and $y \in N^c$. Suppose (Y, J_{dis}) is the discrete interval space with $CardY > 1$. Define the mapping $f : (X, J) \to (Y, J_{dis})$ by

$$f(x) = \begin{cases} a, & x \in N \\ b, & x \notin N. \end{cases}$$

Let $x, y \in X$. If $x, y \in N$ then

$$f^{\rightarrow}(J(x, y)) = f^{\rightarrow}(\{x, y\}) = \{f(x), f(y)\} = \{a\}$$

and

$$J_{dis}(f(x), f(y)) = \{f(x), f(y)\} = \{a\},$$

and consequently,

$$f^{\rightarrow}(J(x, y)) = J_{dis}(f(x), f(y)).$$

Thus f is an interval preserving mapping. Similarly, if $x, y \in N^c$, then f is also an interval preserving mapping.

Now, let $x \in N$ and $y \in N^c$ (resp. $y \in N$ and $x \in N^c$). Note that

$$f^{\rightarrow}(J(x, y)) = \{f(t) \mid t \in J(x, y) = \{x, y\}\} = \{f(x), f(y)\} = \{a, b\}$$

and $J_{dis}(f(x), f(y)) = \{f(x), f(y)\} = \{a, b\}$. Thus, $f^{\rightarrow}((J(x, y)) = J_{dis}(f(x), f(y))$. Hence, f is an interval preserving mapping, but it is not constant, which is a contradiction.

Conversely, suppose that the condition holds. Let (Y, J_{dis}) be a discrete interval space and $f : (X, J) \to (Y, J_{dis})$ be an interval preserving mapping.

If $Card Y = 1$, then f is constant. Suppose that $Card Y > 1$, and f is not constant. Then there exist $x, y \in X$ with $x \neq y$ such that $f(x) \neq f(y)$ and let $N = f^{\leftarrow}\{f(x)\}$. Note that N is a proper subset of X. By our assumption $\{x, y\} \subset J(x, y)$ for some $x \in N$ and $y \notin N$, we have

$$\{f(x), f(y)\} = f^{\rightarrow}(\{x, y\}) \subset f^{\rightarrow}((J(x, y)) \subseteq J_{dis}(f(x), f(y)).$$

By Lemma 2.2 (1), it follows that f is not an interval-preserving mapping, which is a contradiction. Thus f must be constant and, by Definition 4.3, (X, J) is D-connected. $\square$

Theorem 4.3. *Let (X, J_X) and (Y, J_Y) be interval spaces, and let $f : (X, J_X) \longrightarrow (Y, J_Y)$ be an interval preserving mapping. If (X, J_X) is D-connected and f is surjective, then (Y, J_Y) is D-connected.*

Proof. Let $f(x), f(y) \in f(X)$ with $f(x) \neq f(y)$. Since f is an interval preserving mapping, it follows that $f^{\rightarrow}(J_X(x, y)) \subseteq J_Y((f(x), f(y))$. The assumption that there exists a proper subset N of X such that $\{x, y\} \subset J(x, y)$ for some $x \in N$ and $y \notin N$ implies that

$$\{f(x), f(y)\} = f^{\rightarrow}(\{x, y\}) \subset f^{\rightarrow}(J_X(x, y)) \subseteq J_Y((f(x), f(y)),$$

and consequently, $\{f(x), f(y)\} \subset J_Y((f(x), f(y))$ for some $f(x) \in f(N)$ and $f(y) \notin f(N)$. Therefore, $f(X)$ is D-connected. Since f is surjective, it follows that $f(X) = Y$ is D-connected. $\square$

5. Zero-dimensionality in interval spaces

In 1997, Stine [36] gave an alternative characterization of the zero-dimensional space (X, τ) that is, (X, τ) is a zero-dimensional space provided that for all $i \in I$, there exists a family $(X_i, \tau_{i_{dis}})$ and there exists $f_i : (X, \tau) \longrightarrow (X_i, \tau_{i_{dis}})$ such that τ is the initial topology by $(X_i, \tau_{i_{dis}})$ via f_i, where $(X_i, \tau_{i_{dis}})$ is the family of discrete topological spaces. Considering the categorical counterparts, we have the following definition, as given in [37].

Definition 5.1. *(cf. [37]) Let* $\mathcal{U} : C \longrightarrow \mathcal{E}$ *be a topological and* $\mathcal{D} : \mathcal{E} \longrightarrow C$ *be a discrete functor. Any object* $X \in Obj(C)$ *is called a zero-dimensional object provided that for all* $i \in I$, *there exists* $A_i \in Obj(\mathcal{E})$ *and the morphisms* $f_i : \mathcal{U}(X) \longrightarrow A_i$ *such that* $(\overline{f}_i : X \longrightarrow \mathcal{D}(A_i))_{i \in I}$ *is the initial lift of* $(f_i : \mathcal{U}(X) \longrightarrow \mathcal{U}(\mathcal{D}(A_i)) = A_i)_{i \in I}$.

Remark 5.1. *(1) For* $C = \textbf{Top}$ *and* $\mathcal{E} = \textbf{Set}$, *by Theorem 4.3.1 of [36], Definition 5.1 reduces to the usual zero-dimensional topological space.*

(2) If $\mathcal{U} : C \longrightarrow \mathcal{E}$ *is a normalized topological functor, by Theorem 4.3.4 and 5.3.1 of [37], then every indiscrete object in* C *is a zero-dimensional object.*

Theorem 5.1. *Every discrete and indiscrete interval space* (X, J) *is zero-dimensional.*

Proof. Suppose (X, J) is an interval space and $X = \{x\}$ or $X = \{x, y\}$. Then $J_{dis} = J_{ind} = J$. By Remarks 2.1 and 5.1, it is zero-dimensional.

Let $CardX \geq 3$, and $J = J_{dis}$. Consider $f_i(x) = x$ (identity mapping) and $X = X_i$ for all $i \in I$. Clearly, $f_i : X \longrightarrow X_i$ is an interval preserving mapping and $f_i : X \longrightarrow X_i$ is the initial lift of $f_i : (X, J) \longrightarrow (X_i, J_{idis})$. Thus, by Definition 5.1, (X, J) is zero-dimensional.

Now, let $J = J_{ind}$ and take $f_i(x) = c$ (constant mapping) for all $i \in I$. Clearly, $f_i : X \longrightarrow X_i$ is an interval preserving mapping which is the initial lift of $f_i : (X, J) \longrightarrow (X_i, J_{idis})$. Therefore, by Definition 5.1, (X, J) is zero-dimensional. □

Corollary 5.1. *Every interval space (except for a discrete interval space) is D-connected.*

Corollary 5.2. *Every D-disconnected (not D-connected) interval space with cardinality greater than 2 is zero-dimensional.*

Proof. Let (X, J) be a D-disconnected interval space with cardinality greater than 2. By Theorem 4.2, for all $x, y \in X$ with $x \neq y$, $J(x, y) = \{x, y\}$ and consequently, (X, J) is discrete. Thus, by Theorem 5.1, (X, J) is zero-dimensional. □

6. Conclusions

First, we characterized separated, $\textbf{T}_0$, T_0, T_1, pre-T_2, T_2, NT_2 and ST_2 interval spaces and showed that separated = $\textbf{T}_0 \implies T_0 = T_1$ and $T_2 \implies T_0 = T_1$ but the converse is not true in general. Also, we proved that in any interval space with cardinality at most one point, $NT_2 = ST_2$. Further, we showed that every singleton set is closed and every interval space (except for a discrete interval space) is D-connected. Finally, we characterized zero-dimensionality in interval spaces and showed that every discrete and indiscrete interval space is zero-dimensional. Considering these results, the followings can be treated as open research problems:

(1) How can one characterize sobriety, ultraconnectedness and irreducibility in the category **IS**?

(2) Can one characterize pre-T_2, zero-dimensionality and separatedness for quantale generalization of interval spaces, and what would be their relation to the classical ones?

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Conflict of interest

The authors declare that they have no conflicts of interest.

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