Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 3747–3762.
DOI: 10.3934/math.2023187
Received: 03 September 2022
Revised: 04 November 2022
Accepted: 20 November 2022
Published: 28 November 2022

Research article

A characterization and implementation of corank one map germs from 2-space to 3-space in the computer algebra system SINGULAR

Ying Wang[1,2], **Muhammad Ahsan Binyamin**[3,*], **Tauqeer Iqbal**[3], **Saima Aslam**[3] **and Adnan Aslam**[4]

[1] Software Engineering Institute of Guangzhou, Guangzhou 510980, China
[2] Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China
[3] Department of Mathematics, GC University, Faisalabad, Pakistan
[4] Department of Natural Sciences and Humanities, University of Engineering and Technology, Lahore (RCET), Pakistan

* **Correspondence:** Email: ahsanbanyamin@gmail.com.

Abstract: The classification and the geometry of corank one map germs from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ have been studied by Mond [1, 2]. In this paper we characterize the classification of map germs of corank at most 1, in terms of certain invariants. Moreover, by using this characterization, we develop an algorithm to compute the type of map germs with out computing the normal form. Also, we give its implementation in the computer algebra system SINGULAR [15].

Keywords: corank 1; $\mathcal{A}$-equivalence; $\mathcal{A}$-codimension
Mathematics Subject Classification: 58Q05,14H20

1. Introduction

It is very interesting to discuss the recognition problem for the classification of singularities due to some equivalence relations. Classification for map germs under some equivalence relation, means finding a list of map germs and showing that all map germs satisfying certain conditions are equivalent to a map germ in the list. Recognition means finding some criteria that describe when a given map germ is equivalent to a map germ belonging to some class or list. Classification and recognition of singularities of map germs are well understood terms and have been subjects of a large number of investigations in the literature [1, 3–14].

Throughout this paper we deal with $\mathcal{A}$-finite map germs from plane to space of corank at most one at origin. Any map germ f from $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ has a corank one if its Jacobian matrix at 0 has rank equal to $\min(n, p) - 1$. Let m_2 be a maximal ideal in the ring of holomorphic function germ in two variables, denoted by O_2, and $p, q \in m_2^2$ be function germs. A map germ having corank at most 1 is $\mathcal{A}$-equivalent to $(x, p(x, y), q(x, y))$. Mond [1, 2] studied the geometry and $\mathcal{A}$-classification of corank one map germs from surfaces to 3-spaces. He classified the map germs of corank at most 1 from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ having codimension less than or equal to 6. In [2], three analytic invariants $C(f), T(f)$ and $N(f)$ are associated to a map germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^3, 0)$, and together they constitute a complete set of invariants for the $\mathcal{A}$-classification of map germs in Table 1 of [1]. In this paper, we present extensive results to characterize the $\mathcal{A}$-finite map germs of corank at most one given in Table 1 of [2].

2. Preliminaries and some invariants

We use the notation $\mathbb{C}$ to denote the set of complex numbers. Let $A(2,3) =< x, y > \mathbb{C}[[x, y]]^3$ be the set of map germs from plane to space at origin. Let $\mathcal{A} = Aut_{\mathbb{C}}(\mathbb{C}^2, 0) \times Aut_{\mathbb{C}}(\mathbb{C}^3, 0)$. Then, we have a canonical action of $\mathcal{A}$ on $A(2, 3)$ defined by

$$\mathcal{A} \times A(2,3) \to A(2,3)$$

satisfying $((\varphi, \psi), f) \mapsto \psi \circ f \circ \varphi^{-1}$.

Definition 2.1. Let $f_1, f_2 \in A(2, 3)$. Then, f_1 is said to be $\mathcal{A}$-equivalent to f_2, if they lie in the same orbit under the action of $\mathcal{A}$, i.e., there exist two diffeomorphisms, $\psi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ and $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, satisfying $\psi \circ f_1 = f_2 \circ \varphi$. We use the notation $f_1 \sim_{\mathcal{A}} f_2$, if f_1 is $\mathcal{A}$-equivalent to f_2.

Definition 2.2. Let $f \in A(2, 3)$. The orbit map $\theta_f : \mathcal{A} \to A(2, 3)$ is defined as $\theta_f(\varphi, \psi) = \psi \circ f \circ \varphi^{-1}$. Exceptionally, we have $\theta_f(id) = f$. Let $\mathcal{A}_f := Im(\theta_f)$, and then the orbit of f under the action of $\mathcal{A}$ is the image of θ_f. The extended tangent space to the orbit at f, $\mathcal{T}_{\mathcal{A}^e_f, f}$, is defined as

$$\mathcal{T}_{\mathcal{A}^e_f, f} =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >_{\mathbb{C}[[x,y]]} + \mathbb{C}[[f_1, f_2, f_3]]^3.$$

Definition 2.3. Let $f \in A(2, 3)$ with tangent space $\mathcal{T}_{\mathcal{A}^e_f, f}$. The extended $\mathcal{A}$-codimension at f is defined as:

$$c_e(f) = cod_{\mathcal{A}}{}^e(f) := dim_{\mathbb{C}} \frac{\mathcal{A}(2,3)}{\mathcal{T}_{\mathcal{A}^e_f, f}}.$$

$c_e(f)$ exist with the restriction that f should be $\mathcal{A}$-finite and has been implemented in the computer algebra system SINGULAR (see library "`classifyMapGerms.lib`").

Lemma 2.4. *Let f be a map germ from the plane to space at origin. If the corank of f is at most* 1 *at* 0*, then $f \sim_{\mathcal{A}} (x, p(x, y), q(x, y))$ with $p(x, 0) = 0, q(x, 0) = 0$.*

Definition 2.5. Let f be a map germ. Then, the multiplicity $m(f)$ of f is defined by

$$m(f) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{f^* m_3 \mathbb{C}[[x, y]]}.$$

If $f \sim_{\mathcal{A}} (x, p(x, y), q(x, y))$, then

$$m(f) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{< x, p, q >}.$$

Definition 2.6. Let f be a map germ from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of corank 1 at 0. We define $C(f)$ to be the number of crosscaps which appear on the image of a stable perturbation. If $f \sim_{\mathcal{A}} (x, p(x, y), q(x, y))$, then (cf. [16])

$$C(f) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{< \frac{\partial p}{\partial y}, \frac{\partial q}{\partial y} >}.$$

Definition 2.7. Let f be a map germ from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of corank 1 at 0. We define $T(f)$ to be the number of triple points which appear on the image of a stable perturbation. If $f \sim_{\mathcal{A}} (x, p(x, y), q(x, y))$, then (cf. [16])

$$T(f) = \frac{1}{6} dim_{\mathbb{C}} \frac{O_4}{I_3(f)},$$

where $I_3(f)$ is the ideal in $\mathbb{C} \times \mathbb{C}^3 (= \{(x, y, u, v) : x, y, u, v \in \mathbb{C}\}$ generated by the following 4 functions:

$$\frac{p(x,y) - p(x,u)}{y - u}, \frac{q(x,y) - q(x,u)}{y - u}, \frac{1}{u - v}\{\frac{p(x,y) - p(x,u)}{y - u} - \frac{p(x,y) - p(x,v)}{y - v}\},$$

$$\frac{1}{u - v}\{\frac{q(x,y) - q(x,u)}{y - u} - \frac{q(x,y) - q(x,v)}{y - v}\}.$$

3. Characterization of map germs from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^3, 0)$ of corank at most 1

In this section we characterize simple map germs of corank 1 in terms of numerical invariants. Table 1 depicts all map germs from plane to space of corank at most 1, codimension ≤ 6 at origin, plus some others (e.g, S_k, B_k, C_k and H_k, where $k > 6$). The following definition can be found in [1].

Definition 3.1. A map germ f from $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is said to be $\mathcal{A}$-simple if there is a finite number of equivalence classes such that if f is embedded in any family $F : (\mathbb{C}^2 \times p, (0, p_0)) \to (\mathbb{C}^3, 0)$, then, for every $(0, p)$ in a sufficiently small neighbourhood of $(0, p_0)$, the germ of $F(x, y, p)$ lies in one of these equivalence classes.

Proposition 3.2. *The map germs f from plane to space of corank ≤ 1 at origin of codimension ≤ 6, plus some others (e.g, S_k, B_k, C_k and H_k, where $k > 6$), are given in Table 1.*

Proof. For the proof, see [1, 2]. □

Proposition 3.3. *Let f be a map germ from plane to space of corank 1 at origin. Then, $j^2 f$ is $\mathcal{A}$-equivalent to one of the following type:*

$$(x, y^2, xy), (x, y^2, 0), (x, xy, 0), (x, 0, 0).$$

Proof. For the proof, see Proposition 4:2 [1]. □

Table 1. Map germs from plane to space at origin.

Type	Normal form	Conditions
S	$(x, y, 0)$	–
S_0	(x, y^2, xy)	–
S_k	$(x, y^2, y^3 + x^{k+1}y)$	$k \geq 1$
B_k	$(x, y^2, x^2y + y^{2k+1})$	$k \geq 2$
C_k	$(x, y^2, xy^3 + x^ky)$	$k \geq 3$
F_4	$(x, y^2, x^3y + y^5$	–
H_k	$(x, xy + y^{3k-1}, y^3)$	$k \geq 2$
P_3	$(x, xy + y^3, xy^2 + cy^4)$	$c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$
$P_4(\frac{1}{2})$	$(x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^5)$	–
$P_4(\frac{3}{2})$	$(x, xy + y^3, xy^2 + \frac{3}{2}y^4 + y^5)$	–
$P_4(1)$	$(x, xy + y^3, xy^2 + y^4 \pm y^6)$	–
Q_k	$(x, xy + y^3, xy^2 + y^{3k-5})$	$k \geq 4$
R_4	$(x, xy + y^6 + by^7, xy^2 + y^4 + cy^6)$	–
T_4	$(x, xy + y^3, y^4)$	–
X_4	$(x, y^3, x^2y + xy^2 + y^4)$	–

The extended $\mathcal{A}$-codimension, number of crosscaps and number of triple points play an important role in the characterization of map germs from plane to space. These invariants are given in Table 2.

Table 2. The Invariants used for the characterization of the classification of map germs given in Table 1.

Type	$\mathcal{A}_e - codimension = c_e(f)$	$C(f)$	$T(f)$
S	0	0	0
S_0	0	1	0
S_k	k	$k+1$	0
B_k	k	2	0
C_k	k	k	0
F_4	4	3	0
H_k	k	2	$k-1$
P_3	4	3	1
$P_4(\frac{1}{2})$	4	3	1
$P_4(\frac{3}{2})$	4	4	1
$P_4(1)$	4	3	2
Q_k	k	3	$k-2$
R_4	6	3	4
T_4	4	3	1
X_4	4	4	1

In the following propositions, we characterize the 2-jets of the map germ f from plane to space with corank ≤ 1 at origin in terms of number of crosscaps, $C(f)$, and multiplicity, $m(f)$.

Proposition 3.4. *Let f be a map germ from plane to space with corank 0 at origin. If $C(f) = 0$, then $j^2 f$ is of type $(x, y, 0)$.*

Proof. Let $f = (x, p(x, y), q(x, y))$. If

$$C(f) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{< \frac{\partial p}{\partial y}, \frac{\partial q}{\partial y} >} = 0,$$

then $\frac{\partial p}{\partial y}$ or $\frac{\partial q}{\partial y}$ is a unit. We may assume that $\frac{\partial p}{\partial y}$ is a unit. Using the inverse function theorem to the map $\mathbb{C}^2 \to \mathbb{C}^2$ defined by (x, p), we may assume that $p(x, y) = y$. Then, obviously, $(x, y, q(x, y)) \sim_{\mathcal{A}} (x, y, 0)$. □

Lemma 3.5. *Let f be a map germ from plane to space with corank 1 at origin.*

(1) If $C(f) = 1$, then $m(f) = 2$.
(2) If $m(f) > 2$, then $C(f) > 2$.

Proof. We may assume that $f = (x, p(x, y), q(x, y))$ with

$$p(x, y) = b_{01} y + \sum_{\substack{i+j \geq 2 \\ i \geq 0, j \geq 1}} b_{i,j} x^i y^j,$$

and

$$q(x, y) = c_{01} y + \sum_{\substack{i+j \geq 2 \\ i \geq 0, j \geq 1}} c_{i,j} x^i y^j.$$

If $b_{01} \neq 0$ or $c_{01} \neq 0$, then $\frac{\partial p}{\partial y}$ resp. $\frac{\partial q}{\partial y}$ is a unit. This implies

$$C(f) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{< \frac{\partial p}{\partial y}, \frac{\partial q}{\partial y} >} = 0.$$

Since $C(f) = 1$, therefore we have $< \frac{\partial p}{\partial y}, \frac{\partial q}{\partial y} > = < x, y >$. This implies that $b_{02} \neq 0$ or $c_{02} \neq 0$. Assume that $b_{02} \neq 0$, and then $p(0, y) = b_{02} y^2 +$ terms of higher order. This implies that

$$m(f) = dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{< x, p, q >} = 2.$$

The second statement can be proved similarly. □

Proposition 3.6. *Let f be a map germ from plane to space with corank 1 at origin. Then,*

(1) if $C(f) = 1$, then type of $j^2 f$ is (x, y^2, xy);
(2) if $C(f) \geq 2$ and $m(f) = 2$, then type of $j^2 f$ is $(x, y^2, 0)$;
(3) if $C(f) \geq 2$ and $m(f) > 2$, then type of $j^2 f$ is $(x, xy, 0)$ or $(x, 0, 0)$. *

*If $C(f) > 1$, then the image of a stable perturbation is not a cone on its boundary. Thus, the stable perturbation is not equivalent to a representative of the original germ, and hence the original germ was not stable.

Proof. Let f be a map germ from plane to space with corank 1 at origin. Then, f can be written as

$$f(x,y) \sim_{\mathcal{A}} (x, \sum_{i+j\geq 1} b_{i,j}x^iy^j, \sum_{i+j\geq 1} c_{i,j}x^iy^j).$$

By using the left coordinate change $Y_1 = Y - b_{i0}X, Z_1 = Z - c_{i0}X$, we get

$$f(x,y) \sim_{\mathcal{A}} (x, b_{01}y + \sum_{\substack{i+j\geq 2\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j, c_{01}y + \sum_{\substack{i+j\geq 2\\ i\geq 0, j\geq 1}} c_{i,j}x^iy^j).$$

(1) If $m(f) = 2$, then we may assume that $b_{01} = c_{01} = 0$ and $b_{02} \neq 0$. By using the left coordinate change $Z_2 = Z_1 - \frac{c_{02}}{b_{02}}Y_1$, we get

$$f(x,y) \sim_{\mathcal{A}} (x, b_{11}xy + b_{02}y^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} d_{i,j}x^iy^j, \alpha xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} e_{i,j}x^iy^j),$$

where, $\alpha = c_{11} - \frac{b_{11}c_{02}}{b_{02}}$. Now, if $C(f) = 1$, then $\begin{vmatrix} b_{11} & b_{02} \\ c_{11} & c_{02} \end{vmatrix}; \neq 0$ otherwise, $C(f) \neq 1$. This gives $c_{11} \neq \frac{b_{11}c_{02}}{b_{02}}$ and therefore $\alpha \neq 0$. Now, by using the left coordinate change $Y_3 = Y_2 - \frac{b_{11}}{\alpha}Z_2$, we get

$$f(x,y) \sim_{\mathcal{A}} (x, b_{02}y^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} h_{i,j}x^iy^j, \alpha xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} e_{i,j}x^iy^j).$$

This implies j^2f is of type (x, y^2, xy).

(2) If $m(f) = 2$, then from (1) we get

$$f(x,y) \sim_{\mathcal{A}} (x, b_{11}xy + b_{02}y^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j, (c_{11} - \frac{b_{11}c_{02}}{b_{02}})xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} d_{i,j}x^iy^j).$$

Now, if $C(f) \geq 2$, then $\begin{vmatrix} b_{11} & b_{02} \\ c_{11} & c_{02} \end{vmatrix}; = 0$ otherwise, $C(f) < 2$. This gives $c_{11} = \frac{b_{11}c_{02}}{b_{02}}$, so we have

$$f(x,y) \sim_{\mathcal{A}} (x, b_{11}xy + b_{02}y^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j, \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} d_{i,j}x^iy^j).$$

This can be written as

$$f(x,y) \sim_{\mathcal{A}} (x, (y + \frac{b_{11}}{2}x)^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j, \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} d_{i,j}x^iy^j).$$

By using the transformation $y \to y - \frac{b_{11}}{2}x$, we get

$$f(x,y) \sim_{\mathcal{A}} (x, y^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j, \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} d_{i,j}x^iy^j),$$

and this gives that j^2f is of type $(x, y^2, 0)$.

(3) If $m(f) > 2$, then we may assume that $b_{01} = c_{01} = b_{02} = c_{02} = 0$, and thus

$$f(x,y) \sim_{\mathcal{A}} (x, b_{11}xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j, c_{11}xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} c_{i,j}x^iy^j).$$

If $b_{11} \neq 0$, then, the left coordinate change $Z_1 = Z - \frac{c_{11}}{b_{11}}Y$ gives,

$$f(x,y) \sim_{\mathcal{A}} (x, b_{11}xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} b_{i,j}x^iy^j, \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} d_{i,j}x^iy^j).$$

This implies that, j^2f is of type $(x, xy, 0)$. If $c_{11} = b_{11} = 0$, then j^2f is of type $(x, 0, 0)$.

□

Proposition 3.7. *Let f be a map germ from plane to space of corank 0 at origin. If $j^2f \sim_{\mathcal{A}} (x, y, 0)$, then f is of type S.*

Proof. This can be proved in a similar way as we proved Proposition 3.4. □

Proposition 3.8. *Let f be a map germ from plane to space with corank 1 at origin and $j^2f \sim_{\mathcal{A}} (x, y^2, xy)$. Then, f is of type $S_\circ$.*

Proof. If $j^2f \sim_{\mathcal{A}} (x, y^2, xy)$, then

$$f(x,y) \sim_{\mathcal{A}} (x, y^2 + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} h_{i,j}x^iy^j, xy + \sum_{\substack{i+j\geq 3\\ i\geq 0, j\geq 1}} e_{i,j}x^iy^j).$$

Since f is 2-determined (see Theorem 4 : 3 [1]),

$$f(x,y) \sim_{\mathcal{A}} (x, y^2, xy).$$

□

Proposition 3.9. *Let f be a map germ from plane to space with corank 1 at origin, $j^2f \sim_{\mathcal{A}} (x, y^2, 0)$, and $C(f) = 2$. Then,*

(1) if $c_e(f) = 1$, then type of f is S_1; †
(2) if $c_e(f) = k$, $k \geq 2$, then type of f is B_k

Proof. If $j^2f \sim_{\mathcal{A}} (x, y^2, 0)$, then by Theorem 4.1 : 1 [1],

$$f(x,y) \sim_{\mathcal{A}} (x, y^2, \sum_{i+j\geq 1} a_{i,j}x^iy^{2j+1})).$$

We can write it as

$$f(x,y) \sim_{\mathcal{A}} (x, y^2, a_{0,1}y^3 + a_{2,0}x^2y + a_{1,1}xy^3 + a_{0,2}y^5 + \sum_{i+j\geq 3} a_{i,j}x^iy^{2j+1})).$$

†Proposition 4.1 : 16(i) of [1] implies that $c_e(f) = 1$ implies that f is of type S_1. Here, the assumption $C(f) = 2$ is not needed.

If $C(f) = 2$, then $a_{2,0} \neq 0$, otherwise, $C(f) \neq 2$. Take $a_{2,0} = 1$, and then

$$f(x, y) \sim_{\mathcal{A}} (x, y^2, x^2y + a_{0,1}y^3 + a_{1,1}xy^3 + a_{0,2}y^5 + \sum_{i+j\geq 3} a_{i,j}x^i y^{2j+1})).$$

(1) If $c_e(f) = 1$, then $a_{0,1} \neq 0$, otherwise, $c_e(f) \neq 1$. Since f is 3-determined, $f(x, y) \sim_{\mathcal{A}} (x, y^2, y^3 + x^2y)$.

(2) If $c_e(f) \geq 2$, then $a_{0,1} = 0$. This gives,

$$f(x, y) \sim_{\mathcal{A}} (x, y^2, x^2y + a_{1,1}xy^3 + a_{0,2}y^5 + \sum_{i+j\geq 3} a_{i,j}x^i y^{2j+1})).$$

Now, if $c_e(f) = k < \infty$ and $j^3 f \sim_{\mathcal{A}} (x, y^2, x^2y)$, then it follows from Mond's classification that,

$$f(x, y) \sim_{\mathcal{A}} (x, y^2, x^2y + y^{2k+1}).$$

□

Proposition 3.10. *Let f be a map germ from plane to space with corank 1 at origin, $j^2 f \sim_{\mathcal{A}} (x, y^2, 0)$, and $C(f) = 3$. Then,*

(1) if $c_e(f) = 2$, then type of f is S_2;
(2) if $c_e(f) = 3$, then type of f is C_3;
(3) if $c_e(f) = 4$, then type of f is F_4.

Proof. If $j^2 f \sim_{\mathcal{A}} (x, y^2, 0)$, then by Theorem 4.1 : 1 [1], we can write it as

$$f(x, y) \sim_{\mathcal{A}} (x, y^2, \sum_{i+j\geq 1} a_{i,j}x^i y^{2j+1})).$$

We can write it as

$$f(x, y) \sim_{\mathcal{A}} (x, y^2, a_{0,1}y^3 + a_{2,0}x^2y + a_{1,1}xy^3 + a_{0,2}y^5 + a_{3,0}x^3y + a_{2,1}x^2y^3 + a_{1,2}xy^5 + a_{0,3}y^7 + \sum_{i+j\geq 4} a_{i,j}x^i y^{2j+1}).$$

If $C(f) = 3$, then $a_{2,0} = 0$ and $a_{3,0} \neq 0$, otherwise, $C(f) \neq 3$. So,

$$f(x, y) \sim_{\mathcal{A}} (x, y^2, a_{0,1}y^3 + a_{1,1}xy^3 + a_{3,0}x^3y + a_{0,2}y^5 + a_{2,1}x^2y^3 + a_{1,2}xy^5 + a_{0,3}y^7 + \sum_{i+j\geq 4} a_{i,j}x^i y^{2j+1})).$$

(1) If $c_e(f) = 2$, then $a_{0,1} \neq 0$. So, by using left coordinate change $\overline{X} = X$, $\overline{Y} = Y$, $\overline{Z} = Z - \frac{a_{1,1}}{a_{0,1}}XZ$, we get

$$f(x, y) \sim_{\mathcal{A}} (x, y^2, a_{0,1}y^3 + a_{3,0}x^3y + a_{0,2}y^5 + (a_{2,1} - \frac{a_{1,1}^2}{a_{0,1}})x^2y^3 - \frac{a_{3,0}a_{1,1}}{a_{0,1}}x^4y$$
$$+(a_{1,2} - \frac{a_{1,1}a_{0,2}}{a_{0,1}})xy^5 - \frac{a_{1,1}a_{2,1}}{a_{0,1}}x^3y^3 + a_{0,3}y^7 - \frac{a_{1,1}a_{1,2}}{a_{0,1}} + \sum_{i+j\geq 4} a_{i,j}x^i y^{2j+1}).$$

Since f is 4-determined,

$$f(x, y) \sim_{\mathcal{A}} (x, y^2, y^3 + x^3y).$$

(2) If $c_e(f) = 3$, then $a_{0,1} = 0$ and $a_{1,1} \neq 0$. Also, $a_{3,0} \neq 0$, otherwise, f will not be $\mathcal{A}$-finite. So, we have

$$f(x,y) \sim_{\mathcal{A}} (x, y^2, a_{1,1}xy^3 + a_{3,0}x^3y + a_{0,2}y^5 + a_{2,1}x^2y^3 + a_{1,2}xy^5 + a_{0,3}y^7 + \sum_{i+j\geq 4} a_{i,j}x^iy^{2j+1}).$$

Since f is 4-determined, $f(x,y) \sim_{\mathcal{A}} (x, y^2, xy^3 + x^3y)$.

(3) If $c_e(f) = 4$, then $a_{0,1} = a_{1,1} = 0$ and $a_{0,2} \neq 0$. Also, $a_{3,0} \neq 0$, otherwise, f will not be $\mathcal{A}$-finite. So, we have

$$f(x,y) \sim_{\mathcal{A}} (x, y^2, a_{3,0}x^3y + a_{0,2}y^5 + a_{2,1}x^2y^3 + a_{1,2}xy^5 + a_{0,3}y^7 + \sum_{i+j\geq 4} a_{i,j}x^iy^{2j+1}).$$

Since f is 5-determined, $f(x,y) \sim_{\mathcal{A}} (x, y^2, x^3y + y^5)$. □

Proposition 3.11. *Let f be a map germ from plane to space with corank 1 at origin, $j^2f \sim_{\mathcal{A}} (x, y^2, 0)$, and $C(f) = k, 3 < k < \infty$. Then,*

(1) if $c_e(f) = C(f) - 1$, then f is of type S_{k-1};
(2) if $c_e(f) = C(f)$, then f is of type C_k.

Proof. Since $j^2f \sim_{\mathcal{A}} (x, y^2, 0)$,

$$f(x,y) \sim_{\mathcal{A}} (x, y^2, \sum_{i+j\geq 1} a_{i,j}x^iy^{2j+1})).$$

We can write it as

$$f(x,y) \sim_{\mathcal{A}} (x, y^2, a_{0,1}y^3 + a_{2,0}x^2y + a_{1,1}xy^3 + a_{0,2}y^5 + a_{3,0}x^3y + a_{2,1}x^2y^3 + a_{1,2}xy^5 + a_{0,3}y^7 + \sum_{i+j\geq 4} a_{i,j}x^iy^{2j+1}).$$

If $C(f) = k$, then $a_{2,0} = \cdots = a_{k-1,0} = 0$ and $a_{k,0} \neq 0$. Now, if $c_e(f) = C(f) - 1$, then $a_{0,1} \neq 0$, otherwise, f will not be $\mathcal{A}$-finite. This gives $j^3f \sim_{\mathcal{A}} (x, y^2, y^3)$. Then, by Mond's classification f is of type $f(x,y) \sim_{\mathcal{A}} (x, y^2, y^3 + x^{k+1}y)$.

If $c_e(f) = C(f)$, then $a_{0,1} = 0$ and $a_{1,1} \neq 0$, otherwise, f will not be $\mathcal{A}$-finite. This gives $j^3f \sim_{\mathcal{A}} (x, y^2, 0)$. Thus, it follows from Mond's classification that f is of type $f(x,y) \sim_{\mathcal{A}} (x, y^2, xy^3 + x^ky)$. □

Proposition 3.12. *Let f be a map germ from plane to space with corank 1 at origin, $j^2f \sim_{\mathcal{A}} (x, xy, 0)$, and $C(f) = 2$. If $c_e(f) = k$, $k \geq 2$, then f is of type H_k.*

Proof. Since $j^2f \sim_{\mathcal{A}} (x, xy, 0)$,

$$f(x,y) \sim_{\mathcal{A}} (x, xy + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + \sum_{i+j\geq 4} a_{i,j}x^iy^j, b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3 + \sum_{i+j\geq 4} b_{i,j}x^iy^j).$$

By using suitable left coordinate changes, we get

$$f(x,y) \sim_{\mathcal{A}} (x, xy + \sum_{i+j\geq 4} a_{i,j}x^iy^j, b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3 + \sum_{i+j\geq 4} b_{i,j}x^iy^j)^*.$$

If $C(f) = 2$, then $b_{0,3} \neq 0$. Take $b_{0,3} = 1$, we have

$$f(x, y) \sim_{\mathcal{A}} (x, xy + \sum_{i+j\geq 4} a_{i,j}x^i y^j, y^3 + b_{2,1}x^2 y + b_{1,2}xy^2 + \sum_{i+j\geq 4} b_{i,j}x^i y^j).$$

It is easy to see that

$$f(x, y) \sim_{\mathcal{A}} (x, xy + \sum_{i+j\geq 4} a_{i,j}x^i y^j, (y + \frac{b_{1,2}}{3}x)^3 + \sum_{i+j\geq 4} b_{i,j}x^i y^j).$$

Now, by using transformation $y \to y - \frac{b_{1,2}}{3}x$, we get

$$f(x, y) \sim_{\mathcal{A}} (x, xy + \sum_{i+j\geq 4} a_{i,j}x^i y^j, y^3 + \sum_{i+j\geq 4} b_{i,j}x^i y^j).$$

This gives $j^3 f \sim_{\mathcal{A}} (x, xy, y^3)$. If $c_e(f) \geq 2$, then by using Theorem 4.2.1 : $2(a)$ [1], we have

$$f(x, y) \sim_{\mathcal{A}} (x, xy + y^{3k-1}, y^3).$$

□

Proposition 3.13. *Let f be a map germ from plane to space with corank 1 at origin, $j^2 f \sim_{\mathcal{A}} (x, xy, 0)$, and $C(f) = 3$. Then,*

(1) if $c_e(f) = 4$ and $T(f) = 1$, then f is of type P_3 or $P_4(\frac{1}{2})$ or T_4;
(2) if $c_e(f) = 4$ and $T(f) = 2$, then f is of type $P_4(1)$ or Q_4;
(3) if $c_e(f) = 5$ and $T(f) = 3$, then f is of type Q_5;
(4) if $c_e(f) = 6$ and $T(f) = 4$, then f is of type Q_6 or R_4;
(5) if $c_e(f) = k$ and $T(f) = k - 2$, then f is of type Q_k, $k \geq 7$.

Proof. Since $j^2 f \sim_{\mathcal{A}} (x, xy, 0)$,

$$f(x, y) = (x, xy + a_{2,1}x^2 y + a_{1,2}xy^2 + a_{0,3}y^3 + \sum_{i+j\geq 4} a_{i,j}x^i y^j, b_{2,1}x^2 y + b_{1,2}xy^2 + b_{0,3}y^3 + \sum_{i+j\geq 4} b_{i,j}x^i y^j).$$

If $c_e(f) = 4$, then $b_{0,3} = 0$. Then, the coordinate changes $\overline{X} = X, \overline{Y} = Y - a_{2,1}XY, \overline{Z} = Z - b_{2,1}XY$ gives

$$f(x, y) \sim_{\mathcal{A}} (x, xy + a_{1,2}xy^2 + a_{0,3}y^3 + \sum_{i+j\geq 4} a_{i,j}x^i y^j, b_{1,2}xy^2 + \sum_{i+j\geq 4} b_{i,j}x^i y^j).$$

If $b_{1,2} = 0$, then the transformation $x \to$, $y \to y + a_{1,2}y^2$, gives

$$f(x, y) \sim_{\mathcal{A}} (x, xy + a_{0,3}y^3 + \sum_{i+j\geq 4} a_{i,j}x^i y^j, b_{3,1}x^3 y + b_{2,2}x^2 y^2 + b_{1,3}xy^3 + y^4 + \sum_{i+j\geq 4} b_{i,j}x^i y^j).$$

Thus, $j^3 f \sim_{\mathcal{A}} (x, xy + a_{0,3}y^3, 0)$. Now, if $C(f) = 3$, then it follows from the Proposition 4.2.4 : 1 [1] f must have a 4-jet equivalent to $(x, xy + y^3, y^4)$. Since f is 4-determined (see Theorem 4.2.4 : 2 [1]),

$$f(x, y) \sim_{\mathcal{A}} (x, xy + y^3, y^4).$$

If $b_{1,2} \neq 0$, then the coordinate change $\overline{X} = X, \overline{Y} = Y - (\frac{a_{1,2}}{b_{1,2}})Z, \overline{Z} = Z$ gives

$$f(x,y) \sim_{\mathcal{A}} (x, xy + a_{0,3}y^3 + \sum_{i+j\geq 4} a_{i,j}x^iy^j, b_{1,2}xy^2 + \sum_{i+j\geq 4} b_{i,j}x^iy^j).$$

This implies

$$j^3 f \sim_{\mathcal{A}} (x, xy + a_{0,3}y^3, b_{1,2}xy^2).$$

Now, by the Lemma 4.2.2 : 1 [1], this type of f must have a 4-jet equivalent to $(x, xy + y^3, xy^2 + cy^4)$.

(1) If $c_e(f) = 4$ and $T(f) = 1$, then $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + cy^4)$, $c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$ or $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^5)$. These two types can be differentiated by computing their normal forms of order 4 and finding c[‡]. If $c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$, then f is of type $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + cy^4)$, and if $c = \frac{1}{2}$, then $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^5)$.

(2) If $c_e(f) = 4$ and $T(f) = 2$, then $f(x,y) \sim_{\mathcal{A}} (x, xy+y^3, xy^2+y^4+y^6)$ or $f(x,y) \sim_{\mathcal{A}} (x, xy+y^3, xy^2 + y^7)$. These two types can be differentiated by computing their normal forms of order 4 and finding c. If $c = 1$, then $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + y^4 + y^6)$ and, if $c = 0$, then $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + y^7)$.

(3) It is straightforward to prove.

(4) We have $j^4 f \sim_{\mathcal{A}} (x, xy + y^3, xy^2)$. If $c_e(f) = 6$, then $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + y^{13})$ or $f(x,y) \sim_{\mathcal{A}} (x, xy + y^6 + by^7, xy^2 + y^4 + cy^6)$. These two types can be differentiated by computing their normal forms of order 3. If $j^3 f \sim_{\mathcal{A}} (x, xy + y^3, xy^2)$, then $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + y^{13})$, and if $j^3 f \sim_{\mathcal{A}} (x, xy, xy^2)$, then $f(x,y) \sim_{\mathcal{A}} (x, xy + y^6 + by^7, xy^2 + y^4 + cy^6)$.

(5) Now if $c_e(f) > 6$, and $T(f) > 4$, then from Theorem 4.2.2 : 7 [1] $f(x,y) \sim_{\mathcal{A}} (x, xy + y^3, xy^2 + y^{3k-5})$. □

The next algorithm is useful to differentiate the types of P_3 and $P_4(\frac{1}{2})$.

Algorithm 1 To differentiate P_3 and $P_4(\frac{1}{2})$.

Input: Ideal $I = f, g, h$.

Output: Ideal $I = h_1, h_2, h_3$, the required normal form.

Define a map φ, $\varphi(x) = \sum_{1\leq i+j\leq p+2} a_{ij}x^iy^j$, $\varphi(y) = \sum_{1\leq i+j\leq p+2} b_{ijk}x^iy^j$, with parameters having degree up to $p + 2$.

Define a 3×3-matrix T with entries as polynomials of degree $p+1$ having parameters coefficients, *i.e.*, $T = (t_{ij})$, $t_{ij} = \sum_{0\leq l+m\leq 4} a_{ij,l,m}x^ly^m$, $a_{ij,l,m}$ are parameters.

Define the expected normal form $(x^2 + y^3, xy^p, 0)$.

Define $J := \varphi(I) - T \begin{pmatrix} x \\ xy + y^3 \\ xy^2 + y^4 \end{pmatrix}$.

Let K be the ideal generated by all the coefficients of J with respect to x, y.

Compute S, the standard basis of K.

if $S =< 1 >$ **then**

  return$((x, xy + y^3, xy^2 + y^4))$;

if $S \neq< 1 >$ **then**

  return$((x, xy + y^3, xy^2 + \frac{1}{2}y^4 + y^5))$.

[‡]This can be done by using **Algorithm** 1.

Proposition 3.14. *Let f be a map germ from the plane to space with corank 1 at origin, $j^2f \sim_{\mathcal{A}} (x, xy, 0)$, and $C(f) = 4$. If $c_e(f) = 4$ and $T(f) = 1$, then f is of type $P_4(\frac{3}{2})$.*

Proof. The proof is similar to the Proposition 3.13(2). □

Proposition 3.15. *Let f be a map germ from plane to space with corank 1 at origin, $j^2f \sim_{\mathcal{A}} (x, 0, 0)$, and $C(f) = 4$. If $c_e(f) = 4$ and $T(f) = 1$, then f is of type X_4.*

Proof. Since $j^2f \sim_{\mathcal{A}} (x, 0, 0)$,

$$f(x, y) = (x, a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + h.o.t., b_{2,1}x^2y + b_{1,2}xy^2 + b_{0,3}y^3 + h.o.t.).$$

If $C(f) = 4$ and $T(f) = 1$, then $b_{0,3} = 0$; thus,

$$f(x, y) = (x, a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + h.o.t., b_{2,1}x^2y + b_{1,2}xy^2 + h.o.t.).$$

By using the coordinate changes $\overline{X} = X, \overline{Y} = Y - Z, \overline{Z} = Z$, we get

$$f(x, y) \sim_{\mathcal{A}} (x, a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + h.o.t., b_{2,1}x^2y + b_{1,2}xy^2 + h.o.t.).$$

Thus,

$$j^3f \sim_{\mathcal{A}} (x, y^3, xy^2 + xy^2).$$

Now, if $T(f) = 1$, then by using Proposition 4.3 : 2 [1], we get $j^4f \sim_{\mathcal{A}} (x, y^3, xy^2 + xy^2 + y^4)$. As f is 4-determined, $f(x, y) \sim_{\mathcal{A}} (x, y^3, xy^2 + xy^2 + y^4)$. □

4. Singular examples

We have implemented the characterization in the computer algebra system Singular [15]. The code can be downloaded from https://www.mathcity.org/files/ahsan/Proc-classifycoRank1Maps.txt. We give some examples. The examples are constructed from the normals form by applying a generic $\mathcal{A}$-equivalence.

```
ring r=0,(x,y),(c,ds).
```

In the first example we have as an input the map $f(x, y) = (f_1, f_2, f_3)$, where

$$\begin{aligned}
f_1 =& x - y,\\
f_2 =& xy - y^2 - x^2y + xy^2 + y^6 - 6xy^6 + y^7 + 15x^2y^6 - 7xy^7 - 20x^3y^6 + 21x^2y^7 + 15x^4y^6\\
&- 35x^3y^7 - 6x^5y^6 + 35x^4y^7 + x^6y^6 - 21x^5y^7 + 7x^6y^7 - x^7y^7,\\
f_3 =& xy^2 - y^3 - 2x^2y^2 + 2xy^3 + y^4 + x^3y^2 - x^2y^3 - 4xy^4 + 6x^2y^4 + y^6 - 4x^3y^4 - 6xy^6\\
&+ x^4y^4 + 15x^2y^6 - 20x^3y^6 + 15x^4y^6 - 6x^5y^6 + x^6y^6.
\end{aligned}$$

In SINGULAR this can be written as

```
ideal I=x-y,xy-y2-x2y+xy2+y6-6xy6+y7+15x2y6-7xy7-20x3y6+21x2y7+15x4y6
      -35x3y7-6x5y6+35x4y7+x6y6-21x5y7+7x6y7-x7y7, xy2-y3-2x2y2+2xy3
       +y4+x3y2-x2y3-4xy4+6x2y4+y6-4x3y4-6xy6+x4y4+15x2y6-20x3y6+
       15x4y6-6x5y6+x6y6.
```

To compute the required type of map germs, we use the procedure:

```
classifycoRank1Maps(I);
f is of type R_4.
```

In the second example we have as an input the map $f(x, y) = (f_1, f_2, f_3)$, where

$$
\begin{aligned}
f_1 =& x - 3y + 3xy - y^2 - x^2y + xy^2,\\
f_2 =& y^2 - 6xy^2 + 6y^3 + 15x^2y^2 - 32xy^3 + 13y^4 - 20x^3y^2 + 70x^2y^3 - 60xy^4 + 12y^5\\
&+ 15x^4y^2 - 80x^3y^3 + 111x^2y^4 - 46xy^5 + 4y^6 - 6x^5y^2 + 50x^4y^3 - 104x^3y^4 + 68x^2y^5 - 12xy^6 + x^6y^2\\
&- 16x^5y^3 + 51x^4y^4 - 48x^3y^5 + 13x^2y^6 + 2x^6y^3 - 12x^5y^4 + 16x^4y^5 - 6x^3y^6 + x^6y^4 - 2x^5y^5 + x^4y^6,\\
f_3 =& x^3y - 9x^2y^2 + 27xy^3 - 27y^4 - 3x^4y + 39x^3y^2 - 165x^2y^3 + 261xy^4 - 107y^5\\
&+ 3x^5y - 64x^4y^2 + 392x^3y^3 - 900x^2y^4 + 690xy^5 - 129y^6 - x^6y + 50x^5y^2\\
&- 480x^4y^3 + 1584x^3y^4 - 1788x^2y^5 + 559xy^6 + 18y^7 - 19x^6y^2 + 334x^5y^3 - 1632x^4y^4\\
&+ 2344x^3y^5 - 353x^2y^6 - 966xy^7 + 369y^8 + 3x^7y^2 - 135x^6y^3 + 1044x^5y^4 - 1170x^4y^5\\
&- 3599x^3y^6 + 7827x^2y^7 - 4953xy^8 + 983y^9 + 30x^7y^3 - 421x^6y^4 - 1536x^5y^5 + 14993x^4y^6\\
&- 32850x^3y^7 + 30105x^2y^8 - 12036xy^9 + 1683y^{10} - 3x^8y^3 + 105x^7y^4 + 4459x^6y^5 - 34111x^5y^6\\
&+ 90219x^4y^7 - 111363x^3y^8 + 67989x^2y^9 - 19335xy^{10} + 1970y^{11} - 15x^8y^4 - 6308x^7y^5\\
&+ 54768x^6y^6 - 177768x^5y^7 + 281724x^4y^8 - 234871x^3y^9 + 102165x^2y^{10} - 21135xy^{11}\\
&+ 1560y^{12} + x^9y^4 + 6418x^8y^5 - 66444x^7y^6 + 262344x^6y^7 - 516858x^5y^8 + 554289x^4y^9\\
&- 328825x^3y^{10} + 103815x^2y^{11} - 15520xy^{12} + 800y^{13} - 5004x^9y^5 + 62201x^8y^6\\
&- 296004x^7y^7 + 710046x^6y^8 - 945174x^5y^9 + 719565x^4y^{10} - 309010x^3y^{11} + 70310x^2y^{12}\\
&- 7320xy^{13} + 240y^{14} + 3003x^{10}y^5 - 45045x^9y^6 + 257400x^8y^7 - 742830x^7y^8\\
&+ 1200815x^6y^9 - 1131417x^5y^{10} + 621640x^4y^{11} - 191810x^3y^{12} + 30300x^2y^{13} - 2000xy^{14} + 32y^{15}\\
&- 1365x^{1}1y^5 + 25025x^{10}y^6 - 172315x^9y^7 + 595650x^8y^8 - 1154835x^7y^9 + 1316875x^6y^{10}\\
&- 892355x^5y^{11} + 351070x^4y^{12} - 74950x^3y^{13} + 7480x^2y^{14} - 240xy^{15} + 455x^{12}y^5\\
&- 10465x^{11}y^6 + 87945x^{10}y^7 - 365310x^9y^8 + 845295x^8y^9 - 1151415x^7y^{10} + 940455x^6y^{11}\\
&- 454530x^5y^{12} + 123255x^4y^{13} - 16560x^3y^{14} + 800x^2y^{15} - 105x^{13}y^5 + 3185x^{12}y^6\\
&- 33540x^{11}y^7 + 169510x^{10}y^8 - 469315x^9y^9 + 759285x^8y^{10} - 737220x^7y^{11} + 427500x^6y^{12}\\
&- 141795x^5y^{13} + 24095x^4y^{14} - 1560x^3y^{15} + 15x^{14}y^5 - 665x^{13}y^6 + 9230x^{1}2y^7\\
&- 58250x^{11}y^8 + 195195x^{10}y^9 - 375585x^9y^{10} + 430530x^8y^{11} - 295140x^7y^{12} + 116865x^6y^{13}\\
&- 24175x^5y^{14} + 1970x^4y^{15} - x^{15}y^5 + 85x^{14}y^6 - 1725x^{13}y^7 + 14290x^{12}y^8\\
&- 59395x^{11}y^9 + 137235x^{10}y^{10} - 185645x^9y^{11} + 149220x^8y^{12} - 69435x^7y^{13} + 17055x^6y^{1}4\\
&- 1683x^5y^{15} - 5x^{15}y^6 + 195x^{14}y^7 - 2350x^{13}y^8 + 12705x^{12}y^9 - 36015x^{11}y^{10}\\
&+ 57885x^{10}y^{11} - 54420x^9y^{12} + 29475x^8y^{13} - 8455x^7y^{14} + 985x^6y^{15} - 10x^{15}y^7\\
&+ 230x^{14}y^8 - 1785x^{13}y^9 + 6475x^{12}y^{10} - 12570x^{11}y^{11} + 13870x^{10}y^{12} - 8705x^9y^{13}\\
&+ 2885x^8y^{14} - 390x^7y^{15} - 10x^{15}y^8 + 145x^{14}y^9 - 735x^{13}y^{10} + 1780x^{12}y^{11}\\
&- 2330x^{11}y^{12} + 1695x^{10}y^{13} - 645x^9y^{14} + 100x^8y^{15} - 5x^{15}y^9 + 45x^{14}y^{10}
\end{aligned}
$$

$$- 145x^{13}y^{11} + 230x^{12}y^{12} - 195x^{11}y^{13} + 85x^{10}y^{14} - 15x^9y^{15} - x^{15}y^{10}$$
$$+ 5x^{14}y^{11} - 10x^{13}y^{12} + 10x^{12}y^{13} - 5x^{11}y^{14} + x^{10}y^{15}.$$

In SINGULAR this can be written as

```
ideal I=x-3y+3xy-y2-x2y+xy2,y2-6xy2+6y3+15x2y2-32xy3+13y4-20x3y2+70x2y3
        -60xy4+12y5+15x4y2-80x3y3+111x2y4-46xy5+4y6-6x5y2+50x4y3-104x3y4
        +68x2y5-12xy6+x6y2-16x5y3+51x4y4-48x3y5+13x2y6+2x6y3-12x5y4+16x4y5
        -6x3y6+x6y4-2x5y5+x4y6,x3y-9x2y2+27xy3-27y4-3x4y+39x3y2-165x2y3
        +261xy4-107y5+3x5y-64x4y2+392x3y3-900x2y4+690xy5-129y6-x6y
        +50x5y2-480x4y3+1584x3y4-1788x2y5+559xy6+18y7-19x6y2+334x5y3
        -1632x4y4+2344x3y5-353x2y6-966xy7+369y8+3x7y2-135x6y3+1044x5y4
        -1170x4y5-3599x3y6+7827x2y7-4953xy8+983y9+30x7y3-421x6y4-1536x5y5
        +14993x4y6-32850x3y7+30105x2y8-12036xy9+1683y10-3x8y3+105x7y4
        +4459x6y5-34111x5y6+90219x4y7-111363x3y8+67989x2y9-19335xy10+1970
        y11-15x8y4-6308x7y5+54768x6y6-177768x5y7+281724x4y8-234871x3y9
        +102165x2y10-21135xy11+1560y12+x9y4+6418x8y5-66444x7y6+262344x6y7
        -516858x5y8+554289x4y9-328825x3y10+103815x2y11-15520xy12+800y13
        -5004x9y5+62201x8y6-296004x7y7+710046x6y8-945174x5y9+719565x4y10
        -309010x3y11+70310x2y12-7320xy13+240y14+3003x10y5-45045x9y6
        +257400x8y7-742830x7y8+1200815x6y9-1131417x5y10+621640x4y11-191810
        x3y12+30300x2y13-2000xy14+32y15-1365x11y5+25025x10y6-172315x9y7
        +595650x8y8-1154835x7y9+1316875x6y10-892355x5y11+351070x4y12-74950
        x3y13+7480x2y14-240xy15+455x12y5-10465x11y6+87945x10y7-365310x9y8
        +845295x8y9-1151415x7y10+940455x6y11-454530x5y12+123255x4y13-16560
        x3y14+800x2y15-105x13y5+3185x12y6-33540x11y7+169510x10y8-469315x9y9
        +759285x8y10-737220x7y11+427500x6y12-141795x5y13+24095x4y14-1560
        x3y15+15x14y5-665x13y6+9230x12y7-58250x11y8+195195x10y9-375585x9y10
        +430530x8y11-295140x7y12+116865x6y13-24175x5y14+1970x4y15-x15y5
        +85x14y6-1725x13y7+14290x12y8-59395x11y9+137235x10y10-185645x9y11
        +149220x8y12-69435x7y13+17055x6y14-1683x5y15-5x15y6+195x14y7-2350
        x13y8+12705x12y9-36015x11y10+57885x10y11-54420x9y12+29475x8y13
        -8455x7y14+985x6y15-10x15y7+230x14y8-1785x13y9+6475x12y10-12570
        x11y11+13870x10y12-8705x9y13+2885x8y14-390x7y15-10x15y8+145x14y9
        -735x13y10+1780x12y11-2330x11y12+1695x10y13-645x9y14+100x8y15
        -5x15y9+45x14y10-145x13y11+230x12y12-195x11y13+85x10y14-15x9y15
        -x15y10+5x14y11-10x13y12+10x12y13-5x11y14+x10y15;
```

To compute the required type of map germs, we use the procedure:

```
classifycoRank1Maps(I);
f is of type F_4,
```

5. Conclusions

A classifier for map germs from plane to space in terms of codimension, crosscaps and triple points has been given. Moveover, this classifier is implemented in the computer algebra system SINGULAR.

Availability of data and materials

The code used in this paper can be downloaded from: `https://www.mathcity.org/files/ahsan/ProcclassifycoRank1Maps.txt`.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Nos. 62172116, 61972109) and the Guangzhou Academician and Expert Workstation (No. 20200115-9).

The research of the second and third authors is supported by Higher Education Commission of Pakistan by the project No. 7495 /Punjab/NRPU/R&D/HEC/2017.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, *Proc. London Math. Soc.*, **s3-50** (1985), 333–369. https://doi.org/10.1112/plms/s3-50.2.333
2. D. Mond, Some remarks on the geometry and classification of germs of maps from surfaces to 3-space, *Topology*, **26** (1987), 361–383. https://doi.org/10.1016/0040-9383(87)90007-3
3. T. Gaffney, The structure of $T_{\mathcal{A}_f}$ classification and an application to differential geometry, *Proc. Sympos. Pure Math.*, **40** (1983), 409–427.
4. Y. Kabata, Recognition of plane-to-plane map-germs, *Topol. Appl.*, **202** (2016), 216–238. https://doi.org/10.1016/j.topol.2016.01.011
5. J. H. Rieger, Families of maps from the plane to the plane, *J. London Math. Soc.*, **s2-36** (1987), 351–369. https://doi.org/10.1112/jlms/s2-36.2.351
6. J. H. Rieger, A-unimodal map-germs into the plane, *Hokkaido Math. J.*, **33** (2004), 47–64. https://doi.org/10.14492/hokmj/1285766004
7. A. Dimca, C. G. Gibson, Contact unimodular germs from the plane to the plane, *Q. J. Math.*, **34** (1983), 281–295. https://doi.org/10.1093/qmath/34.3.281
8. A. Dimca, C. G. Gibson, Classification of equidumensional contact unimodular map germs, *Math. Scand.*, **56** (1985), 15–28.
9. M. A. Binyamin, S. Aslam, K. Mehmood, Contact unimodal map germs from the plane to the plane, *Comptes Rendus Math.*, **358** (2020), 923–930. https://doi.org/10.5802/crmath.114

10. S. Aslam, M. A. Binyamin, G. Pfister, Recognition of unimodal map germs from the plane to the plane by invariants, *Int. J. Algebra Comput.*, **28** (2018), 1199–1208. https://doi.org/10.1142/S0218196718500534

11. M. A. Binyamin, S. Aslam, K. Mehmood, H. Mahmood, M. Ishaq, Recognition of contact map germs from the plane to the plane by invariants, *C. R. Acad. Bulg. Sci.*, **73** (2020), 758–766. https://doi.org/10.7546/CRABS.2020.06.02

12. S. Aslam, M. A. Binyamin, H. Mahmood, A classifier for equidimensional contact unimodal map germs, *Bull. Math. Soc. Sci. Math. Roumanie*, **64** (2021), 287–302.

13. M. A. Binyamin, H. Mahmood, S. Kanwal, On the classification of simple maps from the plane to the plane, *J. Algebra Appl.*, **16** (2017), 1750199. https://doi.org/10.1142/S0219498817501997

14. M. A. Binyamin, K. Mehmood, G. Pfister, On Mond's classification of simle map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$, *C. R. Acad. Bulg. Sci.*, **74** (2021), 1109–1119. https://doi.org/10.7546/CRABS.2021.08.01

15. W. Decker, G. M. Greuel, G. Pfister, H. Schönemann, SINGULAR. Available from: `http://www.singular.uni-kl.de`.

16. D. Ratcliffe, *On the classification and geometry of finite map-germs*, Ph.D. Thesis, University of Warwick, 1990.