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Research article

Transportation strategy decision-making process using interval-valued complex fuzzy soft information

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Abstract: Transportation is among the more vital economic activities for a business and our daily life actions. At present, transport is one of the key branches playing a crucial role in the development of the economy. Transportation decision-making looks for ways to solve current and anticipated transportation problems while avoiding future problems. An interval-valued complex fuzzy set (IVCFS) is an extended form of fuzzy, interval-valued fuzzy and complex fuzzy sets, and it is used to evaluate complex and inaccurate information in real-world applications. In this research, we aim to examine the novel concept of IVCF soft relations (IVCFSRs) by utilizing the Cartesian product (CP) of two IVCF soft sets (IVCFSSs), which are determined with the help of two different concepts, referred to as IVCF relation and soft sets. Moreover, we investigated various types of relations and also explained them with the help of some appropriate examples. The IVCFSRs have a comprehensive structure discussing due dealing with the degree of interval-valued membership with multidimensional variables. Moreover, IVCFSR-based modeling techniques are included, and they use the score function to select the suitable transportation strategy to improve the value of the analyzed data. Finally, to demonstrate

the effectiveness of the suggested work, comparative analysis with existing methods is performed.

Keywords: interval-valued complex fuzzy set; interval-valued complex fuzzy soft set; interval-valued complex fuzzy soft relation; decision-making problem
Mathematics Subject Classification: 03E72, 03E99, 08A72

1. Introduction

The transportation systems that were previously employed are starting to lose their effectiveness as global urbanization increases. Transportation is essential to human life because it offers so many necessary services. Mouratidis [1] establishes that mobility can be made easier through transportation, which also promotes economic expansion. When used for daily tasks like commuting and traveling, transportation modes substantially impact the quality of life of those who use them. They, therefore, employ private means in addition to public transportation to acquire some independence, and uncertainty may be a familiar element of best-transporting decisions. Uncertainty always occurs when there is a lack of knowledge regarding the outcomes and when the future environment is subject to change, and everything is in a state of transition. The majority of our real-world issues in the fields of medicine, engineering, transport, management, the environment and social sciences frequently contain facts that are not always exact and predictable in nature because of different uncertainties related to these issues. The decision-maker is unaware of all available alternatives and the risks related to the consequences of each alternative. Detecting the difficulty and resolving uncertainty, a new development in mathematics was introduced by Zadeh [2], the fuzzy set (FS). Each element in this set is given a membership degree, ranging from 0 to 1, indicating its quality or effectiveness. FSs are important in human decision-making. Deschrijver and Kerre [3] examined the relationship among numerous FS extensions. Yao [4] related the FSs and rough sets. Chiang and Lin [5] operated on the correlation of FSs. Ragin [6] applies FSs in social sciences. Guiffrida and Nagi [7] used FSs in production management, and Kahraman [8] recommended fuzzy applications in industrial engineering. Mendel [9] introduced the concept of fuzzy relations (FRs). FRs use each element's degree of membership to define the quality of each relationship. If membership is closer to 1, then it demonstrates a good relationship; and if it is closer to 0, then it specifies poor relationships. The FR is a more comprehensive framework than classical relations. FRs were utilized in the clustering analysis by Yeh and Bang [10]. Braae and Rutherford [11] applied FRs in the context setting.

Ramot et al. [12] introduced the idea of a complex fuzzy set (CFS), integrating complex numbers into FS theory. Moreover, they also define the complex fuzzy relation (CFR). The CFSs give the membership degree with two terms: One, amplitude, describes the effectiveness, and the second, phase term describes the duration of effectiveness. Hu et al. [13] recognized the orthogonality relation on CFSs. Zhang et al. [14] studied numerous working effects and δ-equities of CFSs. Li et al. [15] broadly controlled the implementation of CFSs. Tamir et al. [16] suggested some uses of CFS. Khan et al. [17] recognized the CFR in the future commission market. Zadeh [18] announced the idea of interval-valued FSs (IVFSs). The IVFSs are the extended form of FSs. These sets define the membership degree in the form of an interval that is a subset of the unit interval. Bustince and Burillo [19] invented the idea of interval-valued fuzzy relations (IVFRs), which generalize classical relations and FRs. Ashtiani et al. [20] increased the technique of the fuzzy

TOPSIS construct on IVFSs. Zeng et al. [21] gave the entropy of IVFSs and their associations with altered methods. Greenfield et al. [22] illuminated the view of interval-valued complex FSs (IVCFSs). The degrees of membership are complex-valued intervals in the complex plane's unit circle. Dai et al. [23] defined the expanse measure among the IVCFSs.

Humans frequently struggle to select the optimal alternative in the context of all these advancements in decision-making. Many questions and ambiguities are raised in this situation. Molodtsov [24] developed the idea of soft sets (SS) in 1999, which aid in making good choices unpredictably. SSs select the best things on the basis of some taken parameters. Alkhazaleh et al. [25] defined the soft multiset theory, and Yang et al. [26] suggested the generalization of SSs. Maji et al. [27] gave an application of SSs in decision-making problems. Babitha and Sunil [28] gave the idea of soft relations (SRs), the analysis of soft sets. Park et al. [29] explained some effects of equivalence SRs.

Maji et al. [30] introduced the notion of the fuzzy soft set (FSS) by combining the fuzzy set and the soft set. It helps people make better decisions in daily life by reducing uncertainty. Ali [31] wrote a remark on SSs, rough SSs and FSSs. Feng et al. [32] developed a flexible method for FSS decision-making. Yao et al. [33] illuminated the difference between FSSs and soft fuzzy sets. Borah et al. [34] conceived the innovative idea of fuzzy soft relations (FSRs), which are the combination of FRs and SRs. Sut [35] suggested using FSRs in decision-making, and Mockor and Hurtik [36] approximated FSSs using FSRs in association with image processing. In order to tackle problems with periodicity, Thirunavukarasu et al. [37] introduced the concept of the CFS set (CFSS), whose membership degrees take the form of complex numbers. Tamir et al. [38] discussed an overview of the theory and applications of CFSs and complex fuzzy logic. Al-Qudah and Hassan [39] established the concept of a complex multi-fuzzy soft expert set and its application. Yang et al. [40] introduced the interval-valued fuzzy soft set (IVFSS); the IVFSS is a well-known example of an uncertainty model that is more realistic than the FS. Tripathy et al. [41] proposed an application of IVFSSs in group decision-making. Selvachandran et al. [42] established the innovative idea of IVCFSS with an application.

There are many complex difficulties in the human-centered subject of transportation planning that need to be tackled. Services, costs, infrastructure, vehicles and control systems all have characteristics and performances that are often determined by a quantitative examination of their primary effects. Valášková et al. [43] discussed fuzzy logic in decision-making. Pappis and Mamdani [44] solved the practical problem of traffic and transportation by using fuzzy logic. TeodorovicÂ et al. [45] defined the application of transportation and traffic in fuzzy set theory. Tang et al. [46] defined a new multi-attribute decision-making method for a mobile medical app by q-rung orthopair fuzzy information. Tang et al. [47] described q-rung orthopair fuzzy information with the application of a decision theoretic rough set model. Tang et al. [48] introduced the interval type-2 fuzzy programming method with the heterogeneous relationship.

Based on the above information and after going through all the issues, we observed that the indicated issues are faced by professionals everywhere. For instance, existing ideas are widely used by researchers but are limited by poor parameterization techniques, and as a result, experts are unable to give accurate opinions. Additionally, the partial ignorance of the knowledge and its fluctuations at a given point in time cannot be expressed by the notion of FSs and SSs. However, in complex sets, information complexity and ambiguity happen at the same time as changes in the information's phase (periodicity). The enormous amounts of data generated by medical research, databases used by the public and private sectors for biometric and facial recognition and picture segmentation, which can contain huge amounts of difficult and partial data, involve complex and simple type information set. To evaluate the above problems, the main concept of IVCFSRs can simply handle the allocation of parameters as well as manage the phase term (periodicity) in the information, because the proposed concept is a very beneficial and reliable approach for managing the above-mentioned issues.

The traditional fuzzy soft relations (FSRs) only process information in one dimension at a time, which causes experts to lose a lot of information when making decisions. In numerous real-world situations, we have faced various problems where the membership level structure contains two-dimensional information. The loss of information can be prevented by incorporating the second-dimension information in the structure of FSRs. We provide some actual instances to illustrate the significance of the phase term. For example, both the public and private sectors aspire to use biometric technologies to track faculty members' attendance. For this, the institution's leader provides the following details about each system, referred to as the biometric device's model and production data. The traditional FSRs had a very small feature set, which was a crucial factor in their failure. Because the model and manufacturing date of the biometric device systems were represented by the real and imaginary parts of the truth grade, respectively, the theory of IVCFSRs was crucial in correctly evaluating all the concerns mentioned above. The proposed IVCFSR is highly competitive and feasible compared to common or pre-existing fuzzy, soft, complex fuzzy and complex soft sets.

The concept of the interval-valued fuzzy soft set (IVFSS) is a valuable tool in FSS theory for dealing with ambiguity and uncertainty. For the IVFSS, the concept of relations has not yet been established. Therefore, this paper introduces the concept of interval-valued complex fuzzy soft relations (IVCFSRs) based on the CP of two IVFSSs. Moreover, the types of IVCFSRs have been described, including the IVCFS reflexive relation, IVCFS irreflexive relation, IVCFS symmetric relation, IVCFS anti-symmetric relation, IVCFS asymmetric relation, IVCFS complete relation, IVCFS transitive relation, IVCFS equivalence relation, IVCFS partial order relation, IVCFS strict order relation, IVCFS preorder relation and IVCFS equivalence classes. Each definition of IVCFSs has been defined with examples. In addition, several results and properties have been proved for the type of IVCFSR. The innovative concepts of IVCFSRs are superior to the pre-defined structures of SSs, FSSs, CFSSs and IVFSSs and are mathematically and logically verified in a section of comparative analysis. The IVCFSS explains complex-valued membership degrees. The real term of each of the complex-valued functions is called the amplitude, and the imaginary term is called the phase term. This structure has the ability to resolve complex issues with unclear solutions. Additionally, the score functions have also been defined for these novel frameworks, which are obligatory for decision-making processes.

The remainder of this article is organized in the following way: Section 2 contains all of the pre-existing definitions of fuzzy algebra. Section 3 examines the CP of two IVCFSS and the newly defined concept of IVCFSRs. Section 4 proposes an application of transportation strategy by using the study of IVCFSRs. Section 5 compares the proposed structure with predefined frameworks. Section 6 concludes the article.

2. Preliminaries

In this section, we explain pre-existing models of fuzzy algebra, such as FSs, CFSs, IVFSs, IVCFSs, CP of IVFSs, IVFRs, SSs, SRs, FSSs, CFSSs, IVFSSs and IVCFSSs.

Definition 2.1. ([2]) Let $\hat{\mathrm{U}}$ be a nonempty set. An FS $\mathrm{Ҡ}$ is defined as

$$\mathrm{Ҡ} = \{(\upsilon, \dot{\omega}(\upsilon)) : \upsilon \in \hat{\mathrm{U}}\}$$

where $\dot{\omega}: \hat{\mathrm{U}} \to [0,1]$, and $\dot{\omega}(\upsilon)$ is known as the membership degree of υ in $\hat{\mathrm{U}}$.

Definition 2.2. ([12]) Let $\hat{\mathrm{U}}$ be a nonempty set. A CFS $\mathrm{Ҡ}$ is defined as

$$\mathrm{Ҡ} = \{(\upsilon, \dot{r}_{\dot{\omega}}(\upsilon)e^{t_{\dot{\omega}}(\upsilon)2\pi i}) : \upsilon \in \hat{\mathrm{U}}\}$$

where $\dot{r}_{\dot{\omega}}, t_{\dot{\omega}}: \hat{\mathrm{U}} \to [0,1]$, and $\dot{r}_{\dot{\omega}}, t_{\dot{\omega}}$ are known as the amplitude term and phase term of the membership degree, respectively.

Definition 2.3. ([18]) Let $\hat{\mathrm{U}}$ be a nonempty set. An IVFS $\mathrm{Ҡ}$ is defined as

$$\mathrm{Ҡ} = \{(\upsilon, [\dot{\omega}^-(\upsilon), \dot{\omega}^+(\upsilon)]) : \upsilon \in \hat{\mathrm{U}}\}$$

where $\dot{\omega}^-(\upsilon), \dot{\omega}^+(\upsilon): \hat{\mathrm{U}} \to [0,1]$ are the left and right points of the interval, respectively, such that $\dot{\omega}^-(\upsilon) \le \dot{\omega}^+(\upsilon)$.

Definition 2.4. ([22]) Let $\hat{\mathrm{U}}$ be a nonempty set. An IVCFS $\mathrm{Ҡ}$ is defined as

$$\mathrm{Ҡ} = \left\{\left(\upsilon, [\dot{r}^-_{\dot{\omega}}(\upsilon), \dot{r}^+_{\dot{\omega}}(\upsilon)]e^{2\pi[t^-_{\dot{\omega}}(\upsilon), t^+_{\dot{\omega}}(\upsilon)]i}\right) : \upsilon \in \hat{\mathrm{U}}\right\}$$

where $[\dot{r}^-_{\dot{\omega}}(\upsilon), \dot{r}^+_{\dot{\omega}}(\upsilon)] \subseteq [0,1]$ are amplitude terms, and $[t^-_{\dot{\omega}}(\upsilon), t^+_{\dot{\omega}}(\upsilon)] \subseteq [0,1]$ are the phase terms such that $\dot{r}^-_{\dot{\omega}}(\upsilon) \le \dot{r}^+_{\dot{\omega}}(\upsilon)$, and $t^-_{\dot{\omega}}(\upsilon) \le t^+_{\dot{\omega}}(\upsilon)$.

Definition 2.5. ([19]) If $\mathrm{Ҡ}_1 = \{(\upsilon_1, [\dot{\omega}^-(\upsilon_1), \dot{\omega}^+(\upsilon_1)]) : \upsilon_1 \in \hat{\mathrm{U}}\}$ and $\mathrm{Ҡ}_2 = \{(\upsilon_2, [\dot{\omega}^-(\upsilon_2), \dot{\omega}^+(\upsilon_2)]) : \upsilon_2 \in \hat{\mathrm{U}}\}$ are two IVFSs on a non-empty set $\hat{\mathrm{U}}$, then the CP of $\mathrm{Ҡ}_1$ and $\mathrm{Ҡ}_2$ is

$$\mathrm{Ҡ}_1 \times \mathrm{Ҡ}_2 = \left\{\left((\upsilon_1, \upsilon_2), [\dot{\omega}^-_{\mathrm{Ҡ}_1\times\mathrm{Ҡ}_2}(\upsilon_1, \upsilon_2), \dot{\omega}^+_{\mathrm{Ҡ}_1\times\mathrm{Ҡ}_2}(\upsilon_1, \upsilon_2)]\right) : \upsilon_1 \in \mathrm{Ҡ}_1, \upsilon_2 \in \mathrm{Ҡ}_2\right\}$$

where

$$\dot{\omega}^-_{\mathrm{Ҡ}_1\times\mathrm{Ҡ}_2}(\upsilon_1, \upsilon_2) = \min\{\dot{\omega}^-_{\mathrm{Ҡ}_1}(\upsilon_1), \dot{\omega}^-_{\mathrm{Ҡ}_2}(\upsilon_2) : \upsilon_1 \in \mathrm{Ҡ}_1, \upsilon_2 \in \mathrm{Ҡ}_2\},$$

and

$$\dot{\omega}^+_{\mathrm{Ҡ}_1\times\mathrm{Ҡ}_2}(\upsilon_1, \upsilon_2) = \min\{\dot{\omega}^+_{\mathrm{Ҡ}_1}(\upsilon_1), \dot{\omega}^+_{\mathrm{Ҡ}_2}(\upsilon_2) : \upsilon_1 \in \mathrm{Ҡ}_1, \upsilon_2 \in \mathrm{Ҡ}_2\}.$$

$\dot{\omega}^-_{\mathrm{Ҡ}_1\times\mathrm{Ҡ}_2}(\upsilon_1, \upsilon_2)$ and $\dot{\omega}^+_{\mathrm{Ҡ}_1\times\mathrm{Ҡ}_2}(\upsilon_1, \upsilon_2)$ are the left and the right points of the membership interval, respectively, such that $\dot{\omega}^-_{\mathrm{Ҡ}_1\times\mathrm{Ҡ}_2}(\upsilon_1, \upsilon_2) \le \dot{\omega}^+_{\mathrm{Ҡ}_1\times\mathrm{Ҡ}_2}(\upsilon_1, \upsilon_2)$.

Definition 2.6. ([19]) The subset of the CP of the two IVFS is known as the IVFR.

Example 2.7. If $\mathrm{Ҡ}_1 = \{(\upsilon_1, [0.2,0.5]), (\upsilon_2, [0.1,0.8]), (\upsilon_3, [0.4,0.6])\}$, and
$\mathrm{Ҡ}_2 = \{(\dot{x}_1, [0,0.3]), (\dot{x}_2, [0.4,0.7]), (\dot{x}_3, [0.3,0.4])\}$, then the CP of $\mathrm{Ҡ}_1$ and $\mathrm{Ҡ}_2$ is

$$Ҡ_1 \times Ҡ_2 = \begin{Bmatrix} ((\upsilon_1, \dot{x}_1), [0,0.3]), ((\upsilon_1, \dot{x}_2), [0.2,0.5]), ((\upsilon_1, \dot{x}_3), [0.2,0.4]), \\ ((\upsilon_2, \dot{x}_1), [0,0.3]), ((\upsilon_2, \dot{x}_2), [0.1,0.7]), ((\upsilon_2, \dot{x}_3), [0.1,0.4]), \\ ((\upsilon_3, \dot{x}_1), [0,0.3]), ((\upsilon_3, \dot{x}_2), [0.4,0.6]), ((\upsilon_3, \dot{x}_3), [0.3,0.4]) \end{Bmatrix}.$$

The IVFR $\hat{R}$ is

$$\hat{R} = \begin{Bmatrix} ((\upsilon_1, \dot{x}_1), [0,0.3]), ((\upsilon_2, \dot{x}_2), [0.1,0.7]), \\ ((\upsilon_3, \dot{x}_1), [0,0.3]), ((\upsilon_3, \dot{x}_3), [0.3,0.4]) \end{Bmatrix}.$$

Definition 2.8. ([24]) Let $\hat{U}$ be a nonempty set and $\acute{\bar{E}}$ be the set of parameters. Let $Ƥ(\hat{U})$ indicate the power set of $\hat{U}$ and $\bar{A} \subseteq \acute{\bar{E}}$. Then, SS $(Ҡ, \acute{\bar{E}})$ with a mapping $Ҡ: \bar{A} \rightarrow Ƥ(\hat{U})$ is defined by the set of ordered pairs as

$$Ҡ = \{\upsilon, Ҡ(\upsilon), \upsilon \in \acute{\bar{E}}, Ҡ(\upsilon) \in Ƥ(\hat{U})\}.$$

Definition 2.9. ([28]) Let $(Ҡ, \bar{A})$ and $(Ģ, \underline{B})$ be two SSs on $\hat{U}$ and $\bar{A}, \underline{B} \subseteq \acute{\bar{E}}$. Let $(Ҡ, \bar{A}) \times (Ģ, \underline{B}) = (Ḩ, Ç)$ with a mapping $Ḩ: Ç \rightarrow Ƥ(\hat{U})$. Then, the CP of the SSs is denoted and defined as

$$(Ҡ, \bar{A}) \times (Ģ, \underline{B}) = Ḩ(\breve{\imath}, \hat{\jmath}) = \{(\upsilon_{\breve{\imath}}, ķ_{\hat{\jmath}}): \upsilon_{\breve{\imath}} \in (Ҡ, \bar{A}), ķ_{\hat{\jmath}} \in (Ģ, \underline{B})\}.$$

Definition 2.10. ([28]) Let $(Ҡ, \bar{A})$ and $(Ģ, \underline{B})$ be two SSs on $\hat{U}$ and $\bar{A}, \underline{B} \subseteq \acute{\bar{E}}$. Then, a soft relation $\hat{R}$ is any subset of the CP of $(Ҡ, \bar{A}) \times (Ģ, \underline{B})$. It is denoted and defined as

$$\hat{R} = \{(\upsilon_{\breve{\imath}}, ķ_{\hat{\jmath}}): (\upsilon_{\breve{\imath}}, ķ_{\hat{\jmath}}) \in (Ҡ, \bar{A}) \times (Ģ, \underline{B})\}.$$

Definition 2.11. ([30]) Let $\hat{U}$ be a non-empty set and $\acute{\bar{E}}$ be the set of parameters. Let $Ƥ^{\hat{U}}$ indicate the set of all fuzzy subsets of $\hat{U}$ and $\bar{A} \subseteq \acute{\bar{E}}$. Then, an FSS $(Ҡ, \acute{\bar{E}})$ with a mapping $Ҡ: \bar{A} \rightarrow Ƥ^{\hat{U}}$ is given by the set of ordered pair as

$$Ҡ = \{(\upsilon, \dot{\omega}(\upsilon)): \upsilon \in \acute{\bar{E}}, \dot{\omega}(\upsilon) \in Ƥ^{\hat{U}}\},$$

where $\dot{\omega}(\upsilon)$ is known as the membership degree.

Example 2.12. Let $\hat{U}$ be a set of washing machine companies and $\acute{\bar{E}}$ be the set of parameters. Suppose that an FSS $(Ҡ, \acute{\bar{E}})$ illustrates the characteristics of the washing machine in relation to some parameter, and each membership degree is assigned by experts. $\hat{U} = \{\tilde{O}_1, \tilde{O}_2, \tilde{O}_3, \tilde{O}_4\}$, i.e., $\tilde{O}_1 =$ SuperAsia, $\tilde{O}_2 =$ Panasonic, $\tilde{O}_3 =$ Haier, and $\tilde{O}_4 =$ Dawlance. $\acute{\bar{E}} = \{\upsilon_1, \upsilon_2, \upsilon_3\}$, i.e., $\upsilon_1 =$ Efficiency, $\upsilon_2 =$ Size, $\upsilon_3 =$ Spin cycle, and $\upsilon_4 =$ Wash settings.

$$Ҡ(\upsilon_1) = \{\tilde{O}_1 = 0.3, \tilde{O}_2 = 0.5, \tilde{O}_3 = 0.6, \tilde{O}_4 = 0.1\},$$

$$Ҡ(\upsilon_2) = \{\tilde{O}_1 = 0.4, \tilde{O}_2 = 0.6, \tilde{O}_3 = 0.9, \tilde{O}_4 = 0.7\},$$

$$Ҡ(\upsilon_3) = \{\tilde{O}_1 = 0.2, \tilde{O}_2 = 0.8, \tilde{O}_3 = 0.3, \tilde{O}_4 = 0.6\},$$

$$Ƙ(\upsilon_4) = \{\tilde{\acute{O}}_1 = 0.1, \tilde{\acute{O}}_2 = 0.4, \tilde{\acute{O}}_3 = 0.6, \tilde{\acute{O}}_4 = 0.9\}.$$

$(Ƙ, \acute{\bar{E}})$ is a parameterized family $\{Ƙ(\upsilon_i), i = 1,2,3,4\}$.

Definition 2.13. ([37]) Let $\hat{U}$ be a non-empty set and $\acute{\bar{E}}$ be the set of parameters. Assume $C(ƤÛ)$ indicates the set of all complex fuzzy subsets of $\hat{U}$ and $\bar{A} \subseteq \acute{\bar{E}}$. Then, a CFSS $(Ƙ, \acute{\bar{E}})$ with mapping $Ƙ: \bar{A} \to C(Ƥ^{\hat{U}})$ is given by the set of ordered pairs as

$$Ƙ = \{(\upsilon, \dot{r}_{\dot{\omega}}(\upsilon) e^{t_{\dot{\omega}}(\upsilon) 2\pi i}) : \upsilon \in \acute{\bar{E}}\},$$

where $\dot{r}_{\dot{\omega}}$ and $t_{\dot{\omega}}$ are known as the amplitude term and phase term of the membership degree, respectively.

Definition 2.14. ([40]) Let $\hat{U}$ be a non-empty set and $\acute{\bar{E}}$ be the set of parameters. Assume $\hat{I}(Ƥ^{\hat{U}})$ indicates the set of all interval valued fuzzy subsets of $\hat{U}$ and $\bar{A} \subseteq \acute{\bar{E}}$. Then, an IVFSS $(Ƙ, \acute{\bar{E}})$ with a mapping $Ƙ: \bar{A} \to \hat{I}(Ƥ^{\hat{U}})$ is given by the set of ordered pairs as

$$Ƙ = \{(\upsilon, [\dot{\omega}^-(\upsilon), \dot{\omega}^+(\upsilon)]) : \upsilon \in \acute{\bar{E}}\}.$$

Example 2.15. From Example 2.12 assume an IVFSS $(Ƙ, \acute{\bar{E}})$ shows the characteristics of washing machines with reference to some parameter, and each interval valued membership degree is given by experts.

$$Ƙ(\upsilon_1) = \{\tilde{\acute{O}}_1 = [0.1,0.5], \tilde{\acute{O}}_2 = [0.2,0.4], \tilde{\acute{O}}_3 = [0.3,0.6], \tilde{\acute{O}}_4 = [0.4,0.5]\},$$

$$Ƙ(\upsilon_2) = \{\tilde{\acute{O}}_1 = [0.2,0.3], \tilde{\acute{O}}_2 = [0.1,0.4], \tilde{\acute{O}}_3 = [0.2,0.5], \tilde{\acute{O}}_4 = [0.4,0.6]\},$$

$$Ƙ(\upsilon_3) = \{\tilde{\acute{O}}_1 = [0.1,0.5], \tilde{\acute{O}}_2 = [0.2,0.7], \tilde{\acute{O}}_3 = [0.2,0.8], \tilde{\acute{O}}_4 = [0.3,0.4]\},$$

$$Ƙ(\upsilon_4) = \{\tilde{\acute{O}}_1 = [0.1,0.3], \tilde{\acute{O}}_2 = [0.3,0.4], \tilde{\acute{O}}_3 = [0.2,0.6], \tilde{\acute{O}}_4 = [0.4,0.7]\}.$$

Then, the IVFSS $(Ƙ, \acute{\bar{E}})$ is a parameterized family $\{Ƙ(\upsilon_i), i = 1,2,3,4\}$.

Definition 2.16. ([42]) Let $\hat{U}$ be a non-empty set and $\acute{\bar{E}}$ be the set of parameters. Let $C\hat{I}(Ƥ^{\hat{U}})$ indicate the set of all interval valued complex fuzzy subsets of $\hat{U}$ and $\bar{A} \subseteq \acute{\bar{E}}$. Then, an IVCFSS $(Ƙ, \acute{\bar{E}})$ with a mapping $Ƙ: \bar{A} \to C\hat{I}(Ƥ^{\hat{U}})$ is given by the set of ordered pairs as

$$Ƙ = \{(\upsilon, [\dot{\omega}_c^-(\upsilon), \dot{\omega}_c^+(\upsilon)]) : \upsilon \in \acute{\bar{E}}\} = \left\{\left(\upsilon, [\dot{r}_{\dot{\omega}}^-(\upsilon), \dot{r}_{\dot{\omega}}^+(\upsilon)] e^{2\pi [t_{\dot{\omega}}^-(\upsilon), t_{\dot{\omega}}^+(\upsilon)] i}\right) : \upsilon \in \acute{\bar{E}}\right\},$$

where $[\dot{r}_{\dot{\omega}}^-(\upsilon), \dot{r}_{\dot{\omega}}^+(\upsilon)]$ are left and right amplitude terms, and $[t_{\dot{\omega}}^-(\upsilon), t_{\dot{\omega}}^+(\upsilon)]$ are left and right phase terms.

3. Main results

In this section, we aim to demonstrate the novel concept of IVCFSRs by using the CP of two IVCFSSs, which are determined with the help of two different concepts, IVCF relations and soft sets. Moreover, we investigate different types of relations with the help of some suitable examples. The IVCFSR has an inclusive structure because it is discussing the degrees of membership with multidimensional variables.

Definition 3.1. Let $(Ҡ, \bar{A})$ and $(Ģ, \underline{B})$ be two IVCFSSs on $\hat{U}$ and $\bar{A}, \underline{B} \subseteq \acute{\bar{E}}$. Let $(Ҡ, \bar{A}) \times (Ģ, \underline{B}) = (Ḩ, Ç)$ with a mapping $Ҡ: \bar{A} \to C\hat{I}(Ƥ^{\hat{U}})$. Then, the CP of IVCFSSs

$$Ҡ_1 = \left\{\left(\upsilon, \left[\dot{r}^{-}_{\dot{\wp}}(\upsilon), \dot{r}^{+}_{\dot{\wp}}(\upsilon)\right] e^{2\pi\left[ţ^{-}_{\dot{\wp}}(\upsilon), ţ^{+}_{\dot{\wp}}(\upsilon)\right]i}\right) : \upsilon \in \acute{\bar{E}}\right\}$$

and

$$Ҡ_2 = \left\{\left(\breve{\imath}, \left[\dot{r}^{-}_{\dot{\wp}}(\breve{\imath}), \dot{r}^{+}_{\dot{\wp}}(\breve{\imath})\right] e^{2\pi\left[ţ^{-}_{\dot{\wp}}(\breve{\imath}), ţ^{+}_{\dot{\wp}}(\breve{\imath})\right]i}\right) : \breve{\imath} \in \acute{\bar{E}}\right\}$$

is denoted and defined as

$$(Ḩ, Ç) = Ҡ_1 \times Ҡ_2 = \left\{\begin{pmatrix} (\upsilon, \breve{\imath}), \left[\dot{r}^{-}_{(Ҡ_1 \times Ҡ_2)\dot{\wp}}(\upsilon, \breve{\imath}), \dot{r}^{+}_{(Ҡ_1 \times Ҡ_2)\dot{\wp}}(\upsilon, \breve{\imath})\right] \\ e^{2\pi\left[ţ^{-}_{(Ҡ_1 \times Ҡ_2)\dot{\wp}}(\upsilon, \breve{\imath}), ţ^{+}_{(Ҡ_1 \times Ҡ_2)\dot{\wp}}(\upsilon, \breve{\imath})\right]i} \end{pmatrix} : \upsilon, \breve{\imath} \in \acute{\bar{E}}\right\}$$

where $\dot{r}^{-}_{(Ҡ_1 \times Ҡ_2)\dot{\wp}}(\upsilon, \breve{\imath}) = \min\{\dot{r}^{-}_{\dot{\wp}}(\upsilon), \dot{r}^{-}_{\dot{\wp}}(\breve{\imath})\}$, $\dot{r}^{+}_{(Ҡ_1 \times Ҡ_2)\dot{\wp}}(\upsilon, \breve{\imath}) = \min\{\dot{r}^{+}_{\dot{\wp}}(\upsilon), \dot{r}^{+}_{\dot{\wp}}(\breve{\imath})\}$,

$$ţ^{-}_{(Ҡ_1 \times Ҡ_2)\dot{\wp}}(\upsilon, \breve{\imath}) = \min\{ţ^{-}_{\dot{\wp}}(\upsilon), ţ^{-}_{\dot{\wp}}(\breve{\imath})\},\ ţ^{+}_{(Ҡ_1 \times Ҡ_2)\dot{\wp}}(\upsilon, \breve{\imath}) = \min\{ţ^{+}_{\dot{\wp}}(\upsilon), ţ^{+}_{\dot{\wp}}(\breve{\imath})\}.$$

Example 3.2. Let $\hat{U}$ be a set of car companies and $\acute{\bar{E}}$ be the set of parameters. Suppose that an IVCFSS $(Ҡ, \acute{\bar{E}})$ describes the characteristics of a car in relation to some parameters, and each interval valued membership degree is assigned by experts. $\hat{U} = \{\tilde{\acute{O}}_1, \tilde{\acute{O}}_2, \tilde{\acute{O}}_3\}$, i.e., $\tilde{\acute{O}}_1 =$ Toyota, $\tilde{\acute{O}}_2 =$ Suzuki, and $\tilde{\acute{O}}_3 =$ Hyundai. There are three parameters: $\acute{\bar{E}} = \{\upsilon_1, \upsilon_2, \upsilon_3\}$, i.e., $\upsilon_1 =$ Reliable, $\upsilon_2 =$ Attractive, $\upsilon_3 =$ affordable to purchase. Let $(Ҡ, \bar{A})$ and $(Ģ, \underline{B})$ be two IVCFSSs by two experts $\bar{A}$ and $\underline{B}$, respectively, and their correspondent interval valued membership degrees are as follows:

$$(Ҡ, \bar{A}) = \left\{\begin{matrix} \left(\upsilon_1, [0.2e^{0.3\pi i}, 0.6e^{0.4\pi i}], [0.1e^{0.1\pi i}, 0.5e^{0.2\pi i}], [0.3e^{0.3\pi i}, 0.4e^{0.4\pi i}], [0.2e^{0.1\pi i}, 0.7e^{0.5\pi i}]\right), \\ \left(\upsilon_2, [0.3e^{0.1\pi i}, 0.6e^{0.8\pi i}], [0.4e^{0.3\pi i}, 0.5e^{0.6\pi i}], [0.2e^{0.3\pi i}, 0.3e^{0.5\pi i}], [0.1e^{0.3\pi i}, 0.6e^{0.4\pi i}]\right), \\ \left(\upsilon_3, [0.3e^{0.2\pi i}, 0.7e^{0.5\pi i}], [0.2e^{0\pi i}, 0.3e^{0.3\pi i}], [0.4e^{0.1\pi i}, 0.6e^{0.3\pi i}], [0.3e^{0.2\pi i}, 0.5e^{0.6\pi i}]\right) \end{matrix}\right\}$$

and

$$(Ģ, \underline{B}) = \left\{\begin{matrix} \left(\upsilon_1, [0.1e^{0.1\pi i}, 0.4e^{0.3\pi i}], [0.2e^{0.2\pi i}, 0.6e^{0.3\pi i}], [0.3e^{0.4\pi i}, 0.5e^{0.5\pi i}], [0.2e^{0\pi i}, 0.4e^{0.4\pi i}]\right), \\ \left(\upsilon_2, [0.3e^{0.3\pi i}, 0.5e^{0.5\pi i}], [0.3e^{0\pi i}, 0.4e^{0.5\pi i}], [0.2e^{0.2\pi i}, 0.4e^{0.6\pi i}], [0.1e^{0.4\pi i}, 0.5e^{0.5\pi i}]\right), \\ \left(\upsilon_3, [0e^{0.4\pi i}, 0.8e^{0.6\pi i}], [0.1e^{0.1\pi i}, 0.3e^{0.3\pi i}], [0.1e^{0.3\pi i}, 0.7e^{0.4\pi i}], [0.4e^{0.2\pi i}, 0.6e^{0.4\pi i}]\right) \end{matrix}\right\}$$

In the above observations, the first three values of each parameter show the interval valued membership of each company, and the fourth value shows the general belongingness of each parameter to the company. Each row represents the parametric observations. Then, the CP of $(Ҡ, \bar{A})$ and $(Ģ, \underline{B})$ is defined as in Table 1.

Table 1. Cartesian product.

Ordered pair	$\tilde{\mathrm{O}}_1$	$\tilde{\mathrm{O}}_2$	$\tilde{\mathrm{O}}_3$	λ
(υ_1, υ_1)	$[0.1e^{0.1\pi i}, 0.4e^{0.3\pi i}]$	$[0.1e^{0.1\pi i}, 0.5e^{0.2\pi i}]$	$[0.3e^{0.3\pi i}, 0.4e^{0.4\pi i}]$	$[0.2e^{0\pi i}, 0.4e^{0.4\pi i}]$
(υ_1, υ_2)	$[0.2e^{0.3\pi i}, 0.5e^{0.4\pi i}]$	$[0.1e^{0\pi i}, 0.4e^{0.2\pi i}]$	$[0.2e^{0.2\pi i}, 0.4e^{0.4\pi i}]$	$[0.1e^{0.1\pi i}, 0.5e^{0.5\pi i}]$
(υ_1, υ_3)	$[0e^{0.3\pi i}, 0.6e^{0.4\pi i}]$	$[0.1e^{0.1\pi i}, 0.3e^{0.2\pi i}]$	$[0.1e^{0.3\pi i}, 0.4e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.6e^{0.4\pi i}]$
(υ_2, υ_1)	$[0.1e^{0.1\pi i}, 0.4e^{0.3\pi i}]$	$[0.2e^{0.2\pi i}, 0.5e^{0.3\pi i}]$	$[0.2e^{0.3\pi i}, 0.3e^{0.5\pi i}]$	$[0.1e^{0\pi i}, 0.4e^{0.4\pi i}]$
(υ_2, υ_2)	$[0.3e^{0.1\pi i}, 0.5e^{0.5\pi i}]$	$[0.3e^{0\pi i}, 0.4e^{0.5\pi i}]$	$[0.2e^{0.2\pi i}, 0.3e^{0.5\pi i}]$	$[0.1e^{0.3\pi i}, 0.5e^{0.4\pi i}]$
(υ_2, υ_3)	$[0e^{0.1\pi i}, 0.6e^{0.6\pi i}]$	$[0.1e^{0.1\pi i}, 0.3e^{0.3\pi i}]$	$[0.1e^{0.3\pi i}, 0.3e^{0.4\pi i}]$	$[0.1e^{0.2\pi i}, 0.6e^{0.4\pi i}]$
(υ_3, υ_1)	$[0.1e^{0.1\pi i}, 0.4e^{0.3\pi i}]$	$[0.2e^{0\pi i}, 0.3e^{0.3\pi i}]$	$[0.3e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.2e^{0\pi i}, 0.4e^{0.4\pi i}]$
(υ_3, υ_2)	$[0.3e^{0.2\pi i}, 0.5e^{0.5\pi i}]$	$[0.2e^{0\pi i}, 0.3e^{0.3\pi i}]$	$[0.2e^{0.1\pi i}, 0.4e^{0.5\pi i}]$	$[0.1e^{0.2\pi i}, 0.5e^{0.5\pi i}]$
(υ_3, υ_3)	$[0e^{0.2\pi i}, 0.7e^{0.5\pi i}]$	$[0.1e^{0\pi i}, 0.3e^{0.3\pi i}]$	$[0.1e^{0.1\pi i}, 0.6e^{0.3\pi i}]$	$[0.3e^{0.2\pi i}, 0.5e^{0.4\pi i}]$

Definition 3.3. The IVCFSR denoted by $\hat{\mathrm{R}}$ is a subset of the CP of two IVCFSSs, where $\hat{\mathrm{R}}(\bar{\mathrm{A}}, \underline{\mathrm{B}}) \subseteq (\mathrm{Ҡ}, \bar{\mathrm{A}}) \times (\mathrm{Ģ}, \underline{\mathrm{B}})$.

Example 3.4. From Table 1, choose a subset of the CP. Then, the IVCFSR $\hat{\mathrm{R}}$ is

$$\hat{\mathrm{R}} = \left\{ \begin{array}{l} \left((\upsilon_1, \upsilon_2), \begin{bmatrix} 0.2e^{0.3\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.4e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.5e^{0.5\pi i} \end{bmatrix} \right), \\ \left((\upsilon_1, \upsilon_3), \begin{bmatrix} 0e^{0.3\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.3e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.3\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.1\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix} \right), \\ \left((\upsilon_2, \upsilon_2), \begin{bmatrix} 0.3e^{0.1\pi i}, \\ 0.5e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0\pi i}, \\ 0.4e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.3e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.3\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix} \right), \\ \left((\upsilon_3, \upsilon_1), \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.4e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0\pi i}, \\ 0.3e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0.1\pi i}, \\ 0.5e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix} \right), \\ \left((\upsilon_3, \upsilon_3), \begin{bmatrix} 0e^{0.2\pi i}, \\ 0.7e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.3e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.6e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0.2\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix} \right) \end{array} \right\}.$$

Definition 3.5. Suppose that $(\mathrm{Ҡ}, \bar{\mathrm{A}})$ is an IVCFSR on $\hat{\mathrm{U}}$, and

$$\hat{\mathrm{R}} = \left\{ \left((\upsilon, \breve{\imath}), \left[\dot{r}^{-}_{\underset{\cdot}{\omega}}(\upsilon, \breve{\imath}), \dot{r}^{+}_{\underset{\cdot}{\omega}}(\upsilon, \breve{\imath}) \right] e^{2\pi \left[\mathrm{t}^{-}_{\underset{\cdot}{\omega}}(\upsilon, \breve{\imath}), \mathrm{t}^{+}_{\underset{\cdot}{\omega}}(\upsilon, \breve{\imath}) \right] i} \right) : \upsilon, \breve{\imath} \in \hat{\mathrm{R}} \right\}.$$

Then, the inverse of the IVCFSR is denoted by $\hat{\mathrm{R}}^{-1}$ and is defined as

$$\hat{\mathrm{R}}^{-1} = \left\{ \left((\breve{\imath}, \upsilon), \left[\dot{r}^{-}_{\underset{\cdot}{\omega}}(\breve{\imath}, \upsilon), \dot{r}^{+}_{\underset{\cdot}{\omega}}(\breve{\imath}, \upsilon) \right] e^{2\pi \left[\mathrm{t}^{-}_{\underset{\cdot}{\omega}}(\breve{\imath}, \upsilon), \mathrm{t}^{+}_{\underset{\cdot}{\omega}}(\breve{\imath}, \upsilon) \right] i} \right) : \breve{\imath}, \upsilon \in \hat{\mathrm{R}} \right\}.$$

Definition 3.6. Suppose that an IVCFSR $\hat{\mathrm{R}}$ on $(\mathrm{Ҡ}, \bar{\mathrm{A}})$ is known as an IVCFS-reflexive relation if

$$\forall \left((\upsilon), \left[\dot{r}^{-}_{\underset{\cdot}{\omega}}(\upsilon), \dot{r}^{+}_{\underset{\cdot}{\omega}}(\upsilon) \right] e^{2\pi \left[\mathrm{t}^{-}_{\underset{\cdot}{\omega}}(\upsilon), \mathrm{t}^{+}_{\underset{\cdot}{\omega}}(\upsilon) \right] i} \right) \in \hat{\mathrm{R}}$$

$$\Leftrightarrow \forall\left((\upsilon,\upsilon),\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\upsilon),\dot{r}^{+}_{\dot{\omega}}(\upsilon,\upsilon)\right]e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\upsilon),t^{+}_{\dot{\omega}}(\upsilon,\upsilon)\right]i}\right)\in\hat{R}.$$

Definition 3.7. Suppose that an IVCFSR $\hat{R}$ on $(Ꝃ,\bar{A})$ is known as an IVCFS-irreflexive relation if

$$\forall\left((\upsilon),\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon),\dot{r}^{+}_{\dot{\omega}}(\upsilon)\right]e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon),t^{+}_{\dot{\omega}}(\upsilon)\right]i}\right)\in\hat{R}$$

$$\Leftrightarrow \forall\left((\upsilon,\upsilon),\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\upsilon),\dot{r}^{+}_{\dot{\omega}}(\upsilon,\upsilon)\right]e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\upsilon),t^{+}_{\dot{\omega}}(\upsilon,\upsilon)\right]i}\right)\notin\hat{R}.$$

Definition 3.8. An IVCFSR $\hat{R}$ on Ꝃ is called an IVCFS-Symmetric-relation if

$$\forall\left((\upsilon),\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon),\dot{r}^{+}_{\dot{\omega}}(\upsilon)\right]e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon),t^{+}_{\dot{\omega}}(\upsilon)\right]i}\right),\left((\breve{\imath}),\left[\dot{r}^{-}_{\dot{\omega}}(\breve{\imath}),\dot{r}^{+}_{\dot{\omega}}(\breve{\imath})\right]e^{2\pi\left[t^{-}_{\dot{\omega}}(\breve{\imath}),t^{+}_{\dot{\omega}}(\breve{\imath})\right]i}\right)\in\hat{R}\text{ and }\upsilon,\breve{\imath}\in Ꝃ$$

$$\text{if}\left((\upsilon,\breve{\imath}),\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}),\dot{r}^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}),t^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]i}\right)\in\hat{R}$$

$$\Rightarrow\left((\breve{\imath},\upsilon),\left[\dot{r}^{-}_{\dot{\omega}}(\breve{\imath},\upsilon),\dot{r}^{+}_{\dot{\omega}}(\breve{\imath},\upsilon)\right]e^{2\pi\left[t^{-}_{\dot{\omega}}(\breve{\imath},\upsilon),t^{+}_{\dot{\omega}}(\breve{\imath},\upsilon)\right]i}\right)\in\hat{R}\,.$$

Definition 3.9. An IVCFSR $\hat{R}$ on Ꝃ is called an IVCFS-transitive relation if

$$\forall\left((\upsilon),\begin{matrix}\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon),\dot{r}^{+}_{\dot{\omega}}(\upsilon)\right]\\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon),t^{+}_{\dot{\omega}}(\upsilon)\right]i}\end{matrix}\right),\left((\breve{\imath}),\begin{matrix}\left[\dot{r}^{-}_{\dot{\omega}}(\breve{\imath}),\dot{r}^{+}_{\dot{\omega}}(\breve{\imath})\right]\\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\breve{\imath}),t^{+}_{\dot{\omega}}(\breve{\imath})\right]i}\end{matrix}\right),\left((\acute{g}),\begin{matrix}\left[\dot{r}^{-}_{\dot{\omega}}(\acute{g}),\dot{r}^{+}_{\dot{\omega}}(\acute{g})\right]\\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\acute{g}),t^{+}_{\dot{\omega}}(\acute{g})\right]i}\end{matrix}\right)\in\hat{R}\text{ and }\upsilon,\breve{\imath},\acute{g}\in Ꝃ$$

$$\text{Then}\begin{pmatrix}(\upsilon,\breve{\imath}),\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}),\dot{r}^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]\\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}),t^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]i}\end{pmatrix}\in\hat{R}\text{ and}\begin{pmatrix}(\breve{\imath},\acute{g}),\left[\dot{r}^{-}_{\dot{\omega}}(\breve{\imath},\acute{g}),\dot{r}^{+}_{\dot{\omega}}(\breve{\imath},\acute{g})\right]\\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\breve{\imath},\acute{g}),t^{+}_{\dot{\omega}}(\breve{\imath},\acute{g})\right]i}\end{pmatrix}\in\hat{R}$$

$$\Rightarrow\begin{pmatrix}(\upsilon,\acute{g}),\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\acute{g}),\dot{r}^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right]\\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\acute{g}),t^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right]i}\end{pmatrix}\in\hat{R}.$$

Definition 3.10. Suppose that an IVCFSR $\hat{R}$ on Ꝃ is known as an IVCFS-equivalence relation if it is

- ❖ Reflexive,
- ❖ Symmetric,
- ❖ Transitive.

Example 3.11. Table 1 shows the CP of IVCFSRs. The IVCFS-Equivalence- $\hat{R}$ is

$$\hat{R}=\left\{\begin{array}{l}\left((\upsilon_1,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left((\upsilon_1,\upsilon_2),\begin{bmatrix}0.2e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.4e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix}\right),\\ \left((\upsilon_1,\upsilon_3),\begin{bmatrix}0e^{0.3\pi i},\\0.6e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.3e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.1\pi i},\\0.6e^{0.4\pi i}\end{bmatrix}\right)\\ \left((\upsilon_2,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.5e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.3\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left((\upsilon_2,\upsilon_2),\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0\pi i},\\0.4e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right)\\ \left((\upsilon_3,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left((\upsilon_3,\upsilon_3),\begin{bmatrix}0e^{0.2\pi i},\\0.7e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.6e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.2\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right)\end{array}\right\}.$$

Definition 3.12. Assume that an IVCFSR $\hat{R}$ on Ҡ is known as an IVCFS-partial order relation if it is

- ❖ Reflexive,
- ❖ Anti-Symmetric,
- ❖ Transitive.

Example 3.13. Table 1 shows the CP of IVCFSRs. The IVCFS-partial order- $\hat{R}$ is

$$\hat{R}=\left\{\begin{array}{l}\left((\upsilon_1,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left((\upsilon_1,\upsilon_2),\begin{bmatrix}0.2e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.4e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix}\right),\\ \left((\upsilon_2,\upsilon_2),\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0\pi i},\\0.4e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right),\\ (\upsilon_3,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\\ \left((\upsilon_3,\upsilon_2),\begin{bmatrix}0.3e^{0.2\pi i},\\0.5e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.1\pi i},\\0.4e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.2\pi i},\\0.5e^{0.5\pi i}\end{bmatrix}\right)\\ \left((\upsilon_3,\upsilon_3),\begin{bmatrix}0e^{0.2\pi i},\\0.7e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.6e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.2\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right)\end{array}\right\}.$$

Definition 3.14. Assume that IVCFSR $\hat{R}$ on Ҡ is known as an IVCFS-linear order relation if it is

- ❖ Reflexive,
- ❖ Anti-Symmetric,
- ❖ Transitive,
- ❖ Complete.

Example 3.15. Table 1 shows the CP of IVCFSRs. The IVCFS-linear order- $\hat{R}$ is

$$\hat{R} = \left\{\begin{array}{l}
\left((\upsilon_1,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\
\left((\upsilon_2,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.5e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.3\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\
\left((\upsilon_2,\upsilon_2),\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0\pi i},\\0.4e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right),\\
\left((\upsilon_3,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\
\left((\upsilon_3,\upsilon_2),\begin{bmatrix}0.3e^{0.2\pi i},\\0.5e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.1\pi i},\\0.4e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.2\pi i},\\0.5e^{0.5\pi i}\end{bmatrix}\right)\\
\left((\upsilon_3,\upsilon_3),\begin{bmatrix}0e^{0.2\pi i},\\0.7e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.6e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.2\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right)
\end{array}\right\}.$$

Definition 3.16. Assume that an IVCFSR $\hat{R}$ on Ƙ is known as an IVCFS-strict order relation if it is

- Irreflexive,
- Transitive.

Example 3.17. Table 1 shows the CP of IVCFSRs. The IVCFS-strict order- $\hat{R}$ is

$$\hat{R} = \left\{\begin{array}{l}
\left((\upsilon_1,\upsilon_2),\begin{bmatrix}0.2e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.4e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix}\right),\\
\left((\upsilon_1,\upsilon_3),\begin{bmatrix}0e^{0.3\pi i},\\0.6e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.3e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.1\pi i},\\0.6e^{0.4\pi i}\end{bmatrix}\right),\\
\left((\upsilon_2,\upsilon_3),\begin{bmatrix}0e^{0.1\pi i},\\0.6e^{0.6\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.3e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.2\pi i},\\0.6e^{0.4\pi i}\end{bmatrix}\right)
\end{array}\right\}.$$

Definition 3.18. An IVCFSR $\hat{R}$ on Ƙ is called an IVCFS equivalence class of υ modulo $\hat{R}$ and is defined as

$$\hat{R}[\upsilon] = \left\{\left((\upsilon),\begin{matrix}\left[\dot{r}^{-}_{\hat{\omega}}(\upsilon),\dot{r}^{+}_{\hat{\omega}}(\upsilon)\right]\\ e^{2\pi\left[t^{-}_{\hat{\omega}}(\upsilon),t^{+}_{\hat{\omega}}(\upsilon)\right]i}\end{matrix}\right):\left((\breve{\imath},\upsilon),\begin{matrix}\left[\dot{r}^{-}_{\hat{\omega}}(\breve{\imath},\upsilon),\dot{r}^{+}_{\hat{\omega}}(\breve{\imath},\upsilon)\right]\\ e^{2\pi\left[t^{-}_{\hat{\omega}}(\breve{\imath},\upsilon),t^{+}_{\hat{\omega}}(\breve{\imath},\upsilon)\right]i}\end{matrix}\right)\in\hat{R}\right\}.$$

Example 3.19. Table 1 shows the CP of IVCFSRs. The IVCFS-Equivalence-R is

$$\hat{R}=\left\{\begin{array}{l}\left((\upsilon_1,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left((\upsilon_1,\upsilon_2),\begin{bmatrix}0.2e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.4e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix}\right),\\ \left((\upsilon_1,\upsilon_3),\begin{bmatrix}0e^{0.3\pi i},\\0.6e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.3e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.1\pi i},\\0.6e^{0.4\pi i}\end{bmatrix}\right)\\ \left((\upsilon_2,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.5e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.3\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left((\upsilon_2,\upsilon_2),\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0\pi i},\\0.4e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right)\\ \left((\upsilon_3,\upsilon_1),\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left((\upsilon_3,\upsilon_3),\begin{bmatrix}0e^{0.2\pi i},\\0.7e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.6e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.2\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right)\end{array}\right\}.$$

Then, the IVCFS equivalence classes are as follows.

a. υ_1 modulo $\hat{R}$ is given as

$$\hat{R}[\upsilon_1]=\left\{\begin{array}{l}\left(\upsilon_1,\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left(\upsilon_2,\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0\pi i},\\0.4e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right),\\ \left(\upsilon_3,\begin{bmatrix}0e^{0.2\pi i},\\0.7e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.6e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.2\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right)\end{array}\right\}.$$

b. υ_2 modulo $\hat{R}$ is given as

$$\hat{R}[\upsilon_2]=\left\{\begin{array}{l}\left(\upsilon_1,\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left(\upsilon_2,\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0\pi i},\\0.4e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0.2\pi i},\\0.3e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.3\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right),\end{array}\right\}.$$

c. υ_3 modulo $\hat{R}$ is given as

$$\hat{R}[\upsilon_3]=\left\{\begin{array}{l}\left(\upsilon_1,\begin{bmatrix}0.1e^{0.1\pi i},\\0.4e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.5e^{0.2\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.3\pi i},\\0.4e^{0.4\pi i}\end{bmatrix},\begin{bmatrix}0.2e^{0\pi i},\\0.4e^{0.4\pi i}\end{bmatrix}\right),\\ \left(\upsilon_3,\begin{bmatrix}0e^{0.2\pi i},\\0.7e^{0.5\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0\pi i},\\0.3e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.1e^{0.1\pi i},\\0.6e^{0.3\pi i}\end{bmatrix},\begin{bmatrix}0.3e^{0.2\pi i},\\0.5e^{0.4\pi i}\end{bmatrix}\right)\end{array}\right\}.$$

Definition 3.20. For IVCFSRs $\hat{R}_1$ *and* $\hat{R}_2$ *on* K is defined as

$$\begin{pmatrix}(\upsilon,\breve{\imath}),\left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}),\dot{r}^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]\\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}),t^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]i}\end{pmatrix}\in\hat{R}_1 \text{ and } \begin{pmatrix}(\breve{\imath},\acute{g}),\left[\dot{r}^{-}_{\dot{\omega}}(\breve{\imath},\acute{g}),\dot{r}^{+}_{\dot{\omega}}(\breve{\imath},\acute{g})\right]\\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\breve{\imath},\acute{g}),t^{+}_{\dot{\omega}}(\breve{\imath},\acute{g})\right]i}\end{pmatrix}\in\hat{R}_2$$

$$\Rightarrow \begin{pmatrix} (\upsilon, \acute{g}), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon, \acute{g}), \dot{r}^{+}_{\dot{\omega}}(\upsilon, \acute{g})\right] \\ e^{2\pi\left[\dot{t}^{-}_{\dot{\omega}}(\upsilon,\acute{g}), \dot{t}^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right]i} \end{pmatrix} \in \hat{R}_1 \circ \hat{R}_2.$$

$$\hat{R} = \left\{\begin{matrix}
\left((\upsilon_1, \upsilon_1), \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.4e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.5e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0.3\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}\right), \\
\left((\upsilon_1, \upsilon_2), \begin{bmatrix} 0.2e^{0.3\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.4e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.5e^{0.5\pi i} \end{bmatrix}\right), \\
\left((\upsilon_1, \upsilon_3), \begin{bmatrix} 0e^{0.3\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.3e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.3\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.1\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix}\right) \\
\left((\upsilon_2, \upsilon_1), \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.4e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.5e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.3\pi i}, \\ 0.3e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}\right), \\
\left((\upsilon_2, \upsilon_2), \begin{bmatrix} 0.3e^{0.1\pi i}, \\ 0.5e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0\pi i}, \\ 0.4e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.3e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.3\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix}\right) \\
\left((\upsilon_3, \upsilon_1), \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.4e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0\pi i}, \\ 0.3e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0.1\pi i}, \\ 0.5e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}\right), \\
\left((\upsilon_3, \upsilon_3), \begin{bmatrix} 0e^{0.2\pi i}, \\ 0.7e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.3e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.6e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0.2\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix}\right)
\end{matrix}\right\}.$$

Example 3.21. Table 1 shows the CP of IVCFSRs. Two relations $\hat{R}_1$ and $\hat{R}_2$ are

$$\hat{R}_1 = \left\{\begin{matrix}
\left((\upsilon_1, \upsilon_2), \begin{bmatrix} 0.2e^{0.3\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.4e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.5e^{0.5\pi i} \end{bmatrix}\right), \\
\left((\upsilon_2, \upsilon_3), \begin{bmatrix} 0e^{0.1\pi i}, \\ 0.6e^{0.6\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.3e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.3\pi i}, \\ 0.3e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.2\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix}\right), \\
\left((\upsilon_1, \upsilon_3), \begin{bmatrix} 0e^{0.3\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.3e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.3\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.1\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix}\right)
\end{matrix}\right\}$$

and

$$\hat{R}_2 = \left\{\begin{matrix}
\left((\upsilon_2, \upsilon_2), \begin{bmatrix} 0.3e^{0.1\pi i}, \\ 0.5e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0\pi i}, \\ 0.4e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.3e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.3\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix}\right), \\
\left((\upsilon_3, \upsilon_1), \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.4e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0\pi i}, \\ 0.3e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0.1\pi i}, \\ 0.5e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}\right), \\
\left((\upsilon_3, \upsilon_3), \begin{bmatrix} 0e^{0.2\pi i}, \\ 0.7e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.3e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.6e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.3e^{0.2\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix}\right)
\end{matrix}\right\}.$$

Then, the IVCFS-Composite-$\hat{R}$ is given as

$$\hat{R}_1 \circ \hat{R}_2 = \left\{ \begin{array}{l} \left((\upsilon_1, \upsilon_2), \begin{bmatrix} 0.2e^{0.3\pi i}, \\ 0.5e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.4e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.5e^{0.5\pi i} \end{bmatrix} \right), \\ \left((\upsilon_2, \upsilon_1), \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.4e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.2\pi i}, \\ 0.5e^{0.3\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.3\pi i}, \\ 0.3e^{0.5\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix} \right), \\ \left((\upsilon_1, \upsilon_3), \begin{bmatrix} 0e^{0.3\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.1\pi i}, \\ 0.3e^{0.2\pi i} \end{bmatrix}, \begin{bmatrix} 0.1e^{0.3\pi i}, \\ 0.4e^{0.4\pi i} \end{bmatrix}, \begin{bmatrix} 0.2e^{0.1\pi i}, \\ 0.6e^{0.4\pi i} \end{bmatrix} \right) \end{array} \right\}.$$

Theorem 3.22. Assume that IVCFSR $\hat{R}$ is an IVCFS symmetric relation on IVCFSS Ҡ iff $\hat{R} = \hat{R}^c$.

Proof. Suppose that $\hat{R} = \hat{R}^c$. Then, $\begin{pmatrix} (\upsilon, \breve{\imath}), \left[\dot{r}^{-}_{\omega}(\upsilon, \breve{\imath}), \dot{r}^{+}_{\omega}(\upsilon, \breve{\imath})\right] \\ e^{2\pi\left[t^{-}_{\omega}(\upsilon,\breve{\imath}), t^{+}_{\omega}(\upsilon,\breve{\imath})\right] i} \end{pmatrix} \in \hat{R}$

$$\Rightarrow \begin{pmatrix} (\breve{\imath}, \upsilon), \left[\dot{r}^{-}_{\omega}(\breve{\imath}, \upsilon), \dot{r}^{+}_{\omega}(\breve{\imath}, \upsilon)\right] \\ e^{2\pi\left[t^{-}_{\omega}(\breve{\imath},\upsilon), t^{+}_{\omega}(\breve{\imath},\upsilon)\right] i} \end{pmatrix} \in \hat{R}^c$$

$$\Rightarrow \begin{pmatrix} (\breve{\imath}, \upsilon), \left[\dot{r}^{-}_{\omega}(\breve{\imath}, \upsilon), \dot{r}^{+}_{\omega}(\breve{\imath}, \upsilon)\right] \\ e^{2\pi\left[t^{-}_{\omega}(\breve{\imath},\upsilon), t^{+}_{\omega}(\breve{\imath},\upsilon)\right] i} \end{pmatrix} \in \hat{R}.$$

Thus, $\hat{R}$ is an IVCFS-Symmetric relation on an IVCFSS Ҡ.

Conversely, assume that $\hat{R}$ is an IVCFS-symmetric-relation on an IVCFSS Ҡ. Then,

$$\begin{pmatrix} (\upsilon, \breve{\imath}), \left[\dot{r}^{-}_{\omega}(\upsilon, \breve{\imath}), \dot{r}^{+}_{\omega}(\upsilon, \breve{\imath})\right] \\ e^{2\pi\left[t^{-}_{\omega}(\upsilon,\breve{\imath}), t^{+}_{\omega}(\upsilon,\breve{\imath})\right] i} \end{pmatrix} \in \hat{R} \Rightarrow \begin{pmatrix} (\breve{\imath}, \upsilon), \left[\dot{r}^{-}_{\omega}(\breve{\imath}, \upsilon), \dot{r}^{+}_{\omega}(\breve{\imath}, \upsilon)\right] \\ e^{2\pi\left[t^{-}_{\omega}(\breve{\imath},\upsilon), t^{+}_{\omega}(\breve{\imath},\upsilon)\right] i} \end{pmatrix} \in \hat{R}.$$

However, $\begin{pmatrix} (\breve{\imath}, \upsilon), \left[\dot{r}^{-}_{\omega}(\breve{\imath}, \upsilon), \dot{r}^{+}_{\omega}(\breve{\imath}, \upsilon)\right] \\ e^{2\pi\left[t^{-}_{\omega}(\breve{\imath},\upsilon), t^{+}_{\omega}(\breve{\imath},\upsilon)\right] i} \end{pmatrix} \in \hat{R}^c$

$$\Rightarrow \hat{R} = \hat{R}^c.$$

Theorem 3.23. Suppose that an IVCFSR $\hat{R}$ is an IVCFS-transitive relation on IVCFSS Ҡ iff $\hat{R} \circ \hat{R} \subseteq \hat{R}^c$.

Proof. Suppose that $\hat{R}$ is an IVCFS-transitive relation on IVCFSS Ҡ.

Let $\begin{pmatrix} (\upsilon, \acute{g}), \left[\dot{r}^{-}_{\omega}(\upsilon, \acute{g}), \dot{r}^{+}_{\omega}(\upsilon, \acute{g})\right] \\ e^{2\pi\left[t^{-}_{\omega}(\upsilon,\acute{g}), t^{+}_{\omega}(\upsilon,\acute{g})\right] i} \end{pmatrix} \in \hat{R} \circ \hat{R}.$

Then, by the definition of an IVCFS-transitive-relation,

$$\begin{pmatrix} (\upsilon, \breve{\imath}), \left[\dot{r}^{-}_{\omega}(\upsilon, \breve{\imath}), \dot{r}^{+}_{\omega}(\upsilon, \breve{\imath})\right] \\ e^{2\pi\left[t^{-}_{\omega}(\upsilon,\breve{\imath}), t^{+}_{\omega}(\upsilon,\breve{\imath})\right] i} \end{pmatrix} \in \hat{R} \text{ and } \begin{pmatrix} (\breve{\imath}, \acute{g}), \left[\dot{r}^{-}_{\omega}(\breve{\imath}, \acute{g}), \dot{r}^{+}_{\omega}(\breve{\imath}, \acute{g})\right] \\ e^{2\pi\left[t^{-}_{\omega}(\breve{\imath},\acute{g}), t^{+}_{\omega}(\breve{\imath},\acute{g})\right] i} \end{pmatrix} \in \hat{R}$$

$$\begin{pmatrix} (\upsilon, \acute{g}), \left[\dot{r}^{-}_{\omega}(\upsilon, \acute{g}), \dot{r}^{+}_{\omega}(\upsilon, \acute{g})\right] \\ e^{2\pi\left[t^{-}_{\omega}(\upsilon,\acute{g}), t^{+}_{\omega}(\upsilon,\acute{g})\right] i} \end{pmatrix} \in \hat{R}$$

$$\Rightarrow \hat{R} \circ \hat{R} \subseteq \hat{R}.$$

Conversely, assume that $\hat{R} \circ \hat{R} \subseteq \hat{R}$. Then, for $\begin{pmatrix} (\upsilon,\breve{\imath}), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}), \dot{r}^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}), t^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]i} \end{pmatrix} \in \hat{R}$ and $\begin{pmatrix} (\breve{\imath},\acute{g}), \left[\dot{r}^{-}_{\dot{\omega}}(\breve{\imath},\acute{g}), \dot{r}^{+}_{\dot{\omega}}(\breve{\imath},\acute{g})\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\breve{\imath},\acute{g}), t^{+}_{\dot{\omega}}(\breve{\imath},\acute{g})\right]i} \end{pmatrix} \in \hat{R}$,

$$\begin{pmatrix} (\upsilon,\acute{g}), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\acute{g}), \dot{r}^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\acute{g}), t^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right]i} \end{pmatrix} \in \hat{R}\circ\hat{R} \subseteq \hat{R}.$$

$$\begin{pmatrix} (\upsilon,\acute{g}), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\acute{g}), \dot{r}^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\acute{g}), t^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right]i} \end{pmatrix} \in \hat{R}.$$

Thus, $\hat{R}$ is an IVCFS-transitive-relation on IVCFSS Ҡ.

Theorem 3.24. Assume that an IVCFSR $\hat{R}$ is IVCFS-equivalence-relation on IVCFSS Ҡ iff $\hat{R}\circ\hat{R} = \hat{R}$.

Proof. Suppose that

$$\begin{pmatrix} (\upsilon,\breve{\imath}), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}), \dot{r}^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}), t^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]i} \end{pmatrix} \in \hat{R}.$$

Then, by the definition of an IVCFS-symmetric-relation,

$$\begin{pmatrix} (\breve{\imath},\upsilon), \left[\dot{r}^{-}_{\dot{\omega}}(\breve{\imath},\upsilon), \dot{r}^{+}_{\dot{\omega}}(\breve{\imath},\upsilon)\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\breve{\imath},\upsilon), t^{+}_{\dot{\omega}}(\breve{\imath},\upsilon)\right]i} \end{pmatrix} \in \hat{R}.$$

Now, by the definition of an IVCFS-transitive-relation,

$$\begin{pmatrix} (\upsilon,\upsilon), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\upsilon), \dot{r}^{+}_{\dot{\omega}}(\upsilon,\upsilon)\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\upsilon), t^{+}_{\dot{\omega}}(\upsilon,\upsilon)\right]i} \end{pmatrix} \in \hat{R}.$$

However, by the definition of an IVCFS-composite-relation,

$$\begin{pmatrix} (\upsilon,\upsilon), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\upsilon), \dot{r}^{+}_{\dot{\omega}}(\upsilon,\upsilon)\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\upsilon), t^{+}_{\dot{\omega}}(\upsilon,\upsilon)\right]i} \end{pmatrix} \in \hat{R}\circ\hat{R}.$$

Hence,

$$\hat{R} \subseteq \hat{R}\circ\hat{R}. \qquad \text{(i)}$$

Conversely, assume that

$$\begin{pmatrix} (\upsilon,\breve{\imath}), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}), \dot{r}^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\breve{\imath}), t^{+}_{\dot{\omega}}(\upsilon,\breve{\imath})\right]i} \end{pmatrix} \in \hat{R}\circ\hat{R}.$$

Then, there exist

$$\acute{g} \in \text{Ҡ} \ni \begin{pmatrix} (\upsilon,\acute{g}), \left[\dot{r}^{-}_{\dot{\omega}}(\upsilon,\acute{g}), \dot{r}^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\upsilon,\acute{g}), t^{+}_{\dot{\omega}}(\upsilon,\acute{g})\right]i} \end{pmatrix} \in \hat{R} \text{ and } \begin{pmatrix} (\acute{g},\breve{\imath}), \left[\dot{r}^{-}_{\dot{\omega}}(\acute{g},\breve{\imath}), \dot{r}^{+}_{\dot{\omega}}(\acute{g},\breve{\imath})\right] \\ e^{2\pi\left[t^{-}_{\dot{\omega}}(\acute{g},\breve{\imath}), t^{+}_{\dot{\omega}}(\acute{g},\breve{\imath})\right]i} \end{pmatrix} \in \hat{R}.$$

However, $\hat{R}$ is an IVCFS-equivalence-relation on CTSFSS $\mathrm{Ƙ}$, so $\hat{R}$ is also an IVCFS-transitive-relation. Therefore,

$$\begin{pmatrix} (\upsilon,\breve{\imath}), \left[\dot{r}^{-}_{\tilde{\omega}}(\upsilon,\breve{\imath}), \dot{r}^{+}_{\tilde{\omega}}(\upsilon,\breve{\imath})\right] \\ e^{2\pi\left[t^{-}_{\tilde{\omega}}(\upsilon,\breve{\imath}), t^{+}_{\tilde{\omega}}(\upsilon,\breve{\imath})\right]i} \end{pmatrix} \in \hat{R}.$$

$$\Longrightarrow \hat{R} \circ \hat{R} \subseteq \hat{R}. \quad \text{(ii)}$$

Hence, by Eqs (i) and (ii),

$$\hat{R} \circ \hat{R} = \hat{R}.$$

4. Application

In this section, an application of the recommended ideas for the investigation of a transportation strategy decision-making process is presented. The IVCFSRs are the more comprehensive concept because they include both an amplitude and a phase term. The amplitude term shows the effectiveness of the transportation strategy, and the phase term indicates the time duration.

4.1. Transportation strategy decision-making process

Transportation decision-making looks for methods to resolve existing and anticipated transportation problems while avoiding future difficulties. It helps in solving problems on distribution of resources from one place to another. Figure 1 indicates the algorithm of the application.

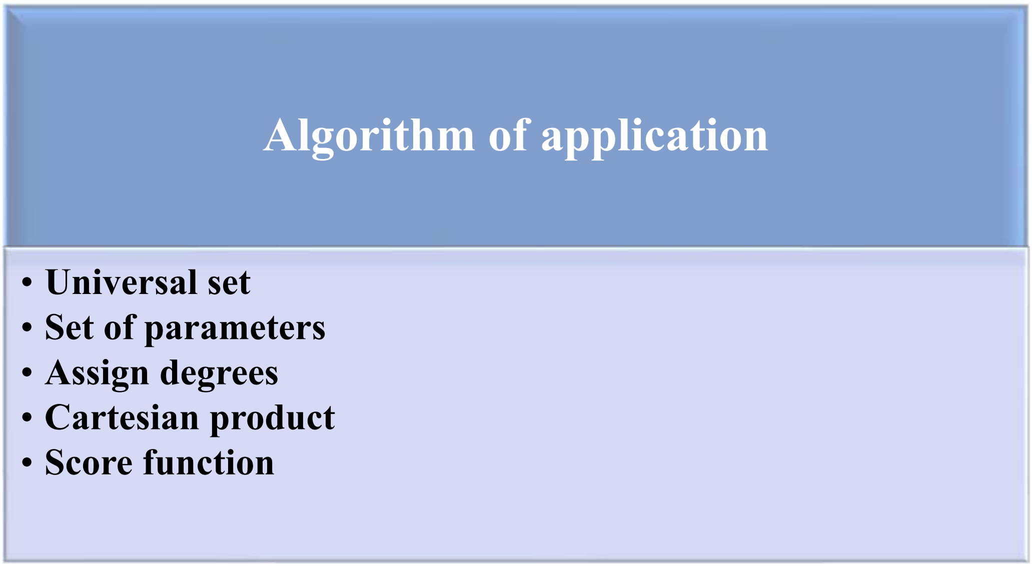

Figure 1. Algorithm of Application.

Furthermore, define the universal set which includes some transportation strategy of the decision-making process. The universal set $\hat{U} = \{\tilde{O}_1, \tilde{O}_2, \tilde{O}_3, \tilde{O}_4\}$ consists of four transportation parameters, i.e., $\tilde{O}_1$ =Transportation mode, $\tilde{O}_2$= Transportation outsourcing/insourcing, $\tilde{O}_3$ = Transportation network and $\tilde{O}_4$= Stakeholder-engagement management. Figure 2 discusses the transportation strategy.

Figure 2. Summary of transportation strategy.

i. Transportation mode

Transportation of products from source to destination can be accomplished by any or a combination of modes of transportation: air, parcel post, truck, rail, water and pipeline.

ii. Transportation outsourcing/insourcing

Managing transportation is a key factor in achieving a more effective and cohesive supply chain. Deciding which parts of the transportation process to outsource and which parts to insource is one of the most important strategic decisions that has a significant effect on the overall profit of the supply chain.

iii. Transportation network

Transportation network design impacts the working of the entire supply chain by implementing the architecture in which multiple transportation decisions are made. It is used to find routes to transport products from multiple sources to multiple destinations, minimizing overall costs without compromising customer responsiveness.

iv. Stakeholder-engagement management

Stakeholder management is an efficient approach to increase the quality of decisions made by managing the conflicting interests of stakeholders and considering the diverse interests of all authorized stakeholders.

4.2. Characteristics of transportation services

Transportation is essential to move people from one place to another. In doing so, however, transportation provides a service, which has some exclusive characteristics. Second, describe the parameters of the transportation strategy $\acute{\hat{E}} = \{\upsilon_1, \upsilon_2, \upsilon_3\}$, i.e., $\upsilon_1 =$ Openness and accessibility, $\upsilon_2 =$ Intangibility, $\upsilon_3 =$ Inseparability, and $\upsilon_4 =$ Variability. The principal characteristics of services may be summarized as in Figure 3.

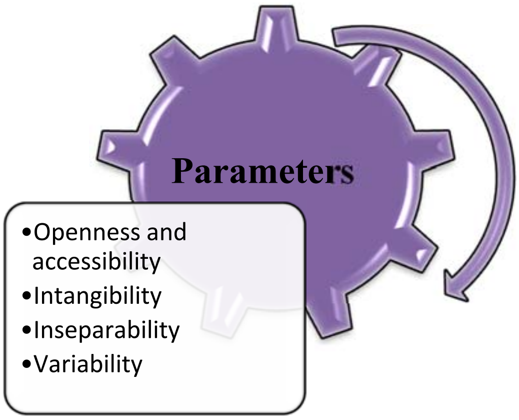

Figure 3. Summary of transportation strategy parameters.

- **Openness and accessibility**

Transportation systems designed and organized to move large numbers of people and goods efficiently, conveniently and quickly require a high level of user access.

- **Intangibility**

Unlike physical products, services cannot be seen, tested, felt, heard or smelled before purchase. Transportation costs can be a significant financial burden, especially for low-income households.

- **Inseparability**

In transportation services, inseparability means that the acts of production and consumption must occur simultaneously. The provision of services requires the active participation of both producers and consumers.

- **Variability**

Transport, which plays a dominant role in services, is highly variable. There are several reasons for service variability. Quality control is limited due to simultaneous production and consumption of services.

Calculations

The expert examines the transportation strategy of all the parameters. Let observations $(\mathcal{H},\mathcal{B})$ be by experts individually. They give the interval valued membership degree on the base of parameters. Suppose that their corresponding interval valued membership matrices are as follows.

$$(\mathcal{H},\mathcal{B})=\begin{pmatrix}
\begin{bmatrix}0.2e^{0.2\pi i},\\0.6e^{0.3\pi i}\end{bmatrix} & \begin{bmatrix}0.1e^{0\pi i},\\0.3e^{0.4\pi i}\end{bmatrix} & \begin{bmatrix}0.2e^{0.1\pi i},\\0.5e^{0.3\pi i}\end{bmatrix} & \begin{bmatrix}0.4e^{0.2\pi i},\\0.7e^{0.5\pi i}\end{bmatrix} & \begin{bmatrix}0.1e^{0.3\pi i},\\0.2e^{0.4\pi i}\end{bmatrix}\\
\begin{bmatrix}0.3e^{0.1\pi i},\\0.5e^{0.4\pi i}\end{bmatrix} & \begin{bmatrix}0.2e^{0.2\pi i},\\0.3e^{0.6\pi i}\end{bmatrix} & \begin{bmatrix}0.3e^{0.1\pi i},\\0.6e^{0.8\pi i}\end{bmatrix} & \begin{bmatrix}0.4e^{0.3\pi i},\\0.5e^{0.7\pi i}\end{bmatrix} & \begin{bmatrix}0.1e^{0.2\pi i},\\0.8e^{0.3\pi i}\end{bmatrix}\\
\begin{bmatrix}0.3e^{0.5\pi i},\\0.7e^{0.7\pi i}\end{bmatrix} & \begin{bmatrix}0.1e^{0.2\pi i},\\0.8e^{0.8\pi i}\end{bmatrix} & \begin{bmatrix}0.4e^{0.5\pi i},\\0.6e^{0.6\pi i}\end{bmatrix} & \begin{bmatrix}0.5e^{0.1\pi i},\\0.6e^{0.4\pi i}\end{bmatrix} & \begin{bmatrix}0.4e^{0.6\pi i},\\0.8e^{0.7\pi i}\end{bmatrix}\\
\begin{bmatrix}0.1e^{0.3\pi i},\\0.7e^{0.5\pi i}\end{bmatrix} & \begin{bmatrix}0.3e^{0.1\pi i},\\0.8e^{0.9\pi i}\end{bmatrix} & \begin{bmatrix}0.4e^{0.2\pi i},\\0.5e^{0.4\pi i}\end{bmatrix} & \begin{bmatrix}0.2e^{0\pi i},\\0.6e^{0.1\pi i}\end{bmatrix} & \begin{bmatrix}0.5e^{0.3\pi i},\\0.6e^{0.9\pi i}\end{bmatrix}
\end{pmatrix}$$

The first value of each parameter shows the interval-valued membership degree assigned by experts to $\tilde{O}_1$, the second value of each parameter shows the interval-valued membership degree assigned by experts to $\tilde{O}_2$, the third value of each parameter shows the interval-valued membership degree assigned by experts to $\tilde{O}_3$, the fourth value of each parameter shows the interval-valued membership degree assigned by experts to $\tilde{O}_4$, and the last value of each parameter shows the general belongingness of each parameter to the transportation strategy and is denoted by λ.

Then, the CP of $(\mathcal{H}, \mathcal{B})$ is shown in Table 2.

Table 2. Cartesian product.

Ordered pair	$\tilde{O}_1$	$\tilde{O}_2$	$\tilde{O}_3$	$\tilde{O}_4$	$\boldsymbol{\lambda}$
(υ_1, υ_1)	$[0.2e^{0.2\pi i}, 0.6e^{0.3\pi i}]$	$[0.1e^{0\pi i}, 0.3e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.4e^{0.2\pi i}, 0.7e^{0.5\pi i}]$	$[0.1e^{0.3\pi i}, 0.2e^{0.4\pi i}]$
(υ_1, υ_2)	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.1e^{0\pi i}, 0.3e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.4e^{0.2\pi i}, 0.5e^{0.5\pi i}]$	$[0.1e^{0.2\pi i}, 0.2e^{0.3\pi i}]$
(υ_1, υ_3)	$[0.2e^{0.2\pi i}, 0.6e^{0.3\pi i}]$	$[0.1e^{0\pi i}, 0.3e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.4e^{0.1\pi i}, 0.6e^{0.4\pi i}]$	$[0.1e^{0.3\pi i}, 0.2e^{0.4\pi i}]$
(υ_1, υ_4)	$[0.1e^{0.2\pi i}, 0.6e^{0.3\pi i}]$	$[0.1e^{0\pi i}, 0.3e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.2e^{0\pi i}, 0.6e^{0.1\pi i}]$	$[0.1e^{0.3\pi i}, 0.2e^{0.4\pi i}]$
(υ_2, υ_1)	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.1e^{0\pi i}, 0.3e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.4e^{0.2\pi i}, 0.5e^{0.5\pi i}]$	$[0.1e^{0.2\pi i}, 0.2e^{0.3\pi i}]$
(υ_2, υ_2)	$[0.3e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.2e^{0.2\pi i}, 0.3e^{0.6\pi i}]$	$[0.3e^{0.1\pi i}, 0.6e^{0.8\pi i}]$	$[0.4e^{0.3\pi i}, 0.5e^{0.7\pi i}]$	$[0.1e^{0.2\pi i}, 0.8e^{0.3\pi i}]$
(υ_2, υ_3)	$[0.3e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.1e^{0.2\pi i}, 0.3e^{0.6\pi i}]$	$[0.3e^{0.1\pi i}, 0.6e^{0.6\pi i}]$	$[0.4e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.1e^{0.2\pi i}, 0.8e^{0.3\pi i}]$
(υ_2, υ_4)	$[0.1e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.3e^{0.6\pi i}]$	$[0.3e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.2e^{0\pi i}, 0.5e^{0.1\pi i}]$	$[0.1e^{0.2\pi i}, 0.6e^{0.3\pi i}]$
(υ_3, υ_1)	$[0.2e^{0.2\pi i}, 0.6e^{0.3\pi i}]$	$[0.1e^{0\pi i}, 0.3e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.4e^{0.1\pi i}, 0.6e^{0.4\pi i}]$	$[0.1e^{0.3\pi i}, 0.2e^{0.4\pi i}]$
(υ_3, υ_2)	$[0.3e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.1e^{0.2\pi i}, 0.3e^{0.6\pi i}]$	$[0.3e^{0.1\pi i}, 0.6e^{0.6\pi i}]$	$[0.4e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.1e^{0.2\pi i}, 0.8e^{0.3\pi i}]$
(υ_3, υ_3)	$[0.3e^{0.5\pi i}, 0.7e^{0.7\pi i}]$	$[0.1e^{0.2\pi i}, 0.8e^{0.8\pi i}]$	$[0.4e^{0.5\pi i}, 0.6e^{0.6\pi i}]$	$[0.5e^{0.1\pi i}, 0.6e^{0.4\pi i}]$	$[0.4e^{0.6\pi i}, 0.8e^{0.7\pi i}]$
(υ_3, υ_4)	$[0.1e^{0.3\pi i}, 0.7e^{0.5\pi i}]$	$[0.1e^{0.1\pi i}, 0.8e^{0.8\pi i}]$	$[0.4e^{0.2\pi i}, 0.5e^{0.4\pi i}]$	$[0.2e^{0\pi i}, 0.6e^{0.1\pi i}]$	$[0.4e^{0.3\pi i}, 0.6e^{0.7\pi i}]$
(υ_4, υ_1)	$[0.1e^{0.2\pi i}, 0.6e^{0.3\pi i}]$	$[0.1e^{0\pi i}, 0.3e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.5e^{0.3\pi i}]$	$[0.2e^{0\pi i}, 0.6e^{0.1\pi i}]$	$[0.1e^{0.3\pi i}, 0.2e^{0.4\pi i}]$
(υ_4, υ_2)	$[0.1e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.2e^{0.1\pi i}, 0.3e^{0.6\pi i}]$	$[0.3e^{0.1\pi i}, 0.5e^{0.4\pi i}]$	$[0.2e^{0\pi i}, 0.5e^{0.1\pi i}]$	$[0.1e^{0.2\pi i}, 0.6e^{0.3\pi i}]$
(υ_4, υ_3)	$[0.1e^{0.3\pi i}, 0.7e^{0.5\pi i}]$	$[0.1e^{0.1\pi i}, 0.8e^{0.8\pi i}]$	$[0.4e^{0.2\pi i}, 0.5e^{0.4\pi i}]$	$[0.2e^{0\pi i}, 0.6e^{0.1\pi i}]$	$[0.4e^{0.3\pi i}, 0.6e^{0.7\pi i}]$
(υ_4, υ_4)	$[0.1e^{0.3\pi i}, 0.7e^{0.5\pi i}]$	$[0.3e^{0.1\pi i}, 0.8e^{0.9\pi i}]$	$[0.4e^{0.2\pi i}, 0.5e^{0.4\pi i}]$	$[0.2e^{0\pi i}, 0.6e^{0.1\pi i}]$	$[0.5e^{0.3\pi i}, 0.6e^{0.9\pi i}]$

The CP of two IVCFSSs is shown in the above table. Now, to calculate the score function, convert the complex values into real values. First, convert all exponential values to the form of $a + ib$, i.e., $a + ib = re^{\pi i\theta}$, as $r = \sqrt{a^2 + b^2}$ and $e^{\pi i\theta} = \cos\pi(\theta) + i\sin\pi(\theta)$. Then $a = r\cos\pi(\theta), b = \sin\pi(\theta)$.

cycle of the circle is denoted by π. Take the modulus after converting the polar form to standard form. After all this process, apply the interval-valued membership score formula to $\frac{1}{2}\left(\dot{r}_{\dot{\omega}}^{+2}+t_{\dot{\omega}}^{+2}-\dot{r}_{\dot{\omega}}^{-2}-t_{\dot{\omega}}^{-2}\right)$. $\dot{r}_{\dot{\omega}}^{+}$ shows the positive amplitude of the interval-valued membership degree, $\dot{r}_{\dot{\omega}}^{-}$ indicates the negative amplitude of the interval-valued membership degree, $t_{\dot{\omega}}^{+}$ shows the positive phase term of the interval-valued membership degree, and $t_{\dot{\omega}}^{-}$ indicates the negative phase term of the interval valued membership degree. They are add the positive amplitude square and positive phase term square and then subtract the negative amplitude term square and phase term square. Table 3 indicates the interval-valued membership score formula for the above table values.

Table 3. Interval-valued membership score formula.

Ordered pair	$\tilde{\ddot{o}}_1$	$\tilde{\ddot{o}}_2$	$\tilde{\ddot{o}}_3$	$\tilde{\ddot{o}}_4$	$\boldsymbol{\lambda}$
(υ_1,υ_1)	0.37	0.24	0.29	0.54	0.1
(υ_1,υ_2)	0.29	0.24	0.29	0.3	0.08
(υ_1,υ_3)	0.37	0.24	0.29	0.35	0.1
(υ_1,υ_4)	0.4	0.24	0.29	0.33	0.1
(υ_2,υ_1)	0.29	0.24	0.29	0.3	0.08
(υ_2,υ_2)	0.31	0.37	0.9	0.49	0.68
(υ_2,υ_3)	0.31	0.4	0.62	0.24	0.68
(υ_2,υ_4)	0.39	0.4	0.31	0.22	0.4
(υ_3,υ_1)	0.37	0.24	0.29	0.35	0.1
(υ_3,υ_2)	0.31	0.4	0.62	0.24	0.68
(υ_3,υ_3)	0.64	0.95	0.31	0.26	0.61
(υ_3,υ_4)	0.64	0.98	0.21	0.33	0.6
(υ_4,υ_1)	0.4	0.24	0.29	0.33	0.1
(υ_4,υ_2)	0.39	0.4	0.31	0.22	0.4
(υ_4,υ_3)	0.64	0.98	0.21	0.33	0.6
(υ_4,υ_4)	0.64	0.9	0.21	0.33	0.83

Now, to find the best transporting strategy, take the highest value from each row while ignoring the last column. The last column is the general belongingness of each transporting strategy parameter. Now, every transporting strategy score is calculated by adding the product of these numerical degrees with the corresponding value of λ. The best transporting strategy chosen by any user is the one that gets a greater numerical value than others. We do not study the numerical degree of the same parametric ordered pair's transporting strategy because it is not a unique work to compare with itself. Table 4 shows the calculation of the score function.

Table 4. Calculation of score function.

$\hat{R}$	(υ_1,υ_1)	(υ_1,υ_2)	(υ_1,υ_3)	(υ_1,υ_4)	(υ_2,υ_1)	(υ_2,υ_2)	(υ_2,υ_3)	(υ_2,υ_4)
$\tilde{\tilde{O}}_i$	$\tilde{\tilde{O}}_4$	$\tilde{\tilde{O}}_4$	$\tilde{\tilde{O}}_1$	$\tilde{\tilde{O}}_1$	$\tilde{\tilde{O}}_4$	$\tilde{\tilde{O}}_3$	$\tilde{\tilde{O}}_3$	$\tilde{\tilde{O}}_2$
Highest degree	×	0.3	0.37	0.4	0.3	×	0.62	0.4
$\boldsymbol{\lambda}$	×	0.08	0.1	0.1	0.08	×	0.68	0.4
$\dot{R}$	(υ_3,υ_1)	(υ_3,υ_2)	(υ_3,υ_3)	(υ_3,υ_4)	(υ_4,υ_1)	(υ_4,υ_2)	(υ_4,υ_3)	(υ_4,υ_4)
$\tilde{\tilde{O}}_i$	$\tilde{\tilde{O}}_1$	$\tilde{\tilde{O}}_3$	$\tilde{\tilde{O}}_2$	$\tilde{\tilde{O}}_2$	$\tilde{\tilde{O}}_1$	$\tilde{\tilde{O}}_2$	$\tilde{\tilde{O}}_2$	$\tilde{\tilde{O}}_2$
Highest degree	0.37	0.62	×	0.98	0.4	0.4	0.98	×
$\boldsymbol{\lambda}$	0.1	0.68	×	0.6	0.1	0.4	0.6	×

$$S(\tilde{\tilde{O}}_1) = (0.37 \times 0.1) + (0.4 \times 0.1) + (0.37 \times 0.1) + (0.4 \times 0.1) = 0.154.$$

$$S(\tilde{\tilde{O}}_2) = (0.4 \times 0.4) + (0.98 \times 0.6) + (0.4 \times 0.4) + (0.98 \times 0.6) = 1.496.$$

$$S(\tilde{\tilde{O}}_3) = (0.62 \times 0.68) + (0.62 \times 0.68) = 0.843.$$

$$S(\tilde{\tilde{O}}_4) = (0.3 \times 0.08) + (0.3 \times 0.08) = 0.048.$$

Thus, transportation outsourcing/insourcing is the best transporting strategy as compared to the other transporting strategy.

5. Comparative analysis

Here, the new conception of IVCFSRs is compared to the several pre-defined structures in FSS theory, such as SRs, FSRs, CFSRs and IVFSRs.

Comparison of SRs, FSRs and CFSRs with IVCFSRs

SRs are associated with crisp knowledge and only tell the yes or no situation. The structure of a FSS is explained by a membership degree, which is a fuzzy number, and the associated relations are known as FSRs. The FSRs in an ordered pair show the effectiveness of the first parameter over the second. The FSRs have only one dimension and provide limited information, but IVCFSRs are broader than the SRs and FSRs. They explain the interval-valued membership of the first object over the second in an ordered pair with amplitude and phase terms. The amplitude term shows the effectiveness, and the phase term shows the duration of effectiveness. It easily deals with human decision making. The advantage of IVCFSRs over FSRs is the complex interval-valued membership degree assigned to each parameter of transportation strategy. So, IVCFSRs enable solving multi-dimensional problems. Meanwhile, FSRs do not have the phase term, and they are limited to solving only single dimensional problems. So, pre-existing structures give limited information about any problem. The CFSSs are described by the complex fuzzy numbers, and corresponding relations are called CFSRs. The CFSRs associate the membership degree with a complex number. The CFSRs are mainly two parts, i.e., an amplitude term and a phase term. An amplitude term characterizes the strength of the particular transportation strategy, and the phase term is used to describe the time period over the certain conditions. Assume that the corresponding membership of CFSRs matrices is as follows.

$$(\mathcal{H},\mathcal{B})=\begin{pmatrix}(0.2e^{0.2\pi i}) & (0.1e^{0\pi i}) & (0.2e^{0.1\pi i}) & (0.4e^{0.2\pi i}) & (0.1e^{0.3\pi i})\\ (0.3e^{0.1\pi i}) & (0.2e^{0.2\pi i}) & (0.3e^{0.1\pi i}) & (0.4e^{0.3\pi i}) & (0.1e^{0.2\pi i})\\ (0.3e^{0.5\pi i}) & (0.1e^{0.2\pi i}) & (0.4e^{0.5\pi i}) & (0.5e^{0.1\pi i}) & (0.4e^{0.6\pi i})\\ (0.1e^{0.3\pi i}) & (0.3e^{0.1\pi i}) & (0.4e^{0.2\pi i}) & (0.2e^{0\pi i}) & (0.5e^{0.3\pi i})\end{pmatrix}$$

The first value of each parameter shows the interval-valued membership degree assigned by experts to $\tilde{\tilde{O}}_1$, the second value of each parameter shows the interval-valued membership degree assigned by experts to $\tilde{\tilde{O}}_2$, the third value of each parameter shows the interval-valued membership degree assigned by experts to $\tilde{\tilde{O}}_3$, the fourth value of each parameter shows the interval-valued membership degree assigned by experts to $\tilde{\tilde{O}}_4$, and the last value of each parameter shows the general belongingness of each parameter to the transportation strategy and is denoted by λ. Table 5 shows the Cartesian product of the above value of CFSRs.

Table 5. Cartesian product of CFSRs.

Ordered pair	$\tilde{\tilde{O}}_1$	$\tilde{\tilde{O}}_2$	$\tilde{\tilde{O}}_3$	$\tilde{\tilde{O}}_4$	λ
(υ_1,υ_1)	$(0.2e^{0.2\pi i})$	$(0.1e^{0\pi i})$	$(0.2e^{0.1\pi i})$	$(0.4e^{0.2\pi i})$	$(0.1e^{0.3\pi i})$
(υ_1,υ_2)	$(0.2e^{0.1\pi i})$	$(0.1e^{0\pi i})$	$(0.2e^{0.1\pi i})$	$(0.4e^{0.2\pi i})$	$(0.1e^{0.2\pi i})$
(υ_1,υ_3)	$(0.2e^{0.2\pi i})$	$(0.1e^{0\pi i})$	$(0.2e^{0.1\pi i})$	$(0.4e^{0.1\pi i})$	$(0.1e^{0.3\pi i})$
(υ_1,υ_4)	$(0.1e^{0.2\pi i})$	$(0.1e^{0\pi i})$	$(0.2e^{0.1\pi i})$	$(0.2e^{0\pi i})$	$(0.1e^{0.3\pi i})$
(υ_2,υ_1)	$(0.2e^{0.1\pi i})$	$(0.1e^{0\pi i})$	$(0.2e^{0.1\pi i})$	$(0.4e^{0.2\pi i})$	$(0.1e^{0.2\pi i})$
(υ_2,υ_2)	$(0.3e^{0.1\pi i})$	$(0.2e^{0.2\pi i})$	$(0.3e^{0.1\pi i})$	$(0.4e^{0.3\pi i})$	$(0.1e^{0.2\pi i})$
(υ_2,υ_3)	$(0.3e^{0.1\pi i})$	$(0.1e^{0.2\pi i})$	$(0.3e^{0.1\pi i})$	$(0.4e^{0.1\pi i})$	$(0.1e^{0.2\pi i})$
(υ_2,υ_4)	$(0.1e^{0.1\pi i})$	$(0.2e^{0.1\pi i})$	$(0.3e^{0.1\pi i})$	$(0.2e^{0\pi i})$	$(0.1e^{0.2\pi i})$
(υ_3,υ_1)	$(0.2e^{0.2\pi i})$	$(0.1e^{0\pi i})$	$(0.2e^{0.1\pi i})$	$(0.4e^{0.1\pi i})$	$(0.1e^{0.3\pi i})$
(υ_3,υ_2)	$(0.3e^{0.1\pi i})$	$(0.1e^{0.2\pi i})$	$(0.3e^{0.1\pi i})$	$(0.4e^{0.1\pi i})$	$(0.1e^{0.2\pi i})$
(υ_3,υ_3)	$(0.3e^{0.5\pi i})$	$(0.1e^{0.2\pi i})$	$(0.4e^{0.5\pi i})$	$(0.5e^{0.1\pi i})$	$(0.4e^{0.6\pi i})$
(υ_3,υ_4)	$(0.1e^{0.3\pi i})$	$(0.1e^{0.1\pi i})$	$(0.4e^{0.2\pi i})$	$(0.2e^{0\pi i})$	$(0.4e^{0.3\pi i})$
(υ_4,υ_1)	$(0.1e^{0.2\pi i})$	$(0.1e^{0\pi i})$	$(0.2e^{0.1\pi i})$	$(0.2e^{0\pi i})$	$(0.1e^{0.3\pi i})$
(υ_4,υ_2)	$(0.1e^{0.1\pi i})$	$(0.2e^{0.1\pi i})$	$(0.3e^{0.1\pi i})$	$(0.2e^{0\pi i})$	$(0.1e^{0.2\pi i})$
(υ_4,υ_3)	$(0.1e^{0.3\pi i})$	$(0.1e^{0.1\pi i})$	$(0.4e^{0.2\pi i})$	$(0.2e^{0\pi i})$	$(0.4e^{0.3\pi i})$
(υ_4,υ_4)	$(0.1e^{0.3\pi i})$	$(0.3e^{0.1\pi i})$	$(0.4e^{0.2\pi i})$	$(0.2e^{0\pi i})$	$(0.5e^{0.3\pi i})$

Comparison of IVFSRs with IVCFSRs

The IVFNs are explained the interval-valued fuzzy number, and the associated relations are known as the IVFSRs. Interval values are preferred because intervals provide accessibility for decision makers in assigning values. They cover the mistakes and the misunderstanding of the decision makers.

Suppose that the corresponding interval-valued membership matrices are as follows.

$$(\mathcal{H},\mathcal{B})=\begin{pmatrix}[0.2,0.6] & [0.1,0.3] & [0.2,0.5] & [0.4,0.7] & [0.1,0.2]\\ [0.3,0.5] & [0.2,0.3] & [0.3,0.6] & [0.4,0.5] & [0.1,0.8]\\ [0.3,0.7] & [0.1,0.8] & [0.4,0.6] & [0.5,0.6] & [0.4,0.8]\\ [0.1,0.7] & [0.3,0.8] & [0.4,0.5] & [0.2,0.6] & [0.5,0.6]\end{pmatrix}$$

The above metrics show the values of the membership degree of IVFSS, but IVFSSs do not discuss the time duration. Therefore, the innovative concept of IVCFSRs is superior to the pre-defined structure because they define the interval-valued membership degree with a complex number. So, it provides comprehensive information on any problem. Table 6 summarizes the comparative study of IVCFSRs with predefined structures.

Table 6. Summary of comparative analysis based on the structure.

Structure	Membership	IV-membership	Multi-dimension
SR	×	×	×
FSR	✓	×	×
CFSR	✓	×	✓
IVFSR	✓	✓	×
IVCFSR	✓	✓	✓

6. Advantages of the proposed method

This article defined the IVCFSRs and their various types due to their numerous advantages over the existing structures. In contrast to the above comparisons, a summarized list of advantages of the introduced framework is given:

- The complex-valued memberships allow for multivariable problems such as periodicity.
- The degree interval-valued functions allow for the modeling of the issue over two different time periods. Therefore, it has the capability to do predictive and prospective analyses.
- A large variety of fuzzy structures, such as SSs, FSs, CFSs, IVFSs, IVCFSs and others, are also generalized by this method.
- The proposed idea is more dominant and superior to the pre-existing ideas, where the presented idea is the modified technique of two different theories, IVCFSs and soft sets (SSs).
- Due to its extensive structure, the proposed work manages uncertainty efficiently compared to other frameworks.

7. Conclusions

The foremost contributions of this paper to the field of fuzzy set theory are the novel relations proposed for interval valued complex fuzzy soft sets (IVCFSSs), as well as the types of these relations, including reflexive, irreflexive, symmetric, anti-symmetric, asymmetric, complete, transitive, equivalence, partial order, strict order, preorder relations and equivalence classes. Some outcomes were proved with appropriate examples. Additionally, the transportation strategy application has been applied in this innovative IVCFSR concept. The goal of this application is to find the most effective transportation strategy. The transportation strategy characterizes the different parameters. The expert gives the interval-valued membership values of each transportation strategy parameter. A score function was constructed for the suggested structures to help in the decision-making processes. Using the score function, they choose the best transportation strategy based on a set of parameters. The score function is used in this article to choose the best objective or anything based on some parameters. Finally, logical justifications are used to mathematically prove that IVCFSRs are preferable to pre-defined structures. The capacity to parametrically handle uncertainty as well as periodicity are the main advantages of IVCFSRs.

The suggested work is more comprehensive than the collection of existing concepts, as fuzzy relations, soft relations, complex fuzzy relations, fuzzy soft relations, complex fuzzy soft relations, interval valued fuzzy relations, interval valued fuzzy soft relations, interval valued complex fuzzy relations are all special cases of the established relations. These ideas will eventually be expanded to include the further generalization of FSSs, leading to the creation of novel structures that could benefit a number of scientific and practical domains. For superior methodology in the future, we aim to apply the novel notion in different types of operators and use their various applications in the field of medical diagnosis, network signals, etc.

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Conflict of interest

The authors declare that they have no conflict of interest.

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