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On some dynamic inequalities of Hilbert's-type on time scales

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Abstract: In this article, we will prove some new conformable fractional Hilbert-type dynamic inequalities on time scales. These inequalities generalize some known dynamic inequalities on time scales, unify and extend some continuous inequalities and their corresponding discrete analogues. Our results will be proved by using some algebraic inequalities, conformable fractional Hölder inequalities, and conformable fractional Jensen's inequalities on time scales.

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1. Introduction

The celebrated Hardy-Hilbert's integral inequality with powers p and q [1] is

$$\int_0^\infty \int_0^\infty \frac{F(\vartheta)g(\varsigma)}{\vartheta+\varsigma}\mathrm{d}\vartheta \mathrm{d}\varsigma \le \frac{\pi}{\sin\frac{\pi}{p}}\left[\int_0^\infty F^p(\vartheta)\mathrm{d}\vartheta\right]^{\frac{1}{p}}\left[\int_0^\infty g^q(\varsigma)\mathrm{d}\varsigma\right]^{\frac{1}{q}}, \tag{1.1}$$

where $p>1$. Putting $p=q=2$, we get:

$$\int_0^\infty \int_0^\infty \frac{F(\vartheta)g(\varsigma)}{\vartheta+\varsigma}\mathrm{d}\vartheta \mathrm{d}\varsigma \le \pi\left[\int_0^\infty F^2(\vartheta)\mathrm{d}\vartheta\right]^{\frac{1}{2}}\left[\int_0^\infty g^2(\varsigma)\mathrm{d}\varsigma\right]^{\frac{1}{2}}, \tag{1.2}$$

where the coefficient π is best possible.

Pachpatte [2] proved the following two inequalities:

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\Phi(a_m)\Psi(b_n)}{m+n}\leqslant M(k,r)\Big(\sum_{m=1}^{k}(k-m+1)\Big(p_m\Phi\Big(\frac{\nabla a_m}{p_m}\Big)^2\Big)^{\frac{1}{2}}\times\Big(\sum_{n=1}^{r}(r-n+1)\Big(q_n\Psi\Big(\frac{\nabla b_n}{q_n}\Big)^2\Big)^{\frac{1}{2}}, \tag{1.3}$$

where

$$M(k,r)=\frac{1}{2}\Big(\sum_{m=1}^{k}\Big(\frac{\Phi(P_m)}{p_m}\Big)^2\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{r}\Big(\frac{\Psi(Q_n)}{Q_n}\Big)^2\Big)^{\frac{1}{2}}$$

$$\int_0^{\vartheta}\int_0^{\varsigma}\frac{\Phi(F(s))\Psi(g(\Im))}{s+\Im}dsd\Im\leqslant L(\vartheta,\varsigma)\Big(\int_0^{\vartheta}(\vartheta-s)\Big(p(s)\Phi\Big(\frac{F'(s)}{p(s)}\Big)^2ds\Big)^{\frac{1}{2}}\times\Big(\int_0^{\varsigma}(\varsigma-\Im)\Big(q(\Im)\Psi\Big(\frac{g'(\Im)}{q(\Im)}\Big)^2d\Im\Big)^{\frac{1}{2}} \tag{1.4}$$

where

$$L(\vartheta,\varsigma)=\frac{1}{2}\Big(\int_0^{\vartheta}\Big(\frac{\Phi(P(s))}{P(s)}\Big)^2ds\Big)^{\frac{1}{2}}\Big(\int_0^{\varsigma}\Big(\frac{\Psi(Q(\Im))}{Q(\Im)}\Big)^2d\Im\Big)^{\frac{1}{2}}.$$

Handley et al. [3], extended (1.3) and (1.4) as follows:

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(a_{\ell,m_\ell})}{\Big(\sum_{\ell=1}^{n}\gamma'_\ell m_\ell\Big)^{\gamma'}}\leqslant M(k_1,\ldots,k_n)\prod_{\ell=1}^{n}\Big(\sum_{m_\ell=1}^{k_\ell}(k_\ell-m_\ell+1)\Big(p_{\ell,m_\ell}\Phi_\ell\Big(\frac{\nabla a_{\ell,m_\ell}}{p_{\ell,m_\ell}}\Big)^{\frac{1}{\gamma_\ell}}\Big)^{\gamma_\ell} \tag{1.5}$$

where

$$M(k_1,\ldots,k_n)=\frac{1}{(\gamma')^{\gamma'}}\prod_{\ell=1}^{n}\Big(\sum_{m_\ell=1}^{k_\ell}\Big(\frac{\Phi_\ell(P_{\ell,m_\ell})}{P_{\ell,m_\ell}}\Big)^{\frac{1}{\gamma'_\ell}}\Big)^{\gamma'_\ell},$$

and

$$\int_0^{\vartheta_1}\cdots\int_0^{\vartheta_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(F(s_\ell))}{\Big(\sum_{\ell=1}^{n}\gamma'_\ell s_\ell\Big)^{\gamma'}}ds_1\ldots ds_n\leqslant L(\vartheta_1,\ldots,\vartheta_n)\prod_{\ell=1}^{n}\Big(\int_0^{\vartheta_\ell}(\vartheta_\ell-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell\Big(\frac{F'(s_\ell)}{p(s_\ell)}\Big)^{\frac{1}{\gamma_\ell}}ds_\ell\Big)^{\gamma_\ell}, \tag{1.6}$$

where

$$L(\vartheta_1,\ldots,\vartheta_n)=\frac{1}{(\gamma')^{\gamma'}}\prod_{\ell=1}^{n}\Big(\int_0^{\vartheta_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\Big)^{\frac{1}{\gamma'_\ell}}ds_\ell\Big)^{\gamma'_\ell}.$$

In 2006, Zhao and Cheung [4] gave the reverse versions of the above inequalities, which are more extensive results for this type of inequalities.

$$\int_0^{\vartheta_1}\int_0^{\varsigma_1}\cdots\int_0^{\vartheta_n}\int_0^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell\Im_\ell)\right)^{\gamma'}}ds_1d\Im_1\dots ds_nd\Im_n \tag{1.7}$$
$$\geqslant G(\vartheta_1\varsigma_1,\dots,\vartheta_ny_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_0^{\vartheta_\ell}\int_0^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\Big(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\Big(\frac{D_2D_1F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}ds_\ell d\Im_\ell\Big)^{\gamma_\ell}$$

where

$$G(\vartheta_1\varsigma_1,\dots,\vartheta_ny_n)=\prod_{\ell=1}^{n}\Big(\int_0^{\vartheta_\ell}\int_0^{\varsigma_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big)^{\frac{1}{\gamma'_\ell}}ds_\ell d\Im_\ell\Big)^{\gamma'_\ell}$$

and

$$P_\ell(s_\ell,\Im_\ell)=\int_0^{\Im_\ell}\int_0^{s_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)d\xi_\ell d\tau_\ell. \tag{1.8}$$

In [5], Pachpatte established the following Hilbert type integral inequalities.

$$\int_0^{\vartheta}\int_0^{\varsigma}\frac{F^h(s)G^l(\Im)}{s+\Im}dsd\Im\leqslant\frac{1}{2}hl(xy)^{\frac{1}{2}}\Big(\int_0^{\vartheta}(\vartheta-s)\Big(F^{h-1}(s)F(s)\Big)^2ds\Big)^{\frac{1}{2}}$$
$$\times\Big(\int_0^{\varsigma}(\varsigma-\Im)\Big(G^{l-1}g(\Im)\Big)^2d\Im\Big)^{\frac{1}{2}}, \tag{1.9}$$

and

$$\int_0^{\vartheta}\int_0^{\varsigma}\frac{\Phi(F(s))\Psi(G(\Im))}{s+\Im}dsd\Im\leqslant L(\vartheta,\varsigma)\Big(\int_0^{\vartheta}(\vartheta-s)\Big(p(s)\Phi\Big(\frac{F(s)}{p(s)}\Big)\Big)^2ds\Big)^{\frac{1}{2}}$$
$$\times\Big(\int_0^{\varsigma}(\varsigma-\Im)\Big(q(\Im)\Psi\Big(\frac{g(\Im)}{q(\Im)}\Big)\Big)^2d\Im\Big)^{\frac{1}{2}} \tag{1.10}$$

where

$$L(\vartheta,\varsigma)=\frac{1}{2}\Big(\int_0^{\vartheta}\Big(\frac{\Phi(P(s))}{P(s)}\Big)^2ds\Big)^{\frac{1}{2}}\Big(\int_0^{\varsigma}\Big(\frac{\Psi(Q(\Im))}{Q(\Im)}\Big)^2d\Im\Big)^{\frac{1}{2}}$$

and

$$\int_0^{\vartheta}\int_0^{\varsigma}\frac{P(s)Q(\Im)\Phi(F(s))\Psi(G(\Im))}{s+\Im}dsd\Im\leqslant\frac{1}{2}(xy)^{\frac{1}{2}}\Big(\int_0^{\vartheta}(\vartheta-s)\Big(p(s)\Phi\Big(F(s)\Big)\Big)^2ds\Big)^{\frac{1}{2}}$$
$$\times\Big(\int_0^{\varsigma}(\varsigma-\Im)\Big(q(\Im)\Psi\Big(g(\Im)\Big)\Big)^2d\Im\Big)^{\frac{1}{2}}. \tag{1.11}$$

$$\int_0^{\vartheta_1}\int_0^{\varsigma_1}\cdots\int_0^{\vartheta_n}\int_0^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell)(\Im_\ell)\right)^{\frac{1}{\gamma}}}ds_1d\Im_1\dots ds_nd\Im_n \tag{1.12}$$

$$\geqslant L(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_0^{\vartheta_\ell}\int_0^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\Big(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\Big(\frac{F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell,\Im_\ell)}\Big)\Big)^{\beta_\ell}ds_\ell d\Im_\ell\Big)^{\frac{1}{\beta_\ell}}$$

where

$$L(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)=\prod_{\ell=1}^{n}\Big(\int_0^{\vartheta_\ell}\int_0^{\varsigma_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big)^{\gamma_\ell}ds_\ell d\Im_\ell\Big)^{\frac{1}{\gamma_\ell}}.$$

Over the past decade, a great number of dynamic Hilbert type inequalities on time scales has been established by many researchers who were motivated by some applications, see the papers [6–15, 24–27, 40–43], see also, [16, 18, 22, 23, 28–32]. For more details on time scales calculus see [33].

In this paper, we extend some generalizations of the integral Hardy-Hilbert inequality to a general time scale using conformable fractional. As special cases of our results, we will recover some integral and discrete inequalities known in the literature. This article is arranged as follows: In Section 2, some basic concepts of the calculus on time scales and useful lemmas are introduced. In Section 3, we state and prove the main results. In Section 4, we state the conclusion.

2. Preliminary

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We define jump operators forward and backward $\sigma:\mathbb{T}\to\mathbb{T}$ and $\rho:\mathbb{T}:\to\mathbb{T}$ respectively by

$$\sigma(\Im):=\inf\{s\in\mathbb{T}: s>\Im\},\qquad \Im\in\mathbb{T},$$

$$\rho(\Im):=\sup\{s\in\mathbb{T}: s<\Im\},\qquad \Im\in\mathbb{T}.$$

In the preceding two definitions, we set $\inf\emptyset=\sup\mathbb{T}$ (i.e., if τ is the maximum of $\mathbb{T}$, then $\sigma(\tau)=\tau$) and $\sup\emptyset=\inf\mathbb{T}$ (i.e., if τ is the minimum of $\mathbb{T}$, then $\rho(\tau)=\tau$), where $\emptyset$ denotes the empty set.

Recently, depending just on the basic limit definition of the derivative, Khalil et al. [34] proposed the conformable derivative $T_\alpha(f)(\xi)$ ($\alpha\in(0,1]$) of a function $f:\mathbb{R}^+\to\mathbb{R}$

$$T_\alpha(f)(\xi)=\lim_{\epsilon\to0}\frac{f(\xi+\epsilon\xi^{1-\alpha})-f(\xi)}{\epsilon},$$

for all $\xi>0$, $\alpha\in(0,1]$. The researchers in [34] also suggested a definition for the α-conformable integral of a function η as follows:

$$\int_a^b\eta(\xi)d_\alpha\xi=\int_a^b\eta(\xi)\xi^{\alpha-1}d\xi.$$

After that, Abdeljawad [35] studied extensive research of the newly introduced conformable calculus. In his work, he introduced a generalization of the conformable derivative $T_\alpha^a(f)(\xi)$ definition. For $\xi>a\in\mathbb{R}^+$ as $f:\mathbb{R}^+\to\mathbb{R}$

$$T_\alpha^a(f)(\xi)=\lim_{\epsilon\to0}\frac{f(\xi+\epsilon(\xi-a)^{1-\alpha})-f(\xi)}{\epsilon}.$$

Benkhettou et al. [36] introduced a conformable calculus on an arbitrary time scale, which is a natural extension of the conformable calculus.

However, in the last few decades, many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials, e.g., polymers. Fractional derivatives provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantages of fractional derivatives in comparison with classical integer-order models.

In [37], the authors studied a version of the nabla conformable fractional derivative on arbitrary time scales.

Definition 2.1. Let $\xi : \mathbb{T} \longrightarrow \mathbb{R}$, $\tau \in \mathbb{T}^k$, and $\alpha \in (0,1]$. For $\tau > 0$, we define $T_\alpha^\Delta(\xi)(\tau)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ- neighbrhood $U_\tau \subset \mathbb{T}$ of τ, $\delta > 0$, such that

$$|[\xi(\sigma(\tau)) - \xi(s)]\tau^{1-\alpha} - T_\alpha^\Delta(\xi)(\tau)[\sigma(\tau) - s]| \leq \varepsilon|\sigma(\tau) - s|,$$

for all $s \in U_\tau$. We call $T_\alpha^\Delta(\xi)(\tau)$ the conformable derivative of ξ of order α at τ, and we define conformable derivative on $\mathbb{T}$ at 0, as $T_\alpha^\Delta(\xi)(0) = \lim_{\tau \longrightarrow 0+} T_\alpha^\Delta(\xi)(\tau)$.

Remark 2.1. If $\alpha = 1$ then we obtain from Definition 2.1 the delta derivative of time scales. The conformable derivative of order zero is defined by the identity operator: $T_0^\Delta(\xi) = \xi$.

Remark 2.2. Along the work, we also use the notation $(\xi)^{\Delta_\alpha}(\tau) = T_\alpha^\Delta(\xi)(\tau)$.

Theorem 2.1. Let $\alpha \in (0,1]$ and $\mathbb{T}$ be a time scale. Assume $\xi : \mathbb{T} \rightarrow \mathbb{R}$ and $\tau \in \mathbb{T}^k$. The following properties hold.

(i) If ξ is conformal differentiable of order α at $\tau > 0$, then ξ is continuous at τ.

(ii) If ξ is continuous at τ and τ is right-scattered, then ξ is conformable differentiable of order α at τ with

$$T_\alpha^\Delta(\xi)(\tau) = \frac{\xi(\sigma(\tau)) - \xi(\tau)}{\mu(\tau)}\tau^{1-\alpha}.$$

(iii) If τ is right-dense, then ξ is conformable differentiable of order α at τ if and only if the limit $\lim_{s\longrightarrow\tau} \frac{\xi(\tau)-\xi(s)}{\tau-s}\tau^{1-\alpha}$ exists as a finite number. In this case,

$$T_\alpha^\Delta(\xi)(\tau) = \lim_{s\longrightarrow\tau} \frac{\xi(\tau) - \xi(s)}{\tau - s}\tau^{1-\alpha}.$$

(iv) If ξ is differentiable of order α at τ, then

$$\xi(\sigma(\tau)) = \xi(\tau) + \mu(\tau)\tau^{\alpha-1}T_\alpha^\Delta(\xi)(\tau).$$

Theorem 2.2. Assume $\xi, \varpi : \mathbb{T} \longrightarrow \mathbb{R}$ are conformable differentiable of order $\alpha \in (0,1]$, then following properties are hold:

(i) The sum $\xi + \varpi : \mathbb{T} \longrightarrow \mathbb{R}$ is conformable differentiable with

$$T_\alpha^\Delta(\xi + \varpi) = T_\alpha^\Delta(\xi) + T_\alpha^\Delta(\varpi).$$

(ii) For any $k \in \mathbb{R}$, $k\xi : \mathbb{T} \longrightarrow \mathbb{R}$ is conformable differentiable with

$$T_\alpha^\Delta(k\xi) = kT_\alpha^\Delta(\xi).$$

(iii) If ξ and ϖ are continuous, then the product $\xi\varpi : \mathbb{T} \longrightarrow \mathbb{R}$ is conformable differentiable with

$$T_\alpha^\Delta(\xi\varpi) = T_\alpha^\Delta(\xi)\varpi + \xi^\sigma T_\alpha^\Delta(\varpi) = T_\alpha^\Delta(\xi)\varpi^\sigma + \xi T_\alpha^\Delta(\varpi).$$

(iv) If ξ is continuous, then $1/\xi$ is conformable differentiable with

$$T_\alpha^\Delta\Big(\frac{1}{\xi}\Big) = \frac{-T_\alpha^\Delta(\xi)}{\xi(\xi \circ \sigma)}$$

valid at all points $\tau \in \mathbb{T}^k$ for which $\xi(\xi \circ \sigma) \neq 0$.

(v) If ξ and ϖ are continuous, then ξ/ϖ is conformable differentiable with

$$T_\alpha^\Delta\Big(\frac{\xi}{\varpi}\Big) = \frac{T_\alpha^\Delta(\xi)\varpi - \xi T_\alpha^\Delta(\varpi)}{\varpi\varpi^\sigma}$$

valid $\forall \tau \in \mathbb{T}^k$, for which $\varpi\varpi^\sigma \neq 0$.

Definition 2.2. Let $\xi : \mathbb{T} \to \mathbb{R}$ be regulated function. Then the α-conformable integral of ξ, $0 < \alpha \leqslant 1$, is defined by

$$\int \xi(\tau)\Delta_\alpha\tau = \int \xi(\tau)\tau^{\alpha-1}\Delta\tau.$$

Definition 2.3. Suppose $\xi : \mathbb{T} \to \mathbb{R}$ is a regulated function. Denote the indefinite α-conformable integral of ξ of order α, $\alpha \in (0, 1]$, as follows: $F_\alpha(\tau) = \int \xi(\tau)\Delta_\alpha\tau$. Then, for all $a, b \in \mathbb{T}$ we define the Cauchy α-conformable integral by

$$\int_a^b \xi(\tau)\Delta_\alpha\tau = F_\alpha(b) - F_\alpha(a).$$

Theorem 2.3. Let $\alpha \in (0, 1]$. Then, for any rd-continuous function $\xi : \mathbb{T} \to \mathbb{R}$, there exists a function $F_\alpha : \mathbb{T} \to \mathbb{R}$ such that $T_\alpha^\Delta(F_\alpha)(\tau) = \xi(\tau)$ for all $\tau \in \mathbb{T}^k$. Function F_α is said to be an α-antiderivative of ξ.

The conformable integral satisfying the next properties

Theorem 2.4. Let $\alpha \in (0, 1]$, $a, b, c \in \mathbb{T}$, $\omega \in \mathbb{R}$, and ξ, ϖ be two rd-continuous functions. Then

(i) $\int_a^b [\xi(\tau) + \varpi(\tau)]\Delta_\alpha\tau = \int_a^b \xi(\tau)\Delta_\alpha\tau + \int_a^b \varpi(\tau)\Delta_\alpha\tau$.

(ii) $\int_a^b \omega\xi(\tau)\Delta_\alpha\tau = \omega \int_a^b \xi(\tau)\Delta_\alpha\tau$.

(iii) $\int_a^b \xi(\tau)\Delta_\alpha\tau = -\int_b^a \xi(\tau)\Delta_\alpha\tau$.

(iv) $\int_a^b \xi(\tau)\Delta_\alpha\tau = \int_a^c \xi(\tau)\Delta_\alpha\tau + \int_c^b \xi(\tau)\Delta_\alpha\tau$.

(v) $\int_a^a \xi(\tau)\Delta_\alpha\tau = 0$.

(vi) If there exists $\xi : \mathbb{T} \to \mathbb{R}$ with $|\zeta(\tau)| \leq |\xi(\tau)|$ for all $\tau \in [a, b]$, then $| \int_a^b \zeta(\tau)|\Delta_\alpha\tau \leq \int_a^b |\xi(\tau)|\Delta_\alpha\tau$;

(vii) If $\xi > 0$ for all $\tau \in [a, b]$, then $\int_a^b \xi(\tau)\Delta_\alpha\tau \geq 0$.

We use the following crucial relations between calculus on time scales $\mathbb{T}$ and differential calculus on $\mathbb{R}$ and difference calculus on $\mathbb{Z}$. Note that:

(i) for any time scales $\mathbb{T}$, we have

$$(\xi)^{\Delta_\alpha}(\tau) = (\xi)^\Delta(\tau)\tau^{1-\alpha}, \quad \int_a^b \xi(\tau)\Delta_\alpha\tau = \int_a^b \xi(\tau)\tau^{\alpha-1}\Delta\tau.$$

(ii) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(\tau) = \tau, \quad \mu(\tau) = 0, \quad f^{\Delta}(\tau) = f'(\tau), \quad \int_a^b f(\tau)\Delta\tau = \int_a^b f(\tau)d\tau. \tag{2.1}$$

(iii) If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\tau) = \tau + 1, \quad \mu(\tau) = 1, \quad f^{\Delta}(\tau) = \Delta f(\tau), \quad \int_a^b f(\tau)\Delta\tau = \sum_{\tau=a}^{b-1} f(\tau). \tag{2.2}$$

Next, we write Hölder's inequality and Jensen's inequality on time scales.

Lemma 2.1. Suppose u, $v \in \mathbb{T}$ with $u < v$. Assume F, $g \in CC^1_{rd}([u,v]_{\mathbb{T}} \times [u,v]_{\mathbb{T}}, \mathbb{R})$ be integrable functions and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ then

$$\begin{aligned}\int_u^v \int_u^v |F^*(r^*, \mathfrak{I}^*)g^*(r^*, \mathfrak{I}^*)|\Delta^\alpha r^* \Delta^\alpha \mathfrak{I}^* \quad &\leq \quad \Big[\int_u^v \int_u^v |F^*(r^*, \mathfrak{I}^*)|^p \Delta^\alpha r^* \Delta^\alpha \mathfrak{I}^*\Big]^{\frac{1}{p}} \\ &\times \Big[\int_u^v \int_u^v |g^*(r^*, \mathfrak{I}^*)|^q \Delta^\alpha r^* \Delta^\alpha \mathfrak{I}^*\Big]^{\frac{1}{q}}. \end{aligned} \tag{2.3}$$

This inequality is reversed if $0 < p < 1$ and if $p < 0$ or $q < 0$.

Lemma 2.2. Let r^*, $\mathfrak{I}^* \in R$ and $-\infty \leqslant m^*, n^* \leqslant \infty$. If $F \in CC^1_{rd}(\mathbb{R}, (m^*, n^*))$. and $\Phi : (m^*, n^*) \longrightarrow \mathbb{R}$ is convex then

$$\phi\Big(\frac{\int_u^v \int_\omega^s F^*(r^*, \mathfrak{I}^*)\Delta_1^\alpha r^* \Delta_2^\alpha \mathfrak{I}^*}{\int_u^v \int_\omega^s \Delta_1^\alpha r^* \Delta_2^\alpha \mathfrak{I}^*}\Big) \leqslant \frac{\int_u^v \int_\omega^s \phi(F^*(r^*, \mathfrak{I}^*))\Delta_1^\alpha r^* \Delta_2^\alpha \mathfrak{I}^*}{\int_u^v \int_\omega^s \Delta_1^\alpha r^* \Delta_2^\alpha \mathfrak{I}^*}. \tag{2.4}$$

This inequality is reversed if $\phi \in C_{rd}((c,d), \mathbb{R})$ is concave.

Theorem 2.5. (Chain rule on time scales [33]) Let $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is Δ^α-differentiable on $\mathfrak{I}^\kappa$, and $F : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists $c \in [\mathfrak{I}, \sigma(\mathfrak{I})]$ with

$$(F \circ g)^{\Delta^\alpha}(\mathfrak{I}) = F'(g(c))(g)^{\Delta^\alpha}(\mathfrak{I}). \tag{2.5}$$

Definition 2.4. Φ is called a supermultiplicative function on $[0, \infty)$ if

$$\Phi(xy) \geqslant \Phi(\vartheta)\Phi(\varsigma), \quad \text{for all } \vartheta, \varsigma \geqslant 0. \tag{2.6}$$

Next, we write Fubini's theorem on time scales.

Lemma 2.3. (Fubini's Thoerem, see [38]) Assume that $(\vartheta, \Sigma_1, \mu_\Delta)$ and $(\varsigma, \Sigma_2, \nu_\Delta)$ are two finite-dimensional time scales measure spaces. Moreover, suppose that $F : \vartheta \times \varsigma \to \mathbb{R}$ is a delta integrable function and define the functions

$$\phi(\varsigma) = \int_\vartheta F(\vartheta, \varsigma)d\mu_\Delta(\vartheta), \quad \varsigma \in \varsigma,$$

and

$$\psi(\vartheta) = \int_\varsigma F(\vartheta, \varsigma)d\nu_\Delta(\varsigma), \quad \vartheta \in \vartheta.$$

Then ϕ is delta integrable on ς and ψ is delta integrable on ϑ and

$$\int_\vartheta d\mu_\Delta(\vartheta) \int_\varsigma F(\vartheta, \varsigma)d\nu_\Delta(\varsigma) = \int_\varsigma d\nu_\Delta(\varsigma) \int_\vartheta F(\vartheta, \varsigma)d\mu_\Delta(\vartheta).$$

Now we are ready to state and prove our main results.

3. Main results

First, we enlist the following assumptions for the proofs of our main results:

(S_1) $\mathbb{T}$ be time scales with $\Im_0$, ϑ_ℓ, ς_ℓ, s_ℓ, $\Im_\ell \in \mathbb{T}$, $(\ell = 1, \ldots, n)$.
(S_2) $F_\ell(s_\ell, \Im_\ell)$ are nonnegative, right-dense continuous functions defined on $[\Im_0, \vartheta_\ell)_\mathbb{T} \times [\Im_0, \varsigma_\ell)_\mathbb{T}$ $(\ell = 1, \ldots, n)$.
(S_3) $F_\ell(s_\ell, \Im_\ell)$ have a partial Δ^α- derivatives $F_\ell^{\Delta_1^\alpha}(s_\ell, \Im_\ell)$ and $F_\ell^{\Delta_2^\alpha}(s_\ell, \Im_\ell)$ with respect s_ℓ and $\Im_\ell$ respectively.
(S_4) $F_\ell(s_\ell, \Im_\ell) \in C_{rd}^2([\Im_0, \vartheta_\ell)_\mathbb{T} \times [\Im_0, \varsigma_\ell)_\mathbb{T}, [0, \infty))$ $(\ell = 1, \ldots, n)$ are increasing.
(S_5) $F_\ell(s_\ell, \Im_\ell) \in C_{rd}^2([\Im_0, \vartheta_\ell)_\mathbb{T} \times [\Im_0, \varsigma_\ell)_\mathbb{T}, [0, \infty))$ $(\ell = 1, \ldots, n)$.
(S_6) $p_\ell(\xi_\ell, \tau_\ell)$ are n positive right-dense continuous functions defined for $\xi_\ell \in (\Im_0, s_\ell)_\mathbb{T}$, $\tau_\ell \in (\Im_0, \Im_\ell)_\mathbb{T}$.
(S_7) $p_\ell(\xi_\ell)$ and $q_\ell(\tau_\ell)$ are positive right-dense continuous functions defined for $\xi_\ell \in (\Im_0, s_\ell)_\mathbb{T}$, $\tau_\ell \in (\Im_0, \Im_\ell)_\mathbb{T}$.
(S_8) Φ_ℓ $(\ell = 1, \ldots, n)$ are n real-valued nonnegative concave and supermultiplicative functions defined on $(0, \infty)$.
(S_9) ϑ_ℓ and ς_ℓ are positive real numbers.
(S_{10}) $s_\ell \in [\Im_0, \vartheta_\ell)_\mathbb{T}$ and $\Im_\ell \in [\Im_0, \varsigma_\ell)_\mathbb{T}$.
(S_{11}) $F_\ell(\Im_0, \Im_\ell) = F_\ell(s_\ell, \Im_0) = 0$, $(\ell = 1, \ldots, n)$.
(S_{12}) $F_\ell^{\Delta_1^\alpha \Delta_2^\alpha}(s_\ell, \Im_\ell) = F_\ell^{\Delta_2^\alpha \Delta_1^\alpha}(s_\ell, \Im_\ell)$.
(S_{13}) $P_\ell(s_\ell, \Im_\ell) = \int_{\Im_0}^{\Im_\ell} \int_{\Im_0}^{s_\ell} p_\ell(\xi_\ell) q_\ell(\tau_\ell) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell$.
(S_{14}) $\tilde{F}_\ell(s_\ell, \Im_\ell) = \int_{\Im_0}^{s_\ell} \int_{\Im_0}^{\Im_\ell} F_\ell(\xi_\ell, \tau_\ell) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell$.
(S_{15}) $P_\ell(s_\ell, \Im_\ell) = \int_{\Im_0}^{s_\ell} \int_{\Im_0}^{\Im_\ell} p_\ell(\xi_\ell, \tau_\ell) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell$.
(S_{16}) $\tilde{F}_\ell(s_\ell, \Im_\ell) = \frac{1}{P_\ell(\xi_\ell, \tau_\ell)} \int_{\Im_0}^{s_\ell} \int_{\Im_0}^{\Im_\ell} p_\ell(\xi_\ell, \tau_\ell) F_\ell(\xi_\ell, \tau_\ell) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell$.
(S_{17}) $\gamma_\ell \in (1, \infty)$, $\gamma'_\ell = 1 - \gamma_\ell$, $\gamma = \sum_{\ell=1}^n \gamma_\ell$, and $\gamma' = \sum_{\ell=1}^n \gamma'_\ell = n - \gamma$, $(\ell = 1, \ldots, n)$.
(S_{18}) $0 < \beta_\ell < 1$.
(S_{19}) $h_\ell \geqslant 2$.
(S_{20}) $\sum_{\ell=1}^n \frac{1}{\gamma_\ell} = \frac{1}{\gamma}$.
(S_{21}) $h_\ell \geqslant 1$.
(S_{22}) $F_\ell(\xi_\ell) \in C_{rd}^1[\Im_0, \vartheta_\ell]_\mathbb{T}$, $(\ell = 1, \ldots, n)$.
(S_{23}) ϑ_ℓ is positive real number.
(S_{24}) $\tilde{F}_\ell(s_\ell) = \int_{\Im_0}^{s_\ell} F_\ell(\xi_\ell) \Delta^\alpha \xi_\ell$.
(S_{25}) $s_\ell \in [\Im_0, \vartheta_\ell)_\mathbb{T}$.
(S_{26}) $p_\ell(\xi_\ell)$ are n positive functions.
(S_{27}) $P_\ell(s_\ell) = \int_{\Im_0}^{s_\ell} p_\ell(\xi_\ell) \Delta^\alpha \xi_\ell$.
(S_{28}) $\tilde{F}_\ell(s_\ell) = \frac{1}{P_\ell(s_\ell)} \int_{\Im_0}^{s_\ell} p(\xi_\ell) F(\xi_\ell) \Delta^\alpha \xi_\ell$.
(S_{29}) $F_\ell(\Im_0) = 0$.

Now, we are ready to state and prove the main results that extend several results in the literature.

Theorem 3.1. Let S_1, S_2, S_9, S_{11}, S_7, S_{13}, S_3, S_{12}, S_8 and S_{17} be satisfied. Then for S_{10} we have that

$$\int_{\Im_0}^{\vartheta_1} \int_{\Im_0}^{\varsigma_1} \cdots \int_{\Im_0}^{\vartheta_n} \int_{\Im_0}^{\varsigma_n} \prod_{\ell=1}^n \frac{\Phi_\ell(F_\ell(s_\ell, \Im_\ell))}{\left(\frac{1}{\gamma'} \sum_{\ell=1}^n \gamma'_\ell (s_\ell - \Im_0)(\Im_\ell - \Im_0)\right)^{\gamma'}} \Delta^\alpha s_1 \Delta^\alpha \Im_1 \ldots \Delta^\alpha s_n \Delta^\alpha \Im_n \tag{3.1}$$

$$\geqslant G(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\Big(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\gamma_\ell}$$

where

$$G(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)=\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big)^{\frac{1}{\gamma'_\ell}}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\gamma'_\ell}.$$

Proof. From the hypotheses of Theorem 3.1, we obtain

$$F_\ell(s_\ell,\Im_\ell)=\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell. \tag{3.2}$$

From (3.2) and S_8, it is easy to observe that

$$\begin{aligned}\Phi_\ell(F_\ell(s_\ell,\Im_\ell)) &= \Phi_\ell\Big(\frac{P_\ell(s_\ell,\Im_\ell)\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\big)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}\Big)\\ &\geqslant \Phi_\ell(P_\ell(s_\ell,\Im_\ell))\Phi_\ell\Big(\frac{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\big)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}\Big).\end{aligned} \tag{3.3}$$

By using inverse Jensen's dynamic inequality, we get that

$$\Phi_\ell(F_\ell(s_\ell,\Im_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\Big)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell. \tag{3.4}$$

Applying inverse Hölder's inequality on the right hand side of (3.4) with indices $1/\gamma_\ell$ and $1/\gamma'_\ell$, we obtain

$$\begin{aligned}\Phi_\ell(F_\ell(s_\ell,\Im_\ell)) &\geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\gamma'_\ell}\\ &\quad\times\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\gamma_\ell}.\end{aligned} \tag{3.5}$$

Using the following inequality on the term $[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\gamma'_\ell}$ where $\gamma'_\ell<0$ and $\lambda_\ell>0$.

$$\prod_{\ell=1}^{n}\lambda_\ell^{\gamma'_\ell}\geqslant\Big(\frac{1}{\gamma'}\Big(\sum_{\ell=1}^{n}\gamma'_\ell\lambda_\ell\Big)\Big)^{\gamma'}. \tag{3.6}$$

We obtain that

$$\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell)) \geqslant \prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\gamma'}$$

$$\times\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\gamma_\ell}. \tag{3.7}$$

From (3.7), we have that

$$\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\gamma'}}$$
$$\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\gamma_\ell}. \tag{3.8}$$

Integrating both sides of (3.8) over s_ℓ, $\Im_\ell$ from $\Im_0$ to ϑ_ℓ, ς_ℓ ($\ell=1,\dots,n$), we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\gamma'}}\Delta^\alpha s_1\Delta^\alpha\Im_1\dots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\gamma_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell. \tag{3.9}$$

Applying inverse Hölder's inequality on the right hand side of (3.9) with indices $1/\gamma_\ell$ and $1/\gamma'_\ell$, we obtain

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\gamma'}}\Delta^\alpha s_1\Delta^\alpha\Im_1\dots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big)^{\frac{1}{\gamma'_\ell}}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\gamma'_\ell} \tag{3.10}$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\gamma_\ell}.$$

By using Fubini's theorem, we observe that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\gamma'}}\Delta^\alpha s_1\Delta^\alpha\Im_1\dots\Delta^\alpha s_n\Delta^\alpha\Im_n \tag{3.11}$$
$$\geqslant G(\vartheta_1\varsigma_1,\dots,\vartheta_n y_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\Big(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\gamma_\ell}.$$

By using the fact $\vartheta_\ell\geqslant\rho(\vartheta_\ell)$, and $\varsigma_\ell\geqslant\rho(\varsigma_\ell)$, we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\gamma'}}\Delta^\alpha s_1\Delta^\alpha\Im_1\dots\Delta^\alpha s_n\Delta^\alpha\Im_n$$

$$\geqslant G(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\Big(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\gamma_\ell}.$$

This completes the proof. □

Remark 3.1. In Theorem 3.1, if $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ we get the result due to Zhao et al. [4, Theorem 1.5].
Remark 3.2. In Theorem 3.1, if we take $\mathbb{T}=\mathbb{R}$, $\alpha=1$ we get inequality 1.7.
Remark 3.3. Let S_1, S_2, S_9, S_{11}, S_7, S_{13}, S_3 and S_{12} be satisfied and let Φ_ℓ, γ_ℓ, γ'_ℓ, γ, and γ' be as in inequality 1.7. Similar to proof of Theorem 3.1, we have

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\gamma'}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\leqslant G^*(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\sigma(\vartheta_\ell)-s_\ell)(\sigma(\varsigma_\ell)-\Im_\ell)\Big(p_\ell(s_\ell)q_\ell(\Im_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta_2^\alpha\Delta_1^\alpha}(s_\ell,\Im_\ell)}{p_\ell(s_\ell)q_\ell(\Im_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\gamma_\ell}$$

where

$$G^*(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)=\frac{1}{(\gamma')^{\gamma'}}\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big)^{\frac{1}{\gamma'_\ell}}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\gamma'_\ell}.$$

This is an inverse form of the inequality (3.1).

Corollary 2.1. Let S_{22}, S_{23}, S_{25}, S_{26}, S_{27}, S_{29}, S_{17} and S_8 be satisfied. Then we have that

$$\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(F_\ell(s_\ell))}{\Big(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)\Big)^{\gamma'}}\Delta^\alpha s_1\ldots\Delta^\alpha s_n \tag{3.12}$$
$$\geqslant G^{**}(\vartheta_1,\ldots,\vartheta_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell\Big(\frac{F_\ell^{\Delta^\alpha}(s_\ell)}{p_\ell(s_\ell)}\Big)\Big)^{\frac{1}{\gamma_\ell}}\Delta^\alpha s_\ell\Big)^{\gamma_\ell}.$$

where

$$G^{**}(\vartheta_1,\ldots,\vartheta_n)=\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\Big)^{\frac{1}{\gamma'_\ell}}\Delta^\alpha s_\ell\Big)^{\gamma'_\ell}.$$

Remark 3.4. In Corollary 2.1, if we take $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ we get an inverse form of inequality (1.5), which was given by Handley et al.
Remark 3.5. In Corollary 2.1, if we take $\mathbb{T}=\mathbb{R}$, $\alpha=1$ we get an inverse form of inequality (1.6), which was given by Handley et al.
Remark 3.6. In inequality (3.12) taking $n=2$, $\gamma_1=\gamma_2=2$, then $\gamma'_1=\gamma'_2=-1$,, we have

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\vartheta_2}\prod_{\ell=1}^{n}\frac{\Phi_1(F_1(s_1))\Phi_1(F_2(s_2))}{\Big((s_1-\Im_0)+(s_2-\Im_0)\Big)^{-2}}\Delta^\alpha s_1\Delta^\alpha s_2 \tag{3.13}$$

$$\geqslant D(\vartheta_1,\vartheta_2)\Big(\int_{\Im_0}^{\vartheta_1}(\rho(\vartheta_1)-s_1)\Big(p_1(s_1)\Phi_1\Big(\frac{F_1^{\Delta^\alpha}(s_1)}{p_1(s_1)}\Big)\Big)^{\frac{1}{2}}\Delta^\alpha s_1\Big)^2$$
$$\times\Big(\int_{\Im_0}^{\vartheta_2}(\rho(\vartheta_2)-s_2)\Big(p_2(s_2)\Phi_2\Big(\frac{F_2^{\Delta^\alpha}(s_2)}{p_2(s_2)}\Big)\Big)^{\frac{1}{2}}\Delta^\alpha s_2\Big)^2$$

where

$$D(\vartheta_1,\vartheta_2)=4\Big(\int_{\Im_0}^{\vartheta_1}\Big(\frac{\Phi_1(P_1(s_1))}{P_2(s_1)}\Big)^{-1}\Delta^\alpha s_1\Big)^{-1}\Big(\int_{\Im_0}^{\vartheta_2}\Big(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\Big)^{-1}\Delta^\alpha s_2\Big)^{-1}.$$

Remark 3.7. If we take $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ the inequality (3.13) is an inverse of inequality of (1.3), which was given by Pachpatte.

Remark 3.8. If we take $\mathbb{T}=\mathbb{R}$, $\alpha=1$ the inequality (3.13) is an inverse of inequality of (1.4), which was given by Pachpatte.

Theorem 3.2. Let S_1, S_4, S_9, and S_{14} be satisfied. Then for S_{10}, S_{18}, S_{19} and S_{20} we have that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^n F_\ell^{h_\ell}(s_\ell,\Im_\ell)}{\Big(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^n h_\ell\Big[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\Big]^{\frac{1}{\gamma_\ell}}$$
$$\times\Big\{\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\Big(H(h_\ell,s_\ell,\Im_\ell)+F_\ell^{h_\ell-1}(s_\ell,\sigma(\Im_\ell))F_\ell(s_\ell,\Im_\ell)\Big)^{\beta_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big\}^{\frac{1}{\beta_\ell}}$$

where

$$H(h_\ell,\xi_\ell,\tau_\ell)=(h_\ell-1)F_\ell^{h_\ell-2}(\xi_\ell,\tau_\ell)\frac{\partial F_\ell}{\Delta^\alpha\tau_\ell}(\xi_\ell,\tau_\ell)\frac{\partial F_\ell}{\Delta^\alpha\xi_\ell}(\xi_\ell,\tau_\ell).$$

Proof. From the hypotheses and by using the chain rule on time scales, we have that

$$\begin{aligned}F_\ell^{h_\ell}(s_\ell,\Im_\ell)&\geqslant\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}h_\ell\Big(H(h_\ell,\xi_\ell,\tau_\ell)+F_\ell^{h_\ell-1}(\xi_\ell,\sigma(\tau_\ell)).\frac{\partial^2F_\ell}{\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}(\xi_\ell,\tau_\ell)\Big)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\\&=h_\ell\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(H(h_\ell,\xi_\ell,\tau_\ell)+F_\ell^{h_\ell-1}(\xi_\ell,\sigma(\tau_\ell))F_\ell(\xi_\ell,\tau_\ell)\Big)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\end{aligned}\tag{3.14}$$

where

$$H(h_\ell,\xi_\ell,\tau_\ell)=(h_\ell-1)F_\ell^{h_\ell-2}(\xi_\ell,\tau_\ell)\frac{\partial F_\ell}{\Delta^\alpha\tau_\ell}(\xi_\ell,\tau_\ell)\frac{\partial F_\ell}{\Delta^\alpha\xi_\ell}(\xi_\ell,\tau_\ell).$$

Applying inverse Hölder's inequality on the right hand side of (3.14) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\begin{aligned}F_\ell^{h_\ell}(s_\ell,\Im_\ell)&\geqslant h_\ell[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\frac{1}{\gamma_\ell}}\\&\quad\times\Big\{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(H(h_\ell,\xi_\ell,\tau_\ell)+F_\ell^{h_\ell-1}(\xi_\ell,\sigma(\tau_\ell))F_\ell(\xi_\ell,\tau_\ell)\Big)^{\beta_\ell}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big\}^{\frac{1}{\beta_\ell}}.\end{aligned}\tag{3.15}$$

Let us note the following means inequality

$$\prod_{\ell=1}^{n} m_\ell^{\frac{1}{\gamma_\ell}} \geqslant \left(\gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell} m_\ell\right)^{\frac{1}{\gamma}} \tag{3.16}$$

we obtain that

$$\begin{aligned}&\frac{\prod_{\ell=1}^{n} F_\ell^{h_\ell}(s_\ell, \mathfrak{I}_\ell)}{\left(\gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell}(s_\ell - \mathfrak{I}_0)(\mathfrak{I}_\ell - \mathfrak{I}_0)\right)^{\frac{1}{\gamma}}} \\ &\geqslant \prod_{\ell=1}^{n} h_\ell \left\{ \int_{\mathfrak{I}_0}^{s_\ell} \int_{\mathfrak{I}_0}^{\mathfrak{I}_\ell} \left(H(h_\ell, \xi_\ell, \tau_\ell) + F_\ell^{h_\ell - 1}(\xi_\ell, \sigma(\tau_\ell)) F_\ell(\xi_\ell, \tau_\ell) \right)^{\beta_\ell} \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell \right\}^{\frac{1}{\beta_\ell}}.\end{aligned} \tag{3.17}$$

Integrating both sides of (3.17) over s_ℓ, $\mathfrak{I}_\ell$ from $\mathfrak{I}_0$ to ϑ_ℓ, ς_ℓ ($\ell = 1, 2, \ldots, n$), we get that

$$\begin{aligned}&\int_{\mathfrak{I}_0}^{\vartheta_1} \int_{\mathfrak{I}_0}^{\varsigma_1} \cdots \int_{\mathfrak{I}_0}^{\vartheta_n} \int_{\mathfrak{I}_0}^{\varsigma_n} \frac{\prod_{\ell=1}^{n} F_\ell^{h_\ell}(s_\ell, \mathfrak{I}_\ell)}{\left(\gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell}(s_\ell - \mathfrak{I}_0)(\mathfrak{I}_\ell - \mathfrak{I}_0)\right)^{\frac{1}{\gamma}}} \Delta^\alpha s_1 \Delta^\alpha \mathfrak{I}_1 \ldots \Delta^\alpha s_n \Delta^\alpha \mathfrak{I}_n \geqslant \prod_{\ell=1}^{n} h_\ell \\ &\times \int_{\mathfrak{I}_0}^{\vartheta_\ell} \int_{\mathfrak{I}_0}^{\varsigma_\ell} \left\{ \int_{\mathfrak{I}_0}^{s_\ell} \int_{\mathfrak{I}_0}^{\mathfrak{I}_\ell} \left(H(h_\ell, \xi_\ell, \tau_\ell) + F_\ell^{h_\ell - 1}(\xi_\ell, \sigma(\tau_\ell)) F_\ell(\xi_\ell, \tau_\ell) \right)^{\beta_\ell} \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell \right\}^{\frac{1}{\beta_\ell}} \Delta^\alpha s_\ell \Delta^\alpha \mathfrak{I}_\ell.\end{aligned} \tag{3.18}$$

Applying inverse Hölder's inequality on the right hand side of (3.18) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\begin{aligned}&\int_{\mathfrak{I}_0}^{\vartheta_1} \int_{\mathfrak{I}_0}^{\varsigma_1} \cdots \int_{\mathfrak{I}_0}^{\vartheta_n} \int_{\mathfrak{I}_0}^{\varsigma_n} \frac{\prod_{\ell=1}^{n} F_\ell^{h_\ell}(s_\ell, \mathfrak{I}_\ell)}{\left(\gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell}(s_\ell - \mathfrak{I}_0)(\mathfrak{I}_\ell - \mathfrak{I}_0)\right)^{\frac{1}{\gamma}}} \Delta^\alpha s_1 \Delta^\alpha \mathfrak{I}_1 \ldots \Delta^\alpha s_n \Delta^\alpha \mathfrak{I}_n \\ &\geqslant \prod_{\ell=1}^{n} h_\ell \left[(\vartheta_\ell - \mathfrak{I}_0)(\varsigma_\ell - \mathfrak{I}_0) \right]^{\frac{1}{\gamma_\ell}} \\ &\times \left\{ \int_{\mathfrak{I}_0}^{\vartheta_\ell} \int_{\mathfrak{I}_0}^{\varsigma_\ell} \left\{ \int_{\mathfrak{I}_0}^{s_\ell} \int_{\mathfrak{I}_0}^{\mathfrak{I}_\ell} \left(H(h_\ell, \xi_\ell, \tau_\ell) + F_\ell^{h_\ell - 1}(\xi_\ell, \sigma(\tau_\ell)) F_\ell(\xi_\ell, \tau_\ell) \right)^{\beta_\ell} \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell \right\} \Delta^\alpha s_\ell \Delta^\alpha \mathfrak{I}_\ell \right\}^{\frac{1}{\beta_\ell}}.\end{aligned}$$

By using Fubini's theorem, we observe that

$$\begin{aligned}&\int_{\mathfrak{I}_0}^{\vartheta_1} \int_{\mathfrak{I}_0}^{\varsigma_1} \cdots \int_{\mathfrak{I}_0}^{\vartheta_n} \int_{\mathfrak{I}_0}^{\varsigma_n} \frac{\prod_{\ell=1}^{n} F_\ell^{h_\ell}(s_\ell, \mathfrak{I}_\ell)}{\left(\gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell}(s_\ell - \mathfrak{I}_0)(\mathfrak{I}_\ell - \mathfrak{I}_0)\right)^{\frac{1}{\gamma}}} \Delta^\alpha s_1 \Delta^\alpha \mathfrak{I}_1 \ldots \Delta^\alpha s_n \Delta^\alpha \mathfrak{I}_n \\ &= \prod_{\ell=1}^{n} h_\ell \left[(\vartheta_\ell - \mathfrak{I}_0)(\varsigma_\ell - \mathfrak{I}_0) \right]^{\frac{1}{\gamma_\ell}} \\ &\times \left\{ \int_{\mathfrak{I}_0}^{\vartheta_\ell} \int_{\mathfrak{I}_0}^{\varsigma_\ell} (\vartheta_\ell - s_\ell)(\varsigma_\ell - \mathfrak{I}_\ell) \left(H(h_\ell, s_\ell, \mathfrak{I}_\ell) + F_\ell^{h_\ell - 1}(s_\ell, \sigma(\mathfrak{I}_\ell)) F_\ell(s_\ell, \mathfrak{I}_\ell) \right)^{\beta_\ell} \Delta^\alpha s_\ell \Delta^\alpha \mathfrak{I}_\ell \right\}^{\frac{1}{\beta_\ell}}.\end{aligned}$$

By using the fact $\vartheta_\ell \geqslant \rho(\vartheta_\ell)$, and $\varsigma_\ell \geqslant \rho(\varsigma_\ell)$, we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}F_\ell^{h_\ell}(s_\ell,\Im_\ell)}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$

$$\geqslant\prod_{\ell=1}^{n}h_\ell\Big[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\Big]^{\frac{1}{\gamma_\ell}}$$

$$\times\left\{\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\Big(H(h_\ell,s_\ell,\Im_\ell)+F_\ell^{h_\ell-1}(s_\ell,\sigma(\Im_\ell))F_\ell(s_\ell,\Im_\ell)\Big)^{\beta_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\right\}^{\frac{1}{\beta_\ell}}.$$

This completes the proof. □

As a special case of Theorem 3.2, when $\mathbb{T}=\mathbb{R}$, $\alpha=1$ we have $\rho(n)=n$, we get the following result.

Corollary 3.2. Let $F_\ell(\xi_\ell,\tau_\ell)\in C^2[(0,\vartheta_\ell)\times(0,\varsigma_\ell),(0,\infty)]$, $\ell=1,\ldots,n$, where ϑ_ℓ and ς_ℓ are positive real numbers and define $F(s_\ell,\Im_\ell)=\int_0^{\vartheta_\ell}\int_0^{\varsigma_\ell}F(\xi_\ell,\tau_\ell)d\xi_\ell d\tau_\ell$, for $s_\ell\in(0,\vartheta_\ell)$, $\Im_\ell\in(0,\varsigma_\ell)$. Then

$$\int_0^{\vartheta_1}\int_0^{\varsigma_1}\cdots\int_0^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}F_\ell^{h_\ell}(s_\ell,\Im_\ell)}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell\Im_\ell)\right)^{\frac{1}{\gamma}}}ds_1d\Im_1\ldots ds_nd\Im_n\geqslant\prod_{\ell=1}^{n}h_\ell[\vartheta_\ell\varsigma_\ell]^{\frac{1}{\gamma_\ell}}$$

$$\times\left\{\int_0^{\vartheta_\ell}\int_0^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\Big(H(h_\ell,s_\ell,\Im_\ell)+F_\ell^{h_\ell-1}(s_\ell,\Im_\ell)F_\ell(s_\ell,\Im_\ell)\Big)^{\beta_\ell}ds_\ell d\Im_\ell\right\}^{\frac{1}{\beta_\ell}}$$

where

$$H(h_\ell,\xi_\ell,\tau_\ell)=(h_\ell-1)F_\ell^{h_\ell-2}(\xi_\ell,\tau_\ell)\frac{\partial F_\ell}{\partial\tau_\ell}(\xi_\ell,\tau_\ell)\frac{\partial F_\ell}{\partial\xi_\ell}(\xi_\ell,\tau_\ell).$$

As a special case of Theorem 3.2, when $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ we have $\rho(n)=n-1$, we get the following result.

Corollary 3.3. Let $\{a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$ $(\ell=1,\ldots,n)$ be n sequences of nonnegative numbers defined for $m_{s_\ell}=1,\ldots,k_{s_\ell}$, and $m_{\Im_\ell}=1,\ldots,k_{\Im_\ell}$, and define

$$A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}=\sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\Im_\ell}}a_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}.$$

Then

$$\sum_{m_{s_1}}^{k_{s_1}}\sum_{m_{\Im_1}}^{k_{\Im_1}}\cdots\sum_{m_{s_n}}^{k_{s_n}}\sum_{m_{\Im_n}}^{k_{\Im_n}}\frac{\prod_{\ell=1}^{n}A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}^{h_\ell}}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(m_{s_\ell}m_{\Im_\ell})\right)^{\frac{1}{\gamma}}}\geqslant h_\ell(k_{s_\ell}k_{\Im_\ell})^{\frac{1}{\gamma_\ell}}$$

$$\times\prod_{\ell=1}^{n}\left(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{\Im_\ell}}^{k_{\Im_\ell}}(k_{s_\ell}-(m_{s_\ell}-1))(k_{\Im_\ell}-(m_{\Im_\ell}-1))\Big(H_{s_\ell,\Im_\ell}+A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}^{h_\ell-1}.a_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}\Big)^{\beta_\ell}\right)^{\frac{1}{\beta_\ell}}$$

where

$$H_{s_\ell,\Im_\ell}=(h_\ell-1)A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}^{h_\ell-2}\nabla_1A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\nabla_2A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}.$$

$$\nabla_1 A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} = A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} - A_{s_\ell,\Im_\ell,m_{s_\ell}-1,m_{\Im_\ell}}.$$
$$\nabla_2 A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} = A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} - A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}-1}.$$

Remark 3.9. Let $F_\ell(\xi_\ell,\tau_\ell)$ and $F_\ell(s_\ell,\Im_\ell) = \int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell} F_\ell(\xi_\ell,\tau_\ell)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell$ change to $F_\ell(\xi_\ell)$ and $F_\ell(s_\ell) = \int_{\Im_0}^{s_\ell} F(\xi_\ell)\Delta^\alpha\xi_\ell$, respectively and with suitable changes, we have
Corollary 3.4. Let S_{21}, S_{22}, S_{23} and S_{24} be satisfied. Then S_{25}, S_{18} and S_{20} we have that

$$\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n} \frac{\prod_{\ell=1}^n F_\ell^{h_\ell}(s_\ell)}{\left(\gamma\sum_{\ell=1}^n \frac{1}{\gamma_\ell}(s_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\ldots\Delta^\alpha s_n$$
$$\geqslant \prod_{\ell=1}^n h_\ell[\vartheta_\ell-\Im_0]^{\frac{1}{\gamma_\ell}}\left\{\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\left(F_\ell^{h_\ell-1}(s_\ell)F_\ell(s_\ell)\right)^{\beta_\ell}\Delta^\alpha s_\ell\right\}^{\frac{1}{\beta_\ell}}. \tag{3.19}$$

Corollary 3.5. In Corollary 3.4, if we take $n = 2, \beta_\ell = \frac{1}{2}$ then the inequality (3.19) changes to

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\vartheta_2}\frac{F_1^{h_1}(s_1)F_2^{h_2}(s_2)}{(s_1+s_2)^{-2}}\Delta^\alpha s_1\Delta^\alpha s_2 \geqslant 4h_1h_2[(\vartheta_1-\Im_0)(\vartheta_2-\Im_0)]^{-1} \tag{3.20}$$
$$\times\left(\int_{\Im_0}^{\vartheta_1}(\rho(\vartheta_1)-s_1)\left(F_1^{h_1-1}(s_1)F_1(s_1)\right)^2\Delta^\alpha s_1\right)^{\frac{1}{2}}\left(\int_{\Im_0}^{\vartheta_2}(\rho(\vartheta_2)-s_2)\left(F_2^{h_2-1}(s_2)F_2(s_2)\right)^2\Delta^\alpha s_2\right)^{\frac{1}{2}}.$$

Remark 3.10. In Corollary 3.5, if we take $\mathbb{T} = \mathbb{R}$, then the inequality (3.20) changes to

$$\int_0^{\vartheta_1}\int_0^{\vartheta_2}\frac{F_1^{h_1}(s_1)F_2^{h_2}(s_2)}{(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4h_1h_2(\vartheta_1\vartheta_2)^{-1}\left(\int_0^{\vartheta_1}(\vartheta_1-s_1)\left(F_1^{h_1-1}(s_1)F_1(s_1)\right)^2 ds_1\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^{\vartheta_2}(\vartheta_2-s_2)\left(F_2^{h_2-1}(s_2)F_2(s_2)\right)^2 ds_2\right)^{\frac{1}{2}}. \tag{3.21}$$

This is an inverse of the inequality (1.9) which was proved by Pachaptte [5].

Corollary 3.6. In Corollary 3.4, if we take $\beta_\ell = \frac{n-1}{n}$, the inequality (3.19) becomes

$$\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n}\frac{\prod_{\ell=1}^n F_\ell^{h_\ell}(s_\ell)}{\left(\sum_{\ell=1}^n(s_\ell-\Im_0)\right)^{\frac{-n}{n-1}}}\Delta^\alpha s_1\ldots\Delta^\alpha s_n$$
$$\geqslant n^{\frac{n}{n-1}}\prod_{\ell=1}^n h_\ell[\vartheta_\ell-\Im_0]^{\frac{-1}{n-1}}\left\{\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\left(F_\ell^{h_\ell-1}(s_\ell)F_\ell(s_\ell)\right)^{\frac{n-1}{n}}\Delta^\alpha s_\ell\right\}^{\frac{n}{n-1}}.$$

Theorem 3.3. Let S_1, S_5, S_{14}, S_6, S_{15}. and S_8 be satisfied. Then for S_{10}, S_{18} and S_{20} we have that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^n\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n \tag{3.22}$$
$$\geqslant L(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)$$

$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\Big(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\Big(\frac{F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell,\Im_\ell)}\Big)\Big)^{\beta_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\frac{1}{\beta_\ell}}$$

where

$$L(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)=\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big)^{\gamma_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\frac{1}{\gamma_\ell}}.$$

Proof. From the hypotheses of Theorem 3.3, S_{14}, S_{15}, and S_8, it is easy to observe that

$$\begin{aligned}\Phi_\ell(F_\ell(s_\ell,\Im_\ell)) &= \Phi_\ell\Big(\frac{P_\ell(s_\ell,\Im_\ell)\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)})\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}\Big)\\ &\geqslant \Phi_\ell(P_\ell(s_\ell,\Im_\ell))\Phi_\ell\Big(\frac{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)})\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}{\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell}\Big).\end{aligned}\tag{3.23}$$

By using inverse Jensen dynamic inequality, we obtain that

$$\Phi_\ell(F_\ell(s_\ell,\Im_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\Big(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\Big)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell.\tag{3.24}$$

Applying inverse Hölder's inequality on the right hand side of (3.24) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(F_\ell(s_\ell,\Im_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\frac{1}{\gamma_\ell}}\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\Big(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\Big)\Big)^{\beta_\ell}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\frac{1}{\beta_\ell}}.\tag{3.25}$$

By using inequality (3.16), on the term $[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\frac{1}{\gamma_\ell}}$, we get that

$$\begin{aligned}&\frac{\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\frac{1}{\gamma}}}\\ &\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\Big(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\Big)\Big)^{\frac{1}{\beta_\ell}}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\frac{1}{\beta_\ell}}.\end{aligned}\tag{3.26}$$

Integrating both sides of (3.26) over s_ℓ, $\Im_\ell$ from $\Im_0$ to ϑ_ℓ, ς_ℓ ($\ell=1,\ldots,n$), we obtain that

$$\begin{aligned}&\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n \qquad (3.27)\\ &\geqslant\prod_{\ell=1}^{n}\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\Big(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\Big)\Big)^{\beta_\ell}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\frac{1}{\beta_\ell}}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell.\end{aligned}$$

Applying inverse Hölder's inequality on the right hand side of (3.27) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell,\Im_\ell))}{P_\ell(s_\ell,\Im_\ell)}\Big)^{\gamma_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\frac{1}{\gamma_\ell}} \tag{3.28}$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\Big(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\Big(\frac{F_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\Big)\Big)^{\beta_\ell}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\frac{1}{\beta_\ell}}.$$

Using Fubini's theorem, we observe that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant L(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\Big(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\Big(\frac{F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell,\Im_\ell)}\Big)\Big)^{\beta_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\frac{1}{\beta_\ell}}.$$

By using the fact $\vartheta_\ell\geqslant\rho(\vartheta_\ell)$, and $\varsigma_\ell\geqslant\rho(\varsigma_\ell)$, we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant L(\vartheta_1\varsigma_1,\ldots,\vartheta_n y_n)$$
$$\times\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\Big(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\Big(\frac{F_\ell(s_\ell,\Im_\ell)}{p_\ell(s_\ell,\Im_\ell)}\Big)\Big)^{\beta_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\frac{1}{\beta_\ell}}.$$

This completes the proof. □

Remark 3.11. In Theorem 3.3, if $\mathbb{T}=\mathbb{R}$, $\alpha=1$ we get the result due to Zhao et al. [39, Theorem 2].

As a special case of Theorem 3.3, when $\mathbb{T}=\mathbb{Z}$, $\alpha=1$ we have $\rho(n)=n-1$, we get the following result.

Corollary 3.6. Let $\{a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$ and $\{p_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$, $(\ell=1,\ldots,n)$ be n sequences of nonnegative numbers defined for $m_{s_\ell}=1,\ldots,k_{s_\ell}$, and $m_{\Im_\ell}=1,\ldots,k_{\Im_\ell}$, and define

$$A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}=\sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\Im_\ell}}a_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}$$
$$P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}=\sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\Im_\ell}}p_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}. \tag{3.29}$$

Then

$$\sum_{m_{s_1}}^{k_{s_1}}\sum_{m_{\Im_1}}^{k_{\Im_1}}\cdots\sum_{m_{s_n}}^{k_{s_n}}\sum_{m_{\Im_n}}^{k_{\Im_n}}\frac{\prod_{\ell=1}^{n}\Phi_\ell(A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}})}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(m_{s_\ell}m_{\Im_\ell})\right)^{\frac{1}{\gamma}}}$$
$$\geqslant C(k_{s_1}k_{\Im_1},\ldots,k_{s_n}k_{\Im_n})$$
$$\times\prod_{\ell=1}^{n}\Big(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{\Im_\ell}}^{k_{\Im_\ell}}(k_{s_\ell}-(m_{s_\ell}-1))(k_{\Im_\ell}-(m_{\Im_\ell}-1))\Big(P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\Phi_\ell\Big(\frac{a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}}{P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}}\Big)\Big)^{\beta_\ell}\Big)^{\frac{1}{\beta_\ell}}$$

where

$$C(k_{s_1}k_{\Im_1},\ldots,k_{s_n}k_{\Im_n})=\prod_{\ell=1}^{n}\Big(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{\Im_\ell}}^{k_{\Im_\ell}}\Big(\frac{\Phi_\ell(P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}})}{P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}}\Big)\Big)^{\beta_\ell}\Big)^{\frac{1}{\beta_\ell}}.$$

Remark 3.12. Let $F_\ell(\xi_\ell,\tau_\ell)$, $p_\ell(\xi_\ell,\tau_\ell)$, $P_\ell(\xi_\ell,\tau_\ell)$, and $F_\ell(\xi_\ell,\tau_\ell)$ change to $F_\ell(\xi_\ell)$, $p_\ell(\xi_\ell)$, $P_\ell(s_\ell)$ and $F_\ell(s_\ell)$, respectively and with suitable changes, we have

Corollary 3.7. Let S_{22}, S_{23}, S_{24}, S_{26}, S_{27} and S_8 be satisfied. Then for S_{18}, S_{20} and S_{25} we have that

$$\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(F_\ell(s_\ell)}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\ldots\Delta^\alpha s_n \tag{3.30}$$
$$\geqslant L^*(\vartheta_1,\ldots,\vartheta_n)\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell\Big(\frac{F_\ell(s_\ell)}{p_\ell(s_\ell)}\Big)\Big)^{\beta_\ell}\Delta^\alpha s_\ell\Big)^{\frac{1}{\beta_\ell}}$$

where

$$L^*(\vartheta_1,\ldots,\vartheta_n)=\prod_{\ell=1}^{n}\Big(\int_{\Im_0}^{\vartheta_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\Big)^{\gamma_\ell}\Delta^\alpha s_\ell\Big)^{\frac{1}{\gamma_\ell}}.$$

Corollary 3.8. In Corollary 3.7, if we take $n=2, \beta_\ell=\frac{1}{2}$ then the inequality (3.30) changes to

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\vartheta_1}\frac{\Phi_1(F_1(s_1))\Phi_2(F_2(s_2))}{((s_1-\Im_0)+(s_2-\Im_0))^{-2}}\Delta^\alpha s_1\Delta^\alpha s_2\geqslant L^{**}(\vartheta_1,\vartheta_2)\Big(\int_{\Im_0}^{\vartheta_1}(\rho(\vartheta_1)-s_1)\Big(p_1(s_1)\Phi\Big(\frac{F_1(s_1)}{p_1(s_1)}\Big)\Big)^2\Delta^\alpha s_1\Big)^{\frac{1}{2}}$$
$$\times\Big(\int_{\Im_0}^{\vartheta_2}(\rho(\vartheta_2)-s_2)\Big(p_2(s_2)\Psi\Big(\frac{F_2(s_2)}{p_2(s_2)}\Big)\Big)^2\Delta^\alpha s_2\Big)^{\frac{1}{2}} \tag{3.31}$$

where

$$L^{**}(\vartheta_1,\vartheta_2)=4\Big(\int_{\Im_0}^{\vartheta_1}\Big(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)}\Big)^{-1}\Delta^\alpha s_1\Big)^{-1}\Big(\int_{\Im_0}^{\vartheta_2}\Big(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\Big)^{-1}\Delta^\alpha s_2\Big)^{-1}$$

Remark 3.13. In Corollary 3.8, if we take $\mathbb{T}=\mathbb{R}$, then the inequality (3.31) changes to

$$\int_0^{\vartheta_1}\int_0^{\vartheta_1}\frac{\Phi_1(F_1(s_1))\Phi_2(F_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2\geqslant L^{**}(\vartheta_1,\vartheta_2)\Big(\int_0^{\vartheta_1}(\vartheta_1-s_1)\Big(p_1(s_1)\Phi\Big(\frac{F_1(s_1)}{p_1(s_1)}\Big)\Big)^2ds_1\Big)^{\frac{1}{2}}$$
$$\times\Big(\int_0^{\vartheta_2}(\vartheta_2-s_2)\Big(p_2(s_2)\Psi\Big(\frac{F_2(s_2)}{p_2(s_2)}\Big)\Big)^2ds_2\Big)^{\frac{1}{2}} \tag{3.32}$$

where

$$L^{**}(\vartheta_1,\vartheta_2)=4\Big(\int_0^{\vartheta_1}\Big(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)}\Big)^{-1}ds_1\Big)^{-1}\Big(\int_0^{\vartheta_2}\Big(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\Big)^{-1}ds_2\Big)^{-1}.$$

This is an inverse of the inequality (1.10) which was proved by Pachaptte [5].

Corollary 3.9. In Corollary 3.7, if we take $\beta_\ell=\frac{n-1}{n}$ the inequality (3.30) becomes

$$\begin{aligned}&\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n}\frac{\prod_{\ell=1}^n\Phi_\ell(F_\ell(s_\ell)}{\Big(\sum_{\ell=1}^n(s_\ell-\Im_0)\Big)^{\frac{-n}{n-1}}}\Delta^\alpha s_1\ldots\Delta^\alpha s_n\\&\geqslant L^*(\vartheta_1,\ldots,\vartheta_n)\prod_{\ell=1}^n\Big(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell\Big(\frac{F_\ell(s_\ell)}{p_\ell(s_\ell)}\Big)\Big)^{\frac{n-1}{n}}\Delta^\alpha s_\ell\Big)^{\frac{n}{n-1}}\end{aligned}$$

where

$$L^*(\vartheta_1,\ldots,\vartheta_n)=n^{\frac{n}{n-1}}\prod_{\ell=1}^n\Big(\int_{\Im_0}^{\vartheta_\ell}\Big(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\Big)^{-(n-1)}\Delta^\alpha s_\ell\Big)^{\frac{-1}{n-1}}.$$

Theorem 3.4. Let S_1, S_5, S_6, S_9, S_{15}, and S_{16} be satisfied. Then for S_{10}, S_{18} and S_{20} we have that

$$\begin{aligned}&\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n \qquad (3.33)\\&\geqslant\prod_{\ell=1}^n\Big[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\Big]^{\frac{1}{\gamma_\ell}}\Big(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)(p_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell)))^{\beta_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\Big)^{\frac{1}{\beta_\ell}}.\end{aligned}$$

Proof. From the hypotheses of Theorem 3.4, and by using inverse Jensen dynamic inequality, we have

$$\begin{aligned}\Phi_\ell(F_\ell(s_\ell,\Im_\ell))&=\Phi_\ell\Big(\frac{1}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)F_\ell(\xi_\ell,\tau_\ell)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)\\&\geqslant\frac{1}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\sigma_\ell,\tau_\ell)\Phi_\ell(F_\ell(\xi_\ell,\tau_\ell))\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell. \qquad (3.34)\end{aligned}$$

Applying inverse Hölder's inequality on the right hand side of (3.34) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(F_\ell(s_\ell,\Im_\ell))\geqslant\frac{1}{P_\ell(s_\ell,\Im_\ell)}[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\frac{1}{\gamma_\ell}}\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell(F_\ell(\xi_\ell,\tau_\ell)))^{\beta_\ell}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\frac{1}{\beta_\ell}}.$$

By using the inequality (3.16), on the term $[(s_\ell-\Im_0)(\Im_\ell-\Im_0)]^{\frac{1}{\gamma_\ell}}$ we get that

$$\frac{P_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\Big(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\Big)^{\frac{1}{\gamma}}}\geqslant\Big(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell(F_\ell(\xi_\ell,\tau_\ell)))^{\beta_\ell}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Big)^{\frac{1}{\beta_\ell}} \qquad (3.35)$$

Integrating both sides of (3.35) over s_ℓ, $\Im_\ell$ from $\Im_0$ to ϑ_ℓ, ς_ℓ ($\ell = 1, \ldots, n$), we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell(F_\ell(\xi_\ell,\tau_\ell))\right)^{\beta_\ell}\Delta^\alpha\sigma_\ell\Delta^\alpha\tau_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Applying inverse Hölder's inequality on the right hand side of (3.36) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n \tag{3.36}$$
$$\geqslant\prod_{\ell=1}^{n}\left[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell(F_\ell(\xi_\ell,\tau_\ell))\right)^{\beta_\ell}\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using Fubini's theorem, we observe that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\left[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\vartheta_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell))\right)^{\beta_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using the fact $\vartheta_\ell \geqslant \rho(\vartheta_\ell)$, and $\varsigma_\ell \geqslant \rho(\varsigma_\ell)$, we get that

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\vartheta_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\Delta^\alpha\Im_1\ldots\Delta^\alpha s_n\Delta^\alpha\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\left[(\vartheta_\ell-\Im_0)(\varsigma_\ell-\Im_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}\int_{\Im_0}^{\varsigma_\ell}(\rho(\vartheta_\ell)-s_\ell)(\rho(\varsigma_\ell)-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell(F_\ell(s_\ell,\Im_\ell))\right)^{\beta_\ell}\Delta^\alpha s_\ell\Delta^\alpha\Im_\ell\right)^{\frac{1}{\beta_\ell}}.$$

This completes the proof. □

Remark 3.14. In Theorem 3.4, if $\mathbb{T} = \mathbb{R}$, $\alpha = 1$ we get the result due to Zhao et al. [39, Theorem 3].

As a special case of Theorem 3.4, when $\mathbb{T} = \mathbb{Z}$, $\alpha = 1$ we have $\rho(n) = n - 1$, we get the following result.

Corollary 3.10. Let $\{a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$ and $\{p_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\}$, ($\ell = 1, \ldots, n$) be n sequences of nonnegative numbers defined for $m_{s_\ell} = 1, \ldots, k_{s_\ell}$, and $m_{\Im_\ell} = 1, \ldots, k_{\Im_\ell}$, and define

$$A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} = \frac{1}{P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}}\sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\Im_\ell}}a_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}p_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}},$$

$$P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}} \;=\; \sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\Im_\ell}} p_{s_\ell,\Im_\ell,m_{\xi_\ell},m_{\eta_\ell}}. \tag{3.37}$$

Then

$$\sum_{m_{s_1}}^{k_{s_1}}\sum_{m_{\Im_1}}^{k_{\Im_1}}\cdots\sum_{m_{s_n}}^{k_{s_n}}\sum_{m_{\Im_n}}^{k_{\Im_n}} \frac{\prod_{\ell=1}^{n} P_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\Phi_\ell(A_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}})}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(m_{s_\ell}m_{\Im_\ell})\right)^{\frac{1}{\gamma}}}$$
$$\geqslant \prod_{\ell=1}^{n}(k_{s_\ell}k_{\Im_\ell})^{\frac{1}{\gamma_\ell}}\left(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{\Im_\ell}}^{k_{\Im_\ell}}(k_{s_\ell}-(m_{s_\ell}-1))(k_{\Im_\ell}-(m_{\Im_\ell}-1))\Big(p_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\Phi_\ell\Big(a_{s_\ell,\Im_\ell,m_{s_\ell},m_{\Im_\ell}}\Big)\Big)^{\beta_\ell}\right)^{\frac{1}{\beta_\ell}}.$$

Remark 3.15. Let $F_\ell(\xi_\ell,\tau_\ell)$, $p_\ell(\xi_\ell,\tau_\ell)$, $P_\ell(\xi_\ell,\tau_\ell)$ and

$$F_\ell(s_\ell,\Im_\ell) = \frac{1}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_0^{\Im_\ell} p_\ell(\xi_\ell,\tau_\ell)F_\ell(\xi_\ell,\tau_\ell)\Delta^\alpha\xi_\ell\Delta^\alpha\tau_\ell$$

changes to $F_\ell(\xi_\ell)$, $p_\ell(\xi_\ell)$, $P_\ell(s_\ell)$, and

$$F_\ell(s_\ell) = \frac{1}{P_\ell(s_\ell)}\int_{\Im_0}^{s_\ell} p_\ell(\xi_\ell)F_\ell(\xi_\ell)\Delta^\alpha\xi_\ell$$

respectively and with suitable changes, we have

Corollary 3.11. Let S_{22}, S_{23}, S_{26}, S_{27} and S_{28} be satisfied. Then for S_{18}, S_{20} and S_{25} we have that

$$\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n}\frac{\prod_{\ell=1}^{n} P_\ell(s_\ell)\Phi_\ell(F_\ell(s_\ell)}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta^\alpha s_1\ldots\Delta^\alpha s_n \tag{3.38}$$
$$\geqslant \prod_{\ell=1}^{n}(\vartheta_\ell-\Im_0)^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell(F_\ell(s_\ell))\Big)^{\beta_\ell}\Delta^\alpha s_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Corollary 3.12. In Corollary 3.11, if we take $n=2, \beta_\ell=\frac{1}{2}$ then the inequality (3.30) changes to

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\vartheta_1}\frac{P_1(s_1)P_2(s_2)\Phi_1(F_1(s_1))\Phi_2(F_2(s_2))}{((s_1-\Im_0)+(s_2-\Im_0))^{-2}}\Delta^\alpha s_1\Delta^\alpha s_2 \geqslant 4[(\vartheta_1-\Im_0)(\vartheta_1-\Im_0)]^{-1} \tag{3.39}$$
$$\times\left(\int_{\Im_0}^{\vartheta_1}(\rho(\vartheta_1)-s_1)\Big(p_1(s_1)\Phi_1(F_1(s_1))\Big)^2\Delta^\alpha s_1\right)^{\frac{1}{2}}\left(\int_{\Im_0}^{\vartheta_2}(\rho(\vartheta_2)-s_2)\Big(p_2(s_2)\Phi_2\Big(F_2(s_2)\Big)\Big)^2\Delta^\alpha s_2\right)^{\frac{1}{2}}.$$

Remark 3.16. In Corollary 3.12, if we take $\mathbb{T}=\mathbb{R}$, then the inequality (3.39) changes to

$$\int_0^{\vartheta_1}\int_0^{\vartheta_1}\frac{P_1(s_1)P_2(s_2)\Phi_1(F_1(s_1))\Phi_2(F_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4[\vartheta_1\vartheta_1]^{-1} \tag{3.40}$$
$$\times\left(\int_0^{\vartheta_1}(\vartheta_1-s_1)\Big(p_1(s_1)\Phi_1(F_1(s_1))\Big)^2 ds_1\right)^{\frac{1}{2}}\left(\int_0^{\vartheta_2}(\vartheta_2-s_2)\Big(p_2(s_2)\Phi_2\Big(F_2(s_2)\Big)\Big)^2 ds_2\right)^{\frac{1}{2}}.$$

This is an inverse of the inequality (1.11) which was proved by Pachaptte [5].
Corollary 3.13. In Corollary 3.12, let $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore the inequality (3.39) change to

$$\int_{\Im_0}^{\vartheta_1}\int_{\Im_0}^{\vartheta_1} \frac{\Phi_1(F_1(s_1))\Phi_2(F_2(s_2))}{(s_1 s_2)^{-1}((s_1-\Im_0)+(s_2-\Im_0))^{-2}}\Delta^\alpha s_1\Delta^\alpha s_2 \geqslant 4[(\vartheta_1-\Im_0)(\vartheta_1-\Im_0)]^{-1} \times\Big(\int_{\Im_0}^{\vartheta_1}(\rho(\vartheta_1)-s_1)\Big(\Phi_1(F_1(s_1))\Big)^2\Delta^\alpha s_1\Big)^{\frac{1}{2}}\Big(\int_{\Im_0}^{\vartheta_2}(\rho(\vartheta_2)-s_2)\Big(\Phi_2\Big(F_2(s_2)\Big)\Big)^2\Delta^\alpha s_2\Big)^{\frac{1}{2}}. \quad (3.41)$$

Remark 3.17. In Corollary 3.13, if we take $\mathbb{T}=\mathbb{R}$, then the inequality (3.41) change to

$$\int_0^{\vartheta_1}\int_0^{\vartheta_1} \frac{\Phi_1(F_1(s_1))\Phi_2(F_2(s_2))}{(s_1 s_2)^{-1}(s_1+s_2)^{-2}}ds_1 ds_2 \geqslant 4[\vartheta_1\vartheta_1]^{-1} \times\Big(\int_0^{\vartheta_1}(\vartheta_1-s_1)\Big(\Phi_1(F_1(s_1))\Big)^2 ds_1\Big)^{\frac{1}{2}}\Big(\int_0^{\vartheta_2}(\vartheta_2-s_2)\Big(\Phi_2\Big(F_2(s_2)\Big)\Big)^2 ds_2\Big)^{\frac{1}{2}}.$$

This is an inverse inequality of the following inequality which was proved by Pachpatte [39].

$$\int_0^{\vartheta}\int_0^{\varsigma} \frac{\Phi(F(s))\Psi(G(\Im))}{(s\Im)^{-1}(s+\Im)}ds d\Im \leqslant \frac{1}{2}[\vartheta\varsigma]^{\frac{1}{2}} \times\Big(\int_0^{\vartheta}(\vartheta-s_1)\Big(\Phi(F(s))\Big)^2 ds\Big)^{\frac{1}{2}}\Big(\int_0^{\varsigma}(\varsigma-\Im)\Big(\Psi\Big(g(\Im)\Big)\Big)^2 d\Im\Big)^{\frac{1}{2}}.$$

Corollary 3.14. In Corollary 3.11, if we take $\beta_\ell = \frac{n-1}{n}$ $(\ell = 1,\dots,n)$ the inequality (3.38)

$$\int_{\Im_0}^{\vartheta_1}\cdots\int_{\Im_0}^{\vartheta_n} \frac{\prod_{\ell=1}^n P_\ell(s_\ell)\Phi_\ell(F_\ell(s_\ell)}{\Big(\sum_{\ell=1}^n(s_\ell-\Im_0)\Big)^{\frac{-n}{n-1}}}\Delta^\alpha s_1\dots\Delta^\alpha s_n \geqslant n^{\frac{n}{n-1}}\prod_{\ell=1}^n(\vartheta_\ell-\Im_0)^{\frac{-1}{n-1}}\Big(\int_{\Im_0}^{\vartheta_\ell}(\rho(\vartheta_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell(F_\ell(s_\ell))\Big)^{\frac{n-1}{n}}\Delta^\alpha s_\ell\Big)^{\frac{n}{n-1}}.$$

4. Conclusions

In this manuscript, by employing the conformable fractional Hölder inequalities and conformable fractional Jensen's inequalities on time scales, several generalizations of the conformable fractional Hardy-Hilbert inequality on time scales are introduced. Beside that, we also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases.

Conflict of interest

The authors declare that there is no competing interest.

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