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Research article

Existence and multiplicity of solutions for critical Choquard-Kirchhoff type equations with variable growth

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Abstract: We prove the existence and multiplicity of solutions for a class of Choquard-Kirchhoff type equations with variable exponents and critical reaction. Because the appearance of the critical reaction, we deal with the lack of compactness by using the concentration-compactness principle. In particular, we discuss the main results in non-degenerate and degenerate cases. And we apply combination of Krasnoselskii genus and the Hardy-Littlewood-Sobolev inequality to get the results of existence and multiplicity.

Keywords: Choquard equation; critical nonlinearity; concentration-compactness principles; variational methods
Mathematics Subject Classification: 35J20, 35J60, 35J62

1. Introduction

We study the critical nonlocal Choquard equations with variable exponents of the form:

$$\begin{cases} K(\mathcal{T}_{p(x)}(u))((-\Delta)_{p(x)}u + V(x)|u|^{p(x)-2}u) = \lambda \left(\int_{\mathbb{R}^N} \frac{G(y,u(y))}{|x-y|^{\alpha(x,y)}}dy\right) g(x,u) + |u|^{p^*(x)-2}u \quad \text{in } \mathbb{R}^N, \\ u \in W_V^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where

$$\mathcal{T}_{p(x)}(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})dx,$$

K: $\mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the Kirchhoff function, $V \in C\left(\mathbb{R}^N, \mathbb{R}^+\right)$, $\alpha : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^+$, f is a continuous function, λ is a real parameter, $p : \mathbb{R}^N \to \mathbb{R}$ is a function and $p^*(x) = Np(x)/(N - p(x))$ is the critical Sobolev exponent.

In the sequel, if $h_1, h_2 \in C\left(\mathbb{R}^N\right)$, we say that $h_1 \ll h_2$ if $\inf\left\{h_2(x) - h_1(x) : x \in \mathbb{R}^N\right\} > 0$. And $C > 0$ may represent different constants.

Throughout this paper, we consider the following hypotheses:

($\mathcal{P}$) $p : \mathbb{R}^N \to \mathbb{R}$ is continuous and p satisfies

$$1 < p^- := \inf_{x\in\mathbb{R}^N} p(x) \leq p(x) \leqslant p^+ := \sup_{x\in\mathbb{R}^N} p(x) < N.$$

($\mathcal{V}$) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, with V_0 being a positive constant. Moreover, for any $\mathcal{D} > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq \mathcal{D}\} < \infty$, where meas$(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

($\mathcal{K}$) (K_1) $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and there exists $k_0 > 0$ satisfying $\inf_{t\geq 0} K(t) = k_0$.
(K_2) For all $t \geq 0$, there exists $\sigma \in [1, p^*(x)/2p^+)$ such that $\sigma\mathcal{K}(t) \geq K(t)t$, where$\mathcal{K}(t) = \int_0^t K(s)ds$.
(K_3) For all $t \in \mathbb{R}^+$, there exists $k_1 > 0$ such that $K(t) \geq k_1 t^{\sigma-1}$ and $K(0) = 0$.

($\mathcal{G}$) (g_1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and g is odd for the second variable.
(g_2) $r \in C(\mathbb{R}^N)$ and there exist $r(x) \geq 0$ satisfying $p(x) \ll r(x)q^- \leq r(x)q^+ \ll p^*(x)$. There exist $a \geq 0$ such that

$$0 \leq a \in L^\infty(\mathbb{R}^N) \cap L^{\frac{p^*(x)q^+}{p^*(x)-r(x)q^+}}(\mathbb{R}^N) \cap L^{\frac{p^*(x)q^-}{p^*(x)-r(x)q^-}}(\mathbb{R}^N)$$

and

$$|g(x,t)| \leq a(x)|t|^{r(x)-2}t \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

For all $x, y \in \mathbb{R}^N$,

$$\frac{1}{q(x)} + \frac{\alpha(x,y)}{N} + \frac{1}{q(y)} = 2$$

where

$$0 < \alpha^- := \inf_{x,y\in\mathbb{R}^N} \alpha(x,y) \leq \lambda^+ := \sup_{x,y\in\mathbb{R}^N} \alpha(x,y) < N.$$

(g_3) For all $t \in \mathbb{R}^+, g(x,t)$ and $G(x,t) = \int_0^t g(x,s)ds$, there exists θ satisfying $0 < \theta G(x,t) \leq 2g(x,t)t$ where $p^+/\sigma < \theta < p^*(x)$.

In 1931, variable exponents Lebesgue spaces appeared in [34]. It is known that the $p(\cdot)$-Laplacian is derived from the p-Laplacian, especially to the Laplacian ($p = 2$). From a practical point of view, variable exponents problem has many applications in the life, such as in image processing [11] and electrorheological fluids [40]. For these reasons, many authors have begun to study the existence of solutions to variable exponents problem, such as the books of Rădulescu-Repovš [39] and Diening et al. [13]. When it comes to critical problem, we know Brézis and Nirenberg studied in [8] at first in 1983 and then it is a nature extensions of [8]. However, many critical problems are confronted with the lack of compactness. In 1984, Lions in [26, 27] initially introduce the concentration-compactness principles. And in [5, 6, 10], authors show that there exists a minimizing or a (PS) sequence at infinity. In recent decades, it is nature for many scholars to consider more results for critical exponents $p(x)$-Laplacian equations. In [7, 15], they study the variable exponents second concentration-compactness principles in Ω. Moreover, there are much more results regarding $p(x)$-Laplacian and fractional $p(x)$-Laplacian equations, such as [1, 16, 18, 20–22, 29].

On the other hand, the study of the Choquard equation began with Fröhlich [17] and Pekar [35] who dealt with the following quantum polaron model:

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u \text{ in } \mathbb{R}^3. \tag{1.2}$$

Then in the following Choquard equation:

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x-y|^\lambda}\right) |u|^{p-2}u \text{ in } \mathbb{R}^N. \tag{1.3}$$

In particular, when $N = 3, p = 2$ and $\lambda = 1$, Lieb in [25] used problem (1.3) to get some significant results about plasma. As is known to all, Penrose [31, 36] applied Eq (1.3) as the model to solve gravity problem. Recently, more and more works pay attention to the problem (1.3) of existence and multiplicity of solutions. When it comes to the whole domain $\mathbb{R}^N$ of Choquard equations, we can cite [32, 33, 43] to get more details. For the critical case in bounded domains Ω, Gao and Yang in [45] considered about the following critical Choquard problem:

$$-\Delta u = \lambda u + \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right) |u|^{2^*_\mu - 2}u \text{ in } \Omega.$$

Then in [46], we got the existence of solutions for a series of equations by using variational methods. When it comes to the Choquard problems with variable exponents, we found there is lack of relevant results. Therefore, we call attention to [28], it is the first time to consider the nonhomogeneous Choquard equation with $p(x)$-Laplacian operator by using variational methods. Secondly, in combination with the truncation function and Krasnoselskii's genus, they found the multiplicity of solutions for the Choquard-type $p(x)$-Laplacian equations with non-degenerate Kirchhoff term. In [41], the authors proved the existence of at least two nontrivial solutions for nonhomogeneous Choquard equations by using of Nehari manifold and minimax methods.

Recently, Alves and Tavares [2] considered the following quasilinear variable exponent Choquard equations:

$$(-\Delta)_{p(x)}u + V(x)|u|^{p(x)-2}u = \int_{\mathbb{R}^N} \frac{G(y, u(y))}{|x-y|^{\alpha(x,y)}} dy g(x, u) \text{ in } \mathbb{R}^N. \tag{1.4}$$

The existence of solutions for Eq (1.4) was deduced from using the Hardy-Littlewood-Sobolev inequality together with variational methods. Zhang et al. in [47] considered the following equation:

$$\begin{cases} -\Delta_{p(x)}u + \mu|u|^{p(x)-2}u = \displaystyle\int_{\mathbb{R}^N} \frac{G(y, u(y))}{|x-y|^{\alpha(x,y)}} dy g(x, u) + \beta(x)|u|^{p^*(x)-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where $p : \mathbb{R}^N \to \mathbb{R}$ is radially symmetric and $\mu > 0$. The existence of infinitely many solutions for problem (1.5) was obtained by variational methods, Hardy-Littlewood-Sobolev inequality and the concentration-compactness principle. The results of critical Choquard-Kirchhoff equations with variable exponents Eq (1.1) does not obtain, especially for the degenerate cases.

The research complete and improve results for the critical Choquard-Kirchhoff type equations involving variable exponents. Especially we discuss the results in non-degenerate and degenerate cases which are treated in many papers, for example, see a well-known paper [12]. In the recent decades, more and more attention were paied to degenerate Kirchhoff problem. For example, in 2015, Autuori et al. in [4] used the mountain pass theorem to demonstrate the asymptotic behavior of non-negative solutions for Kirchhoff equations. Then Pucci et al. in [37] considered entire solutions for the stationary Kirchhoff equations. Not long after, in 2016, Caponi and Pucci in [9] also investigate existence of entire solutions for a class of Kirchhoff fractional equations. And we can refer to [23, 24, 30, 42, 44] to get related content and details.

And $u \in W_V^{1,p(x)}\left(\mathbb{R}^N\right)$ is a weak solution of Eq (1.1) if

$$\begin{aligned} &K(\mathcal{T}_{p(x)}(u)) \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + V(x)|u|^{p(x)-2}uv\right) dx \\ &= \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(y,u(y))g(x,u(x))v(x)}{|x-y|^{\alpha(x,y)}} dxdy + \int_{\mathbb{R}^N} |u|^{p^*(x)-2}uvdx \end{aligned} \tag{1.6}$$

for all $v \in W_V^{1,p(x)}\left(\mathbb{R}^N\right)$. The space $W_V^{1,p(x)}\left(\mathbb{R}^N\right)$ will be introduced in Section 2.

Now we are in a position to give the main theorems of this paper.

Theorem 1.1. *Assume p, V, K and g respectively satisfy $(\mathcal{P})$, $(\mathcal{V})$, (K_1), (K_2) and (g_1)–(g_3), respectively. In $W_V^{1,p(x)}(\mathbb{R}^N)$, there exists $\lambda_1 > 0$ and $\lambda \geq \lambda_1$, Eq (1.1) admits a nontrivial solution.*

Theorem 1.2. *Assume p, V, K and g satisfy $(\mathcal{P})$, $(\mathcal{V})$, (K_1), (K_2) and (g_1)–(g_3), respectively. In $W_V^{1,p(x)}(\mathbb{R}^N)$, there exists constant $\lambda_2 > 0$ and $\lambda > \lambda_2$, Eq (1.1) admits at least s pairs of nontrivial solutions.*

Therefore, we obtain similar results in the degenerate case.

Theorem 1.3. *Assume p, V, K and g satisfy $(\mathcal{P})$, $(\mathcal{V})$, (K_2), (K_3) and (g_1)–(g_3), respectively. In $W_V^{1,p(x)}(\mathbb{R}^N)$, there exists $\lambda_3 > 0$ and $\lambda \geq \lambda_3$, Eq (1.1) admits a nontrivial solution in $W_V^{1,p(x)}(\mathbb{R}^N)$.*

Theorem 1.4. *Assume p, V, K and g satisfy $(\mathcal{P})$, $(\mathcal{V})$, (K_2), (K_3) and (g_1)–(g_3), respectively. In $W_V^{1,p(x)}(\mathbb{R}^N)$, there exists constant $\lambda_4 > 0$ and $\lambda \geq \lambda_4$, Eq (1.1) admits at least s pairs of nontrivial solutions in $W_V^{1,p(x)}(\mathbb{R}^N)$.*

The paper is organized as follows. Section 2 contains fundamental knowledge of spaces with variable exponents. In Section 3, we verify the $(PS)_c$ condition. Section 4 and Section 5 respectively prove Theorems 1.1–1.4.

2. Preliminaries

In this section, we give fundamental knowledge on the Lebesgue spaces and the Sobolev spaces with variable exponents. We refer to [13, 14] for more details.

Assume Ω be a bounded domain of $\mathbb{R}^N$, and

$$C_+(\bar{\Omega}) = \{f \in C(\bar{\Omega}) : f(x) > 1 \text{ for all } x \in \bar{\Omega}\}.$$

We define

$$f^- = \min_{x\in\bar{\Omega}} f(x), f^+ = \max_{x\in\bar{\Omega}} f(x).$$

And we define the variable exponent Lebesgue space as

$$L^{p(x)}(\Omega)=\left\{u:\Omega\to\mathbb{R}\mid u\text{ is measurable and }\int_\Omega|u(x)|^{p(x)}dx<+\infty\right\}$$

endowed with the norm

$$|u|_{L^{p(x)}(\Omega)}=|u|_{p(x)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le 1\right\}.$$

The Lebesgue-Sobolev space with variable exponents $W^{1,p(x)}\left(\mathbb{R}^N\right)$ is defined by:

$$W^{1,p(x)}\left(\mathbb{R}^N\right)=\left\{u\in L^{p(x)}\left(\mathbb{R}^N\right):|\nabla u|\in L^{p(x)}\left(\mathbb{R}^N\right)\right\},$$

with the norm

$$\|u\|_{W^{1,p(x)}\left(\mathbb{R}^N\right)}=|u|_{p(x)}+|\nabla u|_{p(x)}.$$

For problem (1.1), we study in $W_V^{1,p(x)}\left(\mathbb{R}^N\right)$ which is more suitable, with the norm

$$\|u\|_{W_V^{1,p(x)}\left(\mathbb{R}^N\right)}=\|\nabla u\|_{L^{p(x)}\left(\mathbb{R}^N\right)}+\|u\|_{L_V^{p(x)}\left(\mathbb{R}^N\right)}$$

where

$$\|u\|_{L_V^{p(x)}\left(\mathbb{R}^N\right)}=\inf\left\{\eta>0:\int_{\mathbb{R}^N}V(x)\left|\frac{u}{\eta}\right|^{p(x)}dx\le 1\right\}.$$

Proposition 2.1 ([14]). (1) *Denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$, there holds*

$$\int_\Omega|uv|dx\le\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)},\ u\in L^{p(x)}(\Omega),v\in L^{p'(x)}(\Omega).$$

(2) *$\rho:L^{p(x)}(\Omega)\to\mathbb{R}$ and $\rho(u)=\int_\Omega|u|^{p(x)}dx$,*

$$|u|_{p(x)}<1(=1,>1)\Leftrightarrow\rho(u)<1(=1,>1),$$
$$|u|_{p(x)}>1\Rightarrow|u|_{p(x)}^{p^-}\le\rho(u)\le|u|_{p(x)}^{p^+},$$
$$|u|_{p(x)}<1\Rightarrow|u|_{p(x)}^{p^+}\le\rho(u)\le|u|_{p(x)}^{p^-}.$$

Proposition 2.2 ([2]). *Assume $p,q\in C^+\left(\mathbb{R}^N\right)$, $w\in L^{p^+}\left(\mathbb{R}^N\right)\cap L^{p^-}\left(\mathbb{R}^N\right)$, $z\in L^{q^+}\left(\mathbb{R}^N\right)\cap L^{q^-}\left(\mathbb{R}^N\right)$, and $\alpha:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ be a continuous function satisfying $0<\alpha^-:=\inf_{x\in\mathbb{R}^N}\alpha(x)\le\alpha^+:=\sup_{x\in\mathbb{R}^N}\alpha(x)<N$ and for $\forall x,y\in\mathbb{R}^N$, there is*

$$\frac{1}{p(x)}+\frac{\alpha(x,y)}{N}+\frac{1}{q(y)}=2.$$

Then, we have

$$\left|\iint_{\mathbb{R}^{2N}}\frac{w(x)z(y)}{|x-y|^{\alpha(x,y)}}dxdy\right|\le C\left(|w|_{L^{p^+}\left(\mathbb{R}^N\right)}|z|_{L^{q^+}\left(\mathbb{R}^N\right)}+|w|_{L^{p^-}\left(\mathbb{R}^N\right)}|z|_{L^{q^-}\left(\mathbb{R}^N\right)}\right)$$

where $C>0$ is irrelevant w and z.

Corollary 2.1. *For $w(x) = z(x) = |v(x)|^{\tau(x)} \in L^{r^+}(\mathbb{R}^N) \cap L^{r^-}(\mathbb{R}^N)$, there exists $C > 0$ which is irrelevant r such that*

$$\left| \iint_{\mathbb{R}^{2N}} \frac{|v(x)|^{\tau(x)} |v(y)|^{\tau(y)}}{|x-y|^{\alpha(x,y)}} dxdy \right| \le C \left(\left| |v|^{\tau(\cdot)} \right|^2_{L^{r^+}(\mathbb{R}^N)} + \left| |v|^{\tau(\cdot)} \right|^2_{L^{r^-}(\mathbb{R}^N)} \right),$$

τ, $r \in C_+(\overline{\mathbb{R}^N})$ satisfying $1 < \tau^- r^- \le \tau(x) r^- \le \tau(x) r^+ < p^*(x)$.

Remark 2.1. *If $(\mathcal{P})$ and $(\mathcal{V})$ hold, then for all $s \in C^+\left(\mathbb{R}^N\right)$ and $p(x) \le s(x) \le p^*(x), \forall x \in \mathbb{R}^N$,*

$$W_V^{1,p(x)}\left(\mathbb{R}^N\right) \hookrightarrow L^{s(x)}\left(\mathbb{R}^N\right)$$

is compact embedding. Hence,

$$\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \le S\,|u|_{L^{s(x)}(\mathbb{R}^N)},$$

where S is the best Sobolev constant.

Remark 2.2. *We can find that there is $b > 0$ satisfying*

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right) dx \ge b \left(\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx \right).$$

3. $(PS)_c$ condition

Let's first recall the definition of the $(PS)_c$ condition. The functional J_λ satisfies the $(PS)_c$ condition if any sequence $J_\lambda(u_n) \to c$ and $J'_\lambda(u_n) \to 0$ has a convergent subsequence. In this section, we will prove the functional J_λ satisfies the $(PS)_c$ condition.

The energy functional $J_\lambda : W_V^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ is

$$J_\lambda(u) = \mathcal{K}\left(\mathcal{T}_{p(x)}(u)\right) - \lambda \Lambda(u) - \int_{\mathbb{R}^N} \frac{1}{p^*(x)} |u|^{p^*(x)} dx, \tag{3.1}$$

where

$$\Lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(x, u(x)) G(y, u(y))}{|x-y|^{\alpha(x,y)}} dxdy,$$

$G(x,t) = \int_0^t g(x,s) ds$. Obviously, $J_\lambda \in C^1(W_V^{1,p(x)}(\mathbb{R}^N))$. Moreover, for all $u, v \in W_V^{1,p(x)}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned} \langle J'_\lambda(u), v\rangle =& K(\mathcal{T}_{p(x)}(u)) \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + V(x)|u|^{p(x)-2} uv\right) dx \\ &- \lambda \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(y,u(y)) g(x,u(x)) v(x)}{|x-y|^{\alpha(x,y)}} dxdy - \int_{\mathbb{R}^N} |u|^{p^*(x)-2} uv dx. \end{aligned} \tag{3.2}$$

Hence, the solutions of problem (1.1) are the critical points of J_λ.

Lemma 3.1. *Assume $(\mathcal{P})$, $(\mathcal{V})$, $(\mathcal{G})$, (K_1) and (K_2) hold. Let $(u_n)_n \subset W_V^{1,p(x)}(\mathbb{R}^N)$ be a (PS) sequence of J_λ, then*

$$J_\lambda(u_n) \to c_\lambda \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{in} \quad (W_V^{1,p(x)}(\mathbb{R}^N))' \tag{3.3}$$

as $n \to \infty$, where $(W_V^{1,p(x)}(\mathbb{R}^N))'$ is the dual of $W_V^{1,p(x)}(\mathbb{R}^N)$. If there is $\tau(x) = \frac{p^*(x)}{p^*(x)-p^+}$ such that

$$c_\lambda < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right) \min\left\{ \left(k_0 S^{p^+}\right)^{\tau^+}, \left(k_0 S^{p^+}\right)^{\tau^-} \right\}, \tag{3.4}$$

where S is Sobolev constant. In $W_V^{1,p(x)}(\mathbb{R}^N)$, $(u_n)_n \to u$ strongly.

Proof. We prove $(u_n)_n$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$.

Assume $(u_n)n$ and c_λ satisfy (3.3) and (3.4), respectively. Then, from (f_3), we can deduce that

$$\begin{aligned}
&c_\lambda + 1 + o(1)\|u_n\| \\
&\geq J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\
&\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right)\frac{k_0}{p^+}\left[\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}dx\right] + \int_{\mathbb{R}^N}\left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)|u_n|^{p^*(x)}dx \\
&\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right)\frac{k_0}{p^+}\|u_n\|^{p^-}.
\end{aligned} \tag{3.5}$$

So $(u_n)_n$ is bounded in $W_V^{1,p(x)}(\mathbb{R}^N)$.

Then we need to demonstrate

$$\langle \Lambda'(u_n) - \Lambda'(u), u_n - u\rangle \to 0 \text{ as } n \to \infty.$$

In fact, since $u_n \to u$ weakly in $W_V^{1,p(x)}\left(\mathbb{R}^N\right)$ as $n \to \infty$, when $\Lambda'(u) \in \left(W_V^{1,p(x)}\left(\mathbb{R}^N\right)\right)'$, we yield that

$$\langle \Lambda'(u), u_n - u\rangle \to 0 \text{ as } n \to \infty.$$

So we only need to prove that

$$\langle \Lambda'(u_n), u_n - u\rangle \to 0 \text{ as } n \to \infty.$$

We deduce from Proposition 2.2 that

$$\begin{aligned}
|\langle \Lambda'(u_n), u_n - u\rangle| \leq& C\,\|G(x, u_n)\|_{L^{p^+}(\mathbb{R}^N)}\,\|g(x, u_n)(u_n - u)\|_{L^{q^+}(\mathbb{R}^N)} \\
&+ C\,\|G(x, u_n)\|_{L^{p^-}(\mathbb{R}^N)}\,\|g(x, u_n)(u_n - u)\|_{L^{q^-}(\mathbb{R}^N)}.
\end{aligned} \tag{3.6}$$

Combining (g_2) and $(u_n)_n$,

$$\begin{aligned}
\|G(x, u_n)\|_{L^{p^+}(\mathbb{R}^N)} \leq& C\left(\int_{\mathbb{R}^N}\left(|u_n|^{p^+r(x)}\right)dx\right)^{\frac{1}{p^+}} \\
\leq& C\max\left\{\|u_n\|^{r^+}_{L^{p^+r(x)}(\mathbb{R}^N)}, \|u_n\|^{r^-}_{L^{p^+r(x)}(\mathbb{R}^N)}\right\} \\
\leq& C
\end{aligned} \tag{3.7}$$

and

$$\|G(x, u_n)\|_{L^{p^-}(\mathbb{R}^N)} \leq C\max\left\{\|u_n\|^{r^+}_{L^{p^-r(x)}(\mathbb{R}^N)}, \|u_n\|^{r^-}_{L^{p^-r(x)}(\mathbb{R}^N)}\right\} \leq C. \tag{3.8}$$

According to (g_2) and Remark 2.1, we can yield that

$$\begin{aligned}
&\|g(x, u_n)(u_n - u)\|^{q^+}_{L^{q^+}(\mathbb{R}^N)} \\
&\leq C\left\||u_n|^{q^+(r(x)-1)}\right\|_{L^{\frac{r(x)}{r(x)-1}}(\mathbb{R}^N)}\left\||u_n - u|^{q^+}\right\|_{L^{r(x)}(\mathbb{R}^N)} \\
&\leq C\max\left\{\|u_n - u\|^{q^+}_{L^{q^+r(x)}(\mathbb{R}^N)}, \|u_n - u\|^{\frac{q^+r^-}{r^+}}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\} \\
&\quad + C\max\left\{\|u_n - u\|^{\frac{q^+r^+}{r^-}}_{L^{q^+r(x)}(\mathbb{R}^N)}, \|u_n - u\|^{q^+}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\} \\
&= o_n(1) \text{ as } n \to \infty
\end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
&\|g(\cdot,u_n)(u_n-u)\|_{L^{q^-}(\mathbb{R}^N)}^{q^-}\\
&\le C\left\||u_n|^{q^-(r(\cdot)-1)}\right\|_{L^{\frac{r(x)}{r(x)-1}}(\mathbb{R}^N)}\left\||u_n-u|^{q^-}\right\|_{L^{r(x)}(\mathbb{R}^N)}\\
&\le C\max\left\{\|u_n-u\|_{L^{q^-r(x)}(\mathbb{R}^N)}^{q^-},\|u_n-u\|_{L^{q^-r(x)}(\mathbb{R}^N)}^{\frac{q^-r^-}{r^+}}\right\}\\
&\quad+C\max\left\{\|u_n-u\|_{L^{q^-r(x)}(\mathbb{R}^N)}^{\frac{q^-r^+}{r^-}},\|u_n-u\|_{L^{q^-r(x)}(\mathbb{R}^N)}^{q^-}\right\}\\
&=o_n(1)\quad\text{as }\ n\to\infty.
\end{aligned}
\tag{3.10}
$$

Combining (3.6)–(3.10), we can obtain $\langle\Lambda'(u_n),u_n-u\rangle\to 0$ as $n\to\infty$. So we deduce that

$$\langle\Lambda'(u_n)-\Lambda'(u),u_n-u\rangle\to 0,\ \text{as } n\to\infty.$$

Then, in view of the concentration-compactness principle for variable exponents in [19], we get

$$
\begin{aligned}
&u_n\to u\quad\text{a.e. in } \mathbb{R}^N,\\
&u_n\rightharpoonup u\quad\text{in}\quad W_V^{1,p(x)}(\mathbb{R}^N),\\
&U_n(x)\overset{*}{\rightharpoonup}\mu\ge U(x)+\sum_{i\in I}\delta_{x_i}\mu_i,\\
&|u_n|^{p^*(x)}\overset{*}{\rightharpoonup}\nu=|u|^{p^*(x)}+\sum_{i\in I}\delta_{x_i}\nu_i,\\
&S\nu_i^{\frac{1}{p^*(x)}}\le\mu_i^{\frac{1}{p(x)}}\quad\text{for}\quad i\in I,
\end{aligned}
\tag{3.11}
$$

where

$$U_n(x):=|\nabla u_n(x)|^{p(x)}+V(x)|u_n(x)|^{p(x)}$$

and

$$U(x):=|\nabla u(x)|^{p(x)}+V(x)|u(x)|^{p(x)}.$$

Furthermore, we yield that

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}U_n(x)dx=\mu(\mathbb{R}^N)+\mu_\infty,\\
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*(x)}dx=\nu(\mathbb{R}^N)+\nu_\infty,\\
&S\nu_\infty^{1/p_\infty^*}\le\mu_\infty^{1/p_\infty},
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
&\mu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{|x|>R\}}(|\nabla u_n(x)|^{p(x)}+V(x)|u_n(x)|^{p(x)})dx,\\
&\nu_\infty=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{|x|>R\}}|u_n|^{p^*(x)}dx,\\
&p_\infty=\lim_{|x|\to\infty}p(x)\quad\text{and}\quad p_\infty^*=\lim_{|x|\to\infty}p^*(x).
\end{aligned}
$$

Now we demonstrate

$$I=\emptyset\quad\text{and}\quad\nu_\infty=0.$$

We assume that $I \neq \emptyset$. For any $i \in I$ and any $\varepsilon > 0$ small, we define a function $\phi_{\varepsilon,i}$ centered at z_i satisfying

$$0 \le \phi_{\varepsilon,i}(x) \le 1,\ \phi_{\varepsilon,i}(x) = 1 \text{ in } B_{2\varepsilon}(z_i),\ \phi_{\varepsilon,i}(x) = 0 \text{ in } B_{\varepsilon}(z_i)^c,\ |\nabla \phi_{\varepsilon,i}(x)| \le 2/\varepsilon.$$

Combining with $\langle J'_\lambda(u_n), u_n\phi_{\varepsilon,i}\rangle \to 0$, we deduce that

$$\begin{aligned} &K(\mathcal{T}_{p(x)}(u_n)) \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)}\phi_{\varepsilon,i} + V(x)|u_n|^{p(x)}\phi_{\varepsilon,i} + |\nabla u_n|^{p(x)-2}\nabla u_n \nabla \phi_{\varepsilon,i} u_n\right) dx \\ &= \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(y,u_n(y))g(x,u_n(x))u_n\phi_{\varepsilon,i}}{|x-y|^{\alpha(x,y)}} dxdy + \int_{\mathbb{R}^N} |u_n|^{p^*(x)}\phi_{\varepsilon,i} dx + o_n(1). \end{aligned} \tag{3.13}$$

We deduce from $u_n \to u$ in $L^{p(x)}\left(B_{2\varepsilon}(z_i)\right)$ that

$$\left\|\nabla \phi_{\varepsilon,i} u_n\right\|_{L^{p(x)}(\mathbb{R}^N)} \to \|\nabla \phi_{\varepsilon,i} u\|_{L^{p(x)}(\mathbb{R}^N)} \quad \text{as } n \to \infty.$$

So,

$$\begin{aligned} &\lim_{n\to\infty} \left|\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi_{\varepsilon,i} u_n dx\right| \\ &\le \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-1} |\nabla \phi_{\varepsilon,i} u_n| dx \\ &\le \limsup_{n\to\infty} C \left\| |\nabla u_n|^{p(x)-1}\right\|_{L^{\frac{p(x)}{p(x)-1}}(\mathbb{R}^N)} \left\|\nabla\phi_{\varepsilon,i} u_n\right\|_{L^{p(x)}(\mathbb{R}^N)} \\ &\le C \|\nabla \phi_{\varepsilon,i} u\|_{L^{p(x)}(\mathbb{R}^N)} \end{aligned} \tag{3.14}$$

and in $\mathbb{R}^N$, we can choose w_N to be the unit sphere,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \phi_{\varepsilon,i} u|^{p(x)} dx \\ &= \int_{B_{2\varepsilon}(z_i)} |\nabla \phi_{\varepsilon,i} u|^{p(x)} dx \le C \left\| |\nabla \phi_{\varepsilon,i}|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p^*(x)-p(x)}}(B_{2\varepsilon}(z_i))} \left\| |u|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\varepsilon}(z_i))} \\ &\le C \max\left\{ \left(\int_{B_{2\varepsilon}(z_i)} |\nabla\phi_{\varepsilon,i}|^N dx\right)^{\frac{p^+}{N}}, \left(\int_{B_{2\varepsilon}(z_i)} |\nabla\phi_{\varepsilon,i}|^N dx\right)^{\frac{p^-}{N}} \right\} \left\| |u|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\varepsilon}(z_i))} \\ &\le C \max\left\{ \left(\frac{4^N w_N}{N}\right)^{\frac{p^+}{N}}, \left(\frac{4^N w_N}{N}\right)^{\frac{p^-}{N}} \right\} \left\| |u|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p(x)}}(B_{2\varepsilon}(z_i))} \\ &= o_\varepsilon(1) \ \text{ as } \varepsilon \to 0. \end{aligned} \tag{3.15}$$

Next, as $n \to \infty$, we claim

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(y,u_n(y))g(x,u_n(x))u_n\phi_{\varepsilon,i}}{|x-y|^{\alpha(x,y)}} dxdy = \left\langle \Lambda'\left(u_n\right), u_n\phi_{\varepsilon,i}\right\rangle$$

$$\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(y,u(y))g(x,u(x))u(x)\phi_{\varepsilon,i}(x)}{|x-y|^{\alpha(x,y)}} dxdy = \left\langle \Phi'(u), u\phi_{\varepsilon,i}\right\rangle.$$

According to Proposition 2.2 and the Lebesgue dominated convergence theorem,

$$\begin{aligned}
&\left|\left\langle \Lambda'\left(u_n\right), u_n\phi_{\varepsilon,i}\right\rangle - \left\langle \Lambda'(u), u\phi_{\varepsilon,i}\right\rangle\right| \\
&\leq \left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G\left(y, u_n(y)\right)\left(g\left(x, u_n(x)\right)u_n(x) - g(x, u(x))u(x)\right)}{|x-y|^{\alpha(x,y)}}dxdy\right| \\
&\quad + \left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left(G\left(y, u_n(y)\right) - G(y, u(y))\right)g(x, u(x))u(x)}{|x-y|^{\alpha(x,y)}}dxdy\right| \\
&= o_n(1) \ \text{ as } n \to \infty.
\end{aligned} \tag{3.16}$$

And

$$\begin{aligned}
\left|\left\langle \Phi'(u), u\Lambda_{\varepsilon,i}\right\rangle\right| &\leq C\|g(x,u)u\phi_{\varepsilon,i}\|_{L^{q^+}(\mathbb{R}^N)} + C\|g(x,u)u\phi_{\varepsilon,i}\|_{L^{q^-}(\mathbb{R}^N)} \\
&\leq C\left(\int_{B_{2\varepsilon}(z_i)} |u|^{r(x)q^+}dx\right)^{\frac{1}{q^+}} + C\left(\int_{B_{2\varepsilon}(z_i)} |u|^{r(x)q^-}dx\right)^{\frac{1}{q^-}} \\
&= o_\varepsilon(1) \ \text{ as } \varepsilon \to 0.
\end{aligned} \tag{3.17}$$

Combining (3.13)–(3.17), we deduce that

$$K(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}\phi_{\varepsilon,i} + V(x)|u_n|^{p(x)}u_n\phi_{\varepsilon,i}\right)dx = \int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_{\varepsilon,i}dx + o_n(1). \tag{3.18}$$

Since $\phi_{\varepsilon,i}$ has compact support and (K_1), choosing $n \to \infty$ and $\varepsilon \to 0$ in (3.18), we get

$$k_0\mu_i \leq \nu_i.$$

In view of (3.11), we yield that

$$\nu_i \geq \left(k_0 S^{p^+}\right)^{\frac{p^*(z_i)}{p^*(z_i)-p^+}} \geq \min\left\{\left(k_0 S^{p^+}\right)^{\tau^+}, \left(k_0 S^{p^+}\right)^{\tau^-}\right\}, \tag{3.19}$$

where $\tau(x) = \frac{p^*(x)}{p^*(x)-p^+(x)}$. On account of (3.3), (3.19) and (3.5),

$$\begin{aligned}
c_\lambda &= \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle\right) \\
&\geq \int_{\mathbb{R}^N}\left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)|u_n|^{p^*(x)}dx \\
&\geq \left(\frac{1}{\theta} - \frac{1}{p^+}\right)\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_{\varepsilon,i}dx \geq \left(\frac{1}{\theta} - \frac{1}{p^+}\right)\nu_i \\
&\geq \left(\frac{1}{\theta} - \frac{1}{p^+}\right)\min\left\{\left(m_0 S^{p^+}\right)^{\tau^+}, \left(m_0 S^{p^+}\right)^{\tau^-}\right\} > c_\lambda.
\end{aligned} \tag{3.20}$$

We get a contradiction, so $I = \emptyset$.

Then we show $\nu_\infty = 0$. In the first, we assume $\nu_\infty > 0$. Similarly, we define $\phi_R \in C_0^\infty(\mathbb{R}^N)$ satisfying $\phi_R(x) = 0$ in B_R and $\phi_R(x) = 1$ in B_{R+1}^c. According to $\langle J'_\lambda(u_n), u_n\phi_R\rangle \to 0$ as $n \to \infty$, we deduce

$$\begin{aligned}
&K(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}\phi_R + V(x)|u_n|^{p(x)}u_n\phi_R + |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_R u_n\right)dx \\
&= \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y, u_n(y))g(x, u_n(x))u_n\phi_R}{|x-y|^{\alpha(x,y)}}dxdy + \int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_R dx + o_n(1).
\end{aligned} \tag{3.21}$$

We have

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_R u_n dx\right| = 0$$

and

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y,u_n(y))g(x,u_n(x))u_n\phi_R}{|x-y|^{\alpha(x,y)}}dxdy = \langle\Lambda'(u_n), u_n\phi_R\rangle = 0.$$

So we get

$$K(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}\phi_R + V(x)|u_n|^{p(x)}u_n\phi_R\right)dx = \int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_R dx + o_n(1). \tag{3.22}$$

Letting $R\to\infty$ in (3.22), we deduce

$$k_0\mu_\infty \le \nu_\infty. \tag{3.23}$$

According to (3.12) and (3.24), we can also infer

$$\begin{aligned} c_\lambda &= \lim_{n\to\infty}J_\lambda(u_n) = \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle\right)\\ &\ge \int_{\mathbb{R}^N}\left(\frac{1}{\theta}-\frac{1}{p^*(x)}\right)|u_n|^{p^*(x)}dx\\ &\ge \left(\frac{1}{\theta}-\frac{1}{p^+}\right)\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_R dx \ge \left(\frac{1}{\theta}-\frac{1}{p^+}\right)\nu_\infty\\ &\ge \left(\frac{1}{\theta}-\frac{1}{p^+}\right)\min\left\{\left(k_0 S^{p^+}\right)^{\tau^+}, \left(k_0 S^{p^+}\right)^{\tau^-}\right\} > c_\lambda. \end{aligned} \tag{3.24}$$

Then we get a contradiction, so $\nu_\infty = 0$.

Therefore, combination of $I=\emptyset$ and $\nu_\infty = 0$,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*(x)}dx = \int_{\mathbb{R}^N}|u|^{p^*(x)}dx.$$

According to the Brézis-Lieb type lemma, we get

$$\int_{\mathbb{R}^N}|u_n-u|^{p^*(x)}dx \to 0,$$

thus $\|u_n-u\|_{L^{p^*(x)}(\mathbb{R}^N)}\to 0$. Consequently, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|u_n|^{p^*(x)-2}u_n - |u|^{p^*(x)-2}u\right)(u_n-u)\,dx = 0. \tag{3.25}$$

Then we get

$$\lim_{n\to\infty}\left(K\left(\mathcal{T}_{p(x)}(u_n)\right) - K\left(\mathcal{T}_{p(x)}(u)\right)\right)\langle L(u), u_n-u\rangle = 0, \tag{3.26}$$

where $w\in W_V^{1,p(x)}(\mathbb{R}^N)$, $L(v)$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ was

$$\begin{aligned} \langle L(v), w\rangle &= \int_{\mathbb{R}^N}\left(|\nabla v|^{p(x)-2}\nabla v\nabla w + V(x)|v|^{p(x)-2}vw\right)dx\\ &= \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y,v(y))g(x,v(x))w(x)}{|x-y|^{\alpha(x,y)}}dxdy + \int_{\mathbb{R}^N}|v|^{p^*(x)-2}vwdx. \end{aligned} \tag{3.27}$$

Combining the weak convergence of $(u_n)_n$ in $W_V^{1,p(x)}(\mathbb{R}^N)$ with the boundedness of $(K(T_{p(x)}(u_n)) - K(T_{p(x)}(u)))_n$ in $\mathbb{R}^N$, we obtain that

$$\lim_{n\to\infty}\left(K\left(\mathcal{T}_{p(x)}(u_n)\right) - K\left(\mathcal{T}_{p(x)}(u)\right)\right)\langle L(u), u_n - u\rangle = 0. \tag{3.28}$$

In view of $\langle J_\lambda(u_n), u_n - u\rangle \to 0$ $(n \to \infty)$,

$$\begin{aligned}
o(1) &= \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \\
&= K\left(\mathcal{T}_{p(x)}(u_n)\right)[\langle L(u_n), u_n - u\rangle - \langle L(u), u_n - u\rangle] + [K\left(\mathcal{T}_{p(x)}(u_n)\right) - K\left(\mathcal{T}_{p(x)}(u)\right)]\langle L(u), u_n - u\rangle \\
&\quad - \lambda\langle \Lambda'(u_n) - \Lambda'(u), u_n - u\rangle - \int_{\mathbb{R}^N}\left(|u_n|^{p^*(x)-2}u_n - |u|^{p^*(x)-2}u\right)(u_n - u)\,dx \\
&= K\left(\mathcal{T}_{p(x)}(u_n)\right)[\langle L(u_n), u_n - u\rangle - \langle L(u), u_n - u\rangle] + o(1).
\end{aligned} \tag{3.29}$$

Hence, we obtain that

$$\int_{\mathbb{R}^N}\left(|\nabla(u_n - u)|^{p(x)} + V(x)|u_n - u|^{p(x)}\right)dx = 0.$$

Therefore, in $W_V^{1,p(x)}(\mathbb{R}^N)$, we get $(u_n)_n \to u$ strongly. □

4. Non-degenerate case for Eq (1.1)

We respectively demonstrate Theorems 1.1 and 1.2 by using the mountain pass theorem [3] and the Krasnoselskii genus [38].

4.1. Proof of Theorem 1.1

First of all, we demonstrate there exists mountain pass structure for J_λ.

Lemma 4.1. *Let E is a real Banach space, and $J_\lambda \in C^1(E)$, with $J_\lambda(0) = 0$. If the following conditions hold true:*

(1) *For any $u \in E$ and $\|u\|_E = \rho$, there is $\rho, \chi > 0$ satisfying $J_\lambda(u) \geq \chi$;*
(2) *For any $\|\omega\|_E > \rho$, there is $\omega \in E$ satisfying $J_\lambda(\omega) < 0$.*

We can define $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 1, \gamma(1) = \omega\}$. Hence,

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} J_\lambda(\gamma(t)) \geq \chi$$

and $(u_n)_n \subset E$.

Proof. First, we verify condition (1) of Lemma 4.1. In view of (K_1), we have $\inf_{t\geq 0} K(t) = k_0$. Then according to Remark 2.1, we get $\|u\|_{W_V^{1,p(x)}(\mathbb{R}^N)} \leq S\,|u|_{L^{s(x)}(\mathbb{R}^N)}$. So,

$$
\begin{aligned}
J_\lambda(u) &= \mathcal{K}\left(\mathcal{T}_{p(x)}(u)\right) - \lambda\Lambda(u) - \int_{\mathbb{R}^N} \frac{1}{p^*(x)}|u|^{p^*(x)}dx \\
&\geq \frac{k_0}{p^+}\|u\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - C\|G(x,u)\|^2_{L^{q^+}(\mathbb{R}^N)} - C\|G(x,u)\|^2_{L^{q^-}(\mathbb{R}^N)} \\
&\geq \frac{k_0}{p^+}\|u\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - C\max\left\{\|u\|^{2r^+}_{L^{q^+r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\} \\
&\quad - C\max\left\{\|u\|^{2r^+}_{L^{q^-r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^-r(x)}(\mathbb{R}^N)}\right\} \\
&\geq \frac{k_0}{p^+}\|u\|^{p^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{2C}{S^{2r^-}}\|u\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)} \text{ for any } u \in W_V^{1,p(x)}(\mathbb{R}^N),
\end{aligned}
$$

where $\|u\| \leq 1$. Hence, let $\rho, \chi > 0$, $\|u\| = \rho$ and the fact $p^- \leq q^-r(x) \leq q^+r(x) \ll p^*$ satisfying $J_\lambda(u) \geq \chi$. So we prove (1) of Lemma 4.1.

In order to prove the conclusion (2) of Lemma 4.1, we choose $\psi \in C_0^\infty(\mathbb{R}^N)$ and $\psi > 0$, combination with (K_2), for all $t \geq 1$, we get

$$
\mathcal{K}(t) \leq \mathcal{K}(1)t^\sigma. \tag{4.1}
$$

According to the conditions of f, we obtain that

$$
\begin{aligned}
J_\lambda(t\psi) &= \mathcal{K}\left(\mathcal{T}_{p(x)}(t\psi)\right) - \lambda\Lambda(t\psi) - \int_{\mathbb{R}^N} \frac{1}{p^*(x)}|t\psi|^{p^*(x)}dx \\
&\leq \mathcal{K}\left(\mathcal{T}_{p(x)}(t\psi)\right) - \int_{\mathbb{R}^N} \frac{1}{p^*(x)}|t\psi|^{p^*(x)}dx \\
&\leq \mathcal{K}(1)t^{\sigma p^+}T_{p(x)}(\psi) - \frac{t^{p^*(x)}}{p^*(x)}\int_{\mathbb{R}^N}|\psi|^{p^*(x)}dx \quad \text{for all } t > 1.
\end{aligned}
\tag{4.2}
$$

In view of $\sigma p^+ < p^*(x)$ and for t_0 large enough, we obtain $J_\lambda(t_0\psi) < 0$ and $t_0\|\psi\| > \rho$. Set $\omega = t_0\psi$, so e satisfies the conditions and the conclusion (2) is true. □

Proof of Theorem 1.1. Next, if λ large enough, we prove that

$$
c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)\min\left\{\left(k_0S^{p^+}\right)^{\tau^+}, \left(k_0S^{p^+}\right)^{\tau^-}\right\}. \tag{4.3}
$$

Combining (4.3), Lemmas 3.1 and 4.1, it is obvious that we can deduce J_λ exists nontrivial critical points. So we need to demonstrate (4.3). Let $v_0 \in W_V^{1,p(x)}(\mathbb{R}^N)$ such that

$$
\mathcal{T}_{p(x)}(v_0) = 1 \quad \text{and} \quad \lim_{t\to\infty} J_\lambda(tv_0) = -\infty.
$$

Hence, for some $t_\lambda > 0$, we get $\sup_{t\geq 0} J_\lambda(tv_0) = J_\lambda(t_\lambda v_0)$. And

$$
\begin{aligned}
&K(\mathcal{T}_{p(x)}(tv_0))\int_{\mathbb{R}^N}\left(|\nabla tv_0|^{p(x)} + V(x)|tv_0|^{p(x)}\right)dx \\
&= \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y,tv_0(y))g(x,tv_0(x))tv_0}{|x-y|^{\alpha(x,y)}}dxdy + \int_{\mathbb{R}^N}|tv_0|^{p^*(x)}dx.
\end{aligned}
\tag{4.4}
$$

Next, we need to prove the boundedness of $\{t_\lambda\}_{\lambda>0}$. First, for any $\lambda > 0$, let $t_\lambda \geq 1$. According to (4.4), we obtain that

$$\begin{aligned} p^+\sigma\mathcal{K}(1)t_\lambda^{2p^+\sigma} &\geq p^+\sigma\mathcal{K}(1)\left(\mathcal{T}_{p(x)}(t_\lambda v_0)\right)^\sigma \\ &\geq p^+M\left(\mathcal{T}_{p(x)}(t_\lambda v_0)\right)T_{p(x)}(t_\lambda v_0) \\ &\geq K\left(\mathcal{T}_{p(x)}(t_\lambda v_0)\right)\left[|\nabla t_\lambda v_0|^{p(x)} + V(x)|t_\lambda v_0|^{p(x)}\right] \\ &\geq t_\lambda^{p^*(x)}\int_{\mathbb{R}^N}|v_0|^{p^*(x)}dx. \end{aligned} \tag{4.5}$$

Since $\sigma \in [1, p^*(x)/2p^+)$, so $2p^+\sigma < p^*(x)$ and (4.5), hence we get the boundedness of $\{t_\lambda\}_\lambda$.

The next step is to demonstrate $t_\lambda \to 0$ as $\lambda \to \infty$. Similarly, we get $\lambda_n \to \infty$ and $t_{\lambda_n} \to t_0$. we yield

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y, t_{\lambda_n}v_0(y))g(x, t_{\lambda_n}v_0(x))t_{\lambda_n}v_0}{|x-y|^{\alpha(x,y)}}dxdy \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y, t_0v_0(y))g(x, t_0v_0(x))t_0v_0}{|x-y|^{\alpha(x,y)}}dxdy$$

as $n \to \infty$. And

$$\lambda_n\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y, t_{\lambda_n}v_0(y))g(x, t_{\lambda_n}v_0(x))t_{\lambda_n}v_0}{|x-y|^{\alpha(x,y)}}dxdy \to \infty \quad \text{as} \quad n \to \infty.$$

So it results $K\left(T_{p(x)}(t_0v_0)\right) = \infty$ thanks to (4.4). Hence, $t_\lambda \to 0$ as $\lambda \to \infty$. Then we have

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^N}\frac{G(y, t_\lambda v_0(y))g(x, t_\lambda v_0(x))t_\lambda v_0}{|x-y|^{\alpha(x,y)}}dxdy = 0$$

and

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^N}|t_\lambda v_0|^{p^*(x)}dx = 0.$$

Moreover, an easy computation gives that

$$\lim_{\lambda\to\infty}\left(\sup_{t\geq 0}J_\lambda(tv_0)\right) = \lim_{\lambda\to\infty}J_\lambda(t_\lambda v_0) = 0.$$

Then we can find $\lambda_1 > 0$ satisfying for any $\lambda \geq \lambda_1$, we have

$$\sup_{t\geq 0}J_\lambda(tv_0) < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)\min\left\{\left(k_0S^{p^+}\right)^{\tau^+}, \left(k_0S^{p^+}\right)^{\tau^-}\right\}.$$

Let $\omega = \tau v_0$ satisfying $J_\lambda(\omega) < 0$, so choosing$\gamma(t) = t\tau v_0$, we get

$$c_\lambda \leq \max_{t\in[0,1]}J_\lambda(\gamma(t)).$$

Finally, if λ large enough, we obtain

$$c_\lambda \leq \sup_{t\geq 0}J_\lambda(tv_0) < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)\min\left\{\left(k_0S^{p^+}\right)^{\tau^+}, \left(k_0S^{p^+}\right)^{\tau^-}\right\}.$$

Hence, Eq (1.1) admits a nontrivial solution.

4.2. Proof of Theorem 1.2

In this part, we demonstrate Theorem 1.2 by using similar method in [38]. Assume Π is a set which includes closed subsets A. A are symmetric and A are subsets of $X\setminus\{0\}$ which is an infinite dimensional Banach space.

Lemma 4.2 ([38]). *X is shown above. There exists U satisfying $X = U \oplus V$. Assume $J_\lambda \in C^1(X)$ be an even functional and $J_\lambda(0) = 0$. J_λ meets*

- (I_1) *for any $u \in \partial B_\rho \bigcap Z$, there is constant $\rho, \chi > 0$ satisfying $J_\lambda(u) \geq \chi$;*
- (I_2) *for any c and $c \in (0, \xi)$, there is constant $\xi > 0$, J_λ satisfying the $(PS)_c$ condition;*
- (I_3) *there exists $R = R(\widetilde{X}) > 0$ satisfying $J_\lambda(u) \leq 0$ on $\widetilde{X} \setminus B_R$ where $\widetilde{X} \subset X$ ia any finite dimensional subspace.*

Let U is s dimensional Banach space and we give the definition of $U = span\{u_1, \cdots, u_s\}$. When $n \geq s$, we have $u_{n+1} \notin Q_n = span\{u_1, \cdots, u_n\}$. So we assume $R_n = R(Q_n)$ and $\Omega_n = B_{R_n} \bigcap Q_n$. Then we give the definition of Q_n, that is

$$Z_n = \{\eta \in C(\Omega_n, X) \,:\, \eta|_{\partial B_{R_n} \bigcap Q_n} = id, \psi is\ odd\}.$$

We have

$$\Gamma_t = \left\{\eta(\overline{\Omega_n \setminus V}) \,:\, \eta \in Z_n,\ n \geq t,\ D \in \Pi,\ \gamma(D) \leq n - t\right\},$$

where $\gamma(D)$ is genus of D. Let

$$c_t = \inf_{E\in\Gamma_t} \max_{u\in E} J_\lambda(u) \quad t \in \mathbb{N}.$$

Therefore, when $t > s$, we find $0 \leq c_t \leq c_{t+1}$,with $c_t < \xi$. For any $t > s$ such that $c_t = c_{t+1} = \cdots = c_{t+\alpha} = c < \xi$, T_c denote the set of critical points in X and we have $\gamma(T_c) \geq \alpha + 1$.

Proof of Theorem 1.2. According to Lemma 4.2, we can prove Theorem 1.2.

First, we verify conditions. Since $J_\lambda \in C^1(W_V^{1,p(x)}(\mathbb{R}^N))$. According to (3.2), we have $J_\lambda(0) = 0$. Therefore, the demonstration is analogous to (1) and (2) in Lemma 4.1. Since J_λ meets conditions (I_1) and (I_3) of Lemma 4.2.

Therefore, we can demonstrate there exists $(\Upsilon_n)_n \subset \mathbb{R}^+$, and $\Upsilon_n \leq \Upsilon_{n+1}$, satisfying

$$c_n^\lambda = \inf_{E\in\Gamma_n} \max_{u\in E} J_\lambda(u) < \Upsilon_n.$$

According to the definition of c_n^λ, we deduce that

$$c_n^\lambda = \inf_{E\in\Gamma_n} \max_{u\in E} J_\lambda(u) \leq \inf_{E\in\Gamma_n} \max_{u\in E} \left\{\mathcal{K}\left(\mathcal{T}_{p(x)}(u)\right) - \frac{1}{p^*(x)} \int_{\mathbb{R}^N} |u|^{p^*(x)} dx\right\}.$$

And

$$\Upsilon_n = \inf_{E\in\Gamma_n} \max_{u\in E} \left\{\mathcal{K}\left(\mathcal{T}_{p(x)}(u)\right) - \frac{1}{p^*(x)} \int_{\mathbb{R}^N} |u|^{p^*(x)} dx\right\},$$

so that $\Upsilon_n < \infty$ and $\Upsilon_n \leq \Upsilon_{n+1}$. We show that Eq (1.1) has at least s pairs of solutions. And we have two possibilities:

(I) In the first case, we assume $\lambda > 0$. There exists k_0 such that

$$\sup_n \Upsilon_n < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right) \min\left\{\left(k_0 S^{p^+}\right)^{\tau^+}, \left(k_0 S^{p^+}\right)^{\tau^-}\right\}.$$

(II) In the second case, similarly, in (4.3) and $\lambda > \lambda_2$, we deduce from $\lambda_2 > 0$ that

$$c_n^\lambda \leq \Upsilon_n < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right) \min\left\{\left(m_0 S^{p^+}\right)^{\tau^+}, \left(m_0 S^{p^+}\right)^{\tau^-}\right\}.$$

Hence, we yield

$$0 < c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_t^\lambda < \Upsilon_n < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right) \min\left\{\left(k_0 S^{p^+}\right)^{\tau^+}, \left(k_0 S^{p^+}\right)^{\tau^-}\right\}.$$

By means of Proposition 9.30 in [38], we get that J_λ has many critical values. And there exists $t = 1, 2, \cdots, s-1$ such that$c_t^\lambda = c_{t+1}^\lambda$, $T_{c_t^\lambda}$ has a lot of critical points. So we demonstrate Theorem 1.2.

5. Degenerate case for Eq (1.1)

In this part, we consider Eq (1.1) in the degenerate case. We give a crucial lemma at first.

Lemma 5.1. *J_λ has a (PS) sequence $(u_n)_n$ in $W_V^{1,p(x)}(\mathbb{R}^N)$. Let $\tau_\sigma(x) := \frac{p^*(x)}{p^*(x)-p^+\sigma}$,*

$$c_\lambda < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right) \min\left\{\left(p^- k_1 S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^- k_1 S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}, \tag{5.1}$$

then $(u_n)_n \to u$ strongly.

Proof. If $\inf_{n\geq 1} \|u_n\| = 0$, then there is a subsequence of $(u_n)_n$ such that $u_n \to 0$ as $n \to \infty$. Hence, we suppose $d := \inf_{n\geq 1} \|u_n\| > 0$ and $\|u_n\| > 1$. In view of the definition of the (PS) sequence, we get

$$\begin{aligned}
&c_\lambda + 1 + o(1)\|u_n\| \\
&\geq J_\lambda(u_n) - \frac{1}{\theta}\langle J_\lambda'(u_n), u_n\rangle \\
&= \mathcal{K}\left(\mathcal{T}_{p(x)}(u_n)\right) - \frac{1}{\theta} K\left(\mathcal{T}_{p(x)}(u_n)\right)\left[\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} dx\right] \\
&\quad + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)|u_n|^{p^*(x)} dx + \lambda \iint_{\mathbb{R}^N} \frac{G(y, u_n(y))}{|x-y|^{\alpha(x,y)}}\left(\frac{g(x, u_n)u_n}{\theta} - \frac{G(x, u_n)}{2}\right) dxdy.
\end{aligned}$$

According to the conditions of (K_2), (K_3) and (g_3), we can deduce

$$\begin{aligned}
c_\lambda + 1 + o(1)\|u_n\| &\geq J_\lambda(u_n) - \frac{1}{\theta}\langle J_\lambda'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right) K\left(\mathcal{T}_{p(x)}(u_n)\right)\mathcal{T}_{p(x)}(u_n) \\
&\geq \left(\frac{1}{\sigma} - \frac{p^+}{\theta}\right) k_1 \|u_n\|^{p^+\sigma}.
\end{aligned} \tag{5.2}$$

Since $p^+\sigma > 1$, we deduce $(u_n)_n$ is bounded.

Refer to Lemma 3.1, we can assume $I \neq \emptyset$. For any $i \in I$ and any $\epsilon > 0$ small, we define $\phi_{\epsilon,i}$ as Lemma 3.1. In view of $\langle J_\lambda'(u_n), u_n\phi_{\epsilon,i}\rangle \to 0$ as $n \to \infty$. Thus we have

$$\begin{aligned}&K(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}\phi_{\varepsilon,i} + V(x)|u_n|^{p(x)}u_n\phi_{\varepsilon,i} + |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_{\varepsilon,i}u_n\right)dx\\&= \lambda\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y,u_n(y))g(x,u_n(x))u_n\phi_{\varepsilon,i}}{|x-y|^{\alpha(x,y)}}dxdy + \int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_{\varepsilon,i}dx + o_n(1).\end{aligned} \tag{5.3}$$

Similarly,

$$\limsup_{\varepsilon\to0}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_{\varepsilon,i}u_n dx\right| = 0$$

and

$$\limsup_{\varepsilon\to0}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(y,u_n(y))g(x,u_n(x))u_n\phi_{\varepsilon,i}}{|x-y|^{\alpha(x,y)}}dxdy = \langle\Lambda'(u_n), u_n\phi_{\varepsilon,i}\rangle = 0.$$

By (K_3) and (5.3), we have

$$\begin{aligned}&K(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}\phi_{\varepsilon,i} + V(x)|u_n|^{p(x)}u_n\phi_{\varepsilon,i}\right)dx\\&= K(\mathcal{T}_{p(x)}(u_n))\int_{\mathbb{R}^N}U_n(x)\phi_{\varepsilon,i}dx \geq K(\mathcal{T}_{p(x)}(u_n\phi_{\varepsilon,i}))\int_{\mathbb{R}^N}U_n(x)\phi_{\varepsilon,i}dx\\&\geq p^-\left(\mathcal{T}_{p(x)}(u_n\phi_{\varepsilon,i})\right)\mathcal{T}_{p(x)}(u_n\phi_{\varepsilon,i}) \geq p^-k_1\left(T_{p(x)}(u_n\phi_{\varepsilon,i})\right)^\sigma\\&\geq (p^-)^{\sigma+1}k_1\left(\int_{\mathbb{R}^N}U_n(x)\phi_{\varepsilon,i}dx\right)^\sigma\\&\geq (p^-)^{\sigma+1}k_1\mu_i^\sigma.\end{aligned} \tag{5.4}$$

According to (5.3), we get

$$(p^-)^{\sigma+1}k_1\mu_i^\sigma \leq \nu_i.$$

Then we find that either $\nu_i = 0$ or

$$\nu_i \geq \left(p^-k_1S^{p^+\sigma}\right)^{\frac{p^*(z_i)}{p^*(z_i)-p^+\sigma}} \geq \min\left\{\left(p^-k_1S^{p^+\sigma}\right)^{\tau_\sigma^+},\ \left(p^-k_1S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}. \tag{5.5}$$

We deduce from (K_2), (g_3) and (5.5) that

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\phi_{\epsilon,i}dx. \tag{5.6}$$

So we get

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)\min\left\{\left(p^-k_1S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^-k_1S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}.$$

Thus we get a contradiction with (5.1). Hence $\nu_i = 0$.

Next, we claim that $\nu_\infty = 0$. Similarly, we give a smooth cut-off function ϕ_R and because of $\langle J'_\lambda(u_n), u_n\phi_R\rangle \to 0$,

$$\begin{aligned} &K(\mathcal{T}_{p(x)}(u_n)) \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)}\phi_R + V(x)|u_n|^{p(x)}u_n\phi_R + |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi_R u_n \right) dx \\ &= \lambda \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(y,u_n(y))g(x,u_n(x))u_n\phi_R}{|x-y|^{\alpha(x,y)}} dxdy + \int_{\mathbb{R}^N} |u_n|^{p^*(x)}\phi_R dx + o_n(1). \end{aligned} \tag{5.7}$$

Similarly, we get

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi_R u_n dx\right| = 0$$

and

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(y,u_n(y))g(x,u_n(x))u_n\phi_R}{|x-y|^{\alpha(x,y)}} dxdy = \langle \Lambda'(u_n), u_n\phi_R\rangle = 0.$$

According to (5.5), we obtain that

$$K(\mathcal{T}_{p(x)}(u_n)) \int_{\mathbb{R}^N} U_n(x)\phi_R dx \geq (p^-)^{\sigma+1}k_1\mu_\infty^\sigma \tag{5.8}$$

and

$$(p^-)^{\sigma+1}k_1\mu_\infty^\sigma \leq \nu_\infty.$$

Combining with (3.12), we find that either $\nu_\infty = 0$ or

$$\nu_\infty \geq \left(p^- k_1 S^{p^+\sigma}\right)^{\frac{p^*_\infty}{p^*_\infty - p^+\sigma}} \geq \min\left\{\left(p^- k_1 S^{p^+\sigma}\right)^{\tau_\sigma^+},\ \left(p^- k_1 S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}. \tag{5.9}$$

Then we prove (5.9). Similarly, in view of (3.21), we deduce that

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)\min\left\{\left(p^- k_1 S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^- k_1 S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}. \tag{5.10}$$

By (5.1), it is an obvious contradiction. Thus $\nu_\infty = 0$.

Hence, we have $I = \emptyset$ and

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p^*(x)}dx = \int_{\mathbb{R}^N} |u|^{p^*(x)}dx.$$

According to a Brézis-Lieb lemma with variable exponents (see [19], Lemma 3.9) and last equality, we get

$$\int_{\mathbb{R}^N} |u_n - u|^{p^*(x)}dx \to 0$$

and

$$\int_{\mathbb{R}^N} |u_n|^{p^*(x)-2}u_n(u_n - u)dx \to 0.$$

In view of (3.27), (3.29) and $\langle J_\lambda(u_n), u_n - u\rangle \to 0$, we deduce

$$\int_{\mathbb{R}^N} \left(|\nabla(u_n - u)|^{p(x)} + V(x)|u_n - u|^{p(x)}\right)dx = 0.$$

So $(u_n)_n \to u$ strongly where $u \in W_V^{1,p(x)}(\mathbb{R}^N)$. □

Lemma 5.2. *J_λ satisfies the conditions* (1) *and* (2) *of Lemma* 4.1.

Proof. By (K_3), (g_2) and Remark 2.1, for any $\lambda > 0$, $u \in W_V^{1,p(x)}(\mathbb{R}^N)$ and $\|u\| < 1$, we deduce

$$\begin{aligned} J_\lambda(u) =& \mathcal{K}\left(\mathcal{T}_{p(x)}(u)\right) - \lambda\Lambda(u) - \int_{\mathbb{R}^N} \frac{1}{p^*(x)}|u|^{p^*(x)}dx \\ \geq& \frac{1}{\sigma}K\left(\mathcal{T}_{p(x)}(u)\right)\mathcal{T}_{p(x)}(u) - -\frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - C\|G(\cdot,u)\|^2_{L^{q^+}(\mathbb{R}^N)} - C\|G(\cdot,u)\|^2_{L^{q^-}(\mathbb{R}^N)} \\ \geq& \frac{k_1}{\sigma(p^+)^\sigma}\|u\|^{p^+\sigma}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - C\max\left\{\|u\|^{2r^+}_{L^{q^+r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^+r(x)}(\mathbb{R}^N)}\right\} \\ &- C\max\left\{\|u\|^{2r^+}_{L^{q^-r(x)}(\mathbb{R}^N)}, \|u\|^{2r^-}_{L^{q^-r(x)}(\mathbb{R}^N)}\right\} \\ \geq& \frac{k_1}{\sigma(p^+)^\sigma}\|u\|^{p^+\sigma}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{S}{p^*}\|u\|^{p^*}_{W_V^{1,p(x)}(\mathbb{R}^N)} - \frac{2C}{S^{2r^-}}\|u\|^{2r^-}_{W_V^{1,p(x)}(\mathbb{R}^N)}. \end{aligned} \tag{5.11}$$

Since $p^+\sigma < p^*(x)$ and $p(x) \ll rq^- \leq rq^+ \ll p^*(x)$, choosing $\rho, \chi > 0$ and $\|u\| = \rho$, we have $J_\lambda(u) \geq \chi$. Thus we prove (1) in Lemma 4.1. Similarly, we prove (2) of Lemma 4.1 is true. □

Proof of Theorem 1.3. The proof is analogous to Lemma 4.1, we can obtain

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) < \left(\frac{1}{\theta} - \frac{1}{p^*(x)}\right)\min\left\{\left(p^-k_1S^{p^+\sigma}\right)^{\tau_\sigma^+}, \left(p^-k_1S^{p^+\sigma}\right)^{\tau_\sigma^-}\right\}.$$

The remaining steps are analogous to Theorem1.1.

Proof of Theorem 1.4. The proof of Theorem 1.4 is analogous to Theorem 1.2.

6. Conclusions

In this paper, we consider the critical nonlocal Choquard-Kirchhoff type equations with variable exponents in the degenerate cases and non-degenerate cases. Because of the existence of critical reaction, we apply the concentration-compactness principle to overcome the lack of compactness. By using the variational methods, the Hardy-Littlewood-Sobolev inequality and Krasnoselskii genus, we get the results of existence and multiplicity of solutions for a class of critical Choquard-Kirchhoff type equations.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. C. O. Alves, J. L. P. Barreiro, Existence and multiplicity of solutions for a $p(x)$-Laplacian equation with critical growth, *J. Math. Anal. Appl.*, **403** (2013), 143–154. https://doi.org/10.1016/j.jmaa.2013.02.025

2. C. O. Alves, L. S. Tavares, A Hardy-Littlewood-Sobolev-type inequality for variable exponents and applications to quasilinear Choquard equations involving variable exponent, *Mediterr. J. Math.*, **16** (2019), 1–27. https://doi.org/10.1007/S00009-019-1316-Z

3. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7

4. G. Autuori, A. Fiscella, P. Pucci, Stationary Kirchhoff problems involving a fractional operator and a critical nonlinearity, *Nonlinear Anal.*, **125** (2015), 699–714. http://doi.org/10.1016/j.na.2015.06.014

5. A. K. Ben-Naouma, C. Troestler, M. Willem, Extrema problems with critical Sobolev exponents on unbounded domains, *Nonlinear Anal.*, **26** (1996), 823–833. https://doi.org/10.1016/0362-546X(94)00324-B

6. G. Bianchi, J. Chabrowski, A. Szulkin, Symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent, *Nonlinear Anal.*, **25** (1995), 41–59. https://doi.org/10.1016/0362-546X(94)E0070-W

7. J. F. Bonder, A. Silva, Concentration-compactness principle for variable exponent spaces and applications, *Electron. J. Differ. Equ.*, **141** (2010), 1–18. https://doi.org/10.48550/arXiv.0906.1922

8. H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.*, **36** (1983), 437–477. https://doi.org/10.1002/cpa.3160360405

9. M. Caponi, P. Pucci, Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations, *Ann. Mat. Pura Appl.*, **195** (2016), 2099–2129. https://doi.org/10.1007/s10231-016-0555-x

10. J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differ. Equ.*, **3** (1995), 493–512. https://doi.org/10.1007/BF01187898

11. Y. Chen, S. Levine, R. Rao, Functionals with $p(x)$ growth in image processing, *SIAM J. Appl. Math.*, **66** (2006), 1383–1406. https://doi.org/10.1137/050624522

12. P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.*, **108** (1992), 247–262. https://doi.org/10.1007/BF02100605

13. L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Springer-Verlag, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8

14. X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617

15. Y. Fu, The principle of concentration compactness in $L^{p(x)}$ spaces and its application, *Nonlinear Anal.*, **71** (2009), 1876–1892. https://doi.org/10.1016/j.na.2009.01.023

16. Y. Fu, X. Zhang, Multiple solutions for a class of $p(x)$-Laplacian equations in $\mathbb{R}^N$ involving the critical exponent, *Proc. Roy. Soc.*, **466** (2010), 1667–1686. https://doi.org/10.1098/rspa.2009.0463

17. H. Fröhlich, Theory of electrical breakdown in ionic crystals, *Proc. Roy. Soc. Edinburgh Sect.*, **160** (1937), 230–241. https://doi.org/10.1098/rspa.1937.0106

18. K. Ho, Y. H. Kim, A-priori bounds and multiplicity of solutions for nonlinear elliptic problems involving the fractional $p(\cdot)$-Laplacian, *Nonlinear Anal.*, **188** (2019), 179–201. https://doi.org/10.1016/j.na.2019.06.001

19. K. Ho, Y. H. Kim, I. Sim, Existence results for Schrödinger $p(\cdot)$-Laplace equations involving critical growth in $\mathbb{R}^N$, *arXiv*, 2018. https://doi.org/10.48550/arXiv.1807.03961

20. S. Liang, J. Zhang, Multiple solutions for a noncooperative $p(x)$-Laplacian equations in $\mathbb{R}^N$ involving the critical exponent, *J. Math. Anal. Appl.*, **403** (2013), 344–356. https://doi.org/10.1016/j.jmaa.2013.01.003

21. S. Liang, J. Zhang, Infinitely many small solutions for the $p(x)$-Laplacian operator with nonlinear boundary conditions, *Ann. Mat. Pura Appl.*, **192** (2013), 1–16. https://doi.org/10.1007/s10231-011-0209-y

22. S. Liang, G. Molica Bisci, B. Zhang, Sign-changing solutions for Kirchhoff-type problems involving variable-order fractional Laplacian and critical exponents, *Nonlinear Anal. Model. Control*, **27** (2022), 1–20. https://doi.org/10.15388/namc.2022.27.26575

23. S. Liang, D. Repovš, B. Zhang, Fractional magnetic Schrödinger-Kirchhoff problems with convolution and critical nonlinearities, *Math. Meth. Appl. Sci.*, **43** (2020), 2473–2490. https://doi.org/10.1002/mma.6057

24. S. Liang, P. Pucci, B. Zhang, Existence and multiplicity of solutions for critical nonlocal equations with variable exponents, *Appl. Anal.*, 2022, 1–24. https://doi.org/10.1080/00036811.2022.2107916

25. E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, *Stud. Appl. Math.,* **57** (1977) 93–105. https://doi.org/10.1002/sapm197757293

26. P. L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case, part I, *Ann. Inst. H. Poincaré Anal. Non Lineaire*, **1** (1984), 109–145. https://doi.org/10.1016/S0294-1449(16)30428-0

27. P. L. Lions, The concentration compactness principle in the calculus of variations. The limit case, part II, *Ann. Inst. H. Poincaré Anal. Non Lineaire*, **1** (1985), 45–121. https://doi.org/10.4171/RMI/12

28. B. B. V. Maia, On a class of $p(x)$-Choquard equations with sign-changing potential and upper critical growth, *Rend. Circ. Mat. Palermo, II. Ser.*, **70** (2021), 1175–1199. https://doi.org/10.1007/s12215-020-00553-y

29. J. Mawhin, G. Molica Bisci, A Brezis-Nirenberg type result for a nonlocal fractional operator, *J. Lond. Math. Soc.*, **95** (2017), 73–93. https://doi.org/10.1112/jlms.12009

30. X. Mingqi, G. Molica Bisci, G. Tian, B. Zhang, Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p-Laplacian, *Nonlinearity*, **29** (2016), 357–374. https://doi.org/10.1088/0951-7715/29/2/357

31. I. M. Moroz, R. Penrose, P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, *Classical Quant. Grav.*, **15** (1998), 2733. https://doi.org/10.1088/0264-9381/15/9/019

32. V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent, *Commun. Contemp. Math.*, **17** (2015), 1550005. https://doi.org/10.1142/S0219199715500054

33. V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, **265** (2013), 153–184. https://doi.org/10.1016/j.jfa.2013.04.007

34. W. Orlicz, Über konjugierte exponentenfolgen, *Studia Math.*, **3** (1931), 200–212. https://doi.org/10.4064/sm-3-1-200-211

35. S. Pekar, *Untersuchung über die elektronentheorie der Kristalle*, Akad. Verlag, 1954. https://doi.org/10.1515/9783112649305

36. R. Penrose, On Gravity's role in quantum state reduction, *Gen. Relat. Gravit.*, **28** (1996), 581–600. https://doi.org/10.1007/BF02105068

37. P. Pucci, M. Xiang, B. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.*, **5** (2016), 27–55. https://doi.org/10.1515/anona-2015-0102

38. P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, American Mathematical Society, 1986. https://doi.org/10.1090/cbms/065

39. V. Rădulescu, D. Repovš, *Partial differential equations with variable exponents*, 1 Ed., CRC Press, Boca Raton, 2015. https://doi.org/10.5772/15954

40. M. Ruzicka, *Electrorheological fluids modeling and mathematical theory*, Springer-Verlag, Berlin, 2000. https://doi.org/10.1007/bfb0104029

41. X. Shi, Y. Zhao, H. Chen, Existence of solutions for nonhomogeneous Choquard equations involving p-Laplacian, *Mathematics*, **7** (2019), 871. https://doi.org/10.3390/math7090871

42. Y. Song, S. Shi, On a degenerate p-fractional Kirchhoff equations involving critical Sobolev-Hardy nonlinearities, *Mediterr. J. Math.*, **15** (2018), 1–18. https://doi.org/10.1007/s00009-017-1062-z

43. X. Sun, Y. Song, S. Liang, On the critical Choquard-Kirchhoff problem on the Heisenberg group, *Adv. Nonlinear Anal.*, **12** (2023), 210–236. https://doi.org/10.1515/anona-2022-0270

44. L. Wang, K. Xie, B. Zhang, Existence and multiplicity of solutions for critical Kirchhoff-type p-Laplacian problems, *J. Math. Anal. Appl.*, **458** (2018), 361–378. https://doi.org/10.1016/j.jmaa.2017.09.008

45. F. Gao, M. Yang, The Brezis-Nirenberg type critical problem for the nonlinear Choquard equation, *Sci. China Math.*, **61** (2018), 1219–1242. https://doi.org/10.1007/s11425-016-9067-5

46. F. Gao, M. Yang, On nonlocal Choquard equations with Hardy–Littlewood–Sobolev critical exponents, *J. Math. Anal. Appl.*, **448** (2017), 1006–1041. https://doi.org/10.1016/j.jmaa.2016.11.015

47. Y. Zhang, X. Tang, V. Rădulescu, High perturbations of Choquard equations with critical reaction and variable growth, *Proc. Amer. Math. Soc.*, **149** (2021), 3819–3835. http://dx.doi.org/10.1090/proc/15469

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