Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 2995–3025.
DOI: 10.3934/math.2023155
Received: 01 October 2022
Revised: 03 November 2022
Accepted: 10 November 2022
Published: 14 November 2022

Research article

New generalization of fuzzy soft sets: (a, b)-Fuzzy soft sets

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Abstract: Many models of uncertain knowledge have been designed that combine expanded views of fuzziness (expressions of partial memberships) with parameterization (multiple subsethood indexed by a parameter set). The standard orthopair fuzzy soft set is a very general example of this successful blend initiated by fuzzy soft sets. It is a mapping from a set of parameters to the family of all orthopair fuzzy sets (which allow for a very general view of acceptable membership and non-membership evaluations). To expand the scope of application of fuzzy soft set theory, the restriction of orthopair fuzzy sets that membership and non-membership must be calibrated with the same power should be removed. To this purpose we introduce the concept of (a, b)-fuzzy soft set, shortened as (a, b)-FSS. They enable us to address situations that impose evaluations with different importances for membership and non-membership degrees, a problem that cannot be modeled by the existing generalizations of intuitionistic fuzzy soft sets. We establish the fundamental set of arithmetic operations for (a, b)-FSSs and explore their main characteristics. Then we define aggregation operators for (a, b)-FSSs and discuss their main properties and the relationships between them. Finally, with the help of suitably defined scores and accuracies we design a multi-criteria decision-making strategy that operates in this novel framework. We also analyze a decision-making problem to endorse the validity of (a, b)-FSSs for decision-making purposes.

Keywords: (a, b)-fuzzy soft set; score and accuracy functions; aggregation operators; multi-criteria decision-making
Mathematics Subject Classification: 03E72, 03E99, 08A72

1. Introduction

The purpose of this paper is to launch a novel model of uncertain knowledge that combines an expanded view of fuzziness (with original expressions of partial memberships and non-memberships) with parameterization abilities (in the standard form of multiple subsethood indexed by a parameter set). In order to fully grasp the expected impact of such an innovation, let us first recall the streamlined sequence of improvements leading to the state-of-the-art.

It is well known that Zadeh [48] expanded crisp sets by allowing the membership of an element to a set to lie in the closed interval $[0, 1]$, rather than being limited to $\{0, 1\}$. He called this type of set a fuzzy set in 1965, and nowadays it has been extensively applied in many areas such as medicinal sciences, engineering, economics, etc. Afterwards Atanassov [16] defined the notion of intuitionistic fuzzy set (IF-set) which handles problems where the non-membership degree does not necessarily derive from the membership degree. Like the case of fuzzy sets, this notion attracted many researchers that investigated its main properties and demonstrated how it can be applied in practical situations [6, 19].

Once the independence of membership and non-membership had been laid out, many authors engaged in the task of enlarging the admissible combinations of membership and non-membership, while keeping mathematical tractability. Two new kinds of sets generalizing IF-sets, namely, Pythagorean fuzzy sets (PF-sets) and Fermatean fuzzy sets (FF-sets), were defined by Yager [43] and Senapati and Yager [39]. Their contribution hinges on the utilization of powers (either 2 in PF-sets or 3 in FF-sets) to calibrate the membership and non-membership evaluations. A wider range of situations producing a satisfactory generalization is encapsulated by q-rung orthopair fuzzy sets (q-ROF sets), introduced by Yager [44]. All the aforementioned extensions of fuzzy sets are particular cases of q-ROF sets: IFSs, PFSs, and FFSs arise by giving the values 1, 2, 3 to q.

However, there are some cases and situations requiring evaluations with different importance (whence calibration) for the membership and non-membership degrees. This requirement cannot be met by the aforementioned generalizations of IF-sets. To achieve this goal, Al-shami [9] defined a family of (a, b)-fuzzy sets (abbreviated (a, b)-FSs), where $a, b \geq 1$. Special cases of this family have been introduced and studied, e.g., (2,1)-fuzzy sets [8], SR-fuzzy sets [11] and (3,2)-fuzzy sets [24]. Aggregation operators, as a prominent technique allowing to approach decision-making issues, have been defined for some of these fuzzy environments [26, 27, 29, 33, 35]

The soft set defines another altogether different scenario to address vagueness and uncertainty [32]. A list of characteristics yields a family of subsets which are considered as approximate descriptions of a concept (one for each viewpoint defined by a property). Soft sets quickly drew the attention of many researchers with different goals [3, 5, 10, 12–14, 20, 25].

With the progression of communication and technology, the solutions to many complicated issues need to resort to more than one analytical tool. In relation with this, Maji et al. [30] showed that fuzzy and soft set theories can work in tandem. They prompted the 'fuzzification of soft sets paradigm' which describes a larger class of phenomena more accurately. These models were further investigated in many articles like [7, 18, 28, 31, 38]. As an expected reaction, researchers developed this type of hybridization further by introducing intuitionistic fuzzy soft sets (IFS-sets) [41], Pythagorean fuzzy soft sets (PFS-sets) [34], Fermatean fuzzy soft sets (FFS-sets) [37] and q-rung orthopair fuzzy soft sets (q-ROFS sets) [23]. Of course, these generalizations of IFS-sets have reasonable motivations that led to studying them.

To complete this landscape and provide a more general instrument for the mathematical analysis of uncertainty, in this article we introduce the novel concept of (a, b)-fuzzy soft sets, abbreviated as (a, b)-FSSs, which is produced by hybridization of (a, b)-fuzzy sets and soft sets.

The motivation for this research is threefold. First, we shall launch a new family of generalized fuzzy soft sets that encompass the IFS-sets. This family called (a, b)-fuzzy soft sets enlarges the space jointly defined by membership and non-membership specifications beyond the space allowed by IFS-sets. We shall also show that this new family does not obtain from the class of q-ROFS sets, i.e., it is a rigthful novel extension of IFS-sets. Secondly, we shall present novel kinds of weighted aggregation operators which have a potential to be applied to practical problems. They are especially adept for those problems that need to be evaluated with different importance for membership and non-membership grades. Finally, we shall design a multi-criteria decision-making method based on the accuracy and score functions for the purpose of choosing optimal alternative(s) in the framework defined by the new model.

The content of the rest of this manuscript is arranged as follows:

(i) In Section 2, we recall some the main concepts of fuzzy set theory and its extensions via classical and soft settings.

(ii) We devote Section 3 to introducing the class of (a, b)-FSSs, which expands the grade space of q-ROFS sets and provides a suitable environment to model some real-life issues. We study the main properties of this class and define a set of operations via this class.

(iii) In Section 4, we present some new operations and aggregation operators on (a, b)-FSSs and show the relationships between them.

(iv) In Section 5, we provide an application showing how the current class is applied to address practical problems. For simplicity and preciseness, we have proceeded in the case where $a = 1$ and $b = 2$.

(v) In the end, we outline the main contributions of the manuscript and propose some future work in Section 6.

2. Preliminaries

In this section we give some antecedents that lay the ground for the presentation of our novel model. We need background on two initially independent areas, namely, fuzzy models and soft set theory. Both approaches to uncertainty have been combined in various forms, and we shall describe the necessary ideas too.

2.1. Classical fuzzification

Many models have succeeded in improving the scope of applications of fuzzy sets. The fundamental fuzzy model is given in the next definition:

Definition 2.1. *[48] The fuzzy set is defined over a universal set U as follows:*

$O = \{\langle \hat{x}, \varphi_O(\hat{x})\rangle : \hat{x} \in U\}$, *where φ_O is a function from U to $[0, 1]$ which represents the membership degree of every $\hat{x} \in U$ to O under the constraint $0 \leq \varphi_O(\hat{x}) \leq 1$.*

A key improvement consisted of the separation of membership from non-membership evaluations. This achievement was first formalized by the next concept:

Definition 2.2. *[16] The IF-set is defined over a universal set U as follows.*

$$O = \{\langle \hat{x}, \varphi_O(\hat{x}), \varpi_O(\hat{x})\rangle : \hat{x} \in U\}, \text{ where the functions } \varphi_O \text{ and } \varpi_O \text{ from } U \text{ to } [0,1] \text{ respectively represent the membership and non-membership degrees of every } \hat{x} \in U \text{ to } O \text{ under the constraint } 0 \leq \varphi_O(\hat{x}) + \varpi_O(\hat{x}) \leq 1.$$

The indeterminacy degree of each $\hat{x} \in U$ with respect to this IFS is given by

$$\zeta_O(\hat{x}) = 1 - (\varphi_O(\hat{x}) + \varpi_O(\hat{x})).$$

One should bear in mind that if an IF set O satisfies $\varphi_O(\hat{x}) = 1 - \varpi_O(\hat{x})$ for every element $\hat{x} \in U$, then O can be identified with a fuzzy set without loss of information.

Remark 2.3. *If no confusion is likely to arise, we shall drop the subindex and simply write φ, ϖ and ζ instead of φ_O, ϖ_O and ζ_O, respectively.*

Afterwards the power of the membership and non-membership degrees was introduced in order to expand the mathematical tractability of wider sets of evaluations. The next concept subsumes other interesting cases:

Definition 2.4. *[44] Fix $q \geq 1$. The q-ROF set is defined over a universal set U as follows:*

$$O = \{\langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle : \hat{x} \in U\}, \text{ where the functions } \varphi \text{ and } \varpi \text{ from } U \text{ into } [0,1] \text{ respectively represent the membership and non-membership degrees of every } \hat{x} \in U \text{ to } O \text{ under the constraint } 0 \leq (\varphi(\hat{x}))^q + (\varpi(\hat{x}))^q \leq 1.$$

The indeterminacy degree of each $\hat{x} \in U$ with respect to this qROFS is given by

$$\zeta(\hat{x}) = \sqrt[q]{1 - ((\varphi(\hat{x}))^q + (\varpi(\hat{x}))^q)}.$$

PF-sets [43] amount to the case $q = 2$, whereas FF-set s [39] consist of the case $q = 3$.

A different procedure for the expansion of the field of admissible membership and non-membership evaluations has been recently introduced in [8]. The key idea is that membership and non-membership evaluations are raised to the power of different indices. As a result, the model that arises lies strictly in between IF-sets and PF-sets:

Definition 2.5. *[8] The (2,1)-FS O over the universal set U is defined as follows.*

$$O = \{\langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle : \hat{x} \in U\}, \text{ where the functions } \varphi \text{ and } \varpi \text{ from } U \text{ into } [0,1] \text{ respectively represent the membership and non-membership degrees of every } \hat{x} \in U \text{ to } O \text{ under the constraint } 0 \leq (\varphi(\hat{x}))^2 + \varpi(\hat{x}) \leq 1.$$

The indeterminacy degree with respect to a (2,1)-FS O is a function $\zeta : U \to [0,1]$ given by

$$\zeta(\hat{x}) = (1 - ((\varphi(\hat{x}))^2 + \varpi(\hat{x})))^{\frac{2}{3}} \text{ for each } \hat{x} \in U.$$

As mentioned above, it is pretty obvious that the next implications hold true:

$$\text{IF-set} \Rightarrow \text{(2,1)-FS} \Rightarrow \text{PF-set} \Rightarrow \text{FF-set} \Rightarrow q\text{-ROF set}.$$

The next example shows that the converse of these implications fail to be true:

Example 2.6. *Let* $O_1 = \{\langle \hat{x}, 0.7, 0.5\rangle, \langle \hat{y}, 0.6, 0.3\rangle\}$, $O_2 = \{\langle \hat{x}, 0.4, 0.1\rangle, \langle \hat{y}, 0.8, 0.4\rangle\}$, $O_3 = \{\langle \hat{x}, 0.9, 0.5\rangle, \langle \hat{y}, 0.5, 0.3\rangle\}$ *and* $O_4 = \{\langle \hat{x}, 0.75, 0.1\rangle, \langle \hat{y}, 0.8, 0.8\rangle\}$ *be defined over* $U = \{\hat{x}, \hat{y}\}$. *Now, by direct calculation we can check that* O_1, O_2, O_3, O_4 *are (2,1)-FS, PF-set, FF-set, and q-ROF set (for all* $q > 3$*), respectively. On the other hand, we have the following:*

(i) O_1 *is not an IF-set because* $\varphi(\hat{x}) + \varpi(\hat{x}) = 1.2 \nleq 1$.

(ii) O_2 *is not a (2,1)-FS because* $(\varphi(\hat{y}))^2 + \varpi(\hat{y}) = 1.13 \nleq 1$.

(iii) O_3 *is not a PF-set because* $(\varphi(\hat{x}))^2 + (\varpi(\hat{x}))^2 = 1.06 \nleq 1$.

(iv) O_4 *is not an FF-set because* $(\varphi(\hat{x}))^3 + (\varpi(\hat{x}))^3 = 1.024 \nleq 1$.

Motivated by the tractability and flexibility of the (2,1)-FS model, a related concept has been defined recently:

Definition 2.7. *[9] Let* a, b *be positive real numbers greater than or equal to one. The* (a, b)*-FS* O *over the universal set U is given as follows.*

$O = \{\langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle : \hat{x} \in U\}$, *where* $\varphi, \varpi : U \to [0, 1]$ *are functions that respectively determine the degrees of membership and non-membership for every* $\hat{x} \in U$ *under the constraint*

$$0 \leq (\varphi(\hat{x}))^a + (\varpi(\hat{x}))^b \leq 1.$$

The degree of indeterminacy with respect to an (a, b)*-FS* O *is a function* $\zeta : U \to [0, 1]$ *given by*

$$\zeta(\hat{x}) = (1 - ((\varphi(\hat{x}))^a + (\varpi(\hat{x})^b))^{\frac{1}{ab}} \text{ for each } \hat{x} \in U.$$

In relation with the later definition, an (a, b)-FN or (a, b)-fuzzy number is a pair $\theta = \langle \varphi, \varpi\rangle$, where $\varphi, \varpi \in [0, 1]$ and $0 \leq \varphi^a + \varpi^b \leq 1$.

Figure 1 illustrates the boundaries of the acceptable evaluations in the IFS, PFS, and FFS models, in addition to the case of a $(2, 5)$-FS. Together with Example 2.6, it is a further evidence that the (a, b)-FS model has distinctive traits as compared to the previous IFS, PFS, FFS, and q-ROFS models.

The set-theoretic operations of subsethood, union and intersection via the fuzzy soft environments were formulated by integrating their counterparts via soft and fuzzy environments. To illustrate this matter, we present these operations via the environment of IF-sets. Recall that:

(i) $\{\langle \hat{x}, \varphi_1(\hat{x}), \varpi_1(\hat{x})\rangle : \hat{x} \in U\} \leq \{\langle \hat{x}, \varphi_2(\hat{x}), \varpi_2(\hat{x})\rangle : \hat{x} \in U\}$ provided that $\varphi_1(\hat{x}) \leq \varphi_2(\hat{x})$ and $\varpi_2(\hat{x}) \geq \varpi_1(\hat{x})$ for each $\hat{x} \in U$.

(ii) Union is defined by the expression $\{\langle \hat{x}, \varphi_1(\hat{x}), \varpi_1(\hat{x})\rangle : \hat{x} \in U\} \cup \{\langle \hat{x}, \varphi_2(\hat{x}), \varpi_2(\hat{x})\rangle : \hat{x} \in U\} =$ $\{\langle \hat{x}, max\{\varphi_1(\hat{x}), \varphi_2(\hat{x})\}, min\{\varpi_1(\hat{x}), \varpi_2(\hat{x})\rangle : \hat{x} \in U\}$.

(iii) Intersection is defined by the expression $\{\langle \hat{x}, \varphi_1(\hat{x}), \varpi_1(\hat{x})\rangle : \hat{x} \in U\} \cap \{\langle \hat{x}, \varphi_2(\hat{x}), \varpi_2(\hat{x})\rangle : \hat{x} \in U\} =$ $\{\langle \hat{x}, min\{\varphi_1(\hat{x}), \varphi_2(\hat{x})\}, max\{\varpi_1(\hat{x}), \varpi_2(\hat{x})\rangle : \hat{x} \in U\}$.

The operations above were generalized to the environments of PF-sets, FF-sets and q-OPF sets. They follow a similar structure.

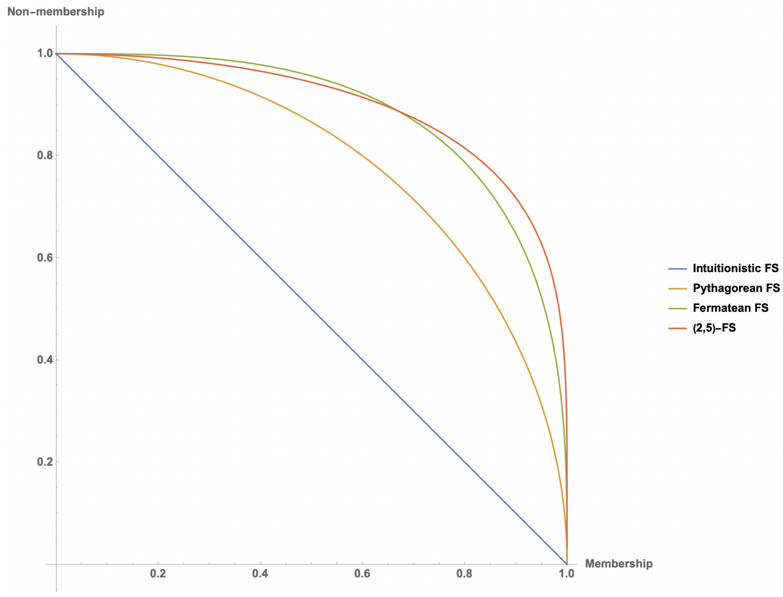

Figure 1. A graphical comparison of the models in Definitions 2.4 and 2.7.

2.2. *Soft fuzzification*

The crude idea of a soft set means an expansion of subsethood of a set that embeds a multiplicity of subsets, indexed by a set of parameters, attributes, or characteristics:

Definition 2.8. *[32] An ordinary mapping f from nonempty set of parameters Δ to the power set 2^U of the universal set of objects U is called a "soft set". It is denoted by the pair (f, Δ) and one can write this as $(f, \Delta) = \{(\sigma, f(\sigma)) : \sigma \in \Delta$ and $f(\sigma) \in 2^U\}$. We call $f(\sigma)$ a σ-component of (f, Δ).*

A deep discussion of the semantical interpretation of this definition has been given in [47], see also [4] for an update view.

We have discussed how subsethood can be understood in generalized fuzzy manners in Section 2.1. Whence these ideas can be naturally added to the notion of a soft set as follows:

Definition 2.9. *Let Δ be a set of parameters and $FS(U), IFS(U), PFS(U)$, $FFS(U)$ and $qROS(U)$ be respectively the families of all fuzzy sets, IF-sets, PF-sets, FF-sets and q-ROF sets defined over the universal set U. Then*

(i) *[30] a mapping f from Δ to $FS(U)$ is called a fuzzy soft set (briefly, FS-set). It can be written as follows:*

$$(f, \Delta) = \{(\sigma, \langle \hat{x}, \varphi(\hat{x}) \rangle) : \sigma \in \Delta \text{ and } \langle \hat{x}, \varphi(\hat{x}) \rangle \in FS(U)\}.$$

(ii) *[41] a mapping f from Δ to $IFS(U)$ is called an intuitionistic fuzzy soft set (briefly, IFS-set). It can be written as follows:*

$$(f, \Delta) = \{(\sigma, \langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x}) \rangle) : \sigma \in \Delta \text{ and } \langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x}) \rangle \in IFS(U)\}.$$

(iii) *[37] a mapping f from Δ to $PFS(U)$ is called a Pythagorean fuzzy soft set (briefly, PFS-set). It can be written as follows:*

$$(f, \Delta) = \{(\sigma, \langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x}) \rangle) : \sigma \in \Delta \text{ and } \langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x}) \rangle \in PFS(U)\}.$$

(iv) *[37] a mapping f from Δ to $FFS(U)$ is called a Fermatean fuzzy soft set (briefly, FFS-set). It can be written as follows:*

$(f,\Delta) = \{(\sigma, \langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle) : \sigma \in \Delta$ *and* $\langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle \in FFS(U)\}$.

(v) *[23] a mapping f from Δ to $qROS(U)$ is called a q-rung orthopair fuzzy soft set (briefly, q-ROFS set). It can be written as follows:*

$(f,\Delta) = \{(\sigma, \langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle) : \sigma \in \Delta$ *and* $\langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle \in qROS(U)\}$.

Put shortly, what is requested in the previous definition is that for each $\sigma \in \Delta$, $f(\sigma)$ must respectively be a fuzzy set, an IF-set, a PS-set, an FF-set, and a q-ROF set.

For our purposes it will suffice to recall the performance of algebraic concepts in the IFS-set setting.

Definition 2.10. *(see [1]) Let (f,Δ) and (h,Ω) be two IFS-sets. We say that (f,Δ) is an IFS subset of (h,Ω), denoted by $(f,\Delta) \sqsubseteq (h,\Omega)$, if $\Delta \subseteq \Omega$ and $f(\sigma) \preceq h(\sigma)$ for each $\sigma \in \Delta$.*

Definition 2.11. *(see [1]) Let (f,Δ) and (h,Ω) be two IFS-sets, where $\Delta \cap \Omega \neq \emptyset$. Then*

(i) *the IFS-union of them, denoted by $(f,\Delta) \sqcup (h,\Omega)$, is an IFS-set $(g, \Delta \cup \Omega)$ given by*

$$g(\theta) = \begin{cases} f(\theta) & \text{when } \theta \in \Delta - \Omega, \\ h(\theta) & \text{when } \theta \in \Omega - \Delta, \\ f(\theta) \bigcup h(\theta) & \text{when } \theta \in \Delta \cap \Omega, \end{cases}$$

(ii) *the IFS-intersection of them, denoted by $(f,\Delta) \sqcup (h,\Omega)$, is an IFS-set $(g, \Delta \cap \Omega)$ given by $g(\theta) = f(\theta) \cap h(\theta)$.*

3. A new model: (a,b)-Fuzzy soft sets

In this section, we shall introduce the novel model that justifies our study. Afterwards in Section 4 we shall define operations and aggregation mechanisms that apply to this framework.

Definition 3.1. *Let Δ be a set of parameters and (a,b)-$FS(U)$ be the family of all (a,b)-FSs defined over the universal set U. Then, a mapping f from Δ to (a,b)-$FS(U)$ is called an (a,b)-fuzzy soft set (briefly, (a,b)-FSS). It can be written as: $(f,\Delta) = \{(\sigma, \langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle) : \sigma \in \Delta$ and $\langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle \in (a,b)$-$FS(U)\}$.*

The degree of indeterminacy with respect to an (a,b)-FSS (f,Δ) is a function $\zeta : U \to [0,1]$ given by: For each $\hat{x} \in U$,

$$\zeta(\hat{x}) = (1 - ((\varphi(\hat{x}))^a + (\varpi(\hat{x})^b))^{\frac{1}{ab}}.$$

It is obvious that this degree of indeterminacy satisfies the equality $(\varphi(\hat{x}))^a + (\varpi(\hat{x}))^b + (\zeta(\hat{x}))^{ab} = 1$. Note that $\zeta(\hat{x}) = 0$ whenever $(\varphi(\hat{x}))^a + (\varpi(\hat{x}))^b = 1$.

Remark 3.2. *For the sake of simplicity, an (a,b)-FSS $(f,\Delta) = \{(\sigma, \langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle) : \sigma \in \Delta$ and $\langle \hat{x}, \varphi(\hat{x}), \varpi(\hat{x})\rangle \in (a,b)$-$FS(U)\}$ is denoted by the symbol $(f,\Delta) = (\sigma, \varphi, \varpi)$. The family of all (a,b)-FSSs defined over U is symbolized by $I^{(a,b)-FSS}$.*

Throughout this manuscript, we deal with (a,b)-FSSs that are defined with respect to a fixed set of parameters Δ.

Remark 3.3. *The family of all (a,b)-FSSs coincides with*

(1) $IFS(U)$ if $a = b = 1$.

(2) $PFS(U)$ if $a = b = 2$.

(3) $FFS(U)$ if $a = b = 3$.

(4) $qROFS(U)$ if $a = b = q$.

In the next result we compare (a,b)-FSS with the previous generalizations of IFSs.

Proposition 3.4. *The following statements hold true:*

(1) Any IFS-set is an (a,b)-FSS.

(2) If $a \geq 2$ and $b \geq 2$, then any PFS-set is an (a,b)-FSS.

(3) If $a \geq 3$ and $b \geq 3$, then any FFS-set is an (a,b)-FSS.

(4) If $a \geq q$ and $b \geq q$, then any q-ROFS set is an (a,b)-FSS.

Proof. Straightforward. □

Example 2.6 demonstrates that the converses of the assertions given in Proposition 3.4 are generally not true.

Now we proceed to define algebraic operations on the new model. We begin with union, intersection, and complement.

Definition 3.5. *Let $(f_1,\Delta) = (\sigma,\varphi_1,\varpi_1)$ and $(f_2,\Delta) = (\sigma,\varphi_2,\varpi_2)$ be (a,b)-FSSs on U. Then*

(1) $(f_1,\Delta) \sqcup (f_2,\Delta) = (\sigma, max\{\varphi_1,\varphi_2\}, min\{\varpi_1,\varpi_2\})$.

(2) $(f_1,\Delta) \sqcap (f_2,\Delta) = (\sigma, min\{\varphi_1,\varphi_2\}, max\{\varpi_1,\varpi_2\})$.

(3) $(f_1,\Delta)^c = (\sigma, \varpi_1^{\frac{b}{a}}, \varphi_1^{\frac{a}{b}})$.

These concepts are well defined. Note that it can be easily seen that the family of (a,b)-FSSs is closed under the operators of $\sqcup$ and $\sqcap$, i.e., $(f_1,\Delta) \sqcup (f_2,\Delta) \in I^{(a,b)-FSS}$ and $(f_1,\Delta) \sqcap (f_2,\Delta) \in I^{(a,b)-FSS}$. Also, $(\varpi^{\frac{b}{a}})^a + (\varphi^{\frac{a}{b}})^b = \varpi^b + \varphi^a \leq 1$, so $(f,\Delta)^c$ is an (a,b)-FSS. It is obvious that $((f,\Delta)^c)^c = (\sigma,\varphi,\varpi) = (f,\Delta)$.

The next example shows how these operators are calculated in practice.

Example 3.6. *Assume that $(f_1,\Delta) = (\sigma, 0.6, 0.45)$ and $(f_2,\Delta) = (\sigma, 0.8, 0.5)$ are $(5,3)$-FSSs on U. Then*

(1) $(f_1,\Delta) \sqcup (f_2,\Delta) = (\sigma, max\{\varphi_1,\varphi_2\}, min\{\varpi_1,\varpi_2\})$

$$= (\sigma, max\{0.6, 0.8\}, min\{0.45, 0.5\}) = (\sigma, 0.8, 0.45).$$

(2) $(f_1,\Delta) \sqcap (f_2,\Delta) = (\sigma, min\{\varphi_1,\varphi_2\}, max\{\varpi_1,\varpi_2\})$

$$= (\sigma, min\{0.6, 0.8\}, max\{0.45, 0.5\}) = (0.6, 0.5).$$

(3) $(f_1^c, \Delta) \approx (\sigma, 0.736, 0.55186)$.

The operators $\sqcup$ and $\sqcap$ given in Definition 3.5, can be generalized for arbitrary numbers of (a, b)-FSSs as follows.

Definition 3.7. *Let* $\{(f_i, \Delta) = (\sigma, \varphi_i, \varpi_i) : i \in I\}$ *be a family of* (a, b)*-FSSs on* U*. Then*

(1) $\underset{i\in I}{\sqcup}(f_i, \Delta) = (\sigma, sup\{\varphi_i : i \in I\}, inf\{\varpi_i : i \in I\})$.

(2) $\underset{i\in I}{\sqcap}(f_i, \Delta) = (\sigma, inf\{\varphi_i : i \in I\}, sup\{\varpi_i : i \in I\})$.

It is straightforward to prove that the operators $\sqcup$ and $\sqcap$ are commutative:

Proposition 3.8. *Let* $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ *and* $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ *be* (a, b)*-FSSs on* U*. Then*

(1) $(f_1, \Delta) \sqcup (f_2, \Delta) = (f_2, \Delta) \sqcup (f_1, \Delta)$.

(2) $(f_1, \Delta) \sqcap (f_2, \Delta) = (f_2, \Delta) \sqcap (f_1, \Delta)$.

With respect to associativity, the next proposition assures that it holds true for the operators $\sqcup$ and $\sqcap$ too:

Proposition 3.9. *Let* $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$, $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ *and* $(f_3, \Delta) = (\sigma, \varphi_3, \varpi_3)$ *be* (a, b)*-FSSs on* U*. Then*

(1) $(f_1, \Delta) \sqcup ((f_2, \Delta) \sqcup (f_3, \Delta)) = ((f_1, \Delta) \sqcup (f_2, \Delta)) \sqcup (f_3, \Delta)$.

(2) $(f_1, \Delta) \sqcap ((f_2, \Delta) \sqcap (f_3, \Delta)) = ((f_1, \Delta) \sqcap (f_2, \Delta)) \sqcap (f_3, \Delta)$.

Proof. For the three (a, b)-FSSs $(f_1, \Delta), (f_2, \Delta)$ and (f_3, Δ) on U, according to Definition 3.5, we obtain:

(1) $(f_1, \Delta) \sqcup ((f_2, \Delta) \sqcup (f_3, \Delta)) = (\sigma, \varphi_1, \varpi_1) \sqcup (\sigma, max\{\varphi_2, \varphi_3\}, min\{\varpi_2, \varpi_3\})$
$= (\sigma, max\{\varphi_1, max\{\varphi_2, \varphi_3\}\}, min\{\varpi_1, min\{\varpi_2, \varpi_3\}\})$
$= (\sigma, max\{max\{\varphi_1, \varphi_2\}, \varphi_3\}, min\{min\{\varpi_1, \varpi_2\}, \varpi_3\})$
$= (\sigma, max\{\varphi_1, \varphi_2\}, min\{\varpi_1, \varpi_2\}) \sqcup (\sigma, \varphi_3, \varpi_3)$
$= ((f_1, \Delta) \sqcap (f_2, \Delta)) \sqcup (f_3, \Delta)$.

(2) Similar to 1 above.

□

Theorem 3.10. Let $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$, $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ and $(f_3, \Delta) = (\sigma, \varphi_3, \varpi_3)$ be (a, b)-FSSs. Then

(1) $((f_1, \Delta) \sqcup (f_2, \Delta)) \sqcap (f_3, \Delta) = ((f_1, \Delta) \sqcap (f_3, \Delta)) \sqcup ((f_2, \Delta) \sqcap (f_3, \Delta))$.

(2) $((f_1, \Delta) \sqcap (f_2, \Delta)) \sqcup (f_3, \Delta) = ((f_1, \Delta) \sqcup (f_3, \Delta)) \sqcap ((f_2, \Delta) \sqcup (f_3, \Delta))$.

Proof. For the three (a, b)-FSSs $(f_1, \Delta), (f_2, \Delta)$ and (f_3, Δ), according to Definition 3.5, we obtain:

(1) $((f_1, \Delta) \sqcup (f_2, \Delta)) \sqcap (f_3, \Delta) = (\sigma, max\{\varphi_1, \varphi_2\}, min\{\varpi_1, \varpi_2\}) \sqcap (\sigma, \varphi_3, \varpi_3)$
$= (\sigma, min\{max\{\varphi_1, \varphi_2\}, \varphi_3\}, max\{min\{\varpi_1, \varpi_2\}, \varpi_3\})$. And
$((f_1, \Delta) \sqcap (f_3, \Delta)) \sqcup ((f_2, \Delta) \sqcap (f_3, \Delta)) =$
$(\sigma, min\{\varphi_1, \varphi_3\}, max\{\varpi_1, \varpi_3\}) \sqcup (\sigma, min\{\varphi_2, \varphi_3\}, max\{\varpi_2, \varpi_3\}) =$
$(\sigma, max\{min\{\varphi_1, \varphi_3\}, min\{\varphi_2, \varphi_3\}\}, min\{max\{\varpi_1, \varpi_3\}, max\{\varpi_2, \varpi_3\}\})$. Then,

$$min\{max\{\varphi_1,\varphi_2\},\varphi_3\} = \begin{cases} \varphi_2 & \text{if } \varphi_1 \leq \varphi_2 \leq \varphi_3, \\ \varphi_1 & \text{if } \varphi_2 \leq \varphi_1 \leq \varphi_3, \\ \varphi_3 & \text{if } \varphi_1 \leq \varphi_3 \leq \varphi_2, \\ \varphi_3 & \text{if } \varphi_3 \leq \varphi_1 \leq \varphi_2, \\ \varphi_3 & \text{if } \varphi_2 \leq \varphi_3 \leq \varphi_1, \\ \varphi_3 & \text{if } \varphi_3 \leq \varphi_2 \leq \varphi_1, \end{cases}$$

$$max\{min\{\varpi_1,\varpi_2\},\varpi_3\} = \begin{cases} \varpi_3 & \text{if } \varpi_1 \leq \varpi_2 \leq \varpi_3, \\ \varpi_3 & \text{if } \varpi_2 \leq \varpi_1 \leq \varpi_3, \\ \varpi_3 & \text{if } \varpi_1 \leq \varpi_3 \leq \varpi_2, \\ \varpi_1 & \text{if } \varpi_3 \leq \varpi_1 \leq \varpi_2, \\ \varpi_3 & \text{if } \varpi_2 \leq \varpi_3 \leq \varpi_1, \\ \varpi_2 & \text{if } \varpi_3 \leq \varpi_2 \leq \varpi_1, \end{cases}$$

$$max\{min\{\varphi_1,\varphi_3\},min\{\varphi_2,\varphi_3\}\} = \begin{cases} \varphi_2 & \text{if } \varphi_1 \leq \varphi_2 \leq \varphi_3, \\ \varphi_1 & \text{if } \varphi_2 \leq \varphi_1 \leq \varphi_3, \\ \varphi_3 & \text{if } \varphi_1 \leq \varphi_3 \leq \varphi_2, \\ \varphi_3 & \text{if } \varphi_3 \leq \varphi_1 \leq \varphi_2, \\ \varphi_3 & \text{if } \varphi_2 \leq \varphi_3 \leq \varphi_1, \\ \varphi_3 & \text{if } \varphi_3 \leq \varphi_2 \leq \varphi_1, \end{cases}$$

$$min\{max\{\varpi_1,\varpi_3\},max\{\varpi_2,\varpi_3\}\} = \begin{cases} \varpi_3 & \text{if } \varpi_1 \leq \varpi_2 \leq \varpi_3, \\ \varpi_3 & \text{if } \varpi_2 \leq \varpi_1 \leq \varpi_3, \\ \varpi_3 & \text{if } \varpi_1 \leq \varpi_3 \leq \varpi_2, \\ \varpi_1 & \text{if } \varpi_3 \leq \varpi_1 \leq \varpi_2, \\ \varpi_3 & \text{if } \varpi_2 \leq \varpi_3 \leq \varpi_1, \\ \varpi_2 & \text{if } \varpi_3 \leq \varpi_2 \leq \varpi_1. \end{cases}$$

It follows that $min\{max\{\varphi_1,\varphi_2\},\varphi_3\} = max\{min\{\varphi_1,\varphi_3\},min\{\varphi_2,\varphi_3\}\}$ and $max\{min\{\varpi_1,\varpi_2\},\varpi_3\} = min\{max\{\varpi_1,\varpi_3\},max\{\varpi_2,\varpi_3\}\}$. Hence, $((f_1,\Delta)\sqcup(f_2,\Delta))\sqcap(f_3,\Delta) = ((f_1,\Delta)\sqcap(f_3,\Delta))\sqcup((f_2,\Delta)\sqcap(f_3,\Delta))$.

(2) $((f_1,\Delta)\sqcap(f_2,\Delta))\sqcup(f_3,\Delta) = (\sigma, min\{\varphi_1,\varphi_2\}, max\{\varpi_1,\varpi_2\})\sqcup(\sigma,\varphi_3,\varpi_3)$
$= (\sigma, max\{min\{\varphi_1,\varphi_2\},\varphi_3\}, min\{max\{\varpi_1,\varpi_2\},\varpi_3\})$. And,
$((f_1,\Delta)\sqcup(f_3,\Delta))\sqcap((f_2,\Delta)\sqcup(f_3,\Delta)) = (\sigma, max\{\varphi_1,\varphi_3\}, min\{\varpi_1,\varpi_3\})\sqcap(\sigma, max\{\varphi_2,\varphi_3\}, min\{\varpi_2,\varpi_3\}) =$
$(\sigma, min\{max\{\varphi_1,\varphi_3\},max\{\varphi_2,\varphi_3\}\}, max\{min\{\varpi_1,\varpi_3\},min\{\varpi_2,\varpi_3\}\})$. Then,

$$max\{min\{\varphi_1,\varphi_2\},\varphi_3\} = \begin{cases} \varphi_3 & \text{if } \varphi_1 \leq \varphi_2 \leq \varphi_3, \\ \varphi_3 & \text{if } \varphi_2 \leq \varphi_1 \leq \varphi_3, \\ \varphi_3 & \text{if } \varphi_1 \leq \varphi_3 \leq \varphi_2, \\ \varphi_1 & \text{if } \varphi_3 \leq \varphi_1 \leq \varphi_2, \\ \varphi_3 & \text{if } \varphi_2 \leq \varphi_3 \leq \varphi_1, \\ \varphi_2 & \text{if } \varphi_3 \leq \varphi_2 \leq \varphi_1, \end{cases}$$

$$min\{max\{\varpi_1,\varpi_2\},\varpi_3\} = \begin{cases} \varpi_2 & \text{if } \varpi_1 \le \varpi_2 \le \varpi_3, \\ \varpi_1 & \text{if } \varpi_2 \le \varpi_1 \le \varpi_3, \\ \varpi_3 & \text{if } \varpi_1 \le \varpi_3 \le \varpi_2, \\ \varpi_3 & \text{if } \varpi_3 \le \varpi_1 \le \varpi_2, \\ \varpi_3 & \text{if } \varpi_2 \le \varpi_3 \le \varpi_1, \\ \varpi_3 & \text{if } \varpi_3 \le \varpi_2 \le \varpi_1, \end{cases}$$

$$min\{max\{\varphi_1,\varphi_3\},max\{\varphi_2,\varphi_3\}\} = \begin{cases} \varphi_3 & \text{if } \varphi_1 \le \varphi_2 \le \varphi_3, \\ \varphi_3 & \text{if } \varphi_2 \le \varphi_1 \le \varphi_3, \\ \varphi_3 & \text{if } \varphi_1 \le \varphi_3 \le \varphi_2, \\ \varphi_1 & \text{if } \varphi_3 \le \varphi_1 \le \varphi_2, \\ \varphi_3 & \text{if } \varphi_2 \le \varphi_3 \le \varphi_1, \\ \varphi_2 & \text{if } \varphi_3 \le \varphi_2 \le \varphi_1, \end{cases}$$

$$max\{min\{\varpi_1,\varpi_3\},min\{\varpi_2,\varpi_3\}\} = \begin{cases} \varpi_2 & \text{if } \varpi_1 \le \varpi_2 \le \varpi_3, \\ \varpi_1 & \text{if } \varpi_2 \le \varpi_1 \le \varpi_3, \\ \varpi_3 & \text{if } \varpi_1 \le \varpi_3 \le \varpi_2, \\ \varpi_3 & \text{if } \varpi_3 \le \varpi_1 \le \varpi_2, \\ \varpi_3 & \text{if } \varpi_2 \le \varpi_3 \le \varpi_1, \\ \varpi_3 & \text{if } \varpi_3 \le \varpi_2 \le \varpi_1. \end{cases}$$

It follows that $max\{min\{\varphi_1,\varphi_2\},\varphi_3\} = min\{max\{\varphi_1,\varphi_3\},max\{\varphi_2,\varphi_3\}\}$ and $min\{max\{\varpi_1,\varpi_2\},\varpi_3\} = max\{min\{\varpi_1,\varpi_3\},min\{\varpi_2,\varpi_3\}\}$. Hence, $((f_1,\Delta)\sqcap(f_2,\Delta))\sqcup(f_3,\Delta) = ((f_1,\Delta)\sqcup(f_3,\Delta))\sqcap((f_2,\Delta)\sqcup(f_3,\Delta))$.

□

Theorem 3.11. *Let* $(f_1,\Delta) = (\sigma,\varphi_1,\varpi_1)$ *and* $(f_2,\Delta) = (\sigma,\varphi_2,\varpi_2)$ *be* (a,b)*-FSSs on* U*. Then*

(1) $((f_1,\Delta)\sqcup(f_2,\Delta))^c = (f_1,\Delta)^c \sqcap (f_2,\Delta)^c$.

(2) $((f_1,\Delta)\sqcap(f_2,\Delta))^c = (f_1,\Delta)^c \sqcup (f_2,\Delta)^c$.

Proof. (1) For the (a,b)-FSSs (f_1,Δ) and (f_2,Δ), according to Definition 3.5, we obtain

$((f_1,\Delta)\sqcup(f_2,\Delta))^c = (\sigma,max\{\varphi_1,\varphi_2\},min\{\varpi_1,\varpi_2\})^c$
$= (\sigma,min\{(\varpi_1)^{\frac{b}{a}},(\varpi_2)^{\frac{b}{a}}\},max\{(\varphi_1)^{\frac{a}{b}},(\varphi_2)^{\frac{a}{b}}\})$
$= (\sigma,(\varpi_1)^{\frac{b}{a}},(\varphi_1)^{\frac{a}{b}}) \sqcap (\sigma,(\varpi_2)^{\frac{b}{a}},(\varphi_2)^{\frac{a}{b}})$
$= (f_1,\Delta)^c \sqcap (f_2,\Delta)^c$.

(2) Similar to 1.

□

To offer tools for comparisons, we define the score and accuracy functions of (a,b)-FSSs which will be helpful to rank (a,b)-FSSs.

Proposition 3.12. *Let* $(f,\Delta) = (\sigma,\varphi,\varpi)$ *be an* (a,b)*-FSS on* U*. Then, the value of* $\varphi^a - \varpi^b$ *lies in the closed interval* $[-1,1]$.

Proof. For any (a,b)-FSS (f,Δ), we have $\varphi^a + \varpi^b \le 1$. This implies that $\varphi^a - \varpi^b \le \varphi^a \le 1$ and $\varphi^a - \varpi^b \ge -\varpi^b \ge -1$. Hence, $-1 \le \varphi^a - \varpi^b \le 1$, as required. □

Definition 3.13. *The score function score* $: I^{(a,b)-FS} \to [-1, 1]$ *of any* (a, b)*-FSS* $(f, \Delta) = (\sigma, \varphi, \varpi)$ *is given by the formula* $score(f, \Delta) = \varphi^a - \varpi^b$.

Definition 3.14. *Let* $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ *and* $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ *be* (a, b)*-FSSs. We say that*

(i) *If* $score((f_1, \Delta)) > score((f_2, \Delta))$*, then* $(f_1, \Delta) \succ (f_2, \Delta)$.

(ii) *If* $score((f_1, \Delta)) < score((f_2, \Delta))$*, then* $(f_1, \Delta) \prec (f_2, \Delta)$.

(iii) *If* $score((f_1, \Delta)) = score((f_2, \Delta))$*, then* $(f_1, \Delta) \simeq (f_2, \Delta)$.

The score function is not a sufficient tool to specify which better (a, b)-FSSs can be chosen in some cases. This occurs when the two (a, b)-FSSs satisfy $\varphi^a = \lambda^b$. But we know that these (a, b)-FSSs may not match with each other. So that, comparison depending on the score function is not acceptable (or appropriate) to address these cases. For this reason, we introduce the concept of accuracy function for (a, b)-FSSs to make a comparison of (a, b)-FSS.

Definition 3.15. *The accuracy function acc* $: I^{(a,b)-FS} \to [0, 1]$ *of an* (a, b)*-FSS* $(f, \Delta) = (\sigma, \varphi, \varpi)$ *is given by the formula* $acc(f, \Delta) = \varphi^a + \varpi^b$.

As is standard in related models, we can make use of the score and accuracy functions to compare between (a, b)-FSSs.

Definition 3.16. *Let* $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ *and* $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ *be* (a, b)*-FSSs, where* $score(f_k, \Delta)$ *and* $acc(f_k, \Delta)$ *respectively denote their evaluations by the score and accuracy functions* $(k = 1, 2)$*. We declare that:*

(i) *If* $score((f_1, \Delta)) > score((f_2, \Delta))$*, then* $(f_1, \Delta) \succ (f_2, \Delta)$.

(ii) *If* $score((f_1, \Delta)) < score((f_2, \Delta))$*, then* $(f_1, \Delta) \prec (f_2, \Delta)$.

(iii) *If* $score((f_1, \Delta)) = score((f_2, \Delta))$*, then*

(1) If $acc((f_1, \Delta)) > acc((f_2, \Delta))$*, then* $(f_1, \Delta) \succ (f_2, \Delta)$.

(2) If $acc((f_1, \Delta)) < acc((f_2, \Delta))$*, then* $(f_1, \Delta) \prec (f_2, \Delta)$.

(3) If $acc((f_1, \Delta)) = acc((f_2, \Delta))$*, then* $(f_1, \Delta) = (f_2, \Delta)$.

Definition 3.17. *Let* $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ *and* $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ *be* (a, b)*-FSSs on* U*. A natural quasi-ordering on the* (a, b)*-FSSs is defined as follows.*

$$(f_1, \Delta) \geq (f_2, \Delta) \text{ iff } \varphi_1 \geq \varphi_2 \text{ and } \varpi_1 \leq \varpi_2.$$

4. Operations on (a, b)-fuzzy soft sets and aggregation procedures

In this section, we do two things. First we deal with some standard arithmetic operations on (a, b)-FSSs. Then we shall start a tentative approach to the combination of (a, b)-FNs.

4.1. Some arithmetic operations on (a, b)-FSSs

Arithmetic operations should at least include the notion of a sum, a product, a product by scalar, and exponentiation. We proceed to define these concepts in the framework of (a, b)-FSSs. Subsequent theorems prove that these operations are well-defined.

Definition 4.1. *Let $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ and $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ be (a, b)-FSSs on U, and δ be a positive real number ($\delta > 0$). We define the following operations.*

(1) $(f_1, \Delta) \oplus (f_2, \Delta) = \left(\sigma, \sqrt[a]{\varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^a}, \varpi_1\varpi_2\right)$.

(2) $(f_1, \Delta) \otimes (f_2, \Delta) = \left(\sigma, \varphi_1\varphi_2, \sqrt[b]{\varpi_1^b + \varpi_2^b - \varpi_1^b\varpi_2^b}\right)$.

(3) $\delta(f_1, \Delta) = \left(\sigma, \sqrt[a]{1 - (1 - \varphi_1^a)^\delta}, \varpi_1^\delta\right)$.

(4) $O_1^\delta = \left(\sigma, \varphi_1^\delta, \sqrt[b]{1 - (1 - \varpi_1^b)^\delta}\right)$.

Theorem 4.2. *If $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ and $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ are (a, b)-FSSs on U, then $(f_1, \Delta) \oplus (f_2, \Delta)$ and $(f_1, \Delta) \otimes (f_2, \Delta)$ are (a, b)-FSSs.*

Proof. Let $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ and $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ be (a, b)-FSSs. Then, for each σ we obtain

$$0 \leq \varphi_1^a + \varpi_1^b \leq 1 \text{ and } 0 \leq \varphi_2^a + \varpi_2^b \leq 1.$$

Then, we have

$$\varphi_1^a \geq \varphi_1^a\varphi_2^a, \varphi_2^a \geq \varphi_1^a\varphi_2^a, 0 \leq \varphi_1^a\varphi_2^a \leq 1$$

and

$$\varpi_1^b \geq \varpi_1^b\varpi_2^b, \varpi_2^b \geq \varpi_1^b\varpi_2^b, 0 \leq \varpi_1^b\varpi_2^b \leq 1.$$

This implies that $\sqrt[a]{\varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^a} \geq 0$.

Since $\varphi_2^a \leq 1$ and $0 \leq 1 - \varphi_1^a$, $\varphi_2^a(1 - \varphi_1^a) \leq (1 - \varphi_1^a)$, we get that $\varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^a \leq 1$. Thus, $\sqrt[a]{\varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^a} \leq 1$. It is clear that $0 \leq \varpi_1^b \leq 1 - \varphi_1^a$ and $0 \leq \varpi_2^b \leq 1 - \varphi_2^a$.

Now, $(\sqrt[a]{\varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^a})^a + \varpi_1^b\varpi_2^b \leq \varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^a + (1 - \varphi_1^a)(1 - \varphi_2^a) = 1$.

Hence, $0 \leq (\sqrt[a]{\varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^2})^a + \varpi_1^b\varpi_2^b \leq 1$ which means that $(f_1, \Delta) \oplus (f_2, \Delta)$ is an (a, b)-FSS.

Following similar arguments, we prove that $(f_1, \Delta) \otimes (f_2, \Delta)$ is an (a, b)-FSS. □

Theorem 4.3. *Let $(f, \Delta) = (\sigma, \varphi, \varpi)$ be an (a, b)-FSS on U and δ be a positive real number. Then, $\delta(f, \Delta)$ and $(f, \Delta)^\delta$ are (a, b)-FSSs.*

Proof. Since $0 \leq \varphi^a \leq 1$, $0 \leq \varpi^b \leq 1$ and $0 \leq (\varphi)^a + \varpi^b \leq 1$ for each σ, we find

$0 \leq \varpi^b \leq 1 - \varphi^a$

$\Rightarrow 0 \leq (1 - \varphi^a)^\delta$

$\Rightarrow 1 - (1 - \varphi^a)^\delta \leq 1$

$\Rightarrow 0 \leq \sqrt[a]{1 - (1 - \varphi^a)^\delta} \leq \sqrt[a]{1} = 1.$

It is clear that $0 \leq \varpi^{\delta} \leq 1$, then we get

$$0 \leq (\sqrt[a]{1-(1-\varphi^{a})^{\delta}})^{a} + (\varpi^{\delta})^{b} \leq 1-(1-\varphi^{a})^{\delta} + (1-\varphi^{a})^{\delta} = 1.$$

Following similar arguments, we obtain

$$0 \leq (\varphi^{\delta})^{a} + 1-(1-\varpi)^{\delta} \leq 1.$$

Hence, $\delta(f, \Delta)$ and $(f, \Delta)^{\delta}$ are (a, b)-FSSs. □

Theorem 4.4. *Let $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ and $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ be (a, b)-FSSs on U. Then*

(1) $(f_1, \Delta) \oplus (f_2, \Delta) = (f_2, \Delta) \oplus (f_1, \Delta)$.

(2) $(f_1, \Delta) \otimes (f_2, \Delta) = (f_2, \Delta) \otimes (f_1, \Delta)$.

Proof. From Definition 4.1, we obtain:

(1) $(f_1, \Delta) \oplus (f_2, \Delta) = \left(\sigma, \sqrt[a]{\varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^a}, \varpi_1\varpi_2\right)$
$\left(\sigma, \sqrt[a]{\varphi_2^a + \varphi_1^a - \varphi_2^a\varphi_1^a}, \varpi_2\varpi_1\right) = (f_2, \Delta) \oplus (f_1, \Delta)$.

(2) $(f_1, \Delta) \otimes (f_2, \Delta) = \left(\sigma, \varphi_1\varphi_2, \sqrt[b]{\varpi_1^b + \varpi_2^b - \varpi_1^b\varpi_2^b}\right)$
$= \left(\sigma, \varphi_2\varphi_1, \sqrt[b]{\varpi_2^b + \varpi_1^b - \varpi_2^b\varpi_1^b}\right) = (f_2, \Delta) \otimes (f_1, \Delta)$.

□

Theorem 4.5. *Let $(f, \Delta) = (\sigma, \varphi, \varpi)$, $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$ and $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ be (a, b)-FSSs on U. Then*

(1) $\delta((f_1, \Delta) \oplus (f_2, \Delta)) = \delta(f_1, \Delta) \oplus \delta(f_2, \Delta)$.

(2) $(\delta_1 + \delta_2)(f, \Delta) = \delta_1(f, \Delta) \oplus \delta_2(f, \Delta)$.

(3) $((f_1, \Delta) \otimes (f_2, \Delta))^{\delta} = (f_1, \Delta)^{\delta} \otimes (f_2, \Delta)^{\delta}$.

(4) $(f, \Delta)^{(\delta_1+\delta_2)} = (f, \Delta)^{\delta_1} \otimes (f, \Delta)^{\delta_2}$.

Proof. (1) $\delta((f_1, \Delta) \oplus (f_2, \Delta)) = \delta\left(\sigma, \sqrt[a]{\varphi_1^a + \varphi_2^a - \varphi_1^a\varphi_2^a}, \varpi_1\varpi_2\right)$
$= \left(\sigma, \sqrt[a]{1-(1-\varphi_1^a - \varphi_2^a + \varphi_1^a\varphi_2^a)^{\delta}}, (\varpi_1\varpi_2)^{\delta}\right)$
$= \left(\sigma, \sqrt[a]{1-(1-\varphi_1^a)^{\delta}(1-\varphi_2^a)^{\delta}}, \varpi_1^{\delta}\varpi_2^{\delta}\right)$.
And $\delta(f_1, \Delta) \oplus \delta(f_2, \Delta) = \left(\sigma, \sqrt[a]{1-(1-\varphi_1^a)^{\delta}}, \varpi_1^{\delta}\right) \oplus \left(\sigma, \sqrt{1-(1-\varphi_2^a)^{\delta}}, \varpi_2^{\delta}\right)$
$= \left(\sigma, \sqrt[a]{1-(1-\varphi_1^a)^{\delta} + 1-(1-\varphi_2^a)^{\delta} - (1-(1-\varphi_1^a)^{\delta})(1-(1-\varphi_2^a)^{\delta})}, \varpi_1^{\delta}\varpi_2^{\delta}\right)$
$= \left(\sigma, \sqrt[a]{1-(1-\varphi_1^a)^{\delta}(1-\varphi_2^a)^{\delta}}, \varpi_1^{\delta}\varpi_2^{\delta}\right) = \delta((f_1, \Delta) \oplus (f_2, \Delta))$.

(2) $(\delta_1 + \delta_2)(f, \Delta) = (\delta_1 + \delta_2)(\sigma, \varphi, \varpi) = \left(\sigma, \sqrt[a]{1-(1-\varphi^a)^{\delta_1+\delta_2}}, \varpi^{\delta_1+\delta_2}\right)$
$= \left(\sigma, \sqrt[a]{1-(1-\varphi^a)^{\delta_1}(1-\varphi^a)^{\delta_2}}, \varpi^{\delta_1+\delta_2}\right)$
$= \left(\sigma, \sqrt[a]{1-(1-\varphi^a)^{\delta_1} + 1-(1-\varphi^a)^{\delta_2} - (1-(1-\varphi^a)^{\delta_1})(1-(1-\varphi^a)^{\delta_2})}, \varpi^{\delta_1}\varpi^{\delta_2}\right)$
$= \left(\sigma, \sqrt[a]{1-(1-\varphi^a)^{\delta_1}}, \varpi^{\delta_1}\right) \oplus \left(\sigma, \sqrt[a]{1-(1-\varphi^a)^{\delta_2}}, \varpi^{\delta_2}\right)$
$= \delta_1(f, \Delta) \oplus \delta_2(f, \Delta)$.

(3) $((f_1,\Delta)\otimes(f_2,\Delta))^\delta = \left(\sigma, \varphi_1\varphi_2, \sqrt[b]{\varpi_1^b+\varpi_2^b-\varpi_1^b\varpi_2^b}\right)^\delta$
$= \left(\sigma, (\varphi_1\varphi_2)^\delta, \sqrt[b]{1-(1-\varpi_1^b-\varpi_2^b+\varpi_1^b\varpi_2^b)^\delta}\right)$
$= \left(\sigma, \varphi_1^\delta\varphi_2^\delta, \sqrt[b]{1-(1-\varpi_1^b)^\delta(1-\varpi_2^b)^\delta}\right)$
$= \left(\sigma, \varphi_1^\delta, \sqrt[b]{1-(1-\varpi_1^b)^\delta}\right)\otimes\left(\sigma, \varphi_2^\delta, \sqrt[b]{1-(1-\varpi_2^b)^\delta}\right)$
$= (f_1,\Delta)^\delta\otimes(f_2,\Delta)^\delta$.

(4) $(f,\Delta)^{\delta_1}\otimes(f,\Delta)^{\delta_2} = \left(\sigma, \varphi^{\delta_1}, \sqrt[b]{1-(1-\varpi^b)^{\delta_1}}\right)\otimes\left(\sigma, \varphi^{\delta_2}, \sqrt[b]{1-(1-\varpi^b)^{\delta_2}}\right)$
$= \left(\sigma, \varphi^{\delta_1+\delta_2}, \sqrt[b]{1-(1-\varpi^b)^{\delta_1}+1-(1-\varpi^b)^{\delta_2}-(1-(1-\varpi^b)^{\delta_1})(1-(1-\varpi^b)^{\delta_2})}\right)$
$= \left(\sigma, \varphi^{\delta_1+\delta_2}, \sqrt[b]{1-(1-\varpi^b)^{\delta_1+\delta_2}}\right)$
$= (f,\Delta)^{(\delta_1+\delta_2)}$.

□

Theorem 4.6. *Let* $(f_1,\Delta)=(\sigma,\varphi_1,\varpi_1)$ *and* $(f_2,\Delta)=(\sigma,\varphi_2,\varpi_2)$ *be* (a,b)*-FSSs on* U*, and* $\delta>0$*. Then*

(1) $\delta((f_1,\Delta)\sqcup(f_2,\Delta)) = \delta(f_1,\Delta)\sqcup\delta(f_2,\Delta)$.

(2) $((f_1,\Delta)\sqcup(f_2,\Delta))^\delta = (f_1,\Delta)^\delta\sqcup(f_2,\Delta)^\delta$.

Proof. For the two (a,b)-FSSs (f_1,Δ) and (f_2,Δ), and $\delta>0$, according to Definitions 3.5 and 4.1, we obtain

(1) $\delta((f_1,\Delta)\sqcup(f_2,\Delta)) = \delta(\sigma, max\{\varphi_1,\varphi_2\}, min\{\varpi_1,\varpi_2\})$
$= \left(\sigma, \sqrt[a]{1-(1-max\{\varphi_1^a,\varphi_2^a\})^\delta}, min\{\varpi_1^\delta,\varpi_2^\delta\}\right)$.
And $\delta(f_1,\Delta)\sqcup\delta(f_2,\Delta) = \left(\sigma, \sqrt[a]{1-(1-\varphi_1^a)^\delta}, \varpi_1^\delta\right)\sqcup\left(\sigma, \sqrt[a]{1-(1-\varphi_2^a)^\delta}, \varpi_2^\delta\right)$
$= \left(\sigma, max\{\sqrt[a]{1-(1-\varphi_1^a)^\delta}, \sqrt{1-(1-\varphi_2^a)^\delta}\}, min\{\varpi_1^\delta,\varpi_2^\delta\}\right)$
$= \left(\sigma, \sqrt{1-(1-max\{\varphi_1^a,\varphi_2^a\})^\delta}, min\{\varpi_1^\delta,\varpi_2^\delta\}\right) = \delta((f_1,\Delta)\sqcup(f_2,\Delta))$.

(2) Similar to the proof of claim 1.

□

Theorem 4.7. *Let* $(f,\Delta)=(\sigma,\varphi,\varpi)$*,* $(f_1,\Delta)=(\sigma,\varphi_1,\varpi_1)$ *and* $(f_2,\Delta)=(\sigma,\varphi_2,\varpi_2)$ *be* (a,b)*-FSSs on* U*, and* $\delta>0$*. Then*

(1) $((f_1,\Delta)\oplus(f_2,\Delta))^c = (f_1,\Delta)^c\otimes(f_2,\Delta)^c$.

(2) $((f_1,\Delta)\otimes(f_2,\Delta))^c = (f_1,\Delta)^c\oplus(f_2,\Delta)^c$.

(3) $((f,\Delta)^c)^\delta = (\delta(f,\Delta))^c$.

(4) $\delta(E)^c = ((f,\Delta)^\delta)^c$.

Proof. (1) $((f_1,\Delta)\oplus(f_2,\Delta))^c = \left(\sigma, \sqrt[a]{\varphi_1^a+\varphi_2^a-\varphi_1^a\varphi_2^a}, \varpi_1\varpi_2\right)^c$
$= \left(\sigma, \sqrt[a]{\varpi_1^b\varpi_2^b}, \sqrt[b]{\varphi_1^a+\varphi_2^a-\varphi_1^a\varphi_2^a}\right)$
$= \left(\sigma, \sqrt[a]{\varpi_1^b}\sqrt[a]{\varpi_2^b}, \sqrt[b]{\varphi_1^a+\varphi_2^a-\varphi_1^a\varphi_2^a}\right)$

$= (\sigma, \sqrt[a]{\varpi_1^b}, \varphi_1^a) \otimes (\sqrt[b]{\varpi_2^b}, \varphi_2^a)$
$= (f_1, \Delta)^c \otimes (f_2, \Delta)^c.$

(2) $((f_1, \Delta) \otimes (f_2, \Delta))^c = \left(\sigma, \varphi_1\varphi_2, \sqrt[b]{\varpi_1^b + \varpi_2^b - \varpi_1^b\varpi_2^b}\right)^c$
$= \left(\sigma, \sqrt[a]{\varpi_1^b + \varpi_2^b - \varpi_1^b\varpi_2^b}, \sqrt[b]{\varphi_1^a\varphi_2^a}\right)$
$= \left(\sigma, \sqrt[a]{\varpi_1^b + \varpi_2^b - \varpi_1^b\varpi_2^b}, \sqrt[b]{\varphi_1^a}\sqrt[b]{\varphi_2^a}\right)$
$= (\sigma, \sqrt[a]{\varpi_1^b}, \sqrt[b]{\varphi_1^a}) \oplus (\sigma, \sqrt[a]{\varpi_{(f,\Delta)_2}^b}, \sqrt[b]{\varphi_{(f,\Delta)_2}^a})$
$= (f_1, \Delta)^c \oplus (f_2, \Delta)^c.$

(3) $((f, \Delta)^c)^\delta = (\sigma, \sqrt[a]{\varpi^b}, \sqrt[b]{\varphi^a})^\delta$
$= \left(\sigma, (\sqrt[a]{\varpi^b})^\delta, \sqrt[b]{1 - (1 - \varphi^a)^\delta}\right)$
$= \left(\sigma, \sqrt[a]{1 - (1 - \varphi^a)^\delta}, \varpi^\delta\right)^c$
$= (\delta(f, \Delta))^c.$

(4) $\delta(f, \Delta)^c = \delta(\sigma, \sqrt[a]{\varpi^b}, \sqrt[b]{\varphi^a}) = \left(\sigma, \sqrt[a]{1 - (1 - \varpi^b)^\delta}, (\sqrt[b]{\varphi^a})^\delta\right) = \left(\sigma, \varphi^\delta, \sqrt[b]{1 - (1 - \varpi^b)^\delta}\right)^c$
$= ((f, \Delta)^\delta)^c.$

□

Theorem 4.8. *Let* $(f_1, \Delta) = (\sigma, \varphi_1, \varpi_1)$, $(f_2, \Delta) = (\sigma, \varphi_2, \varpi_2)$ *and* $(f_3, \Delta) = (\sigma, \varphi_3, \varpi_3)$ *be* (a, b)*-FSSs on* U*. Then*

(1) $((f_1, \Delta) \sqcap (f_2, \Delta)) \oplus (f_3, \Delta) = ((f_1, \Delta) \oplus (f_3, \Delta)) \sqcap ((f_2, \Delta) \oplus (f_3, \Delta)).$

(2) $((f_1, \Delta) \sqcup (f_2, \Delta)) \oplus (f_3, \Delta) = ((f_1, \Delta) \oplus (f_3, \Delta)) \sqcup ((f_2, \Delta) \oplus (f_3, \Delta)).$

(3) $((f_1, \Delta) \sqcap (f_2, \Delta)) \otimes (f_3, \Delta) = ((f_1, \Delta) \otimes (f_3, \Delta)) \sqcap ((f_2, \Delta) \otimes (f_3, \Delta)).$

(4) $((f_1, \Delta) \sqcup (f_2, \Delta)) \otimes (f_3, \Delta) = ((f_1, \Delta) \otimes (f_3, \Delta)) \sqcup ((f_2, \Delta) \otimes (f_3, \Delta)).$

Proof. (1) $((f_1, \Delta) \sqcap (f_2, \Delta)) \oplus (f_3, \Delta) = (\sigma, min\{\varphi_1, \varphi_2\}, max\{\varpi_1, \varpi_2\}) \oplus (\sigma, \varphi_3, \varpi_3)$
$= \left(\sigma, \sqrt[a]{min\{\varphi_1^a, \varphi_2^a\} + \varphi_3^a - \varphi_3^a min\{\varphi_1^a, \varphi_2^a\}}, max\{\varpi_1, \varpi_2\}\varpi_3\right)$
$= \left(\sigma, \sqrt[a]{(1 - \varphi_3^a)min\{\varphi_1^a, \varphi_2^a\} + \varphi_3^a}, max\{\varpi_1\varpi_3, \varpi_2\varpi_3\}\right).$
And $((f_1, \Delta) \oplus (f_3, \Delta)) \sqcap ((f_2, \Delta) \oplus (f_3, \Delta))$
$= \left(\sigma, \sqrt[a]{\varphi_1^a + \varphi_3^a - \varphi_1^a\varphi_3^a}, \varpi_1\varpi_3\right) \sqcap \left(\sigma, \sqrt{\varphi_2^a + \varphi_3^a - \varphi_2^a\varphi_3^a}, \varpi_2\varpi_3\right)$
$= \left(\sigma, min\{\sqrt[a]{\varphi_1^a + \varphi_3^a - \varphi_1^a\varphi_3^a}, \sqrt{\varphi_2^a + \varphi_3^a - \varphi_2^a\varphi_3^a}\}, max\{\varpi_1\varpi_3, \varpi_2\varpi_3\}\right)$
$= \left(\sigma, min\{\sqrt[a]{(1 - \varphi_3^a)\varphi_1^a + \varphi_3^a}, \sqrt[a]{(1 - \varphi_3^a)\varphi_2^a + \varphi_3^a}\}, max\{\varpi_1\varpi_3, \varpi_2\varpi_3\}\right)$
$= \left(\sigma, \sqrt[a]{(1 - \varphi_3^a)min\{\varphi_1^a, \varphi_2^a\} + \varphi_3^a}, max\{\varpi_1\varpi_3, \varpi_2\varpi_3\}\right).$
Hence, $((f_1, \Delta) \sqcap (f_2, \Delta)) \oplus (f_3, \Delta) = ((f_1, \Delta) \oplus (f_3, \Delta)) \sqcap ((f_2, \Delta) \oplus (f_3, \Delta)).$

(2) Similar to the proof of claim 1.

(3) $((f_1, \Delta) \sqcap (f_2, \Delta)) \otimes (f_3, \Delta) = (\sigma, min\{\varphi_1, \varphi_2\}, max\{\varpi_1, \varpi_2\}) \otimes (f_3, \Delta)$
$= \left(\sigma, min\{\varphi_1, \varphi_2\}\varphi_3, \sqrt[b]{max\{\varpi_1^b, \varpi_2^b\} + \varpi_3^b - \varpi_3^b max\{\varpi_1^b, \varpi_2^b\}}\right)$
$= \left(\sigma, min\{\varphi_1\varphi_3, \varphi_2\varphi_3\}, \sqrt[b]{(1 - \varpi_3^b)max\{\varpi_1^b, \varpi_2^b\} + \varpi_3^b}\right).$

And $((f_1,\Delta)\otimes(f_3,\Delta))\sqcap((f_2,\Delta)\otimes(f_3,\Delta)) = \left(\sigma,\varphi_1\varphi_3,\sqrt[b]{\varpi_1^b+\varpi_3^b-\varpi_1^b\varpi_3^b}\right)$
$\sqcap\left(\sigma,\varphi_2\varphi_3,\sqrt[b]{\varpi_2^b+\varpi_3^b-\varpi_2^b\varpi_3^b}\right)$
$=\left(\sigma,\varphi_1\varphi_3,\sqrt[b]{(1-\varpi_3^b)\varpi_1^b+\varpi_3^b}\right)\sqcap\left(\sigma,\varphi_2\varphi_3,\sqrt[b]{(1-\varpi_3^b)\varpi_2^b+\varpi_3^b}\right)$
$=\left(\sigma,min\{\varphi_1\varphi_3,\varphi_2\varphi_3\},max\left\{\sqrt[b]{(1-\varpi_3^b)\varpi_1^b+\varpi_3^b},\sqrt[b]{(1-\varpi_3^b)\varpi_2^b+\varpi_3^b}\right\}\right)$
$=\left(\sigma,min\{\varphi_1\varphi_3,\varphi_2\varphi_3\},\sqrt[b]{(1-\varpi_3^b)max\{\varpi_1^b,\varpi_2^b\}+\varpi_3^b}\right).$
Hence, $((f_1,\Delta)\sqcap(f_2,\Delta))\otimes(f_3,\Delta)=((f_1,\Delta)\otimes(f_3,\Delta))\sqcap((f_2,\Delta)\otimes(f_3,\Delta))$.

(4) Similar to the proof of claim 3.

□

Theorem 4.9. *Let $(f_1,\Delta)=(\sigma,\varphi_1,\varpi_1)$, $(f_2,\Delta)=(\sigma,\varphi_2,\varpi_2)$ and $(f_3,\Delta)=(\sigma,\varphi_3,\varpi_3)$ be (a,b)-FSSs on U. Then*

(1) $(f_1,\Delta)\oplus(f_2,\Delta)\oplus(f_3,\Delta)=(f_1,\Delta)\oplus(f_3,\Delta)\oplus(f_2,\Delta)$.

(2) $(f_1,\Delta)\otimes(f_2,\Delta)\otimes(f_3,\Delta)=(f_1,\Delta)\otimes(f_3,\Delta)\otimes(f_2,\Delta)$.

Proof. (1) $(f_1,\Delta)\oplus(f_2,\Delta)\oplus(f_3,\Delta)$
$=(\sigma,\varphi_1,\varpi_1)\oplus(\varphi_2,\varpi_2)\oplus(\sigma,\varphi_3,\varpi_3)$
$=\left(\sigma,\sqrt[a]{\varphi_1^a+\varphi_2^a-\varphi_1^a\varphi_2^a},\varpi_1\varpi_2\right)\oplus(\sigma,\varphi_3,\varpi_3)$
$=\left(\sigma,\sqrt[a]{\varphi_1^a+\varphi_2^a-\varphi_1^a\varphi_2^a+\varphi_3^a-\varphi_3^a(\varphi_1^a+\varphi_2^a-\varphi_1^a\varphi_2^a)},\varpi_1\varpi_2\varpi_3\right)$
$=\left(\sigma,\sqrt[a]{\varphi_1^a+\varphi_2^a+\varphi_3^a-\varphi_1^a\varphi_2^a-\varphi_1^a\varphi_3^a-\varphi_2^a\varphi_3^a+\varphi_1^a\varphi_2^a\varphi_3^a},\varpi_1\varpi_2\varpi_3\right)$
$=\left(\sigma,\sqrt[a]{\varphi_1^a+\varphi_3^a-\varphi_1^a\varphi_3^a+\varphi_2^a-\varphi_2^a(\varphi_1^a+\varphi_3^a-\varphi_1^a\varphi_3^a)},\varpi_1\varpi_2\varpi_3\right)$
$=\left(\sigma,\sqrt[a]{\varphi_1^a+\varphi_3^a-\varphi_1^a\varphi_3^a},\varpi_1\varpi_3\right)\oplus(\sigma,\varphi_2,\varpi_2)$
$=(f_1,\Delta)\oplus(f_3,\Delta)\oplus(f_2,\Delta)$.

(2) Similar to the proof of claim 1.

□

4.2. Aggregation of (a,b)-fuzzy soft sets

Definition 4.10. *Let $(f_j,\Delta)=(\sigma,\varphi_j,\varpi_j)$ $(j=1,2,...,s)$ be a family of (a,b)-FNs on U, and $w=(\nu_1,\nu_2,...,\nu_s)^T$ be a weight vector of (f_j,Δ) with $\nu_j>0$ and $\sum_{j=1}^s\nu_j=1$. Then*

(1) an (a,b)-fuzzy weighted average ((a,b)-FWA) operator is given by

$$(a,b)\text{-}FWA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))=(\sigma,\textstyle\sum_{j=1}^s\nu_j\varphi_j,\ \sum_{j=1}^s\nu_j\varpi_j).$$

(2) an (a,b)-fuzzy weighted geometric ((a,b)-FWG) operator is given by

$$(a,b)\text{-}FWG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))=(\sigma,\textstyle\prod_{j=1}^s\varphi_j^{\nu_j},\ \prod_{j=1}^s\varpi_j^{\nu_j}).$$

(3) an (a,b)-fuzzy weighted power average ((a,b)-FWPA) operator is given by

$$(a,b)\text{-}FWPA((f_1,\Delta),(f_2,\Delta),...,(f,\Delta)_m)=(\sigma,(\textstyle\sum_{j=1}^s\nu_j\varphi_j^a)^{\frac{1}{a}},\ (\sum_{j=1}^s\nu_j\varpi_j^b)^{\frac{1}{b}}).$$

(4) an (a,b)-fuzzy weighted power geometric ((a,b)-FWPG) operator is given by

$$(a,b)\text{-}FWPG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)) = (\sigma, (1-\prod_{j=1}^{s}(1-\varphi_j^a)^{\nu_j})^{\frac{1}{a}},\ (1-\prod_{j=1}^{s}(1-\varpi_j^b)^{\nu_j})^{\frac{1}{b}}).$$

Remark 4.11. *It should be noted that the values obtained from the operators presented in the above definition need not be an (a,b)-FSS, in general.*

Theorem 4.12. *Let $(f_j,\Delta) = (\sigma,\varphi_j,\varpi_j)(i = 1,2,...,s)$ be a family of (a,b)-FNs on U, $(f,\Delta) = (\sigma,\varphi,\varpi)$ be an (a,b)-FN and $w = (\nu_1,\nu_2,...,\nu_s)^T$ be a weight vector of (f_j,Δ) with $\sum_{j=1}^{s}\nu_j = 1$. Then*

(1) $(a,b)\text{-}FWA((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)\text{-}FWA((f_1,\Delta)\otimes(f,\Delta),(f_2,\Delta)\otimes(f,\Delta),...,(f_s,\Delta)\otimes(f,\Delta))$.

(2) $(a,b)\text{-}FWG((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)\text{-}FWG((f_1,\Delta)\otimes(f,\Delta),(f_2,\Delta)\otimes(f,\Delta),...,(f_s,\Delta)\otimes(f,\Delta))$.

(3) $(a,b)\text{-}FWPA((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)\text{-}FWPA((f_1,\Delta)\otimes(f,\Delta),(f_2,\Delta)\otimes(f,\Delta),...,(f_s,\Delta)\otimes(f,\Delta))$.

(4) $(a,b)\text{-}FWPG((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)\text{-}FWPG((f_1,\Delta)\otimes(f,\Delta),(f_2,\Delta)\otimes(f,\Delta),...,(f_s,\Delta)\otimes(f,\Delta))$.

Proof. We shall give the proofs of 1 and 4. Following similar technique, one can prove the other affirmations.

(1) For any $(f_j,\Delta) = (\sigma,\varphi_j,\varpi_j)$ $(j = 1,2,...,s)$ and $(f,\Delta) = (\sigma,\varphi,\varpi)$, we obtain for each σ

$\sqrt[a]{\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a} \geq \sqrt[a]{2\varphi_j^a\varphi^a-\varphi_j^a\varphi^a} = \varphi_j\varphi$, and

$$\sqrt[b]{\varpi_j^b+\varpi^b-\varpi_j^b\varpi^b} \geq \sqrt[b]{2\varpi_j^b\varpi^b-\varpi_j^b\varpi^b} = \varpi_j\varpi.$$

That is,

$$\sum_{j=1}^{s}\nu_j\sqrt[a]{\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a} \geq \sum_{j=1}^{s}\nu_j\varphi_j\varphi \tag{4.1}$$

and

$$\sum_{j=1}^{s}\nu_j\sqrt[b]{\varpi_j^b+\varpi^b-\varpi_j^b\varpi^b} \geq \sum_{j=1}^{s}\nu_j\varpi_j\varpi. \tag{4.2}$$

According to item 1 of Definition 4.10 and items 1 and 2 of Definition 4.1, we have

$(a,b)\text{-}FWA((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) = (\sigma, \sum_{j=1}^{s}\nu_j\sqrt[a]{\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a}, \sum_{j=1}^{s}\nu_j\varpi_j\varpi)$

and

$(a,b)\text{-}FWA((f_1,\Delta)\otimes(f,\Delta),(f_2,\Delta)\otimes(f,\Delta),...,(f_s,\Delta)\otimes(f,\Delta)) = (\sigma, \sum_{j=1}^{s}\nu_j\varphi_j\varphi, \sum_{j=1}^{s}\nu_j\sqrt[b]{\varpi_j^b+\varpi^b-\varpi_j^b\varpi^b})$.

Hence, from (4.1) and (4.2), we complete the proof.

(4) For any $(f_j,\Delta)=(\sigma,\varphi_j,\varpi_j)(j=1,2,...,s)$ and $(f,\Delta)=(\sigma,\varphi,\varpi)$, we obtain

$\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a\geq 2\varphi_j^a\varphi^a-\varphi_j^a\varphi^a=\varphi_j^a\varphi^a$

$\Rightarrow 1-(\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a)\leq 1-\varphi_j^a\varphi^a$

$\Rightarrow (1-(\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a))^{\nu_j}\leq (1-\varphi_j^a\varphi^a)^{\nu_j}$

$\Rightarrow \prod_{j=1}^{s}(1-(\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a))^{\nu_j}\leq \prod_{j=1}^{s}(1-\varphi_j^a\varphi^a)^{\nu_j}$

$\Rightarrow 1-\prod_{j=1}^{s}(1-(\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a))^{\nu_j}\geq 1-\prod_{j=1}^{s}(1-\varphi_j^a\varphi^a)^{\nu_j}.$

Similarly,

$\Rightarrow 1-\prod_{j=1}^{s}(1-(\varpi_j^b+\varpi^b-\varpi_j^b\varpi^b))^{\nu_j}\geq 1-\prod_{j=1}^{s}(1-\varpi_j^b\varpi^b)^{\nu_j}.$

According to items 1 and 2 of Definition 4.1, we have

$(a,b)\text{-}FWPG((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta))=$
$(\sigma,(1-\prod_{j=1}^{s}(1-(\varphi_j^a+\varphi^a-\varphi_j^a\varphi^a))^{\nu_j})^{\frac{1}{a}},\ (1-\prod_{j=1}^{s}(1-\varpi_j^b\varpi^b)^{\nu_j})^{\frac{1}{b}})$, and

$(a,b)\text{-}FWPG((f_1,\Delta)\otimes(f,\Delta),(f_2,\Delta)\otimes(f,\Delta),...,(f_s,\Delta)\otimes(f,\Delta))=$
$(\sigma,(1-\prod_{j=1}^{s}(1-\varphi_j^a\varphi^a)^{\nu_j})^{\frac{1}{a}},\ (1-\prod_{j=1}^{s}(1-(\varpi_j^b+\varpi^b-\varpi_j^b\varpi^b))^{\nu_j})^{\frac{1}{b}})$.

Hence, $(a,b)\text{-}FWPG((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta))\geq(a,b)\text{-}FWPG((f_1,\Delta)\otimes(f,\Delta),(f_2,\Delta)\otimes(f,\Delta),...,(f_s,\Delta)\otimes(f,\Delta))$, as required. □

Theorem 4.13. *Let $(f_j,\Delta)=(\sigma,\varphi_j,\varpi_j)$ and $(h_j,\Delta)=(\sigma,\alpha_j,\beta_j)$ (with $j=1,2,...,s$) be two families of (a,b)-FSSs on U, and $w=(\nu_1,\nu_2,...,\nu_s)^T$ be a weight vector of them with $\sum_{j=1}^{s}\nu_j=1$. Then*

(1) $(a,b)\text{-}FWA((f_1,\Delta)\oplus(h_1,\Delta),(f_2,\Delta)\oplus(h_2,\Delta),...,(f_s,\Delta)\oplus(h_s,\Delta))\geq(a,b)\text{-}FWA((f_1,\Delta)\otimes(h_1,\Delta),(f_2,\Delta)\otimes(h_2,\Delta),...,(f_s,\Delta)\otimes(h_s,\Delta))$.

(2) $(a,b)\text{-}FWG((f_1,\Delta)\oplus(h_1,\Delta),(f_2,\Delta)\oplus(h_2,\Delta),...,(f_s,\Delta)\oplus(h_s,\Delta))\geq(a,b)\text{-}FWG((f_1,\Delta)\otimes(h_1,\Delta),(f_2,\Delta)\otimes(h_2,\Delta),...,(f_s,\Delta)\otimes(h_s,\Delta))$.

(3) $(a,b)\text{-}FWPA((f_1,\Delta)\oplus(h_1,\Delta),(f_2,\Delta)\oplus(h_2,\Delta),...,(f_s,\Delta)\oplus(h_s,\Delta))\geq(a,b)\text{-}FWPA((f_1,\Delta)\otimes(h_1,\Delta),(f_2,\Delta)\otimes(h_2,\Delta),...,(f_s,\Delta)\otimes(h_s,\Delta))$.

(4) $(a,b)\text{-}FWPG((f_1,\Delta)\oplus(h_1,\Delta),(f_2,\Delta)\oplus(h_2,\Delta),...,(f_s,\Delta)\oplus(h_s,\Delta))\geq(a,b)\text{-}FWPG((f_1,\Delta)\otimes(h_1,\Delta),(f_2,\Delta)\otimes(h_2,\Delta),...,(f_s,\Delta)\otimes(h_s,\Delta))$.

Proof. We shall give the proof for 1. Following similar technique, one can prove the other affirmations.
(1) For any $(f_j,\Delta)=(\sigma,\varphi_j,\varpi_j)$ and $(h_j,\Delta)=(\alpha_j,\beta_j)$ $(j=1,2,...,s)$, we find for each σ

$$\sqrt[a]{\varphi_j^a+\alpha_j^a-\varphi_j^a\alpha_j^a}\geq\sqrt[a]{2\varphi_j^a\alpha_j^a-\varphi_j^a\alpha_j^a}=\varphi_j\alpha_j.$$

That is,

$$\sum_{j=1}^{s}\nu_j\sqrt[a]{\varphi_j^a+\alpha_j^a-\varphi_j^a\alpha_j^a}\geq\sum_{j=1}^{s}\nu_j\varphi_j\alpha_j.$$

Similarly,

$$\sum_{j=1}^{s} \nu_j \sqrt[b]{\varpi_j^b + \beta_j^b - \varpi_j^b\beta_j^b} \geq \sum_{j=1}^{s} \nu_j \varpi_j \beta_j.$$

By items 1 and 2 of Definition 4.1, we have

$(a,b) - FWA((f_1,\Delta) \oplus (h_1,\Delta), (f_2,\Delta) \oplus (h_2,\Delta), ..., (f_s,\Delta) \oplus (h_s,\Delta)) =$
$(\sigma, \sum_{j=1}^{s} \nu_j \sqrt[a]{\varphi_j^a + \alpha_j^a - \varphi_j^a\alpha_j^a}, \sum_{j=1}^{s} \nu_j \varpi_j \beta_j)$

and

$(a,b) - FWA((f_1,\Delta) \otimes (h_1,\Delta), (f_2,\Delta) \otimes (h_2,\Delta), ..., (f_s,\Delta) \otimes (h_s,\Delta)) =$
$(\sigma, \sum_{j=1}^{s} \nu_j \varphi_j \alpha_j, \sum_{j=1}^{s} \nu_j \sqrt[b]{\varpi_j^b + \beta_j^b - \varpi_j^b\beta_j^b})$.

Hence, (a,b)-$FWA((f_1,\Delta) \oplus (h_1,\Delta), (f_2,\Delta) \oplus (h_2,\Delta), ..., (f_s,\Delta) \oplus (h_s,\Delta)) \geq (a,b)$-$FWA((f_1,\Delta) \otimes (h_1,\Delta), (f_2,\Delta) \otimes (h_2,\Delta), ..., (f_s,\Delta) \otimes (h_s,\Delta))$, as required. □

Theorem 4.14. *Let $(f_j,\Delta) = (\sigma, \varphi_j, \varpi_j)(j = 1,2,...,s)$ be a family of (a,b)-FNs on U, and $w = (\nu_1, \nu_2, ..., \nu_s)^T$ be a weight vector of (f_j,Δ) with $\sum_{j=1}^{s} \nu_j = 1$ and $\delta \geq 1$. Then*

(1) (a,b)-$FWA(\delta(f_1,\Delta), \delta(f_2,\Delta), ..., \delta(f_s,\Delta)) \geq (a,b)$-$FWA((f_1,\Delta)^\delta, (f_2,\Delta)^\delta, ..., (f_s,\Delta)^\delta)$.

(2) (a,b)-$FWG(\delta(f_1,\Delta), \delta(f_2,\Delta), ..., \delta(f_s,\Delta)) \geq (a,b)$-$FWG((f_1,\Delta)^\delta, (f_2,\Delta)^\delta, ..., (f_s,\Delta)^\delta)$.

(3) (a,b)-$FWPA(\delta(f_1,\Delta), \delta(f_2,\Delta), ..., \delta(f_s,\Delta)) \geq (a,b)$-$FWPA((f_1,\Delta)^\delta, (f_2,\Delta)^\delta, ..., (f_s,\Delta)^\delta)$.

(4) (a,b)-$FWPG(\delta(f_1,\Delta), \delta(f_2,\Delta), ..., \delta(f_s,\Delta)) \geq (a,b)$-$FWPG((f_1,\Delta)^\delta, (f_2,\Delta)^\delta, ..., (f_s,\Delta)^\delta)$.

Proof. We shall give the proof for claim 1. Following a similar technique, one can prove the other statements.
(1) For any $(f_j,\Delta) = (\sigma, \varphi_j, \varpi_j)$ $(j = 1,2,...,s)$, we have

(a,b)-$FWA(\delta(f_1,\Delta), \delta(f_2,\Delta), ..., \delta(f_s,\Delta)) = (\sigma, \sum_{j=1}^{s} \nu_j \sqrt[a]{1-(1-\varphi_j^a)^\delta}, \sum_{j=1}^{s} \nu_j \varpi_j^\delta)$, and

(a,b)-$FWA((f_1,\Delta)^\delta, (f_2,\Delta)^\delta, ..., (f_s,\Delta)^\delta) = (\sigma, \sum_{j=1}^{s} \nu_j \varphi_j^\delta, \sum_{j=1}^{s} \nu_j \sqrt[b]{1-(1-\varpi_j^b)^\delta})$.

Let $g(\varphi_j) = 1-(1-\varphi_j^a)^\delta - (\varphi_j^a)^\delta$. We demonstrate that $g(\varphi_j) \geq 0$. It follows from the Newton generalized binomial theorem that

$$(1-\varphi_j^a)^\delta + (\varphi_j^a)^\delta \leq (1-\varphi_j^a+\varphi_j^a)^\delta = 1.$$

This means that $g(\varphi_j) \geq 0$. Now,

$$1-(1-\varphi_j^a)^\delta - (\varphi_j^a)^\delta \geq 0$$
$$\Rightarrow 1-(1-\varphi_j^a)^\delta \geq (\varphi_j^a)^\delta$$
$$\Rightarrow \sqrt[a]{1-(1-\varphi_j^a)^\delta} \geq \varphi_j^\delta$$
$$\Rightarrow \sum_{j=1}^{s} \nu_j \sqrt[a]{1-(1-\varphi_j^a)^\delta} \geq \sum_{j=1}^{s} \nu_j \varphi_j^\delta.$$

Similarly,

$$\sum_{j=1}^{s} \nu_j \sqrt[b]{1-(1-\varpi_j^b)^\delta} \geq \sum_{j=1}^{s} \nu_j \varpi_j^\delta.$$

Hence, (a,b)-$FWA(\delta(f_1,\Delta), \delta(f_2,\Delta), ..., \delta(f_s,\Delta)) \geq (a,b)$-$FWA((f_1,\Delta)^\delta, (f_2,\Delta)^\delta, ..., (f_s,\Delta)^\delta)$, as required. □

Theorem 4.15. *Let $(f_j,\Delta) = (\sigma, \varphi_j, \varpi_j)$ $(j = 1,2,...,s)$ be a family of (a,b)-FNs on U, $(f,\Delta) = (\sigma,\varphi,\varpi)$ be an (a,b)-FN on U and $w = (\nu_1,\nu_2,...,\nu_s)^T$ be a weight vector of (f_j,Δ) with $\sum_{j=1}^{s} \nu_j = 1$ and $\delta \geq 1$. Then*

(1) (a,b)-$FWA(\delta(f_1,\Delta) \oplus (f,\Delta), \delta(f_2,\Delta) \oplus (f,\Delta), ..., \delta(f_s,\Delta) \oplus (f,\Delta)) \geq (a,b)$-$FWA((f_1,\Delta)^\delta \otimes (f,\Delta), (f_2,\Delta)^\delta \otimes (f,\Delta), ..., (f_s,\Delta)^\delta \otimes (f,\Delta))$.

(2) (a,b)-$FWG(\delta(f_1,\Delta) \oplus (f,\Delta), \delta(f_2,\Delta) \oplus (f,\Delta), ..., \delta(f_s,\Delta) \oplus (f,\Delta)) \geq (a,b)$-$FWG((f_1,\Delta)^\delta \otimes (f,\Delta), (f_2,\Delta)^\delta \otimes (f,\Delta), ..., (f_s,\Delta)^\delta \otimes (f,\Delta))$.

(3) (a,b)-$FWPA(\delta(f_1,\Delta) \oplus (f,\Delta), \delta(f_2,\Delta) \oplus (f,\Delta), ..., \delta(f_s,\Delta) \oplus (f,\Delta)) \geq (a,b)$-$FWPA((f_1,\Delta)^\delta \otimes (f,\Delta), (f_2,\Delta)^\delta \otimes (f,\Delta), ..., (f_s,\Delta)^\delta \otimes (f,\Delta))$.

(4) (a,b)-$FWPG(\delta(f_1,\Delta) \oplus (f,\Delta), \delta(f_2,\Delta) \oplus (f,\Delta), ..., \delta(f_s,\Delta) \oplus (f,\Delta)) \geq (a,b)$-$FWPG((f_1,\Delta)^\delta \otimes (f,\Delta), (f_2,\Delta)^\delta \otimes (f,\Delta), ..., (f_s,\Delta)^\delta \otimes (f,\Delta))$.

Proof. We shall give the proof for claim 1. Following a similar technique, one can prove the other statements.
(1) For any $(f_j,\Delta) = (\sigma,\varphi_j,\varpi_j)$ $(j = 1,2,...,s)$ and $(f,\Delta) = (\sigma,\varphi,\varpi)$, we have

$$(a,b)\text{-}FWA(\delta(f_1,\Delta) \oplus (f,\Delta), \delta(f_2,\Delta) \oplus (f,\Delta), ..., \delta(f_s,\Delta) \oplus (f,\Delta)) = (\sigma, \sum_{j=1}^{s} \nu_j \sqrt[a]{1-(1-\varphi_j^a)^\delta(1-\varphi^a)}, \sum_{j=1}^{s} \nu_j \varpi_j^\delta \varpi),$$

and

$$(a,b)\text{-}FWA((f_1,\Delta)^\delta \otimes (f,\Delta), (f_2,\Delta)^\delta \otimes (f,\Delta), ..., (f_s,\Delta)^\delta \otimes (f,\Delta)) = (\sigma, \sum_{j=1}^{s} \nu_j \varphi_j^\delta \varphi, \sum_{j=1}^{s} \nu_j \sqrt[b]{1-(1-\varpi_j^b)^\delta(1-\varpi^b)}).$$

Let $g(\varphi_j) = 1-(1-\varphi_j^a)^\delta(1-\varphi^a)-(\varphi_j^a)^\delta\varphi^a$. We demonstrate that $g(\varphi_j) \geq 0$. To do this, let $h(\varphi_j) = (1-\varphi_j^a)^\delta + (\varphi_j^a)^\delta$. Then

$$h'(\varphi_j) = -2\delta\varphi_j(1-\varphi_j^a)^{\delta-1} + 2\delta\varphi_j(\varphi_j^a)^{\delta-1} = 2\delta\varphi_j((\varphi_j^a)^{\delta-1} - (1-\varphi_j^a)^{\delta-1}).$$

Now, if $\varphi_j > \frac{1}{\sqrt[a]{2}}$, then $h(\varphi_j)$ is monotonic increasing and if $\varphi_j < \frac{1}{\sqrt[a]{2}}$, then $h(\varphi_j)$ is monotonic decreasing. Therefore, $h(\varphi_j) \leq h(\varphi_j)_{max} = max\{h(0), h(1)\} = 1$. Note that $(1-\varphi_j^a)^\delta(1-\varphi^a)+(\varphi_j^a)^\delta\varphi^a \leq 1$. This automatically means that

$$g(\varphi_j) = 1-(1-\varphi_j^a)^\delta(1-\varphi^a)-(\varphi_j^a)^\delta\varphi^a \geq 0$$

$$\Rightarrow \sum_{j=1}^{s} \nu_j \sqrt[a]{1-(1-\varphi_j^a)^\delta(1-\varphi^a)} \geq \sum_{j=1}^{s} \nu_j \varphi_j^\delta \varphi.$$

Similarly,

$$\sum_{j=1}^{s} \nu_j \sqrt[b]{1-(1-\varpi_j^b)^\delta(1-\varpi^b)} \geq \sum_{j=1}^{s} \nu_j \varpi_j^\delta \varpi.$$

Hence, (a,b)-$FWA(\delta(f_1,\Delta)\oplus(f,\Delta),\delta(f_2,\Delta)\oplus(f,\Delta),...,\delta(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)$-$FWA((f_1,\Delta)^{\delta}\otimes(f,\Delta),(f_2,\Delta)^{\delta}\otimes(f,\Delta),...,(f_s,\Delta)^{\delta}\otimes(f,\Delta))$. □

According to Remark 4.11, we need to impose a further condition to prove the following three results; this condition is that the values obtained from the operators presented in Definition 4.10 is an (a,b)-FSS.

Theorem 4.16. *Let $(f_j,\Delta)=(\sigma,\varphi_j,\varpi_j)(j=1,2,...,s)$ be a family of (a,b)-FNs on U, $(f,\Delta)=(\sigma,\varphi,\varpi)$ be an (a,b)-FN on U and $w=(\nu_1,\nu_2,...,\nu_s)^T$ be a weight vector of (f_j,Δ) with $\sum_{j=1}^{s}\nu_j=1$. Then*

(1) (a,b)-$FWA((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)$-$FWA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(f,\Delta)$.

(2) (a,b)-$FWA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\oplus(f,\Delta) \geq (a,b)$-$FWA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(f,\Delta)$.

(3) (a,b)-$FWG((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)$-$FWG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(f,\Delta)$.

(4) (a,b)-$FWG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\oplus(f,\Delta) \geq (a,b)$-$FWG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(f,\Delta)$.

(5) (a,b)-$FWPA((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)$-$FWPA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(f,\Delta)$.

(6) (a,b)-$FWPA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\oplus(f,\Delta) \geq (a,b)$-$FWPA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(f,\Delta)$.

(7) (a,b)-$FWPG((f_1,\Delta)\oplus(f,\Delta),(f_2,\Delta)\oplus(f,\Delta),...,(f_s,\Delta)\oplus(f,\Delta)) \geq (a,b)$-$FWPG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(f,\Delta)$.

(8) (a,b)-$FWPG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\oplus(f,\Delta) \geq (a,b)$-$FWPG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(f,\Delta)$.

Proof. Similar to the proof of Theorem 4.12. □

Theorem 4.17. *Let $(f_j,\Delta)=(\sigma,\varphi_j,\varpi_j)$ and $(h_j,\Delta)=(\sigma,\alpha_j,\beta_j)(j=1,2,...,s)$ be two families of (a,b)-FSSs on U, and $w=(\nu_1,\nu_2,...,\nu_s)^T$ be a weight vector of them with $\sum_{j=1}^{s}\nu_j=1$. Then*

(1) (a,b)-$FWA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\oplus(a,b)$-$FWA((h_1,\Delta),(h_2,\Delta),...,(h_s,\Delta)) \geq (a,b)$-$FWA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(a,b)$-$FWA((h_1,\Delta),(h_2,\Delta),...,(h_s,\Delta))$.

(2) (a,b)-$FWG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\oplus(a,b)$-$FWG((h_1,\Delta),(h_2,\Delta),...,(h_s,\Delta)) \geq (a,b)$-$FWG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(a,b)$-$FWG((h_1,\Delta),(h_2,\Delta),...,(h_s,\Delta))$.

(3) (a,b)-$FWPA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\oplus(a,b)$-$FWPA((h_1,\Delta),(h_2,\Delta),...,(h_s,\Delta)) \geq (a,b)$-$FWPA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(a,b)$-$FWPA((h_1,\Delta),(h_2,\Delta),...,(h_s,\Delta))$.

(4) (a,b)-$FWPG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\oplus(a,b)$-$FWPG((h_1,\Delta),(h_2,\Delta),...,(h_s,\Delta)) \geq (a,b)$-$FWPG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta))\otimes(a,b)$-$FWPG((h_1,\Delta),(h_2,\Delta),...,(h_s,\Delta))$.

Proof. Similar to the proof of Theorem 4.13. □

Theorem 4.18. *Let $(f_j,\Delta)=(\sigma,\varphi_j,\varpi_j)(j=1,2,...,s)$ be a family of (a,b)-FNs on U, and $w=(\nu_1,\nu_2,...,\nu_s)^T$ be a weight vector of (f_j,Δ) with $\sum_{j=1}^{s}\nu_j=1$ and $\delta\geq 1$. Then*

(1) $\delta(a,b)$-$FWA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)) \geq ((a,b)$-$FWA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)))^{\delta}$.

(2) $\delta(a,b)$-$FWG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)) \geq ((a,b)$-$FWG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)))^{\delta}$.

(3) $\delta(a,b)$-$FWPA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)) \geq ((a,b)$-$FWPA((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)))^{\delta}$.

(4) $\delta(a,b)$-$FWPG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)) \geq ((a,b)$-$FWPG((f_1,\Delta),(f_2,\Delta),...,(f_s,\Delta)))^{\delta}$.

Proof. Similar to the proof of Theorem 4.14. □

5. Application of (a,b)-FSSs to MCDM problems

In this section, we integrate the tools that have been defined before in order to produce a multi-criteria decision making (or MCDM) methodology for data in the form of (a,b)-FSSs. Then we provide an illustrative example that clarifies the application of this strategy of solution.

5.1. MCDM problems with respect to the environment of (a,b)-FSSs

MCDM concerns the techniques or strategies followed by a decision maker in order to get the best or optimal alternative(s) among a set of feasible options, in such way that its (their) performance(s) with respect to multiple criteria is (are) jointly superior. To illustrate that situation, assume that a set $U = \{\hat{x}_i : i = 1,2,...,n\}$ of n different alternatives have been evaluated by a decision maker under a set of m different criteria $\Delta = \{\sigma_j : j = 1,2,...,m\}$. Let us consider a situation where the decision maker estimates his/her preferences in terms of (a,b)-FNs: $\theta_{ij} = \left\langle \varphi_{ij}, \varpi_{ij} \right\rangle_{i\times j}$, where $0 \leq \varphi_{ij}^{a} + \varpi_{ij}^{b} \leq 1$ and $\varphi_{ij}, \varpi_{ij} \in [0,1]$ for all $i = 1,2,...,n$ and $j = 1,2,...,m$ such that φ_{ij} and ϖ_{ij} respectively represent the degree that the alternative b_i fulfills and does not fulfill the attribute σ_j provided by the decision maker.

We are ready to present the steps used in the proposed methodology for MCDM with the aforesaid information:

Step 1: Describe a MCDM problem under study using (a,b)-FSSs environment.

Step 2: Convert the (a,b)-FSSs environment into the normalized (a,b)-FSSs environment.

Step 3: Produce an (a,b)-FSS for each alternative $\hat{x} \in U$.

Step 4: Assessment of the alternatives using score and accuracy functions for each (a,b)-FSS.

Step 5: Determine the optimal ranking order of the alternatives with respect to the values of score and accuracy functions.

Additionally, we provide Algorithm 1 and Figure 2 in order to show how the optimal alternative(s) is (or are) selected with this methodology.

5.2. Illustrative examples

In this part, a synthetic example will be used to illustrate the application of the methodology described in the previous section.

Example 5.1. *Assume that we intend to establish an import company of laptops. So we ask the administration to evaluate four brands of laptops, namely,* $U = \{Acer, Lenovo, HP, Dell\}$*, in terms of the criteria* $\Delta = \{\sigma_i : i = 1,2,3,4,5\}$*, where:*

- σ_1 *is the processor and dedicated graphics,*
- σ_2 *is RAM and storage capacity,*
- σ_3 *is battery capacity & daily usage*
- σ_4 *is build quality, warranty & support, and*
- σ_5 *is the price.*

After examination and investigation, administration suggested a weight vector corresponding to every criteria as follows $\omega = (0.1, 0.3, 0.2, 0.1, 0.3)^T$. *The performance of these brands is evaluated under a type of an (a,b)-FSSs environment; say, (1,2)-FSSs. Consider the evaluation as displayed in Table 1, where every ordered pair* (φ, ϖ) *represents the degrees of membership (which means to what extent this brand fulfills the corresponding criteria) and non-membership (which means to what extent this brand dissatisfies the corresponding criteria), where* $0 \leq \varphi + \varpi^2 \leq 1$ *and* φ, ϖ *lie in* $[0, 1]$.

Suppose that the way for accessing the optimal brand with appreciation to every criterion is furnished according to the different types of (1,2)-FSS operators introduced in Definition 4.10. Now, we summarize the data given in Table 1 by four (1,2)-FSSs:

$(f_{Acer}, \Delta) = \{(\sigma_1, 0.3, 0.7), (\sigma_2, 0.6, 0.45), (\sigma_3, 0.7, 0.3), (\sigma_4, 0.8, 0.3), (\sigma_5, 0.5, 0.7)\}$;
$(f_{Lenovo}, \Delta) = \{(\sigma_1, 0.5, 0.7), (\sigma_2, 0.7, 0.5), (\sigma_3, 0.4, 0.5), (\sigma_4, 0.8, 0.4), (\sigma_5, 0.9, 0.2)\}$;
$(f_{HP}, \Delta) = \{(\sigma_1, 0.8, 0.3), (\sigma_2, 0.7, 0.6), (\sigma_3, 0.5, 0.6), (\sigma_4, 0.6, 0.6), (\sigma_5, 0.75, 0.4)\}$;
$(f_{Dell}, \Delta) = \{(\sigma_1, 0.7, 0.1), (\sigma_2, 0.9, 0.1), (\sigma_3, 0.85, 0.3), (\sigma_4, 0.9, 0.2), (\sigma_5, 0.5, 0.3)\}$.

To rank these brands we compute their score functions (as given in Table 2), but we deal herein with weighted parameters, so we update the formula given in Definition 3.13 to be as follows.

$$score(f, \Delta) = \sum_{\sigma \in \Delta} w_\sigma(\varphi - \varpi^2).$$

Input : The set of alternatives U and the set of criteria Δ.
Output: The most desirable alternative(s).

```
Describe a MCDM problem under study using (a,b)-FSSs environment for the alternative set
  x̂_i (i = 1, 2, ..., n) with a set of parameters Δ;
Convert (a,b)-FSSs environment into the normalized (a,b)-FSSs environment;
foreach i ≤ n do
    Compose an (a,b)-FSS (f_{x̂_i}, Δ) for each alternative x̂_i ∈ U;
    Compute score function for each alternative x̂_i ∈ U.
end
Let D = {x̂_i : score(f_{x̂_i}, Δ) = max{score(f_{x̂_i}, Δ) : i = 1, 2, ..., n}};
if D is a singleton set, say, x̂_k then
    return x̂_k is the desirable (optimal) alternative.
else
    Compute accuracy function for each alternative x̂_i ∈ D;
    Let E = {x̂_i : acc(f_{x̂_i}, Δ) = max{acc(f_{x̂_i}, Δ) : x̂_i ∈ D}};
    return each x̂_i ∈ E represents a desirable (an optimal) alternative;
end
```

Algorithm 1: The algorithm of selection of optimal alternative(s).

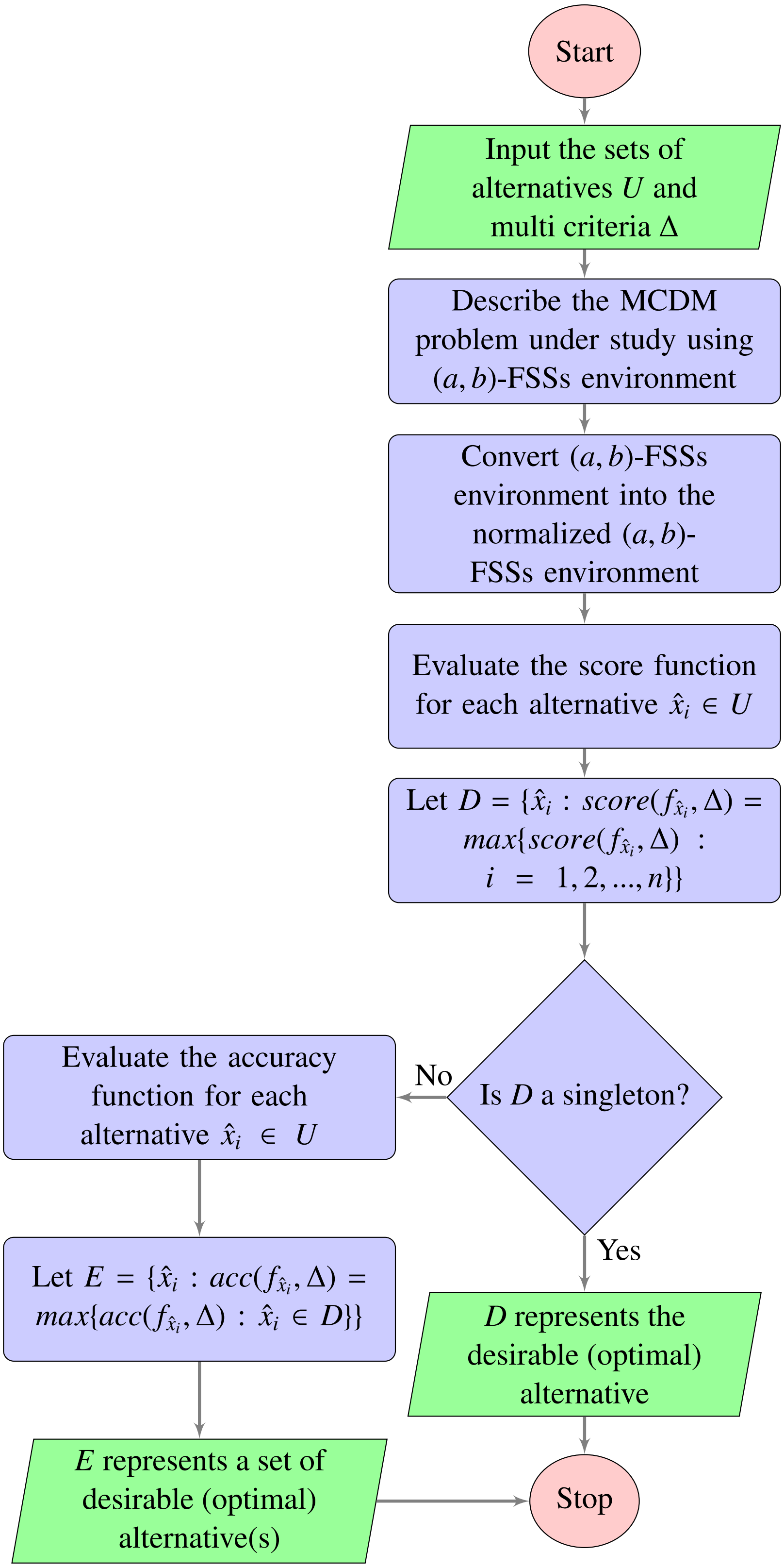

Figure 2. Flow chart explaining the selection of the optimal alternative(s).

Table 1. (1,2)-Fuzzy numbers associated with our case study.

Brands	σ_1	σ_2	σ_3	σ_4	σ_5
Acer	(0.3, 0.7)	(0.6, 0.45)	(0.7, 0.3)	(0.8, 0.3)	(0.5, 0.7)
Lenovo	(0.5, 0.7)	(0.7, 0.5)	(0.4, 0.5)	(0.8, 0.4)	(0.9, 0.2)
HP	(0.8, 0.3)	(0.7, 0.6)	(0.5, 0.6)	(0.6, 0.6)	(0.75, 0.4)
Dell	(0.7, 0.1)	(0.9, 0.1)	(0.85, 0.3)	(0.9, 0.2)	(0.5, 0.3)

Table 2. Evaluations by the score function under the conditions $a = 1$ and $b = 2$.

	(f_{Acer}, Δ)	(f_{Lenovo}, Δ)	(f_{HP}, Δ)	(f_{Dell}, Δ)	Rank
$Score$	0.29625	0.488	0.402	0.697	Dell > Lenovo > HP > Acera

Remark 5.2. *Should some brands have had the same evaluation by the score function, then we would compute their respective accuracies to decide which one(s) is (or are) the most desirable brand(s).*

It can be noted from the above discussion that the selection of the optimal alternative heavily relies on the type of generalization of IFS sets, and on the values of the weight vector.

Remark 5.3. *By the given illustrative example, we remark the following points:*

(i) *The input data of this example cannot be handled by IFS-sets because the sum of membership and non-membership degrees for some parameters is greater than one. Notice that* $f_{Acer}(\sigma_4) = (0.8, 0.3) \notin IFS(U)$.

(ii) *The evaluation followed by the administration of this company gives different importances for the membership and non-membership degrees, which can be considered neither by IFS-sets not by its generalizations in the existing literature.*

6. Conclusions

Fuzzy sets and their generalizations have been a fertile ground for research, both as standalone models and as components of more sophisticated frameworks. Almost without exception, every new blend of properties has fostered additional investigations in order to establish comparisons, yield further generalizations, prove additional properties, or produce applications (possibly with the help of newly designed strategies of solution).

In this paper we have succeeded in producing a novel combination that at the same time, allows for very general expressions of memberships and non-membersips, and parameterized descriptions of the universe of the alternatives. Its fundamental theory has been laid out. It is the basis of a methodology that solves multi-criteria decision making problems whose formulation respects this structure. In addition to the general advantage that the proposed approach provides a more comprehensive instrument for the mathematical analysis of uncertainty, a particular advantage is that it permits to handle situations requiring evaluations with different importances for the membership and non-membership degrees. This feature is peculiar to our model.

In future works we aim at exploring the group decision-making problem associated with the new model. We have prepared the ground with a study of aggregation operators in this framework. In addition to this line of research, we intend to develop the model that arises when we use N-grading for the parameterization of the set of alternatives [20]. This combination should produce (a, b)-fuzzy N-soft sets as a generalization of fuzzy N-soft sets [2], or even more general models (by inspiration of e.g., [21]). Also, we will combine (a, b)-FSSs with rough sets to produce covering approximation spaces like those studied in [45, 46]. Moreover, the fuzzy multigranularity uncertainty measures adopted in [42] can be investigated by making use of (a, b)-FSSs as a generalized expression of fuzziness.

List of abbreviations and symbols

In Tables 3 and 4, we respectively present the main abbreviations and symbols used in this article.

Table 3. Abbreviations of the main concepts mentioned in this work.

Concepts	Abbreviation
intuitionistic fuzzy set	IF-set
Pythagorean fuzzy set	PF-set
Fermatean fuzzy set	FF-set
q-rung orthopair fuzzy set	q-ROF set
(2,1)-Fuzzy set	(2,1)-FS
(a, b)-Fuzzy set	(a, b)-FS
fuzzy soft set	FS-set
intuitionistic fuzzy soft set	IFS-set
Pythagorean fuzzy soft set	PFS-set
Fermatean fuzzy soft set	FFS-set
q-rung orthopair fuzzy soft set	q-ROFS set
(2,1)-Fuzzy soft set	(2,1)-FSS
(a, b)-Fuzzy soft set	(a, b)-FSS
multi-criteria decision-making	MCDM

Table 4. Symbols for the main concepts mentioned in this work.

Concepts	symbols
membership function	φ
non-membership function	ϖ
indeterminacy function	ζ
universal set	U
set of parameters	Δ
soft set	(f, Δ)
family of fuzzy sets over U	$FS(U)$
family of IF-sets over U	$IFS(U)$
family of PF-sets over U	$PFS(U)$
family of FF-sets over U	$FFS(U)$
family of q-ROF sets over U	$qROFS(U)$
family of (a, b)-FSs over U	$(a, b) - FS(U)$
family of (a, b)-FSSs over U	$I^{(a,b)-FSS}$

Acknowledgments

The authors are extremely grateful to the five anonymous referees for their valuable comments which helped to improve the presentation of this manuscript. This project was supported by the Deanship of Scientific Research at Prince Sattam Bin Abdulaziz University under the research project (PSAU2022/01/20167)

Conflicts of interest

The authors declare no conflict of interest.

References

1. B. Ahmad, A. Kharal, On fuzzy soft sets, *Adv. Fuzzy Syst.*, **2009** (2009). https://doi.org/10.1155/2009/586507
2. M. Akram, A. Adeel, J. C. R. Alcantud, Fuzzy *N*-soft sets: A novel model with applications, *J. Intell. Fuzzy Syst.*, **35** (2018), 4757–4771. https://doi.org/10.3233/JIFS-18244
3. J. C. R. Alcantud, Convex soft geometries, *J. Comput. Cognitive Eng.*, **1** (2022), 2–12. https://doi.org/10.47852/bonviewJCCE597820
4. J. C. R. Alcantud, The semantics of *N*-soft sets, their applications, and a coda about three-way decision, *Inform. Sci.*, **606** (2022), 837–852. https://doi.org/10.1016/j.ins.2022.05.084
5. J. C. R. Alcantud, T. M. Al-shami, A. A. Azzam, Caliber and chain conditions in soft topologies, *Mathematics*, **9** (2021), 2349. https://doi.org/10.3390/math9192349
6. J. C. R. Alcantud, A. Z. Khameneh, A. Kilicman, Aggregation of infinite chains of intuitionistic fuzzy sets and their application to choices with temporal intuitionistic fuzzy information, *Inform. Sci.*, **514** (2020), 106–117. https://doi.org/10.1016/j.ins.2019.12.008
7. J. C. R. Alcantud, G. Varela, B. Santos-Buitrago, G. Santos-García, M. F. Jiménez, Analysis of survival for lung cancer resections cases with fuzzy and soft set theory in surgical decision making, *Plos One*, **14** (2019), e0218283. https://doi.org/10.1371/journal.pone.0218283
8. T. M. Al-shami, (2,1)-Fuzzy sets: Properties, weighted aggregated operators and their applications to multi-criteria decision-making methods, *Complex Intell. Syst.*, 2022. https://doi.org/10.1007/s40747-022-00878-4
9. T. M. Al-shami, Generalized umbral for orthopair fuzzy sets: (m,n)-Fuzzy sets and their applications to multi-criteria decision-making methods, *Submitted*.
10. T. M. Al-shami, Soft somewhat open sets: Soft separation axioms and medical application to nutrition, *Comput. Appl. Math.*, **41** (2022), 216. https://doi.org/10.1007/s40314-022-01919-x
11. T. M. Al-shami, H. Z. Ibrahim, A. A. Azzam, A. I. EL-Maghrabi, SR-fuzzy sets and their applications to weighted aggregated operators in decision-making, *J. Funct. Space.*, **2022** (2022). https://doi.org/10.1155/2022/3653225
12. T. M. Al-shami, L. D. R. Kočinac, Almost soft Menger and weakly soft Menger spaces, *Appl. Comput. Math.*, **21** (2022), 35–51. https://doi.org/10.30546/1683-6154.21.1.2022.35

13. T. M. Al-shami, A. Mhemdi, R. Abu-Gdairi, M. E. El-Shafei, Compactness and connectedness via the class of soft somewhat open sets, *AIMS Math.*, **8** (2023), 815–840. https://doi.org/10.3934/math.2023040
14. T. M. Al-shami, A. Mhemdi, A. Rawshdeh, H. Al-jarrah, Soft version of compact and Lindelöf spaces using soft somewhere dense set, *AIMS Math.*, **6** (2021), 8064–8077. https://doi.org/10.3934/math.2021468
15. Z. A. Ameen, T. M. Al-shami, A. A. Azzam, A. Mhemdi, A novel fuzzy structure: Infra-fuzzy topological spaces, *J. Funct. Space.*, **2022** (2022) https://doi.org/10.1155/2022/9778069
16. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Set. Syst.*, **20** (1986), 87–96.
17. M. Atef, M. I. Ali, T. M. Al-shami, Fuzzy soft covering-based multi-granulation fuzzy rough sets and their applications, *Comput. Appl. Math.*, **40** (2021), 115. https://doi.org/10.1007/s40314-021-01501-x
18. N. Cağman, S. Enginoğlu, F. Çitak, Fuzzy soft set theory and its application, *Iran. J. Fuzzy Syst.*, **8** (2011), 137–147.
19. S. K. De, R. B. Akhil, R. Roy, An application of intuitionistic fuzzy sets in medical diagnosis, *Fuzzy Set. Syst.*, **117** (2001), 209–213. https://doi.org/10.1016/S0165-0114(98)00235-8
20. F. Fatimah, D. Rosadi, R. B. F. Hakim, J. C. R. Alcantud, *N*-soft sets and their decision-making algorithms, *Soft Comput.*, **22** (2018), 3829–3842. https://doi.org/10.1007/s00500-017-2838-6
21. F. Fatimah, J. C. R. Alcantud, The multi-fuzzy *N*-soft set and its applications to decision-making, *Neural Comput. Appl.*, **33** (2021), 11437–11446. https://doi.org/10.1007/s00521-020-05647-3
22. M. Gulzar, D. Alghazzawi, M. H. Mateen, N. Kausar, *A certain class of t-intuitionistic fuzzy subgroups*, *IEEE Access*, **8** (2020), 163260–163268. https://doi.org/10.1109/ACCESS.2020.3020366
23. M. T. Hamida, M. Riaz, D. Afzal, Novel MCGDM with q-rung orthopair fuzzy soft sets and TOPSIS approach under q-Rung orthopair fuzzy soft topology, *J. Intell. Fuzzy Syst.*, **39** (2020), 3853–3871. https://doi.org/10.3233/JIFS-192195
24. H. Z. Ibrahim, T. M. Al-shami, O. G. Elbarbary, (3,2)-Fuzzy sets and their applications to topology and optimal choices, *Comput. Intell. Neurosci.*, **2021** (2021). https://doi.org/10.1155/2021/1272266
25. S. J. John, *Soft sets: Theory and applications*, Springer Cham, 2021.
26. H. Liao, Z. Xu, Intuitionistic fuzzy hybrid weighted aggregation operators, *Int. J. Intell. Syst.*, **29** (2014), 971–993. https://doi.org/10.1002/int.21672
27. A. A. Khan, S. Ashraf, S. Abdullah, M. Qiyas, J. Luo, S. U. Khan, Pythagorean fuzzy Dombi aggregation operators and their application in decision support system, *Symmetry*, **11** (2019), 383. https://doi.org/10.3390/sym11030383
28. M. Kirişci, A case study for medical decision making with the fuzzy soft sets, *Afr. Mat.*, **31** (2020), 557–564.
29. M. Kirişci, N. Şimşek, Decision making method related to Pythagorean fuzzy soft sets with infectious diseases application, *J. King Saud Univ.-Com.*, **34** (2022), 5968–5978. https://doi.org/10.1016/j.jksuci.2021.08.010

30. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589–602.
31. A. R. Roy, P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, *J. Comput. Appl. Math.*, **203** (2007), 412–418. https://doi.org/10.1016/j.cam.2006.04.008
32. D. Molodtsov, Soft set theory—first results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
33. X. Peng, H. Yuan, Fundamental properties of Pythagorean fuzzy aggregation operators, *Fund. Inform.*, **147** (2016), 415–446. https://doi.org/10.3233/FI-2016-1415
34. X. Peng, Y. Yang, J. Song, Y. Jiang, Pythagorean fuzzy soft set and its application, *Comput. Eng.*, **41** (2015), 224–229.
35. K. Rahman, S. Abdullah, M. Jamil, M. Y. Khan, Some generalized intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute group decision making, *Int. J. Fuzzy Syst.*, **20** (2018), 1567–1575. https://doi.org/10.1007/s40815-018-0452-0
36. K. Rahman, S. Abdullah, M. S. A. Khan, M. Shakeel, Pythagorean fuzzy hybrid geometric aggregation operator and their applications to multiple attribute decision making, *Int. J. Comput. Sci. Inform. Sec.*, **14** (2016), 837–854.
37. A. Sivadas, S. J. John, *Fermatean fuzzy soft sets and its applications*, Communications in Computer and Information Science, Springer, Singapore, **1345** (2021).
38. S. Saleh, R. Abu-Gdairi, T. M. Al-shami, Mohammed S. Abdo, On categorical property of fuzzy soft topological spaces, *Appl. Math. Inform. Sci.*, **16** (2022), 635–641. http://dx.doi.org/10.18576/amis/160417
39. T. Senapati, R. R. Yager, Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods, *Eng. Appl. Artif. Intell.*, **85** (2019), 112–121. https://doi.org/10.1016/j.engappai.2019.05.012
40. T. Senapati, R. R. Yager, Fermatean fuzzy sets, *J. Amb. Intell. Hum. Comput.*, **11** (2020), 663–674. https://doi.org/10.1007/s12652-019-01377-0
41. Y. J. Xu, Y. K. Sun, D. F. Li, *Intuitionistic fuzzy soft set*, 2010 2nd International Workshop on Intelligent Systems and Applications, 2010, 1–4. https://doi.org/10.1109/IWISA.2010.5473444
42. J. Wan, H. Chen, T. Li, Z. Yuan, J. Liu, W. Huang, Interactive and complementary feature selection via fuzzy multigranularity uncertainty measures, *IEEE T. Cybern.*, 2021. https://doi.org/10.1109/TCYB.2021.3112203
43. R. R. Yager, Pythagorean membership grades in multicriteria decision making, *IEEE T. Fuzzy Syst.*, **22** (2013), 958–965. https://doi.org/10.1109/TFUZZ.2013.2278989
44. R. R. Yager, Generalized orthopair fuzzy sets, *IEEE T. Fuzzy Syst.*, **25** (2017), 1222–1230. https://doi.org/10.1109/TFUZZ.2016.2604005
45. B. Yang, Fuzzy covering-based rough set on two different universes and its application, *Artif. Intell. Rev.*, **55** (2022), 4717–4753. https://doi.org/10.1007/s10462-021-10115-y
46. B. Yang, B. Q. Hu, A fuzzy covering-based rough set model and its generalization over fuzzy lattice, *Inform. Sci.*, **367–368** (2016), 463–486. https://doi.org/10.1016/j.ins.2016.05.053

47. J. Yang, Y. Yao, Semantics of soft sets and three-way decision with soft sets, *Inform. Sci.*, **194** (2020), 105538. https://doi.org/10.1016/j.knosys.2020.105538

48. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353.