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Research article

A new approach to generalized interpolative proximal contractions in non archimedean fuzzy metric spaces

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Abstract: We introduce a new type of interpolative proximal contractive condition that ensures the existence of the best proximity points of fuzzy mappings in the complete non-archimedean fuzzy metric spaces. We establish certain best proximity point theorems for such proximal contractions. We improve and generalize the fuzzy proximal contractions by introducing fuzzy proximal interpolative contractions. The obtained results improve and generalize the best proximity point theorems published in Fuzzy Information and Engineering, 5 (2013), 417–429. Moreover, we provide many nontrivial examples to validate our best proximity point theorem.

Keywords: best proximity point; generalized fuzzy interpolative proximal contractions; fuzzy metric space
Mathematics Subject Classification: 47H10, 26E05, 26E25

1. Introduction

Fixed point theory focuses on the techniques to solve non-linear equations of the kind $p(u) = u$, where p is self-mapping. As a result, the concrete solution of such equations takes into account "fixed point theory". Any approximative solution is also worth examining and can be determined using the best proximity point theory in circumstances where such a problem cannot be solved. Best proximity roughly translates to the smallest value of $d(u, p(u))$ if $p(u)$ is not equal to u. Best proximity theorems, interestingly, are a natural development of fixed point theorems. When the mapping in question is self-mapping, best proximity point becomes a fixed point. The existence of a best proximity point can be determined by analyzing different types of proximal contractions [1–4].

The interpolative contraction principles consist of the product of distances having exponents satisfying some conditions. The term "interpolative contraction" was introduced by the renowned mathematician Erdal Karapinar in his paper [5] published in 2018. The interpolative contraction is defined as follows:

A self-mapping S defined on a metric space $(\mathcal{A}, d)$ is said to be an interpolative contraction, if there exist $\nu \in (0, 1]$ and $K \in [0, 1)$ such that

$$d(Se, Sr) \leq K\,(d(e, r))^{\nu}\,, \forall e, r \in \mathcal{A}.$$

Note that for $\nu = 1$, S is a Banach contraction. If the mapping S defined on a metric space $(\mathcal{A}, d)$ satisfies the following inequalities:

$$\begin{aligned} d(Se, Sr) &\leq K\,(d(e, Se))^{\nu}\,(d(r, Sr))^{1-\nu}\,, \\ d(Se, Sr) &\leq K\,(d(r, Se))^{\nu}\,(d(e, Sr))^{1-\nu}\,, \\ d(Se, Sr) &\leq K\,(d(e, r))^{\eta}\,(d(e, Se))^{\nu}\,(d(r, Sr))^{1-\nu-\eta}\,,\ \nu + \eta < 1 \\ d(Se, Sr) &\leq K\,(d(e, r))^{\nu}\,(d(e, Se))^{\eta}\,(d(r, Sr))^{\gamma}\left(\frac{1}{2}(d(e, Sr) + d(r, Se))\right)^{1-\eta-\nu-\gamma}, \end{aligned}$$

for all $e, r \in \mathcal{A}$, then S is called interpolative Kannan type contraction, interpolative Chatterjea type contraction, interpolative Ćirić-Reich-Rus type contraction and interpolative Hardy Rogers type contraction respectively. Recently, many classical and advanced contractions have been revisited via interpolation (see [6–9]).

Recently, Altun et al. [10], revisited all the interpolative contractions and defined interpolative proximal contractions. They presented the best proximity theorems on such contractions. The aim of this paper is to establish the best proximity point theorems for interpolative proximal contractions to the case of non-self mappings.

The concept of fuzzy sets was given by Zadeh [11]. Schweizer and Sklar [12] defined the notion of continuous t-norms. Gregory and Sapena [13] introduced the notion of fuzzy metric space by using the concept of fuzzy sets, continuous t-norm, and metric space. Pakanazar [15] proved the best proximity point theorems in a fuzzy metric space. The idea of best proximity points of the fuzzy mappings in fuzzy metric space was introduced by Vetro and Salimi [16]. Also, Vetro and Salimi proved the existence and uniqueness of the best proximity point in a non-Archimedean fuzzy metric space.

Many authors have extended this theorem in various directions and in this context Ajeti et al. [17] introduced the notion of coupled best proximity points for some cyclic and semi-cyclic maps in a reflexive Banach space. Gabeleh [18] introduce a new class of non-self mappings, called weak proximal contractions and proved the existence and uniqueness results of the best proximity point for weak proximal contractions. Some utilization of best proximity points has been discussed in [19–21].

Inspired, by these results, we introduce interpolative Kannan type, interpolative Reich-Rus-Ciric type and interpolative Hardy Rogers type in non-Archimedean fuzzy metric space. The aim of this paper is to generalize the interpolative type contraction in a complete non-Archimedean fuzzy metric space. Recently, many nonlinear fuzzy models have appeared in the literature [22] and to show the existence of solutions to such mathematical models, we need generalized fuzzy contractive conditions. In this regard, Hierro et al. [23] and Vetro and Salimi [16] have presented some generalized Lipschitz conditions to obtain the best proximity point theorems. Motivated by the investigations [16,23], in this paper, we suggest various generalized Lipschitz conditions in the fuzzy metric space that can be used to show the existence of fuzzy models of nonlinear systems.

2. Preliminaries

Given two non-empty subsets R and G of a fuzzy metric space, the following notions and notations are used in the sequel.

$$\begin{aligned} F(R,G,\omega) &= \sup\{F(u,v,\omega) : u \in R, v \in G \text{ and } \omega > 0\}, \\ R_0(\omega) &= \{u \in R : F(u,v,\omega) = F(R,G,\omega) \text{ for some } v \in G\}, \\ G_0(\omega) &= \{v \in G : F(u,v,\omega) = F(R,G,\omega) \text{ for some } u \in R\}. \end{aligned}$$

For any $(U,F,*)$ be a fuzzy metric space and R,G be any nonempty subsets of U. We say that G is approximately compact with respect to R, if every sequence $\{u_n\}$ in G satisfying the following condition

$$F(v,u_n,\omega) \to F(v,G,\omega)$$

for some $v \in R$, has a convergent subsequence.

Definition 2.1. *[12] A binary operation $* : I \times I \to I$ is called a continuous t-norm if it satisfies the following axioms:*

(T1) $a * b = b * a$ and $a * (b * c) = (a * b) * c$ for all $a,b,c \in I$;
(T2) $*$ is continuous;
(T3) $a * 1 = a$ for all $a \in I$;
(T4) $a * b \le c * d$ when $a \le c$ and $b \le d$, with $a,b,c,d \in I$.

Definition 2.2. *[23] Let U be arbitrary set, $F : U \times U \times (0,\infty) \to [0,1]$ and $*$ is continuous t-norm then $(U,F,*)$ is said to be a fuzzy metric space if it satisfies the following axioms for all $u,v,w \in U$ and $\omega,\varpi > 0$:*

C1: $F(u,v,\omega) > 0$;
C2: $F(u,v,\omega) = 1 \iff u = v$;
C3: $F(u,v,\omega) = F(v,u,\omega)$;
C4: $F(u,w,\omega+\varpi) \ge F(u,v,\omega) * F(v,w,\varpi)$;
C5: $F(u,v,.) : (0,\infty) \to [0,1]$ is continuous.
If we replace (C4) by
C6: $F(u,w,\max\{\omega,\varpi\}) \ge F(u,v,\omega) * F(v,w,\varpi)$,
then $(U,F,*)$ is said to be non-Archimedean fuzzy metric space. Note that, since (C6) implies (C4), each non-Archimedean fuzzy metric space is a fuzzy metric space.

Definition 2.3. *[23] Let $(U,F,*)$ be a fuzzy metric space. Then*

(i) A sequence $\{u_n\}$ converges to $u \in U$ if and only if $F(u_n, u, \omega) \to 1$ as $n \to +\infty$ for all $\omega > 0$;

(ii) A sequence $\{u_n\}$ in U is a Cauchy sequence if and only if for all $\epsilon \in (0,1)$ and $\omega > 0$, there exits n_0 such that $F(u_n, u_m, \omega) > 1 - \epsilon$ for all $m, n \geq n_0$;

(iii) The fuzzy metric space is complete if every Cauchy sequence converges to some $u \in U$.

Definition 2.4. *[16] Let $(U, F, *)$ be a fuzzy metric space and R, G be any nonempty subsets of U. We say that G is approximately compact with respect to R, if every sequence $\{u_n\}$ in G satisfying the following condition*

$$F(v, u_n, \omega) \to F(v, G, \omega),$$

for some $v \in R$, has a convergent subsequence.

Definition 2.5. *[16] Let $(U, F, *)$ be a fuzzy metric space and R, G be non-empty subsets of U. An element u in R is called a best proximity point of the mapping $\Upsilon : R \to G$, if it satisfies the equation:*

$$F(u, \Upsilon u, \omega) = F(R, G, \omega).$$

A best proximity point of the mapping Υ is not only an approximate solution of the equation $\Upsilon(u) = u$ but also an optimal solution of the minimization problem:

$$\min \{F(u, \Upsilon(u), \omega) : u \in R\}.$$

3. Main results

In this section, we define non-Archimedean fuzzy interpolative contraction mappings and show that it generalizes proximal contraction. We prove the existence of the best proximity points of proximal contraction in a complete non-Archimedean fuzzy metric space followed by supporting examples.

3.1. Interpolative Kannan type proximal contraction in non-Archimedean fuzzy metric space

Definition 3.1. *Let $(U, F, *)$ be a complete non-Archimedean fuzzy metric space and $R, G \subseteq U$. A mapping $\Upsilon : R \to G$ is said to be interpolative Kannan type proximal contraction, if there exist $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that*

$$F(u_1, u_2, \omega) \geq \lambda \left((F(v_1, u_1, \omega))^{\alpha} (F(v_2, u_2, \omega))^{1-\alpha} \right), \tag{3.1}$$

for all $u_1, u_2, v_1, v_2 \in R$, $\omega > 0$ and $u_i \neq v_i$, $i \in \{1, 2\}$ with respect to $F(u_1, \Upsilon v_1, \omega) = F(R, G, \omega)$, $F(u_2, \Upsilon v_2, \omega) = F(R, G, \omega)$ and $F(u, v, \omega) > 0$.

Example 3.2. *Let $U = \mathbb{R} \times \mathbb{R}$ and define the function $F : U \times U \times (0, \infty) \to [0,1]$ by*

$$F(u, v, \omega) = \frac{\omega}{\omega + d((u_1, v_1), (u_2, v_2))},$$

for all $(u_1, v_1), (u_2, v_2) \in U$. Where $d((u_1, v_1), (u_2, v_2)) = \mid u_1 - v_1 \mid + \mid u_2 - v_2 \mid$. Then $(U, F, *)$ is a non-Archimedean fuzzy metric space with $\breve{a} * \bar{e} = \breve{a}\bar{e}$ for all $\breve{a}, \bar{e} \in I$. Let $R, G \subseteq U$ defined by

$$R = \left\{ \left(0, \frac{1}{n}\right); n \in \mathbb{N} \right\} \cup \{(0,0)\},$$

$$G = \{\left(1, \frac{1}{n}\right); n \in \mathbb{N}\} \cup \{(1,0)\}.$$

Define $F(R,G,\omega) = \sup\{F(u,v,\omega) : u \in R, v \in G$ *and* $\omega > 0\}$. *So, we have* $F(R,G,\omega) = \frac{\omega}{\omega+1}$, $R_0(\omega) = R$ *and* $G_0(\omega) = G$. *Define the mapping* $\Upsilon : R \to G$ *by*

$$\Upsilon(u_1, u_2) = \begin{cases} \left(1, \frac{1}{2n}\right), \text{ if } (u_1, u_2) = \left(0, \frac{1}{n}\right) \text{ for all } n \in \mathbb{N} \\ (1, 0), \text{ if } (u_1, u_2) = (0, 0) \end{cases}$$

for all $(u_1, u_2) \in R$. *Then, clearly* $\Upsilon(R_0) \subseteq G_0$. *Now, we show that* Υ *is a interpolative Kannan type contraction. For* $u_1 = \left(0, \frac{1}{2}\right)$, $u_2 = \left(0, \frac{1}{4}\right)$, $v_1 = (0, 1)$, $v_2 = \left(0, \frac{1}{2}\right)$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{3}$ *and* $\omega = 1$.

$$F(u_1, \Upsilon v_1, \omega) = F\left(\left(0, \frac{1}{2}\right), \Upsilon(0,1), 1\right) = F(R, G, \omega),$$

and

$$F(u_2, \Upsilon v_2, \omega) = F\left(\left(0, \frac{1}{4}\right), \Upsilon\left(0, \frac{1}{2}\right), 1\right) = F(R, G, \omega).$$

This implies that,

$$\begin{aligned} F(u_1, u_2, \omega) &= F\left(\left(0, \frac{1}{2}\right), \left(0, \frac{1}{4}\right), 1\right), \\ &\geq \lambda \left(F(v_1, u_1, \omega)\right)^{\alpha} \left(F(v_2, u_2, \omega)\right)^{1-\alpha}, \\ &\geq \lambda \left(F\left((0,1), \left(0, \frac{1}{2}\right), 1\right)\right)^{\frac{1}{2}} \left(F\left(\left(0, \frac{1}{2}\right), \left(0, \frac{1}{4}\right), 1\right)\right)^{1-\frac{1}{2}}, \end{aligned}$$

which yield,

$$0.5714 \geq 0.1826.$$

This shows that Υ *is a interpolative Kannan type contraction. However, for* $u_1 = \left(0, \frac{1}{2}\right)$, $u_2 = \left(0, \frac{1}{4}\right)$, $v_1 = (0, 1)$, $v_2 = \left(0, \frac{1}{2}\right)$, $\lambda = 0.499$ *and* $\omega = 1$. *Now, we have*

$$F(u_1, \Upsilon v_1, \omega) = F\left(\left(0, \frac{1}{2}\right), \Upsilon(0,1), 1\right) = F(R, G, \omega),$$

and

$$F(u_2, \Upsilon v_2, \omega) = F\left(\left(0, \frac{1}{4}\right), \Upsilon\left(0, \frac{1}{2}\right), 1\right) = F(R, G, \omega).$$

Implies,

$$\begin{aligned} (F(u_1, u_2, \omega)) &= F\left(\left(0, \frac{1}{2}\right), \left(0, \frac{1}{4}\right), 1\right) \\ &\geq \lambda \left(\left(F(v_1, u_1, \omega)\right) + \left(F(v_2, u_2, \omega)\right)\right) \end{aligned}$$

$$= \lambda \begin{pmatrix} \left(F\left((0,1), \left(0,\frac{1}{2}\right), 1\right)\right) \\ +F\left(\left(0,\frac{1}{2}\right), \left(0,\frac{1}{4}\right), 1\right) \end{pmatrix},$$

which yield

$$\begin{aligned} 0.5714 &\geq \lambda (0.4 + 0.75), \\ 0.5714 &\ngeq 0.5739. \end{aligned}$$

This is a contradiction. Hence, Υ is not a Kannan type contraction.

Next, we start our main results.

Theorem 3.3. *Let $(U, F, *)$ be a complete non-Archimedean fuzzy metric space and $R, G \subseteq U$ such that G is approximately compact with respect to R. Let $\Upsilon\colon R \to G$ be a interpolative Kannan type contraction. If $R_0 \subseteq R$ such that $\Upsilon(R_0) \subseteq G_0$. Then Υ admits a best proximity point.*

Proof. Let $u_0 \in R_0$. Since $\Upsilon(u_0) \in \Upsilon(R_0) \subseteq G_0$ there exist $u_1 \in R_0$ such that,

$$F(u_1, \Upsilon(u_0), \omega) = F(R, G, \omega).$$

Also, we have $\Upsilon(u_1) \in \Upsilon(R_0) \subseteq G_0$. So, there exist $u_2 \in R_0$ such that,

$$F(u_2, \Upsilon(u_1), \omega) = F(R, G, \omega).$$

This process of existence of point in R_0 implies to have a sequence $\{u_n\} \subseteq R_0$ such that,

$$F(u_{n+1}, \Upsilon(u_n), \omega) = F(R, G, \omega), \tag{3.2}$$

for all $n \in \mathbb{N}$. Observe that, if there exist $n \in \mathbb{N}$ such that $u_n = u_{n+1}$ then from (3.2), the point u_n is a best proximity point of the mapping Υ. On the other hand, if $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Then by (3.2), we have

$$F(u_n, \Upsilon(u_{n-1}), \omega) = F(R, G, \omega),$$

and

$$F(u_{n+1}, \Upsilon(u_n), \omega) = F(R, G, \omega),$$

for all $n \geq 1$. Thus, by (3.1),

$$(F(u_n, u_{n+1}, \omega)) \geq \lambda (F(u_{n-1}, u_n, \omega))^{\alpha} (F(u_n, u_{n+1}, \omega))^{1-\alpha}, \tag{3.3}$$

for all distinct $u_{n-1}, u_n, u_{n+1} \in R$. Since, by (3.3), we have

$$F(u_n, u_{n+1}, \omega) \geq \lambda (F(u_{n-1}, u_n, \omega))^{\alpha} (F(u_n, u_{n+1}, \omega))^{1-\alpha},$$

$$(F(u_n, u_{n+1}, \omega))^{\alpha} \geq \lambda (F(u_{n-1}, u_n, \omega))^{\alpha}. \tag{3.4}$$

So, by (3.4), let $H_n = F(u_n, u_{n+1}, \omega)$. We have $H_{n-1} < H_n$ for all $n \in \mathbb{N}$. This shows that the sequence $\{H_n\}$ is positive and strictly non-decreasing. Thus, it converges to some element $H \geq 1$. Now from (3.4), we have

$$F(u_n, u_{n+1}, \omega) \geq \lambda^{\frac{1}{\alpha}} F(u_{n-1}, u_n, \omega)$$

$$
\begin{aligned}
&\geq \lambda^{\frac{2}{\alpha}} F(u_{n-2}, u_{n-1}, \omega) \\
&\vdots \\
&\geq \lambda^{\frac{n}{\alpha}} F(u_1, u_0, \omega).
\end{aligned}
$$

Then $H_n(\omega) > H_{n-1}(\omega)$, that is the sequence $\{H_n\}$ is non-decreasing sequence for all $\omega > 0$. Consequently, there exist $H(\omega) \leq 1$ such that $\lim_{n\to\infty} H_n(\omega) = H(\omega)$. Now, we claim that $H(\omega) = 1$. Suppose, to the contrary that $0 < H(\omega_0) < 1$ for some $\omega_0 > 0$. Since $H_n(\omega_0) \geq H(\omega_0)$, by taking the limit with $\omega = \omega_0$. We obtain

$$H(\omega_0) \geq \lambda^{\frac{1}{\alpha}} H(\omega_0) > H(\omega_0).$$

Which is contradiction and hence, $H(\omega) = 1$ for all $\omega > 0$. Now, we show $\{u_n\}$ is a cauchy sequence. Assuming this is not true, then there exist $\epsilon \in (0, 1)$ and $\omega_0 > 0$ such that for all $k \in \mathbb{N}$, there are $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$F(u_{m(k)}, u_{n(k)}, \omega_0) \leq 1 - \epsilon.$$

Assume, that $m(k)$ is the least integer exceeding $n(k)$ satisfying the above inequality, that is equivalently,

$$F(u_{m(k)-1}, u_{n(k)}, \omega_0) > 1 - \epsilon,$$

and so for all k we get

$$
\begin{aligned}
1 - \epsilon &\geq F(u_{m(k)}, u_{n(k)}, \omega) \\
&\geq F(u_{m(k)-1}, u_{m(k)}, \omega) * F(u_{m(k)-1}, u_{n(k)}, \omega) \\
&> H_{m(k)}(\omega_0) * (1 - \epsilon).
\end{aligned}
\tag{3.5}
$$

Putting limit $n \to \infty$ in (3.5), we get that

$$\lim_{n\to\infty} F(u_{m(k)}, u_{n(k)}, \omega_0) = 1 - \epsilon,$$

from

$$F(u_{m(k)+1}, u_{n(k)+1}, \omega_0) \geq F(u_{m(k)+1}, u_{m(k)}, \omega_0) * F(u_{m(k)}, u_{n(k)}, \omega_0) * F(u_{n(k)}, u_{n(k)+1}, \omega_0),$$

and

$$F(u_{m(k)}, u_{n(k)}, \omega_0) \geq F(u_{m(k)}, u_{m(k)+1}, \omega_0) * F(u_{m(k)}, u_{n(k)+1}, \omega_0) * F(u_{n(k)+1}, u_{n(k)}, \omega_0),$$

we get

$$\lim_{n\to\infty} F(u_{m(k)+1}, u_{n(k)+1}, \omega_0) = 1 - \epsilon.$$

From Eq (3.2), we know that

$$F(u_{m(k)+1}, \Upsilon u_{m(k)}, \omega_0) = F(R, G, \omega_0)$$

and

$$F(u_{n(k)+1}, \Upsilon u_{n(k)}, \omega_0) = F(R, G, \omega_0).$$

So, by (3.1),

$$F\left(u_{m(k)+1}, u_{n(k)+1}, \omega_0\right) \geq \lambda \left(F\left(u_{m(k)}, u_{m(k)+1}, \omega_0\right)\right)^{\alpha} \left(F\left(u_{n(k)}, u_{n(k)+1}, \omega_0\right)\right)^{1-\alpha},$$

taking lim $k \to \infty$, we get

$$1 - \epsilon \geq \lambda (1 - \epsilon) > 1 - \epsilon.$$

Which is contradiction. Then $\{u_n\}$ is cauchy sequence. Since $(U, F, *)$ is a complete non-Archimedean fuzzy metric space and R is closed subset of U. Then there exist $u \in R$, such that $\lim_{n\to\infty} F(u_n, u, \omega) = 1$. Moreover,

$$\begin{aligned} F(R, G, \omega) &= F(u_{n+1}, \Upsilon(u_n), \omega) \\ &\geq F(u_{n+1}, u, \omega) * F(u, \Upsilon(u_n), \omega) \\ &\geq F(u_{n+1}, u, \omega) * F(u, u_{n+1}, \omega) * F(u_{n+1}, \Upsilon u_n, \omega) \\ &= F(u_{n+1}, u, \omega) * F(u, u_{n+1}, \omega) * F(R, G, \omega). \end{aligned}$$

This implies,

$$\begin{aligned} F(R, G, \omega) &\geq F(u_{n+1}, u, \omega) * F(u, \Upsilon(u_n), \omega) \\ &\geq F(u_{n+1}, u, \omega) * F(u, u_{n+1}, \omega) * F(R, G, \omega). \end{aligned}$$

Applying to limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned} F(R, G, \omega) &\geq 1 * \lim_{n\to\infty} F(u, \Upsilon(u_n), \omega) \\ &\geq 1 * 1 * F(R, G, \omega). \end{aligned}$$

That is,

$$\lim_{n\to\infty} F(u, \Upsilon(u_n), \omega) = F(R, G, \omega).$$

Therefore, $F(u, \Upsilon(u_n), \omega) \to F(u, G, \omega)$ as $n \to \infty$. Since G is approximately compact with respect to R, then there exist $\xi \in R_0(\omega)$ such that,

$$F(\xi, \Upsilon u, \omega) = F(R, G, \omega) = F(u_{n+1}, \Upsilon(u_n), \omega). \tag{3.6}$$

We now show that $u = \xi$. If not, then

$$F(\xi, u_{n+1}, \omega) \geq \lambda (F(u, \xi, \omega))^{\alpha} (F(u_n, u_{n+1}, \omega))^{1-\alpha},$$

on taking limit as $n \to \infty$ gives

$$F(\xi, u, \omega) \geq \lambda (F(u, \xi, \omega))^{\alpha} > (F(u, \xi, \omega))^{\alpha}.$$

Which is contradiction. Hence $F(u, \Upsilon u, \omega) = F(R, G, \omega) = F(\xi, \Upsilon \xi, \omega)$, that is , u is the best proximity point. We show that u is the unique best proximity point of Υ. Assume, on the contrary, that $0 < F(u, v, \omega) < 1$ for all $\omega > 0$ and $v \neq u$ is another best proximity point of Υ, i.e., $F(u, \Upsilon u, \omega) = F(R, G, \omega) = F(v, \Upsilon v, \omega)$ then from (3.1), we have

$$F(u, v, \omega) \geq \lambda (F(u, u, \omega))^{\alpha} (F(v, v, \omega))^{1-\alpha} > 1.$$

Which is contradiction and hence, $F(u, v, \omega) = 1$ for all $\omega > 0$, that is $u = v$. □

3.2. Interpolative Reich-Rus-Ciric type proximal contraction in non-Archimedean fuzzy metric space

Definition 3.4. *Let $(U, F, *)$ be a complete non-Archimedean fuzzy metric space, and $R, G \subseteq U$. A mapping $\Upsilon : R \to G$ is said to be a interpolative Reich-Rus-Ciric type proximal contraction, if there exist $\alpha, \beta \in (0, 1)$ and $\lambda \in [0, 1)$ with $\alpha + \beta < 1$.*

$$F(u_2, u_1, \omega) \geq \lambda (F(v_1, v_2, \omega))^{\alpha} (F(v_1, u_1, \omega))^{\beta} (F(v_2, u_2, \omega))^{1-\alpha-\beta}, \tag{3.7}$$

for all $u_1, u_2, v_1, v_2 \in R$, $\omega > 0$ and $u_i \neq v_i$, $i \in \{1, 2\}$ with respect to $F(u_1, \Upsilon v_1, \omega) = F(R, G, \omega)$, $F(u_2, \Upsilon v_2, \omega) = F(R, G, \omega)$ and $F(u, v, \omega) > 0$.

Example 3.5. *Let $U = \mathbb{R}^2$ and define the function $F : U \times U \times (0, +\infty) \to [0, 1]$ by*

$$F(u, v, \omega) = \frac{\omega}{\omega + d(u, v)},$$

where $d((u_1, v_1), (u_2, v_2)) = \sqrt[2]{(u_2 - u_1)^2 + (v_2 - v_1)^2}$ for all $(u_1, v_1), (u_2, v_2) \in U$. Then $(U, F, *)$ is a non-Archimedean fuzzy metric space with $\breve{a} * \bar{e} = \breve{a}\bar{e}$ for all $\breve{a}, \bar{e} \in I$. Let $R, G \subseteq U$ defined as

$$\begin{aligned} R &= \{(0, u); u \in \mathbb{R}\}, \\ G &= \{(1, u); u \in \mathbb{R}\}. \end{aligned}$$

Define $F(R, G, \omega) = \sup\{F(u, v, \omega) : u \in R, v \in G \text{ and } \omega > 0\}$. So we have $F(R, G, \omega) = \frac{\omega}{\omega+1}$, $R_0(\omega) = R$, $G_0(\omega) = G$. Define the mapping $\Upsilon : R \to G$ by

$$\Upsilon((0, \gamma)) = (1, 2\gamma),$$

for all $(0, \gamma) \in R$. Then clearly $\Upsilon(R_0) \subseteq G_0$. Now, we show that Υ is a interpolative Reich-Rus-Ciric contraction. For $u_1 = (0, 2)$, $v_1 = (0, 1)$, $u_2 = (0, 4)$, $v_2 = (0, 2)$, $\omega = 1, \alpha = \frac{1}{2}, \beta = \frac{1}{3}$ and $\lambda = 0.27$.

$$F(u_1, \Upsilon v_1, \omega) = F((0, 2), \Upsilon(0, 1), 1) = F(R, G, \omega),$$

and

$$F(u_2, \Upsilon v_2, \omega) = F((0, 4), \Upsilon(0, 2), 1) = F(R, G, \omega).$$

This implies that,

$$\begin{aligned} F(u_1, u_2, \omega) &= F((0, 2), (0, 4), 1) \\ &\geq \lambda \left((F(v_1, v_2, \omega))^{\alpha} (F(v_1, u_1, \omega))^{\beta} (F(v_2, u_2, \omega))^{1-\alpha-\beta} \right) \\ &= \lambda \begin{pmatrix} (F((0, 1), (0, 2), \omega))^{\frac{1}{2}} (F((0, 1), (0, 2), 1))^{\frac{1}{3}} \\ (F((0, 2), (0, 4), 1))^{1-\frac{1}{2}-\frac{1}{3}} \end{pmatrix}, \end{aligned}$$

which yield

$$0.3333 \geq 0.1557.$$

This shows that Υ is a interpolative Riech-Rus-Ciric type contraction. However, for $u_1 = (0, 2)$, $v_1 = (0, 1)$ and $u_2 = (0, 4)$, $v_2 = (0, 2)$, $\lambda = 0.27$. Now, we have

$$F(u_1, \Upsilon v_1, \omega) = F((0, 2), \Upsilon(0, 1), 1) = F(R, G, \omega),$$

and

$$F(u_2, \Upsilon v_2, \omega) = F((0,4), \Upsilon(0,2), 1) = F(R, G, \omega).$$

Implies,

$$\begin{aligned} F(u_1, u_2, \omega) &= F((0,2), (0,4), 1) \\ &\geq \lambda \left(F(v_1, v_2, \omega) + F(v_1, u_1, \omega) + F(v_2, u_2, \omega)\right) \\ &= \lambda \begin{pmatrix} F((0,1), (0,2), \omega) + F((0,1), (0,2), 1) + \\ F((0,2), (0,4), 1) \end{pmatrix}, \end{aligned}$$

which yield,

$$0.3333 \ngeq 0.3599.$$

This is a contradiction. Which shows that, Υ is not a Riech-Rus-Ciric type contraction.

Theorem 3.6. *Let $(U, F, *)$ be a complete non-Archimedean fuzzy metric space and $R, G \subseteq U$ such that G approximately compact with respect to R. Let $\Upsilon \colon R \to G$ be a interpolative Reich-Rus-Ciric type contraction. If $R_0 \subseteq R$ such that $\Upsilon(R_0) \subseteq G_0$. Then Υ admits a best proximity point.*

Proof. Let $u_0 \in R_0$. Since $\Upsilon(u_0) \in \Upsilon(R_0) \subseteq G_0$, so there exist $u_1 \in R_0$ such that,

$$F(u_1, \Upsilon(u_0), \omega) = F(R, G, \omega).$$

Also, we have $\Upsilon(u_1) \in \Upsilon(R_0) \subseteq G_0$. So, there exist $u_2 \in R_0$ such that,

$$F(u_2, \Upsilon(u_1), \omega) = F(R, G, \omega).$$

This process of existence of point in R_0 implies to have a sequence $\{u_n\} \subseteq R_0$ such that,

$$F(u_{n+1}, \Upsilon(u_n), \omega) = F(R, G, \omega) \tag{3.8}$$

for all $n \in \mathbb{N}$. Observe that, if there exist $n \in \mathbb{N}$ such that $u_n = u_{n+1}$ then from (3.8), the point u_n is a best proximity point of the mapping Υ. On the other hand, if $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Then by (3.8), we have

$$F(u_n, \Upsilon(u_{n-1}), \omega) = F(R, G, \omega),$$

and

$$F(u_{n+1}, \Upsilon(u_n), \omega) = F(R, G, \omega),$$

for all $n \geq 1$, Thus, by (3.7),

$$(F(u_n, u_{n+1}, \omega)) \geq \lambda \left((F(u_{n-1}, u_n, \omega))^{\alpha} (F(u_{n-1}, u_n, \omega))^{\beta} (F(u_n, u_{n+1}, \omega))^{1-\alpha-\beta} \right), \tag{3.9}$$

for all distinct $u_{n-1}, u_n, u_{n+1} \in R$. Since, by (3.9), we have

$$(F(u_n, u_{n+1}, \omega)) \geq \lambda \left((F(u_{n-1}, u_n, \omega))^{\alpha} (F(u_{n-1}, u_n, \omega))^{\beta} (F(u_n, u_{n+1}, \omega))^{1-\alpha-\beta} \right),$$

$$F(u_n, u_{n+1}, \omega) \geq \lambda \left(F(u_{n-1}, u_n, \omega)\right)^{\alpha+\beta} \left(F(u_n, u_{n+1}, \omega)\right)^{1-\alpha-\beta}. \tag{3.10}$$

So, by (3.10), let $H = F(u_n, u_{n+1}, \omega)$ We have $H_{n-1} < H_n$ for all $n \in \mathbb{N}$. This shows that the sequence $\{H_n\}$ is positive and strictly non-decreasing. Thus, it converges to some element $H \geq 1$. Now, from (3.10), we have

$$\begin{aligned} F(u_n, u_{n+1}, \omega) &\geq \lambda^{\frac{1}{\alpha+\beta}} F(u_{n-1}, u_n, \omega) \\ &\geq \lambda^{\frac{2}{\alpha+\beta}} F(u_{n-2}, u_{n-1}, \omega) \\ &\;\;\vdots \\ &\geq \lambda^{\frac{n}{\alpha+\beta}} F(u_1, u_0, \omega). \end{aligned}$$

Then $H_{n-1}(\omega) < H_n(\omega)$, that is the sequence $\{H_n\}$ is non-decreasing sequence for all $\omega > 0$. Consequently, there exist $H(\omega) \leq 1$ such that $\lim_{n\to\infty} H_n(\omega) = H(\omega)$. Now, we claim that $H(\omega) = 1$. Suppose, to the contrary that $0 < H(\omega_0) < 1$ for some $\omega_0 > 0$. Since $H_n(\omega_0) \geq H(\omega_0)$, by taking the limit with $\omega = \omega_0$. We obtain

$$H(\omega_0) \geq \lambda^{\frac{1}{\alpha+\beta}} H(\omega_0) > H(\omega_0).$$

Which is a contradiction and hence, $H(\omega) = 1$ for all $\omega > 0$. Now, we show $\{u_n\}$ is a cauchy sequence. Assuming this is not true, then there exist $\epsilon \in (0, 1)$ and $\omega_0 > 0$ such that for all $k \in \mathbb{N}$, there are $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$F(u_{m(k)}, u_{n(k)}, \omega_0) \leq 1 - \epsilon.$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the above inequality, that is equivalently,

$$F(u_{m(k)-1}, u_{n(k)}, \omega_0) > 1 - \epsilon,$$

and for all k we get

$$\begin{aligned} 1 - \epsilon &\geq F(u_{m(k)}, u_{n(k)}, \omega) \\ &\geq F(u_{m(k)-1}, u_{m(k)}, \omega) * F(u_{m(k)-1}, u_{n(k)}, \omega) \\ &> H_{m(k)}(\omega_0) * (1 - \epsilon). \end{aligned} \tag{3.11}$$

Putting limit $n \to \infty$ in (3.11), we get that

$$\lim_{n\to\infty} F(u_{m(k)}, u_{n(k)}, \omega_0) = 1 - \epsilon,$$

from

$$F(u_{m(k)+1}, u_{n(k)+1}, \omega_0) \geq F(u_{m(k)+1}, u_{m(k)}, \omega_0) * F(u_{m(k)}, u_{n(k)}, \omega_0) * F(u_{n(k)}, u_{n(k)+1}, \omega_0),$$

and

$$F(u_{m(k)}, u_{n(k)}, \omega_0) \geq F(u_{m(k)}, u_{m(k)+1}, \omega_0) * F(u_{m(k)}, u_{n(k)+1}, \omega_0) * F(u_{n(k)+1}, u_{n(k)}, \omega_0),$$

we get

$$\lim_{n\to\infty} F(u_{m(k)+1}, u_{n(k)+1}, \omega_0) = 1 - \epsilon.$$

From Eq (3.8), we know that

$$F(u_{m(k)+1}, \Upsilon u_{m(k)}, \omega_0) = F(R, G, \omega_0)$$

and

$$F\left(u_{n(k)+1}, \Upsilon u_{n(k)}, \omega_0\right) = F\left(R, G, \omega_0\right).$$

So, by (3.7),

$$F\left(u_{m(k)+1}, u_{n(k)+1}, \omega_0\right) \geq \lambda\left(F\left(u_{m(k)}, u_{n(k)}, \omega\right)\right)^{\alpha}\left(F\left(u_{m(k)}, u_{m(k)+1}, \omega_0\right)\right)^{\beta}\left(F\left(u_{n(k)}, u_{n(k)+1}, \omega_0\right)\right)^{1-\alpha-\beta},$$

taking lim $k \to \infty$ we get

$$1 - \epsilon \geq \lambda(1 - \epsilon) > 1 - \epsilon.$$

Which is contradiction. Then $\{u_n\}$ is cauchy sequence. Since $(U, F, *)$ is a complete non-Archimedean fuzzy metric space and R is closed subset of U. Then there exist $u \in R$, such that $\lim_{n\to\infty} F(u_n, u, \omega) = 1$. Moreover,

$$\begin{aligned} F(R, G, \omega) &= F(u_{n+1}, \Upsilon(u_n), \omega) \geq F(u_{n+1}, u, \omega) * F(u, \Upsilon(u_n), \omega) \\ &\geq F(u_{n+1}, u, \omega) * F(u, u_{n+1}, \omega) * F(u_{n+1}, \Upsilon u_n, \omega) \\ &= F(u_{n+1}, u, \omega) * F(u, u_{n+1}, \omega) * F(R, G, \omega). \end{aligned}$$

This implies,

$$\begin{aligned} F(R, G, \omega) &\geq F(u_{n+1}, u, \omega) * F(u, \Upsilon(u_n), \omega) \\ &\geq F(u_{n+1}, u, \omega) * F(u, u_{n+1}, \omega) * F(R, G, \omega). \end{aligned}$$

Applying to limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned} F(R, G, \omega) &\geq 1 * \lim_{n\to\infty} F(u, \Upsilon(u_n), \omega) \\ &\geq 1 * 1 * F(R, G, \omega). \end{aligned}$$

That is,

$$\lim_{n\to\infty} F(u, \Upsilon(u_n), \omega) = F(R, G, \omega).$$

Therefore, $F(u, \Upsilon(u_n), \omega) \to F(u, G, \omega)$ as $n \to \infty$. Since G is approximately comact with respec to R, there exist $\xi \in R_0(\omega)$ such that

$$F(\xi, \Upsilon u, \omega) = F(R, G, \omega) = F(u_{n+1}, \Upsilon(u_n), \omega). \tag{3.12}$$

We show that $u = \xi$. If not, then

$$F(\xi, u_{n+1}, \omega) \geq \lambda\begin{pmatrix} (F(u, u_n, \omega))^{\alpha}(F(u, \xi, \omega))^{\beta} \\ (F(u_n, u_{n+1}, \omega))^{1-\alpha-\beta} \end{pmatrix},$$

taking limit as $n \to \infty$ gives

$$F(\xi, u, \omega) \geq \lambda(F(u, \xi, \omega))^{\beta} > (F(u, \xi, \omega))^{\beta}.$$

Which is a contradiction. Hence $F(u, \Upsilon u, \omega) = F(R, G, \omega) = F(\xi, \Upsilon\xi, \omega)$ that is, u is the best proximity point. We show that u is the unique best proximity point of Υ. Assume, on the contrary, that $0 < F(u, v, \omega) < 1$ for all $\omega > 0$ and $v \neq u$ is another best proximity point of Υ, i.e., $F(u, \Upsilon u, \omega) = F(R, G, \omega) = F(v, \Upsilon v, \omega)$ then from (3.7) we have

$$F(u, v, \omega) \geq \lambda\left((F(u, v, \omega))^{\alpha}(F(u, u, \omega))^{\beta}(F(v, v, \omega))^{1-\alpha-\beta}\right) > (F(u, v, \omega))^{\alpha}.$$

Which is contradiction and hence $F(u, v, \omega) = 1$ for all $\omega > 0$, that is $u = v$. □

3.3. Interpolative Hardy Rogers proximal contraction in non-Archimedean fuzzy metric space

Definition 3.7. *Let $(U, F, *)$ be a complete non-Archimedean fuzzy metric space, and $R, G \subseteq U$. A mapping $\Upsilon : R \to G$ is said to be interpolative Hardy Rogers type contraction, if there exist $\alpha, \beta, \gamma, \delta \in (0, 1)$ such that $\alpha + \beta + \gamma + \delta < 1$, and $\lambda \in [0, 1)$.*

$$F(u_1, u_2, \omega) \geq \lambda \begin{pmatrix} (F(v_1, v_2, \omega))^{\alpha} (F(v_1, u_1, \omega))^{\beta} (F(v_2, u_2, \omega))^{\gamma} \\ (F(v_1, u_2, \omega))^{\delta} (F(v_2, u_1, \omega))^{1-\alpha-\beta-\gamma-\delta} \end{pmatrix}, \tag{3.13}$$

for all $u_1, u_2, v_1, v_2 \in R$, $\omega > 0$ and $u_i \neq v_i$, $i \in \{1, 2\}$ with respect to $F(u_1, \Upsilon v_1, \omega) = F(R, G, \omega)$, $F(u_2, \Upsilon v_2, \omega) = F(R, G, \omega)$ and $F(u, v, \omega) > 0$.

Example 3.8. *Let $U = \mathbb{R}^2$ and define the function $F : U \times U \times (0, \infty) \to [0, 1]$ by*

$$F(u, v, \omega) = \frac{\omega}{\omega + d((u_1, v_1), (u_2, v_2))},$$

where $d((u_1, v_1), (u_2, v_2)) = \sqrt[2]{(u_2 - u_1)^2 + (v_2 - v_1)^2}$ for all $(u_1, v_1), (u_2, v_2) \in U$. Then $(U, F, *)$ is a non-Archimedean fuzzy metric space with $\breve{a} * \bar{e} = \breve{a}\bar{e}$ for all $\breve{a}, \bar{e} \in I$. Let R,G $\subseteq U$ defined by

$$\begin{aligned} R &= \{(0, u), u \in \mathbb{R}\}, \\ G &= \{(1, u), u \in \mathbb{R}\}. \end{aligned}$$

Define $F(R, G, \omega) = \sup\{F(u, v, \omega) : u \in R, v \in G \text{ and } \omega > 0\}$. Then $F(R, G, \omega) = \frac{\omega}{\omega+1}$, $R_0(\omega) = R$, $G_0(\omega) = G$. Define the mapping $\Upsilon : R \to G$ by

$$\Upsilon(0, u) = \begin{cases} (1, u), & \text{if } s \in [-1, 1], \\ \left(1, u^2\right), & \text{otherwise,} \end{cases}$$

for all $(0, u) \in R$. Then clearly $\Upsilon(R_0) \subseteq G_0$. We show that Υ is interpolative Hardy Rogers type contraction. For $u_1 = (0, 4)$, $v_1 = (0, 2)$, $u_2 = (0, 9)$, $v_2 = (0, 3)$, $\alpha = 0.01$, $\beta = 0.02$, $\gamma = 0.03$, $\delta = 0.04$, $\lambda = \frac{1}{4}$ then we have

$$F(u_1, \Upsilon v_1, \omega) = F((0, 4), \Upsilon(0, 2), 1) = F(R, G, \omega),$$

and

$$F(u_2, \Upsilon v_2, \omega) = F((0, 9), \Upsilon(0, 3), 1) = F(R, G, \omega).$$

This implies that,

$$\begin{aligned} F(u_1, u_2, \omega) &= F((0, 4), (0, 9), 1) \\ &\geq \lambda \begin{pmatrix} (F(v_1, v_2, \omega))^{\alpha} (F(v_1, u_1, \omega))^{\beta} (F(v_2, u_2, \omega))^{\gamma} \\ (F(v_1, u_2, \omega))^{\delta} (F(v_2, u_1, \omega))^{1-\alpha-\beta-\gamma-\delta} \end{pmatrix}, \end{aligned}$$

which yield,

$$0.4082 \geq 0.1129.$$

This shows that Υ *is a interpolative Hardy Rogers type contraction. However, for* $u_1 = (0,4), v_1 = (0,2)$ *and* $u_2 = (0,9), v_2 = (0,3), \lambda = 0.2$ *and* $\omega = 1$. *We know that*

$$F(u_1, \Upsilon v_1, \omega) = F((0,4), \Upsilon(0,2), 1) = F(R, G, \omega),$$

and

$$F(u_2, \Upsilon v_2, \omega) = F((0,9), \Upsilon(0,3), 1) = F(R, G, \omega).$$

Implies

$$\begin{aligned} F(u_1, u_2, \omega) &= F((0,4),(0,9),1) \\ &\geq \lambda \begin{pmatrix} (F(v_1, v_2, \omega)) + (F(v_1, u_1, \omega)) + (F(v_2, u_2, \omega)) \\ + (F(v_1, u_2, \omega)) + (F(v_2, u_2, \omega)) \end{pmatrix}, \end{aligned}$$

which yield,

$$0.1667 \ngeq 0.3201.$$

This is a contradiction. Hence, Υ *is not interpolative Hardy Rogers type contraction.*

Theorem 3.9. *Let* $(U, F, *)$ *be a complete non-Archimedean fuzzy metric space,* $R, G \subseteq U$ *such that* G *is approximately compact with respect to* R. *Let* $\Upsilon: R \to G$ *be a interpolative Hardy Rogers type proximal contraction. If* $R_0 \subseteq R$ *such that* $\Upsilon(R_0) \subseteq G_0$. *Then* Υ *admits a best proximity point.*

Proof. Let $u_0 \in R_0$. Since $\Upsilon(u_0) \in \Upsilon(R_0) \subseteq G_0$, there exist $u_1 \in R_0$ such that,

$$F(u_1, \Upsilon(u_0), \omega) = F(R, G, \omega).$$

Also, we have $\Upsilon(u_1) \in \Upsilon(R_0) \subseteq G_0$, so there exist $u_2 \in R_0$ such that,

$$F(u_2, \Upsilon(u_1), \omega) = F(R, G, \omega).$$

This process of existence of point in R_0 implies to have a sequence $\{u_n\} \subseteq R_0$ such that,

$$F(u_{n+1}, \Upsilon(u_n), \omega) = F(R, G, \omega), \tag{3.14}$$

for all $n \in \mathbb{N}$. Observe that, if there exist $n \in \mathbb{N}$ such that $u_n = u_{n+1}$ then from (3.14), the point u_n is a best proximity point of the mapping Υ. On the other hand, if $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Then by (3.14), we have

$$F(u_n, \Upsilon(u_{n-1}), \omega) = F(R, G, \omega),$$

and

$$F(u_{n+1}, \Upsilon(u_n), \omega) = F(R, G, \omega),$$

for all $n \geq 1$, thus, by (3.13),

$$\begin{aligned} F(u_n, u_{n+1}, \omega) \geq\ & \lambda (F(u_{n-1}, u_n, \omega))^{\alpha} (F(u_{n-1}, u_n, \omega))^{\beta} (F(u_n, u_{n+1}, \omega))^{\gamma}, \\ & (F(u_n, u_n, \omega))^{\delta} (F(u_{n-1}, u_{n+1}, \omega))^{1-\alpha-\beta-\gamma-\delta}, \end{aligned} \tag{3.15}$$

for all distinct $u_{n-1}, u_n, u_{n+1} \in R$. Since, by (3.15), we have

$$\begin{aligned} F(u_n, u_{n+1}, \omega) &\geq \lambda (F(u_{n-1}, u_n, \omega))^{\alpha+\beta} (F(u_n, u_{n+1}, \omega))^{\gamma} (F(u_{n-1}, u_{n+1}, \omega))^{1-\alpha-\beta-\gamma-\delta} \\ &\geq \lambda (F(u_{n-1}, u_n, \omega))^{\alpha+\beta} (F(u_n, u_{n+1}, \omega))^{\gamma} \\ &\quad (F(u_{n-1}, u_n, \omega))^{1-\alpha-\beta-\gamma-\delta} (F(u_n, u_{n+1}, \omega))^{1-\alpha-\beta-\gamma-\delta} \\ &\geq \lambda (F(u_{n-1}, u_n, \omega))^{1-\gamma-\delta} (F(u_n, u_{n+1}, \omega))^{1-\alpha-\beta-\delta} \end{aligned}$$

$$(F(u_n, u_{n+1}, \omega))^{\alpha+\beta+\delta} \geq \lambda (F(u_{n-1}, u_n, \omega))^{1-\gamma-\delta}. \tag{3.16}$$

So, by (3.16), let $H = F(u_n, u_{n+1}, \omega)$, we have $H_{n-1} < H_n$ for all $n \in \mathbb{N}$. This shows that the sequence $\{H_n\}$ is positive and strictly non-decreasing. Thus, it converges to some element $H \geq 1$. Now from (3.16), we have

$$\begin{aligned} F(u_n, u_{n+1}, \omega) &\geq \lambda^{\frac{1}{\alpha+\beta+\delta}} F(u_{n-1}, u_n, \omega)^{\frac{1-\gamma-\delta}{\alpha+\beta+\delta}} \\ &\geq \lambda^{\frac{2}{\alpha+\beta+\delta}} F(u_{n-2}, u_{n-1}, \omega)^{\frac{1-\gamma-\delta}{\alpha+\beta+\delta}} \\ &\vdots \\ &\geq \lambda^{\frac{n}{\alpha+\beta+\delta}} F(u_1, u_0, \omega)^{\frac{1-\gamma-\delta}{\alpha+\beta+\delta}}. \end{aligned}$$

Then $H_{n-1}(\omega) < H_n(\omega)$, that is the sequence $\{H_n\}$ is non-decreasing sequence for all $\omega > 0$. Consequently, there exist $H(\omega) \leq 1$ such that $\lim_{n\to\infty} H_n(\omega) = H(\omega)$. Now, we claim that $H(\omega) = 1$. Suppose, to the contrary that $0 < H(\omega_0) < 1$ for some $\omega_0 > 0$. Since $H_n(\omega_0) \geq H(\omega_0)$, by taking the limit with $\omega = \omega_0$. We obtain

$$H(\omega_0) \geq \lambda^{\frac{1}{\alpha+\beta+\delta}} H(\omega_0) > H(\omega_0).$$

Satisfying the above inequality, that is equivalently,

$$F(u_{m(k)-1}, u_{n(k)}, \omega_0) > 1 - \epsilon,$$

and for all k we get

$$\begin{aligned} 1 - \epsilon &\geq F(u_{m(k)}, u_{n(k)}, \omega) \\ &\geq F(u_{m(k)}, u_{n(k)}, \omega) * F(u_{m(k)}, u_{n(k)}, \omega) \\ &\geq H_{m(k)}(\omega_0) * (1 - \epsilon), \end{aligned} \tag{3.17}$$

putting limit $n \to \infty$ in (3.17), we get that

$$\lim_{n\to\infty} F(u_{m(k)}, u_{n(k)}, \omega_0) = 1 - \epsilon,$$

from

$$F(u_{m(k)+1}, u_{n(k)+1}, \omega_0) \geq F(u_{m(k)+1}, u_{m(k)}, \omega_0) * F(u_{m(k)}, u_{n(k)}, \omega_0) * F(u_{n(k)}, u_{n(k)+1}, \omega_0),$$

and

$$F(u_{m(k)}, u_{n(k)}, \omega_0) \geq F(u_{m(k)}, u_{m(k)+1}, \omega_0) * F(u_{m(k)}, u_{n(k)+1}, \omega_0) * F(u_{n(k)+1}, u_{n(k)}, \omega_0),$$

we get

$$\lim_{n\to\infty} F\left(u_{m(k)+1}, u_{n(k)+1}, \omega_0\right) = 1 - \epsilon.$$

From Eq (3.14), we know that

$$F\left(u_{m(k)+1}, \Upsilon u_{m(k)}, \omega_0\right) = F\left(R, G, \omega_0\right) \text{ and } F\left(u_{n(k)+1}, \Upsilon u_{n(k)}, \omega_0\right) = F\left(R, G, \omega_0\right),$$

so by (3.13),

$$\begin{aligned} F\left(u_{m(k)+1}, u_{n(k)+1}, \omega_0\right) \geq\ & \lambda\left(F\left(u_{m(k)}, u_{n(k)}, \omega\right)\right)^{\alpha}\left(F\left(u_{m(k)}, u_{m(k)+1}, \omega_0\right)\right)^{\beta}\left(F\left(u_{n(k)}, u_{n(k)+1}, \omega_0\right)\right)^{\gamma} \\ & \left(F\left(u_{m(k)}, u_{n(k)+1}, \omega\right)\right)^{\delta}\left(F\left(u_{n(k)}, u_{m(k)+1}, \omega\right)\right)^{1-\alpha-\beta-\gamma-\delta}. \end{aligned}$$

Taking lim $k \to \infty$ we get

$$1 - \epsilon \geq \lambda(1 - \epsilon) > 1 - \epsilon.$$

Which is a contradiction. Then $\{u_n\}$ is cauchy sequence. Since $(U, F, *)$ is a complete non-Archimedean fuzzy metric space and R is closed subset of U. Then there exist $u \in R$, such that $\lim_{n\to\infty} F\left(u_n, u, \omega\right) = 1$. Moreover,

$$\begin{aligned} F(R, G, \omega) &= F(u_{n+1}, \Upsilon(u_n), \omega) \\ &\geq F\left(u_{n+1}, u, \omega\right) * F(u, \Upsilon(u_n), \omega) \\ &\geq F\left(u_{n+1}, u, \omega\right) * F(u, u_{n+1}, \omega) * F\left(u_{n+1}, \Upsilon u_n, \omega\right) \\ &= F\left(u_{n+1}, u, \omega\right) * F(u, u_{n+1}, \omega) * F(R, G, \omega). \end{aligned}$$

This implies,

$$\begin{aligned} F(R, G, \omega) &\geq F\left(u_{n+1}, u, \omega\right) * F(u, \Upsilon(u_n), \omega) \\ &\geq F\left(u_{n+1}, u, \omega\right) * F(u, u_{n+1}, \omega) * F(R, G, \omega). \end{aligned}$$

Applying to limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned} F(R, G, \omega) &\geq 1 * \lim_{n\to\infty} F(u, \Upsilon(u_n), \omega) \\ &\geq 1 * 1 * F(R, G, \omega). \end{aligned}$$

That is,

$$\lim_{n\to\infty} F(u, \Upsilon(u_n), \omega) = F(R, G, \omega).$$

Therefore, $F(u, \Upsilon\left(u_n\right), \omega) \to F(u, G, \omega)$ as $n \to \infty$. Since G is approximately compact with respect to R, there exist $\xi \in R_0\left(\omega\right)$ such that,

$$F\left(\xi, \Upsilon u, \omega\right) = F(R, G, \omega) = F(u_{n+1}, \Upsilon(u_n), \omega). \tag{3.18}$$

We now show that $u = \xi$. If not, then

$$\begin{aligned} F\left(\xi, u_{n+1}, \omega\right) \geq\ & \lambda\left(F\left(u, u_n, \omega\right)\right)^{\alpha}\left(F\left(u, \xi, \omega\right)\right)^{\beta}\left(F\left(u_n, u_{n+1}, \omega\right)\right)^{\gamma} \\ & \left(F\left(u, u_{n+1}, \omega\right)\right)^{\delta}\left(F\left(u_n, \xi, \omega\right)\right)^{1-\alpha-\beta-\gamma-\delta}, \end{aligned}$$

on taking limit as $n \to \infty$ gives

$$F(\xi, u, \omega) \geq \lambda (F(u, \xi, \omega))^{1-\alpha-\gamma-\delta} > (F(u, \xi, \omega))^{1-\alpha-\gamma-\delta}.$$

Which is a contradiction. Hence $F(u, \Upsilon u, \omega) = F(R, G, \omega) = F(\xi, \Upsilon\xi, \omega)$ that is, u is the best proximity point. We show that u is the unique best proximity point of Υ. Assume, on the contrary, that $0 < F(u, v, \omega) < 1$ for all $\omega > 0$ and $v \neq u$ is another best proximity point of Υ, i.e., $F(u, \Upsilon u, \omega) = F(R, G, \omega) = F(v, \Upsilon v, \omega)$ then from (3.13) we have

$$\begin{aligned} F(u, v, \omega) &\geq \lambda (F(u, v, \omega))^{\alpha} (F(u, u, \omega))^{\beta} (F(v, v, \omega))^{\gamma} (F(u, v, \omega))^{\delta} (F(v, u, \omega))^{1-\alpha-\beta-\gamma-\delta} \\ &> (F(u, v, \omega))^{1-\beta-\gamma}. \end{aligned}$$

Which is a contradiction and hence $F(u, v, \omega) = 1$ for all $\omega > 0$, that is $u = v$. This completes the proof. □

4. Conclusions

We have produced several new types of contractive condition that ensures the existence of best proximity points in non-Archimedean complete fuzzy metric spaces. The examples show that the new contractive conditions generalize the corresponding contractions given in earlier works. According to the nature (linear and nonlinear) of contractions (3.1), (3.7) and (3.13), these can be used to show the existence of solutions to fuzzy models of linear and nonlinear dynamic systems. The study carried out in this paper generalizes the valuable research work presented in [5, 14, 23–25].

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Conflict of interest

The authors declare that they have no competing interests.

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