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Research article

Composition operators on Hardy-Sobolev spaces with bounded reproducing kernels

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Abstract: For any real β let H^2_β be the Hardy-Sobolev space on the unit disc $\mathbb{D}$. H^2_β is a reproducing kernel Hilbert space and its reproducing kernel is bounded when $\beta > 1/2$. In this paper, we prove that C_φ has dense range in H^2_β if and only if the polynomials are dense in a certain Dirichlet space of the domain $\varphi(\mathbb{D})$ for $1/2 < \beta < 1$. It follows that if the range of C_φ is dense in H^2_β, then φ is a weak-star generator of H^∞, although the conclusion is false for the classical Dirichlet space $\mathfrak{D}$. Moreover, we study the relation between the density of the range of C_φ and the cyclic vector of the multiplier M^β_φ.

Keywords: Hardy-Sobolev space; composition operator; reproducing kernel; automorphism
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1. Introduction

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. For $f \in H(\mathbb{D})$ we use

$$\mathcal{R}f(z) = z\frac{\partial f}{\partial z}(z)$$

to denote the radial derivative of f at z. If $f(z) = \sum_{k=0}^\infty a_k z^k$ is the Taylor expansion of f, it is easy to see that

$$\mathcal{R}f(z) = \sum_{k=1}^\infty k a_k z^k.$$

More generally, for any real number β and any $f \in H(\mathbb{D})$ with the Taylor expansion above, we define

$$\mathcal{R}^\beta f(z) = \sum_{k=1}^\infty k^\beta a_k z^k$$

and call it the radial derivative of f of order β.

It is clear that these fractional radial differential operators satisfy $\mathcal{R}^{\alpha}\mathcal{R}^{\beta} = \mathcal{R}^{\alpha+\beta}$. When $\beta < 0$, the effect of $\mathcal{R}^{\beta}$ on f is actually "integration" instead of "differentiation". For example, radial differentiation of order -3 is actually radial integration of order 3.

For $\beta \in \mathbb{R}$, the Hardy-Sobolev space H^2_β consists of all analytic functions f on $\mathbb{D}$ such that $\mathcal{R}^{\beta} f$ belongs to the classical Hardy space H^2. It is clear that H^2_β is a Hilbert space with the inner product

$$\langle f, g\rangle_\beta = f(0)\overline{g(0)} + \langle \mathcal{R}^{\beta} f, \mathcal{R}^{\beta} g\rangle_{H^2}.$$

The induced norm in H^2_β is then given by

$$\|f\|^2_\beta = |f(0)|^2 + \|\mathcal{R}^{\beta} f\|^2_{H^2}.$$

Recall that H^2 is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2_{H^2} = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 \, d\sigma(\zeta) < \infty,$$

where $d\sigma$ is the normalized Lebesgue measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$. It is well known that every function $f \in H^2$ has radial limits

$$f(\zeta) = \lim_{r\to 1^-} f(r\zeta)$$

for almost all $\zeta \in \mathbb{T}$. Moreover, the radial limit function $f(\zeta)$ above belongs to $L^2(\mathbb{T}, d\sigma)$. The inner product in H^2 can then be written as

$$\langle f, g\rangle_0 = \langle f, g\rangle_{H^2} = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)} \, d\sigma(\zeta),$$

and its induced norm on H^2 is given by

$$\|f\|^2_0 = \|f\|^2_{H^2} = \int_{\mathbb{T}} |f(\zeta)|^2 \, d\sigma(\zeta).$$

It is well known that a function $f \in H(\mathbb{D})$ belongs to H^2 if and only if

$$\int_{\mathbb{D}} |\mathcal{R} f(z)|^2 (1 - |z|^2) \, dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. See [22, 26, 27]. More generally, for any $t > -1$, we consider the weighted area measure

$$dA_t(z) = (t + 1)(1 - |z|^2)^t \, dA(z),$$

which is a probability measure on $\mathbb{D}$. The spaces

$$A^2_t = L^2(\mathbb{D}, dA_t) \cap H(\mathbb{D})$$

are called weighted Bergman spaces (with standard weights). When $t = 0$, we simply write A^2 for the ordinary Bergman spaces. The following result establishes a natural connection between Hardy-Sobolev spaces and weighted Bergman spaces via fractional derivatives.

Proposition 1. *[6] Suppose $\beta \in \mathbb{R}$ and $f \in H(\mathbb{D})$. Then the following conditions are equivalent.*

(a) $f \in H^2_\beta$.
(b) $\mathcal{R}^{\beta+1} f \in A^2_1$.

If N is a nonnegative integer with $N > \beta$, then the conditions above are also equivalent to

(c) $\mathcal{R}^N f \in A^2_{2(N-\beta)-1}$.

Hardy-Sobolev spaces contain many classical analytic function spaces as special cases. For example, $H^2_{-1/2}$ is the Bergman space A^2, H^2_0 is the Hardy space H^2, and $H^2_{1/2}$ is the Dirichlet space $\mathfrak{D}$ consisting of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty.$$

More generally, for any domain $G \subset \mathbb{C}$ and any positive measure $d\omega$ on G, we will use $A^2(G, d\omega)$ to denote the weighted Bergman space of analytic functions f on G such that

$$\int_G |f(z)|^2\, d\omega(z) < \infty.$$

Similarly, we use $\mathfrak{D}(G, d\omega)$ for the weighted Dirichlet space of analytic functions f on G with

$$\int_G |f'(z)|^2\, d\omega(z) < \infty.$$

When $d\omega$ is ordinary area measure, we will simply write $A^2(G)$ and $\mathfrak{D}(G)$.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map $\mathbb{D}$. For any Hilbert space H of analytic functions on $\mathbb{D}$ we consider the composition operator $C_\varphi : H \to H$ defined by $C_\varphi f = f \circ \varphi$. For $\beta < 1/2$, every composition operator is bounded on H^2_β. However, this is not so for $\beta \geq 1/2$. For example, not every composition operator is bounded on the Dirichlet space. There are conditions (in terms of Carleson type measures, for example) that tell us exactly when C_φ is bounded on $\mathfrak{D}$. See [11,20,29] for example.

The density of the range of a composition operator is an interesting problem. Bourdon and Roan studied the problem for the Hardy space (see [3, 21]) and Cima raised the problem for the Dirichlet space in [9]. In [7], we settled Cima's problem completely:

Theorem 2. *Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, non-constant, and $G = \varphi(\mathbb{D})$. Then the following two conditions are equivalent.*

(i) $C_\varphi : \mathfrak{D} \to \mathfrak{D}$ *is bounded and has dense range.*
(ii) φ *is univalent and the polynomials are dense in* $A^2(G)$.

In [3], Bourdon proved the following result.

Theorem 3. *If $G = \varphi(\mathbb{D})$, where φ is a weak-star generator of H^∞, then the polynomials are dense in $A^2(G)$.*

It is thus natural for us to consider the following problem.

Question 4. *Does the density of polynomials in $A^2(G)$ imply that φ is a weak-star generator of H^∞?*

In general, the answer is no. In fact, Sarason gave a condition in [23] for φ to be a weak-star generator of H^∞, which combined with the Corollary 2 in that paper yields a bounded simply connected domain G such that the polynomials are dense in $A^2(G)$ but any Riemann map $\varphi : \mathbb{D} \to G$ is not a weak-star generator of H^∞; see [3, 17].

In Section 2, we will give a necessary and sufficient condition for composition operators to have dense range on Hardy-Sobolev spaces. Our result shows that if φ is a univalent self-map of $\mathbb{D}$, then the density of polynomials in the weighted Dirichlet spaces

$$\mathfrak{D}\left(\varphi(\mathbb{D}), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA\right), \quad \frac{1}{2} < \beta < 1,$$

implies that φ is a weak-star generator of H^∞.

The density of the range of the composition operator C_φ is relative to the cyclic vectors of the multiplier M_φ^β with symbol φ defined as $M_\varphi^\beta f = \varphi f$ for any $f \in H_\beta^2$. In the last part of this paper, we discuss the relations between the density of C_φ on H_β^2 and the cyclic vectors of M_φ^β for $1/2 < \beta < 1$. Thank you for your cooperation.

2. Weak-star generators and composition operators

In [17], S. N. Mergeljan and A. P. Talmadjan showed that if sufficiently many slits are put in the unit disc then we can obtain a domain G such that the polynomials are dense in $A^2(G)$. By the Riemann mapping theorem, there is an analytic homeomorphism $\varphi : \mathbb{D} \to G$, so C_φ has dense range in $\mathfrak{D}$ by Theorem 2 but φ is not a weak-star generator of H^∞ by Corollary 2 of [23]. However, the boundary of the above domain is not a Jordan curve, the Riemann map may not be continuous up to the boundary, and φ does not belong to the disc algebra $A(\mathbb{D})$. Furthermore, $\varphi \notin \mathfrak{D}_{1-2\beta}$ for $1/2 < \beta < 1$, where

$$\mathfrak{D}_{1-2\beta} = \left\{ f \in H(\mathbb{D}) | f' \in A_{1-2\beta}^2 \right\}$$

is the weighted Dirichlet space with the norm

$$\|f\|_{\mathfrak{D}_{1-2\beta}} = \left[|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-2\beta} dA \right]^{\frac{1}{2}}.$$

Thus, for $\beta < 1$, Proposition 1 shows that $f \in H_\beta^2$ if and only if $Rf \in A_{1-2\beta}^2$ and hence $H_\beta^2 = \mathfrak{D}_{1-2\beta}$, see [6] for more details.

The following result is due to P. Bourdon.

Proposition 5. *(Corollary 3.7 in [3]) Let φ map $\mathbb{D}$ univalently onto $G \subset \mathbb{D}$. If the polynomials are dense in $A^2(G, (1 - |\varphi^{-1}|^2)dA)$, then $C_\varphi : H^2 \to H^2$ has dense range.*

Proposition 5 extends a result of Roan [21] and supplies additional examples of composition operators with dense range. As a special case of our next result, we see that the density of polynomials in $A^2(G, (1 - |\varphi^{-1}|^2)dA)$ is also a necessary condition for the density of the range of C_φ in H_β^2, that is, the converse of Bourdon's result above is also true.

We will use the notion $R(C_\varphi)$ to denote the range of a composition operator. The space on which C_φ acts is usually obvious from the context, or it will be specified whenever there is a possibility for confusion.

Theorem 6. *Suppose $\beta < 1$ and φ is a non-constant analytic self-map of $\mathbb{D}$. Then C_φ has dense range in $H^2_\beta = \mathfrak{D}_{1-2\beta}$ if and only if φ is univalent and the polynomials are dense in $\mathfrak{D}(G, (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$, where $G = \varphi(\mathbb{D})$.*

Proof. First assume that C_φ has dense range in $\mathfrak{D}_{1-2\beta}$. It is easy to see that φ must be univalent. In fact, if there are $z_1, z_2 \in \mathbb{D}$, $z_1 \neq z_2$, such that $\varphi(z_1) = \varphi(z_2)$, then for any $f \in \mathfrak{D}_{1-2\beta}$ we have $C_\varphi f(z_1) = C_\varphi f(z_2)$, which clearly contradicts the assumption that the range of C_φ is dense in $\mathfrak{D}_{1-2\beta}$. To prove that the polynomials are dense in $\mathfrak{D}(G, (1-|\varphi^{-1}(z)|^2)^{1-2\beta}dA)$, fix any $g_0 \in \mathfrak{D}(G, (1-|\varphi^{-1}(z)|^2)^{1-2\beta}dA)$. Since $C_\varphi g_0 \in \mathfrak{D}_{1-2\beta}$ and C_φ has dense range in $\mathfrak{D}_{1-2\beta}$, we can find a sequence $\{p_k\}$ of polynomials such that $\| C_\varphi p_k - C_\varphi g_0 \|_{\mathfrak{D}_{1-2\beta}} \to 0$ in $\mathfrak{D}_{1-2\beta}$. This, by a change of variables, is equivalent to $\|p_k - g_0\| \to 0$ in $\mathfrak{D}(G, (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$.

Conversely, assume that φ is univalent and the polynomials are dense in the space $\mathfrak{D}(G, (1-|\varphi^{-1}|^2)^{1-2\beta})^2dA)$. It is clear that C_φ is an invertible operator from $\mathfrak{D}(G, (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ onto $\mathfrak{D}_{1-2\beta}$, with the inverse being $C_{\varphi^{-1}}$. Thus, for any $g \in \mathfrak{D}_{1-2\beta}$ there is an $f \in \mathfrak{D}(G, (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ such that $C_\varphi f = g$. Let $\{p_k\}$ be a sequence of polynomials such that $p_k \to f$ in $\mathfrak{D}(G, (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$. Then, by a change of variables again,

$$\| C_\varphi p_k - g \|_{\mathfrak{D}_{1-2\beta}} = \| C_\varphi p_k - C_\varphi f \|_{\mathfrak{D}_{1-2\beta}} \to 0$$

in $\mathfrak{D}_{1-2\beta}$. This shows that the range of C_φ is dense in $\mathfrak{D}_{1-2\beta}$. □

However, if the image $\varphi(\mathbb{D})$ has infinite area, even if $\varphi \in A^2(\mathbb{D})$, then the polynomials may not be dense in $A^2(\varphi(\mathbb{D}))$. Here is an example.

Let $f(z) = 1/\sqrt[3]{z}$ be the principal branch of $1/\sqrt[3]{z}$ on $\mathbb{C} \setminus [0, +\infty)$. Then the function

$$\varphi(z) = f(1+z) = \frac{1}{\sqrt[3]{1+z}}$$

is analytic function on $\mathbb{D}$. It is obvious that φ belongs to A^2 and is univalent in the open unit disc, but $\varphi' \notin A^2$, that is, the region $\varphi(\mathbb{D})$ has infinite area. This implies that the polynomials are not dense in $A^2(\varphi(\mathbb{D}))$. In fact, if

$$g(w) = \varphi^{-1}(w) = \frac{1}{w^3} - 1,$$

then $g \notin A^2(\varphi(\mathbb{D}))$, but $g' \in A^2(\varphi(\mathbb{D}))$. However, g' cannot be approximated by polynomials in $A^2(\varphi(\mathbb{D}))$.

This example also implies that the Dirichlet space is not necessarily contained in the Bergman space on a general domain in the complex plane. See [7] and additional references there.

Proposition 7. *Suppose $\beta < 1$ and $\varphi \in \mathfrak{D}_{1-2\beta}$ is univalent. Then C_φ is an invertible operator from $\mathfrak{D}(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ onto $\mathfrak{D}_{1-2\beta}$ with the inverse being $C_{\varphi^{-1}}$. Moreover, C_φ preserves the Dirichlet semi-norms.*

Proof. This follows from an easy change of variables. We leave the routine details to the interested reader.

□

To further characterize the dense range of C_φ on $\mathfrak{D}_{1-2\beta}$ and its relation to weak-star generator of H^∞, we still need the following lemmas.

Lemma 8. *[23] A sequence $\{\psi_n\}_1^\infty$ in H^∞ converges weak-star to the function ψ if and only if it is uniformly bounded and converges piontwise to ψ on $\mathbb{D}$.*

Lemma 9. *Mergelyan's Theorem [24] If K is a compact subset of the plane whose complement is connected, then every complex function that is continuous on K and analytic on its (topological) interior can be uniformly approximated on K by polynomials.*

It follows from Proposition 7 that if $1/2 < \beta < 1$ and $\varphi \in \mathfrak{D}_{1-2\beta}$ is univalent, then

$$\varphi^{-1} \in \mathfrak{D}(\varphi(\mathbb{D}), (1 - |\varphi^{-1}(z)|^2)^{1-2\beta} dA).$$

A standard argument shows that the operators from Proposition 7 satisfy

$$C^*_{\varphi^{-1}} \tilde{K}_w = K_{\varphi^{-1}(w)}, \qquad C^*_\varphi K_z = \tilde{K}_{\varphi(z)},$$

where $\tilde{K}_w(u) = \tilde{K}(u, w)$ and $K_z(v) = K(v, z)$ are the reproducing kernels of $\mathfrak{D}(\varphi(\mathbb{D}), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA)$ at $w \in \varphi(\mathbb{D})$ and of $\mathfrak{D}_{1-2\beta}$ at $z \in \mathbb{D}$, respectively. Since $K(w, z)$ is continuous on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$, we know that $\tilde{K}(u, v)$ is also continuous on $\overline{\varphi(\mathbb{D})} \times \overline{\varphi(\mathbb{D})}$. Hence each function f in $\mathfrak{D}(\varphi(\mathbb{D}), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA)$ is continuous on $\overline{\varphi(\mathbb{D})}$ by properties of the reproducing kernel. In particular, φ^{-1} is continuous on $\overline{\varphi(\mathbb{D})}$. Furthermore, by Lemma 9, φ^{-1} can be uniformly approximated on $\varphi(\mathbb{D})$ by polynomials.

Proposition 10. *Suppose $1/2 < \beta < 1$ and φ is a univalent analytic self-map of $\mathbb{D}$ with $\varphi \in \mathfrak{D}_{1-2\beta}$. Then $Lat(M^\beta_\varphi) = Lat(M^\beta_z)$, where M^β_φ and M^β_z are multiplication operators on the weighted Bergman space $A^2_{1-2\beta}$, and $Lat(M^\beta_\varphi)$ and $Lat(M^\beta_z)$ are their invariant subspace lattices.*

Proof. Since $\varphi \in \mathfrak{D}_{1-2\beta}$, it is clear that M^β_φ is bounded on $A^2_{1-2\beta}$. Lemma 9 implies that there is a sequence $\{p_k\}$ of polynomials such that $p_k(z) \to \varphi^{-1}(z)$ uniformly on $\mathbb{D}$, and this implies that $p_k(\varphi(z)) \to z$ uniformly on $\mathbb{D}$. Thus

$$\int_{\mathbb{D}} |(p_k(\varphi)(z) - z) g(z)|^2 (1 - |z|^2)^{1-2\beta} dA(z) \to 0, \quad g \in A^2_{1-2\beta}.$$

This shows that $M^\beta_{p_k(\varphi)}$ converges to M^β_z in the weak operator topology. Hence, $Lat(M^\beta_\varphi) \subset Lat(M^\beta_z)$. The reversed inclusion is obvious, so we have $Lat(M^\beta_\varphi) = Lat(M^\beta_z)$. □

Corollary 11. *Suppose $1/2 < \beta < 1$ and φ is a univalent analytic self-map of $\mathbb{D}$ with $\varphi \in \mathfrak{D}_{1-2\beta}$. Then C_φ has dense range in $\mathfrak{D}_{1-2\beta}$ if and only if $H^\infty(\varphi(\mathbb{D}))$ is dense in $A^2(\varphi(\mathbb{D}), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA)$.*

Proof. This is a direct consequence of Theorem 6, because every bounded analytic function can be approximated by polynomials in the norm topology of $A^2(\varphi(\mathbb{D}), (1 - |\varphi^{-1}|^2)^{1-2\beta} dA)$. □

Theorem 12. *Suppose $1/2 < \beta < 1$ and φ is an analytic self-map of $\mathbb{D}$ such that C_φ is bounded on $\mathfrak{D}_{1-2\beta}$. If $R(C_\varphi)$ is dense in $\mathfrak{D}_{1-2\beta}$, then φ is a weak-star generator of H^∞.*

Proof. For any $f \in \mathfrak{D}_{1-2\beta}$ there is a sequence $\{p_k\}$ of polynomials such that

$$\|C_\varphi p_k - f\|_{\mathfrak{D}_{1-2\beta}} \to 0.$$

Note that

$$|p_k(\varphi)(z) - f(z)| = |\langle p_k(\varphi) - f, K_z \rangle| \le \|p_k(\varphi) - f\|_{\mathfrak{D}_{1-2\beta}} \|K_z\|_{\mathfrak{D}_{1-2\beta}},$$

where K_z is the reproducing kernel of $\mathfrak{D}_{1-2\beta}$ at z. Since $1/2 < \beta < 1$, the function $z \mapsto \|K_z\|_{\mathfrak{D}_{1-2\beta}} = \sqrt{K(z,z)}$ is bounded on $\mathbb{D}$. Thus, $p_k(\varphi)(z)$ converges uniformly to $f(z)$. Furthermore, $\|p_k(\varphi) - f\|_\infty \to 0$ as $k \to \infty$.

If $f \in H^\infty$, then for any $0 < r < 1$, $f_r(z) = f(rz) \in \mathfrak{D}_{1-2\beta}$. Choose $r_n \in (0,1)$ such that $r_n \to 1$ as $n \to \infty$, then $f_{r_n} \xrightarrow{w^*} f$ in H^∞ by the dominated convergence theorem. For any n, there is a sequence of polynomials $\{p_k^{(n)}\}$ such that $\|p_k^{(n)}(\varphi) - f_{r_n}\|_\infty \to 0$ as $k \to \infty$. Hence, we may find subsequence $\{k_n\}$ such that $p_{k_n}^{(n)}(\varphi) \xrightarrow{w^*} f$ in H^∞. It follows that

$$\{C_\varphi p_k : p_k \text{ is a polynomial}\} = \{p_k(\varphi) : p_k \text{ is a polynomial}\}$$

is weak-star dense in H^∞. □

It is clear that if C_φ maps H^2_β to itself and $1/2 < \beta < 1$, then $\varphi \in H^2_\beta \subset A(\mathbb{D})$. If $\beta \geq 1$, then $z \notin \mathfrak{D}_{1-2\beta}$, so the polynomials cannot be dense in $\mathfrak{D}_{1-2\beta}$. In this case, we need to consider higher order derivatives.

From the discussion above, we see that the density of polynomials in $\mathfrak{D}(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ for $1/2 < \beta < 1$ implies that φ is a weak-star generator of H^∞. On the other hand, φ being a weak-star generator of H^∞ implies that the polynomials are dense in the Dirichlet spaces $\mathfrak{D}$ and $\mathfrak{D}(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ for all $\beta \leq 1/2$.

It is intriguing for us to find some relationship between the density of $R(C_\varphi)$ on two different spaces $\mathfrak{D}_{1-2\beta_1}$ and $\mathfrak{D}_{1-2\beta_2}$ for $1/2 < \beta_1, \beta_2 < 1$. We already know that if C_φ has dense range in $\mathfrak{D}_{1-2\beta}$ for some $1/2 < \beta < 1$, then φ must be a weak-star generator of H^∞, which implies that φ is univalent on the closed unit disc $\overline{\mathbb{D}}$. However, this does not imply that the polynomials are dense in $\mathfrak{D}(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ for all $1/2 < \beta < 1$. In fact, for any given $1/2 < \beta_1 < \beta_2 < 1$ we can find an analytic self-map of $\mathbb{D}$ such that $\varphi \in \mathfrak{D}_{1-2\beta_1} \setminus \mathfrak{D}_{1-2\beta_2}$. Then the polynomials are not dense in $\mathfrak{D}(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta_2}dA)$ but they are dense in $\mathfrak{D}(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta_1}dA)$. Hence, there exists an analytic self-map φ of $\mathbb{D}$ such that C_φ has dense range in $\mathfrak{D}_{1-2\beta_1}$ but does not have dense range in $\mathfrak{D}_{1-2\beta_2}$. This also shows that φ being a weak-star generator of H^∞ does not imply that polynomials are dense in $\mathfrak{D}(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ for all $1/2 < \beta < 1$.

It is well-known that if φ is a weak-star generator of H^∞, then the polynomials are dense in the Bergman space $A^2(\varphi(\mathbb{D}))$, but the converse is not true in general. The following theorem gives a condition for the converse to hold for certain analytic self-maps of $\mathbb{D}$.

Theorem 13. *Suppose $1/2 < \beta < 1$ and $\varphi \in \mathfrak{D}_{1-2\beta}$ is an analytic map-self of $\mathbb{D}$ such that the polynomials are dense in $A^2(\varphi(\mathbb{D}))$. Then the following statements are equivalent to each other:*

(i) $\{C_\varphi p : p$ is a polynomial$\}$ is dense in $A^2_{1-2\beta}$.
(ii) φ is a weak-star generator of H^∞.
(iii) φ is univalent on the open unit disc.

Proof. If $\{C_\varphi p : p$ is a polynomial$\}$ is dense in $A^2_{1-2\beta}$, then φ is clearly univalent on $\mathbb{D}$ by the beginning of the proof of Theorem 6. This shows that (i) implies (iii).

To prove that (iii) implies (ii), assume that $\varphi \in \mathfrak{D}_{1-2\beta}$ is univalent on the open unit disc. Then φ is also univalent on the closed unit disc by Corollary 3.5 in [3] and the continuity of φ on $\overline{\mathbb{D}}$. Thus, φ^{-1} is continuous on $\overline{\varphi(\mathbb{D})}$. By Lemma 9, there is a sequence $\{p_k\}$ of polynomials such that p_k converges uniformly to φ^{-1}. Then $p_k \circ \varphi$ converges uniformly to $f(z) = z$. This implies that φ is a weak-star generator of H^∞ since z is the weak-star generator of H^∞.

Finally, let us assume that (ii) holds. Then for any $f \in H^\infty$ there exists a sequence $\{p_k\}$ of polynomials such that $p_k(\varphi(z))\varphi(z) \to f(z)$ pointwise on $\mathbb{D}$ and $\{\|p_k\|_\infty\}$ is bounded. By the dominated convergence theorem, we have $\|C_\varphi p_k - f\|_{A^2_{1-2\beta}} \to 0$ as $k \to \infty$. This shows that (ii) implies (i) and completes the proof of the theorem. □

3. Cyclic vectors and composition operators

Choosing $\beta = 0$ and $\beta = \pm 1/2$ in Theorem 6, we see that, for univalent functions $\varphi : \mathbb{D} \to \mathbb{D}$, $R(C_\varphi)$ is dense in $A^2(\mathbb{D})$ if and only if the polynomials are dense in $A^2(\varphi(\mathbb{D}), (1 - |\varphi^{-1}|^2)^2 dA)$, $R(C_\varphi)$ is dense in $H^2(\mathbb{D})$ if and only if the polynomials are dense in $A^2(\varphi(\mathbb{D}), (1 - |\varphi^{-1}|^2) dA)$ (see [3]), and $R(C_\varphi)$ is dense in $\mathfrak{D}$ if and only if the polynomials are dense in $A^2(\varphi(\mathbb{D}))$ (see [7]).

Closely related to these discussions, we mention the following result of Hedberg from [25].

Theorem 14. *If f is in the Bergman space A^2 and if f is the derivative of a univalent function, then f is a cyclic vector for A^2. Equivalently, if $\varphi \in \mathfrak{D}$ is univalent, then $H^\infty(\varphi(\mathbb{D}))$ is dense in $A^2(\varphi(\mathbb{D}))$.*

The proof of Theorem 14 in [25] is quite technical. If φ is univalent and $(\varphi^{-1})'$ can be approximated by polynomials on $\varphi(\mathbb{D})$, we will give a simpler proof for the density of $H^\infty(\varphi(\mathbb{D}))$ in $A^2(\varphi(\mathbb{D}))$. The above condition about $(\varphi^{-1})'$ seems natural because, as the (normalized) area of $\mathbb{D} = \varphi^{-1}(\varphi(\mathbb{D}))$, we have

$$\int_{\varphi(\mathbb{D})} |(\varphi^{-1})'|^2 dA = 1.$$

Thus, $(\varphi^{-1})' \in A^2(\varphi(\mathbb{D}))$.

Proposition 15. *Suppose φ is an analytic self-map of $\mathbb{D}$ and $\varphi \in \mathfrak{D}$. Then the function z belongs to $\overline{R(C_\varphi)}$ in $\mathfrak{D}$ if and only if φ is univalent and $(\varphi^{-1})'$ can be approximated by polynomials in $A^2(\varphi(\mathbb{D}))$.*

Proof. If φ is univalent and there is a sequence $\{p_k\}$ of polynomials such that

$$\int_{\varphi(\mathbb{D})} |(p_k - (\varphi^{-1})')(w)|^2 dA(w) \to 0,$$

then

$$\begin{aligned} &\int_{\varphi(\mathbb{D})} |(p_k - (\varphi^{-1})')(w)|^2 dA(w) \\ = &\int_{\mathbb{D}} |p_k(\varphi(z)) - (\varphi^{-1})'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z) \\ = &\int_{\mathbb{D}} |p_k(\varphi(z))\varphi'(z) - (\varphi^{-1})'(\varphi(z))\varphi'(z)|^2 dA(z) \\ = &\int_{\mathbb{D}} |p_k(\varphi(z))\varphi'(z) - 1|^2 dA(z) \to 0 \end{aligned}$$

as $k \to \infty$. Write

$$q_k(z) = \int_0^z p_k(u) du, \qquad k \geq 1.$$

Then q_k is also a polynomial for each k and

$$(C_\varphi q_k)'(z) = \left(\int_0^{\varphi(z)} p_k(u)du\right)' = p_k(\varphi(z))\varphi'(z).$$

Thus

$$\int_{\mathbb{D}} |(C_\varphi q_k)' - 1|^2 dA(z) \to 0, \qquad k \to \infty,$$

so the function z belongs to $\overline{R(C_\varphi)}$ in $\mathfrak{D}$.

Conversely, if the function z is in the closure of $R(C_\varphi)$ in $\mathfrak{D}$, then φ is obviously univalent (see the beginning of the proof of Theorem 6), and reversing the calculations above implies that there is a sequence $\{p_k\}$ of polynomials such that

$$\int_{\varphi(\mathbb{D})} |(p_k - (\varphi^{-1})')(w)|^2 dA(w) \to 0$$

as $k \to \infty$. This ends the proof. □

The following result gives a simpler proof for Hedberg's theorem (i.e., Theorem 14) under an additional assumption.

Proposition 16. *Suppose φ is an analytic self-map of $\mathbb{D}$ and $\varphi \in \mathfrak{D}$. If the function z belongs to $\overline{R(C_\varphi)}$ in $\mathfrak{D}$, then $H^\infty(\varphi(\mathbb{D}))$ is dense in $A^2(\varphi(\mathbb{D}))$.*

Proof. Assume $\tilde{f} \in A^2(\varphi(\mathbb{D}))$. Once again, $z \in \overline{R(C_\varphi)}$ implies that φ is univalent. Thus, there is an $f \in A^2(\mathbb{D})$ such that

$$\tilde{f}(w) = f(\varphi^{-1}(w))(\varphi^{-1})'(w).$$

Let p_k be the k-th partial sum of the Taylor series of f. Then

$$\|p_k - f\|_{A^2} \to 0, \quad k \to \infty.$$

By the formula of changing variables,

$$\|(p_k \circ \varphi^{-1})(\varphi^{-1})' - \tilde{f}\|_{A^2(\varphi(\mathbb{D}))} \to 0, \quad k \to \infty.$$

Since $z \in \overline{R(C_\varphi)}$, it follows from Proposition 15 that there is a sequence $\{q_n\}$ of polynomials such that q_n converges to $(\varphi^{-1})'$ in $A^2(\varphi(\mathbb{D}))$. For any $\epsilon > 0$ choose K_0 such that

$$\|(p_k \circ \varphi^{-1})(\varphi^{-1})' - \tilde{f}\|_{A^2(\varphi(\mathbb{D}))} < \frac{\epsilon}{2} \quad \text{for} \quad k \geq K_0.$$

Choose a positive integer N such that

$$\|(p_{K_0} \circ \varphi^{-1})(q_n - (\varphi^{-1})')\|_{A^2(\varphi(\mathbb{D}))} < \frac{\epsilon}{2} \quad \text{for} \quad n \geq N.$$

Then for $n \geq N$ we have

$$\begin{aligned} & \|(p_{K_0} \circ \varphi^{-1})q_n - \tilde{f}\|_{A^2(\varphi(\mathbb{D}))} \\ \leq \ & \|(p_{K_0} \circ \varphi^{-1})q_n - p_{K_0} \circ \varphi^{-1}(\varphi^{-1})'\|_{A^2(\varphi(\mathbb{D}))} \\ + \ & \|p_{K_0} \circ \varphi^{-1}(\varphi^{-1})' - \tilde{f}\|_{A^2(\varphi(\mathbb{D}))} < \epsilon. \end{aligned}$$

This shows that $H^\infty(\varphi(\mathbb{D}))$ is dense in $A^2(\varphi(\mathbb{D}))$. □

Theorem 17. *Suppose $1/2 < \beta < 1$ and φ is a univalent analytic self-map of $\mathbb{D}$ with $\varphi \in \mathfrak{D}_{1-2\beta}$. If C_φ has dense range in $\mathfrak{D}_{1-2\beta}$, then φ' is a cyclic vector for both M_z^β and M_φ^β on $\mathfrak{D}_{1-2\beta}$.*

Proof. Define
$$E_\varphi : A^2(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta}dA) \to A^2_{1-2\beta}$$
by
$$E_\varphi(f)(z) = (f \circ \varphi)(z)\varphi'(z).$$
Similarly, define
$$E_{\varphi^{-1}} : A^2_{1-2\beta} \to A^2(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$$
by
$$E_{\varphi^{-1}}(f)(w) = (f \circ \varphi^{-1})(w)(\varphi^{-1})'(w).$$
Direct calculation shows that both E_φ and $E_{\varphi^{-1}}$ are isometric operators and
$$E_\varphi E_{\varphi^{-1}} = I_{A^2_{1-2\beta}}, \quad E_{\varphi^{-1}} E_\varphi = I_{A^2(\varphi(\mathbb{D}),(1-|\varphi^{-1}|^2)^{1-2\beta}dA)},$$
are identity operators. Thus, for any function $f \in A^2_{1-2\beta}$ there is a function $\tilde{f} \in A^2(\varphi(\mathbb{D}), (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ such that $f(z) = \tilde{f}(\varphi(z))\varphi'(z)$.

Assume $\{p_k\}$ is a sequence of polynomials such that
$$\int_{\varphi(\mathbb{D})} |p_k(w) - \tilde{f}(w)|^2 (1-|\varphi^{-1}|^2)^{1-2\beta} dA(w) \to 0, \quad k \to \infty.$$
Then
$$\begin{aligned}
&\int_{\mathbb{D}} |p_k(\varphi)(z)\varphi'(z) - f(z)|^2 (1-|z|^2)^{1-2\beta} dA(z) \\
= &\int_{\mathbb{D}} |p_k(\varphi)(z)\varphi'(z) - \tilde{f}(\varphi(z))\varphi'(z)|^2 (1-|z|^2)^{1-2\beta} dA(z) \\
= &\int_{\mathbb{D}} |p_k(\varphi(z)) - \tilde{f}(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^{1-2\beta} dA(z) \\
= &\int_{\varphi(\mathbb{D})} |p_k(w) - \tilde{f}(w)|^2 (1-|\varphi^{-1}|^2)^{1-2\beta} dA(w) \to 0
\end{aligned}$$
as $k \to \infty$. Note $p_k(\varphi)(z)\varphi'(z) = p_k(M_\varphi)(\varphi')(z)$, this shows that φ' is a cyclic vector of M_φ on $\mathfrak{D}_{1-2\beta}$. By Proposition 10, φ' is also a cyclic vector of M_z on $\mathfrak{D}_{1-2\beta}$. □

4. Conclusions

In this paper, we show that C_φ has dense range in H^2_β if and only if the polynomials are dense in a certain Dirichlet space $\mathfrak{D}(G, (1-|\varphi^{-1}|^2)^{1-2\beta}dA)$ for $1/2 < \beta < 1$(see Theorem 6). It follows that if the range of C_φ is dense in H^2_β, then φ is a weak-star generator of H^∞(see Theorems 12 and 13). Moreover, the relation between the density of the range of C_φ and the cyclic vector of the multiplier M_φ^β is studied(see Theorem 17).

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Conflict of interest

The author declares no conflicts of interest in this paper.

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