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Research article

A new approach of soft rough sets and a medical application for the diagnosis of Coronavirus disease

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Abstract: Rough set and soft set theories presents the mathematical foundations for studying decision making problems in different contexts. Some authors have established their own approaches regarding this theory, such as the "soft pre-rough approximation" and "soft β-rough approaximation". In this study, the rationale and results of these two approaches were rigorously analyzed and it was concluded that they are the same. In addition, it was proven that some of the results established with the aforementioned approaches are not true, so we present two proposed modifications to the soft rough approximations, one of which represents an improvement in accuracy with respect to the exposed methods. The approaches addressed in this document were implemented to diagnose COVID-19 in a contextualized situation of a group of patients in Colombia, showing that our proposal obtained the highest accuracy. In addition, an algorithm was designed, which allows analyzing data with a larger universe and set of parameters than those presented in the theoretical and practical examples.

Keywords: soft sets; soft rough sets; soft pre-rough sets; information system; decision making
Mathematics Subject Classification: Primary 26D07, 26D15; Secondary 26A51, 39B62

1. Introduction

In many situations of daily life, making a good decision is often complicated, because depending on the choice, one can succeed or fail. For this reason, several attributes or characteristics must be analyzed in order to make the appropriate decision that will lead us to satisfactorily solve a certain problem in our in our daily lives. For such situations, mathematics provides some tools and methods to determine the best option. Among these tools is rough set theory, which has been extensively studied and applied, as mentioned in [7], and has recently been extended through various approaches derived from combining this theory with others, as can be found in the papers [3–6, 8, 9]. With these approaches, researchers have provided improvements and advantages to address various decision making problems dealing with uncertainty and/or vagueness in real-life information systems, e.g., in [11] the authors proposed an approach for multiple attribute group decision making (MAGDM) problems with different evaluation attribute set based on variable precision diversified attribute multigranulation fuzzy rough set and VIKOR method, in [3] presented a granular computing model by merging soft set and rough set theory with linguistic value information and also a new way of dealing with multiple attribute decision making problems based on the concept linguistic value rough soft set and the VIKOR method, in [8] proposed an algorithmic approach using rough set theory together with two hybridized distance measures applying Hausdorff, Hamming and Euclidean distances under picture fuzzy environment where the evaluating information regarding students, subjects and student's features are given in picture fuzzy numbers, in [9] considered a combination of autoregressive integrated moving average (ARIMA), double exponential smoothing (DES) and Grey model (GM) applying the rough set theory to forecast sugarcane production in India and performed the comparative analysis of the single time series and rough set combination methods by underlying mean absolute percentage error (MAPE) criterion, in [5] presented a method to generate a soft rough approximation as a modification and generalization of the approach of Zhaowen et al., which turned out to be applicable to a decision-making problem related to the symptoms of Coronavirus patients and designed an algorithm that they programmed in MATLAB to obtain the results. Also using a fusion of rough set theory and soft set theory, two mathematical models have been presented that have great utility in making decisions about data sets associated with everyday problems, as can be seen in the studies done by El Sayed et al. [6] and El-Babbly and El Atik [4], where modifications of the soft rough approximations, called soft pre-rough approximations and soft β-rough approximations, respectively, were introduced. The proposals made in [6] and [4] turned out to be applicable to real-life problems, as is the case of detecting people who are more likely to be infected with COVID-19; however, in these paper, some mistakes were made and some fundamental hypotheses were omitted in order to establish certain results.

The main objective of this paper is to provide a new mathematical method based on soft rough approximations, which can be used to address some real life problems considering the characteristics or attributes present in the data related to the information available. First, we present the theoretical details of the soft rough approximations, and second, we review some of the results related to the soft pre-rough approximations and the soft β-rough approximations, and also show that these two approximations are theoretically the same. Later, motivated by the review done, we propose a new approach by using soft rough sets, which we call soft κ-rough approximations, we establish its basic properties and its relationship with the other approaches mentioned above. Subsequently, by using soft pre-rough aproximations, soft κ-rough approximations and the theory of closure spaces, we construct a new modification of soft rough approximations called soft pre-κ-rough approximations, we present the mathematical properties related to this model and show that this constitutes an approach where the accuracy of the approximation is higher than in the other approximations mentioned in this paragraph. The relevance of the present pre-κ-approximations is not only that the boundary regions are reduced or eliminated, but also that they have been obtained from the combination of the theory of soft rough sets and the theory of closure spaces, so they could be extended from a topological point of view by producing new models of soft rough sets in future works, as was done in [1] with rough sets. Finally, we exhibit a practical application of the proposed method in decision making for information systems obtained from the COVID-19 medical diagnosis and present an algorithm to perform the pertinent computations.

2. Preliminaries on soft rough sets

Throughout this work, let U and Ξ be two nonempty sets, called the universal set and the parameter set, respectively. Also, let $\mathcal{P}(X)$ be the power set of X and Λ be a subset of Ξ. Recall that a pair (F, Λ) is a soft set over U, if F is a mapping given by $F : \Lambda \to \mathcal{P}(U)$. In other words, a soft set over U is a parameterized family of subsets of U. For $\lambda \in \Lambda$, $F(\lambda)$ may be considered as the set of λ-approximate elements of the soft set (F, Λ). Observe that for each $\lambda \in \Lambda$, $F(\lambda)$ is a crisp set. If $S = (F, \Lambda)$ is a soft set over U, then the pair $A_s = (U, S)$ is said to be a soft approximation space. We refer to [4, 6] for more details on the definitions and results presented in this section.

Definition 2.1. *[4, 6] Let $S = (F, \Lambda)$ be a soft set over U, $A_S = (U, S)$ be a soft approximation space and X be a subset of U. The soft A_s-lower aproximation and the soft A_s-upper approximation of X are defined, respectively, as follows:*

$$\underline{S}(X) = \{u \in U : \exists \lambda \in \Lambda,\ [u \in F(\lambda) \subseteq X]\}$$

and

$$\overline{S}(X) = \{u \in U : \exists \lambda \in \Lambda,\ [u \in F(\lambda),\ F(\lambda) \cap X \neq \emptyset]\}.$$

Throughout this work, we will refer to $\underline{S}(X)$ and $\overline{S}(X)$ as the soft rough approximations of X with respect to A_S.

Definition 2.2. *[4, 6] Let $A_S = (U, S)$ be a soft approximation space and X be a subset of U. The soft A_s-negative, A_s-positive, A_s-boundary regions and the A_s-accuracy of the soft A_s-approximations of X are defined, respectively, as follows:*

$$\mathrm{NEG}_{A_s}(X) = U - \overline{S}(X),\ \mathrm{POS}_{A_s}(X) = \underline{S}(X),\ \mathrm{BND}_{A_s}(X) = \overline{S}(X) - \underline{S}(X)$$

and

$$\mu_{A_s}(X) = \frac{\left|\underline{S}(X)\right|}{\left|\overline{S}(X)\right|},$$

where $\overline{S}(X) \neq \emptyset$.

Note, that if $\underline{S}(X) = \overline{S}(X) \neq \emptyset$, then $\mathrm{BND}_{A_s}(X) = \emptyset$ and $\mu_{A_s}(X) = 1$. In this case, $X \subseteq U$ is called a soft A_s-exact or soft A_s-definable set. Otherwise, X is said to be a soft A_s-rough set.

Proposition 2.3. *[4, 6] Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ be a soft approximation space. Then, for any subset X of U, we have:*

$$\underline{S}(X) = \bigcup_{\lambda \in \Lambda} \{F(\lambda) : F(\lambda) \subseteq X\} \ \textit{and}\ \overline{S}(X) = \bigcup_{\lambda \in \Lambda} \{F(\lambda) : F(\lambda) \cap X \neq \emptyset\}.$$

Figure 1 illustrates the soft rough approximation of a set. Here we have a geometric and intuitive interpretation of soft rough approximations. Given a parameter $\lambda \in \Lambda$, the set $F(\lambda) \subseteq U$ is represented by a small regular form. The soft A_S-lower approximation is the union of all sets $F(\lambda)$ contained in the target set, as shown in Figure 1 in orange color; while the soft A_S-upper approximation is the union of all sets $F(\lambda)$ that have a nonempty intersection with the target set. The soft A_S-boundary region is the difference between these two approximations, which contains all sets $F(\lambda)$ that cannot be classified with certainty as belonging to the target set or not, as shown in Figure 1 in green color. The soft A_S-negative region is the set of all elements of the universe U that belong neither to the soft A_S-lower approximation nor to the soft A_S-boundary region, as shown in Figure 1 with the part colored white. Note that there may exist elements x in the target set such that $x \notin F(\lambda)$ for each $\lambda \in \Lambda$, some of which are represented in Figure 1 with red dots. Also, it may be noted that, in general, the sets $F(\lambda)$ do not constitute a partition of U, nor a covering of U.

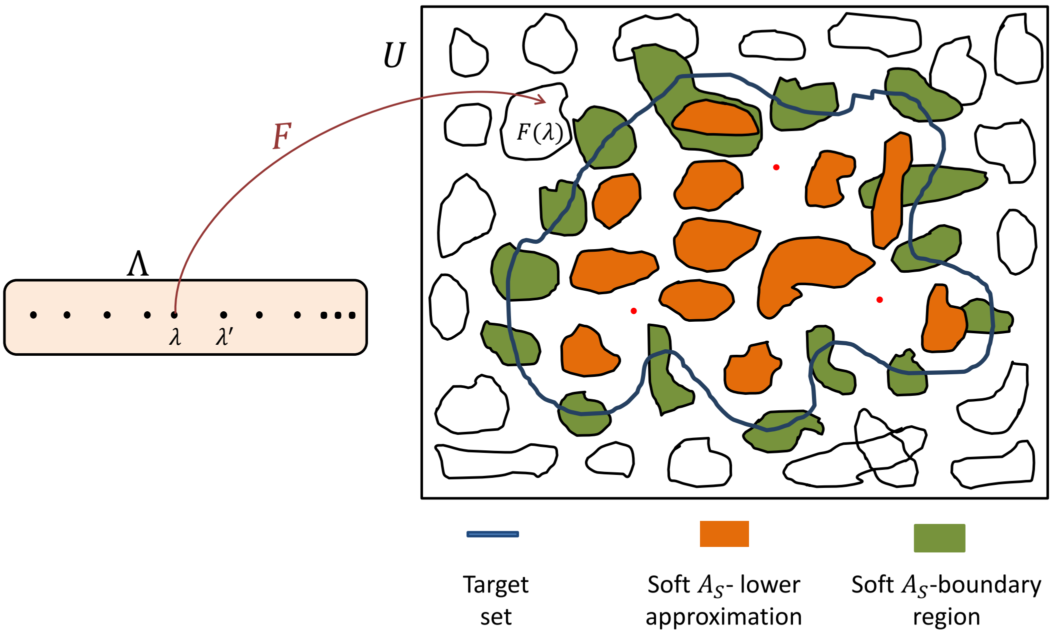

Figure 1. Representation of a soft rough set.

Proposition 2.4. *[4, 6] Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ be a soft approximation space. For any subsets X and Y of U, the following statements hold:*

(1) $\underline{S}(\emptyset) = \overline{S}(\emptyset) = \emptyset$.

(2) $\underline{S}(U) = \overline{S}(U) = \bigcup_{\lambda\in\Lambda} F(\lambda)$.

(3) If $X \subseteq Y$, *then* $\underline{S}(X) \subseteq \underline{S}(Y)$ *and* $\overline{S}(X) \subseteq \overline{S}(Y)$.

(4) $\underline{S}(X \cap Y) \subseteq \underline{S}(X) \cap \underline{S}(Y)$.

(5) $\underline{S}(X) \cup \underline{S}(Y) \subseteq \underline{S}(X \cup Y)$.

(6) $\overline{S}(X \cap Y) \subseteq \overline{S}(X) \cap \overline{S}(Y)$.

(7) $\overline{S}(X \cup Y) = \overline{S}(X) \cup \overline{S}(Y)$.

Proposition 2.5. *[4, 6] Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ be a soft approximation space. A subset X of U is soft A_S-exact if and only if $\overline{S}(X) \subseteq X$.*

Proposition 2.6. *[4, 6] Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ a soft approximation space. Then, for any subset X of U, we have:*

(1) $\underline{S}(\underline{S}(X)) = \underline{S}(X)$.
(2) $\underline{S}(X) \subseteq \overline{S}(\underline{S}(X))$.
(3) $\overline{S}(X) \subseteq \overline{S}(\overline{S}(X))$.
(4) $\underline{S}(\overline{S}(X)) = \overline{S}(X)$.

Definition 2.7. *[4, 6] Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ be a soft approximation space. Then, S is said to be a full soft set, if $U = \bigcup_{\lambda \in \Lambda} F(\lambda)$.*

Note that, if S is a full soft set, then for each $x \in U$, there exists $\lambda \in \Lambda$ such that $x \in F(\lambda)$.

Proposition 2.8. *[4, 6] Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ be a soft approximation space. Then, the following statements are equivalent:*

(1) S is full soft set.
(2) $\underline{S}(U) = U$.
(3) $\overline{S}(U) = U$.
(4) $X \subseteq \overline{S}(X)$, for any $X \subseteq U$.
(5) $\overline{S}(\{x\}) \neq \emptyset$, for any $x \in U$.

Corollary 2.9. *[4, 6] Let $S = (F, \Lambda)$ be a full soft set over U and $A_S = (U, S)$ be a soft approximation space. A subset X of U is soft A_S-exact if and only if $\overline{S}(X) = X$.*

Definition 2.10. *[4, 6] Let $S = (F, \Lambda)$ be a full soft set over U and $A_S = (U, S)$ be a soft approximation space. A subset X of U is said to be:*

(1) Roughly soft A_s-definable, if $\underline{S}(X) \neq \emptyset$ and $\overline{S}(X) \neq U$.
(2) Internally soft A_s-indefinable, if $\underline{S}(X) = \emptyset$ and $\overline{S}(X) \neq U$.
(3) Externally soft A_S-indefinable, if $\underline{S}(X) \neq \emptyset$ and $\overline{S}(X) = U$.
(4) Totally soft A_s-indefinable, if $\underline{S}(X) = \emptyset$ and $\overline{S}(X) = U$.

3. Soft pre-rough approximations and soft β-rough approximations revisited

Employing soft rough approximations, in 2020, El Sayed et al. [6] introduced a new approach to modify and generalize soft rough sets. In particular, they suggest new tools to approximate a set, called soft pre-rough approximations. Very recently, El-Bably and El Atik [4] also introduced other tools to approximate a set, which they called soft β-rough approximations. The main purpose of this section is to show that the soft pre-rough approximations and the soft β-rough approximations are the same, and hence, the approaches presented in [4, 6] are equal. In addition, we will show by means of examples that part (ii) of Theorem 3.1 (resp. Theorem 3.9 in [4]) and part (viii) of Proposition 3.1 in [6] (resp. Proposition 3.3 in [4]) are not true.

To achieve our purpose, we will first define the soft pre-rough approximations.

Definition 3.1. *[6] Let $S = (F, \Lambda)$ be a soft set over U, $A_S = (U, S)$ be a soft approximation space and X be a subset of U. The soft pre-lower aproximation and the soft pre-upper approximation of X are defined, respectively, as follows:*

$$\underline{S_p}(X) = X \cap \underline{S}(\overline{S}(X)) \text{ and } \overline{S_p}(X) = X \cup \overline{S}(\underline{S}(X)).$$

Definition 3.2. *[6] Let $A_S = (U, S)$ be a soft approximation space and X be a subset of U. The soft pre-negative, pre-positive, pre-boundary regions and the pre-accuracy of the soft pre-approximations of X are defined, respectively, as follows:*

$$\mathrm{NEG}_p(X) = U - \overline{S_p}(X),\ \mathrm{POS}_p(X) = \underline{S_p}(X),\ \mathrm{BND}_\kappa(X) = \overline{S_p}(X) - \underline{S_p}(X)$$

and

$$\mu_p(X) = \frac{\left|\underline{S_p}(X)\right|}{\left|\overline{S_p}(X)\right|},$$

where $\overline{S_p}(X) \neq \emptyset$.

Clearly, if $\underline{S_p}(X) = \overline{S_p}(X)$, then $\mathrm{BND}_p(X) = \emptyset$ and $\mu_p(X) = 1$. In this case, X is said to be a soft pre-exact or soft pre-definable set. Otherwise, X is called a soft pre-rough set.

Next, we will present the soft β-rough approximations.

Definition 3.3. *[4] Let $S = (F, \Lambda)$ be a soft set over U, $A_S = (U, S)$ be a soft approximation space and X be a subset of U. The soft β-lower aproximation and the soft β-upper approximation of X are defined, respectively, as follows:*

$$\underline{S_\beta}(X) = X \cap \overline{S}(\underline{S}(\overline{S}(X))) \quad \text{and} \quad \overline{S_\beta}(X) = X \cup \underline{S}(\overline{S}(\underline{S}(X))).$$

Definition 3.4. *[4] Let $A_S = (U, S)$ be a soft approximation space and X be a subset of U. The soft β-negative, β-positive, β-boundary regions and the β-accuracy of the soft pre-approximations of X are defined, respectively, as follows:*

$$\mathrm{NEG}_\beta(X) = U - \overline{S_\beta}(X),\ \mathrm{POS}_\beta(X) = \underline{S_\beta}(X),\ \mathrm{BND}_\beta(X) = \overline{S_\beta}(X) - \underline{S_\beta}(X)$$

and

$$\mu_\beta(X) = \frac{\left|\underline{S_\beta}(X)\right|}{\left|\overline{S_\beta}(X)\right|},$$

where $\overline{S_\beta}(X) \neq \emptyset$.

Observe that, if $\underline{S_\beta}(X) = \overline{S_\beta}(X)$, then $\mathrm{BND}_\beta(X) = \emptyset$ and $\mu_\beta(X) = 1$. In this case, X is called a soft β-exact or soft β-definable set. Otherwise, X is said to be a soft β-rough set.

Lemma 3.5. *Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ be a soft approximation space. For any $X \subseteq U$, the following statements hold:*

(1) $X \cap \overline{S}(X) \subseteq \overline{S}(X \cap \overline{S}(X)) \subseteq \overline{S}(X) \subseteq \overline{S}(\overline{S}(X))$.

(2) $X \cap \overline{S}(X) = X \cap \overline{S}(\overline{S}(X)) = X \cap \overline{S}(X \cap \overline{S}(X))$.

Proof. (1) Let $x \in B = X \cap \overline{S}(X)$. Then, $x \in X$ and $x \in \overline{S}(X)$, which implies that there exists $\lambda \in \Lambda$ such that $x \in X$, $x \in F(\lambda)$ and $F(\lambda) \cap X \neq \emptyset$. Since $x \in F(\lambda) \cap X \cap \overline{S}(X)$, we have $F(\lambda) \cap [X \cap \overline{S}(X)] \neq \emptyset$. Thus, there exists $\lambda \in \Lambda$ such that $x \in F(\lambda)$ and $F(\lambda) \cap B \neq \emptyset$. Therefore, $x \in \overline{S}(B) = \overline{S}(X \cap \overline{S}(X))$ and so, $X \cap \overline{S}(X) \subseteq \overline{S}(X \cap \overline{S}(X))$. On the other hand, by Proposition 2.6(3) and the monotony of $\overline{S}$ applied to the inclusion $X \cap \overline{S}(X) \subseteq X$, we get that $\overline{S}(X \cap \overline{S}(X)) \subseteq \overline{S}(X) \subseteq \overline{S}(\overline{S}(X))$.

(2) By Proposition 2.6(3), we have $X \cap \overline{S}(X) \subseteq X \cap \overline{S}(\overline{S}(X))$ for any $X \subseteq U$. To prove the opposite inclusion $X \cap \overline{S}(\overline{S}(X)) \subseteq X \cap \overline{S}(X)$, let $x \in X \cap \overline{S}(\overline{S}(X))$. Then, $x \in X$ and $x \in \overline{S}(\overline{S}(X))$, which implies that there exists $\lambda \in \Lambda$ such that $x \in X$, $x \in F(\lambda)$ and $F(\lambda) \cap \overline{S}(X) \neq \emptyset$. Thus, there exists $\lambda \in \Lambda$ such that $x \in F(\lambda) \cap X$, i.e. $F(\lambda) \cap X \neq \emptyset$. Therefore, $x \in \overline{S}(X)$ and so $x \in X \cap \overline{S}(X)$. This shows that $X \cap \overline{S}(X) = X \cap \overline{S}(\overline{S}(X))$. The equality $X \cap \overline{S}(\overline{S}(X)) = X \cap \overline{S}(X \cap \overline{S}(X))$ follows easily from part (1). □

Proposition 3.6. *Let $A_S = (U, S)$ be a soft approximation space. For any $X \subseteq U$, the following statements hold:*

(1) $\underline{S_\beta}(X) = \underline{S_p}(X)$.
(2) $\overline{S_\beta}(X) = \overline{S_p}(X)$.

Proof. (1) By Proposition 2.6(4), we have $\underline{S}(\overline{S}(X)) = \overline{S}(X)$ for any $X \subseteq U$. Thus,

$$\underline{S_\beta}(X) = X \cap \overline{S}(\underline{S}(\overline{S}(X))) = X \cap \overline{S}(\overline{S}(X)).$$

Then, by Lemma 3.5(2), it follows that

$$\underline{S_p}(X) = X \cap \underline{S}(\overline{S}(X)) = X \cap \overline{S}(X) = X \cap \overline{S}(\overline{S}(X)) = \underline{S_\beta}(X)$$

for any $X \subseteq U$.

(2) By virtue of Proposition 2.6(4), $\underline{S}(\overline{S}(Y)) = \overline{S}(Y)$ for any $Y \subseteq U$. In particular, for $Y = \underline{S}(X)$, we have $\underline{S}(\overline{S}(\underline{S}(X))) = \overline{S}(\underline{S}(X))$. Therefore,

$$\overline{S_\beta}(X) = X \cup \underline{S}(\overline{S}(\underline{S}(X))) = X \cup \overline{S}(\underline{S}(X)) = \overline{S_p}(X).$$

□

Corollary 3.7. *Let $A_S = (U, S)$ be a soft approximation space and X be a subset of U. Then, we have:*

(1) $\mathrm{NEG}_\beta(X) = \mathrm{NEG}_p(X)$.
(2) $\mathrm{POS}_\beta(X) = \mathrm{POS}_p(X)$.
(3) $\mathrm{BND}_\beta(X) = \mathrm{BND}_p(X)$.
(4) $\mu_\beta(X) = \mu_p(X)$.
(5) *X is a soft β-exact set if and only if it is a soft pre-exact set.*
(6) *X is a soft β-rough set if and only if it is a soft pre-rough set.*

Definition 3.8. *Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ be a soft approximation space. A subset X of U is said to be:*

(1) *Roughly soft pre-definable [6] (resp. roughly soft β-definable [4]), if $\underline{S_p}(X) \neq \emptyset$ and $\overline{S_p}(X) \neq U$ (resp. $\underline{S_\beta}(X) \neq \emptyset$ and $\overline{S_\beta}(X) \neq U$).*

(2) *Internally soft pre-indefinable [6] (resp. internally soft β-indefinable [4]), if $\underline{S_p}(X) = \emptyset$ and $\overline{S_p}(X) \neq U$ (resp. $\underline{S_\beta}(X) = \emptyset$ and $\overline{S_\beta}(X) \neq U$).*
(3) *Externally soft pre-indefinable [6] (resp. externally soft β-indefinable [4]), if $\underline{S_p}(X) \neq \emptyset$ and $\overline{S_p}(X) = U$ (resp. $\underline{S_\beta}(X) \neq \emptyset$ and $\overline{S_\beta}(X) = U$).*
(4) *Totally soft pre-indefinable [6] (resp. totally soft β-indefinable [4]), if $\underline{S_p}(X) = \emptyset$ and $\overline{S_p}(X) = U$ (resp. $\underline{S_p}(X) = \emptyset$ and $\overline{S_p}(X) = U$).*

Remark 3.9. *The notions presented in Definition 3.8 have been slightly modified with respect to the original definitions given in [6] and [4]. Here, the condition that $S = (F, \Lambda)$ be a full soft set over U has been omitted, since in some situations it is not necessary, due to the fact that $X \subseteq \overline{S_p}(X)$ (resp. $X \subseteq \overline{S_\beta}(X)$) for all $X \subseteq U$.*

Corollary 3.10. *Let $S = (F, \Lambda)$ be a soft set over U and $A_S = (U, S)$ be a soft approximation space. If X is a subset of U, then we have:*

(1) *X is a roughly soft β-definable set if and only if it is a roughly soft pre-definable set.*
(2) *X is an internally soft β-indefinable set if and only if it is an internally soft pre-indefinable set.*
(3) *X is an externally soft β-indefinable set if and only if it is an externally soft pre-indefinable set.*
(4) *X is a totally soft β-indefinable set if and only if it is a totally soft pre-indefinable set.*

Remark 3.11. *From Proposition 3.6 we infer that the results and applications presented in the articles [4, 6] correspond to the same approach.*

To finish this section, we turn our attention to show that the statements in part (ii) of Theorem 3.1 in [6] (resp. Theorem 3.9 in [4]) and part (viii) of Proposition 3.1 in [6] (resp. Proposition 3.3 in [4]) are false; which is, the inclusion $\overline{S_p}(X) \subseteq \overline{S}(X)$ (resp. $\overline{S_\beta}(X) \subseteq \overline{S}(X)$) and the equality $\overline{S_p}(X \cup Y) = \overline{S_p}(X) \cup \overline{S_p}(X)$ (resp. $\overline{S_\beta}(X \cup Y) = \overline{S_\beta}(X) \cup \overline{S_\beta}(X)$) are not true. To see this, we present the following two examples.

Example 3.12. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\Xi = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ and $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Let (F, Λ) be a soft set over U given by*

$$F(\lambda_1) = \{x_1, x_6\}, F(\lambda_2) = \{x_3\}, F(\lambda_3) = \emptyset, F(\lambda_4) = \{x_1, x_2, x_5\}.$$

Let us consider the soft approximation space $A_S = (U, S)$ and the subset $X = \{x_3, x_4, x_5\}$ of U. Then, $\overline{S}(X) = \{x_1, x_2, x_3, x_5\}$, which tells us that $X \not\subseteq \overline{S}(X)$. Also, since $\underline{S}(X) = \{x_3\}$, we have $\overline{S}(\underline{S}(X)) = \overline{S}(\{x_3\}) = \{x_3\}$, which implies that $\overline{S_p}(X) = X \cup \overline{S}(\underline{S}(X)) = \{x_3, x_4, x_5\} = X$, and hence $\overline{S_\beta} = \overline{S_p}(X) \not\subseteq \overline{S}(X)$.

Example 3.13. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\Xi = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ and $\Lambda = \{\lambda_1, \lambda_3, \lambda_5\}$. Let (F, Λ) be a soft set over U given by*

$$F(\lambda_1) = \{x_1, x_5\}, F(\lambda_3) = \{x_4\}, F(\lambda_5) = \{x_1, x_6\}.$$

Let us consider the soft approximation space $A_S = (U, S)$ and the subsets $X = \{x_1, x_3\}$ and $Y = \{x_3, x_5\}$ of U. Then,

$$\underline{S}(X) = \underline{S}(\{x_1, x_3\}) = \emptyset,\ \underline{S}(Y) = \underline{S}(\{x_3, x_5\}) = \emptyset,\ \overline{S}(\underline{S}(X)) = \overline{S}(\emptyset) = \emptyset,\ \overline{S}(\underline{S}(Y)) = \overline{S}(\emptyset) = \emptyset,$$
$$\overline{S}(\underline{S}(X \cup Y)) = \overline{S}(\underline{S}(\{x_1, x_3, x_5\})) = \overline{S}(\{x_1, x_5\}) = \{x_1, x_5, x_6\}.$$

Thus,

$$\overline{S_p}(X) \cup \overline{S_p}(Y) = [X \cup \overline{S}(\underline{S}(X))] \cup [Y \cup \overline{S}(\underline{S}(Y))] = (X \cup \emptyset) \cup (Y \cup \emptyset) = X \cup Y = \{x_1, x_3, x_5\}$$

and

$$\overline{S_p}(X \cup Y) = (X \cup Y) \cup \overline{S}(\underline{S}(X \cup Y)) = \{x_1, x_3, x_5\} \cup \{x_1, x_5, x_6\} = \{x_1, x_3, x_5, x_6\}.$$

Therefore,

$$\overline{S_p}(X \cup Y) = \{x_1, x_3, x_5, x_6\} \neq \{x_1, x_3, x_5\} = \overline{S_p}(X) \cup \overline{S_p}(Y).$$

4. New generalized soft rough aproximations

In this section, we propose a new approach to perform studies similar to those performed in [4, 6]. This new approach has some properties that will allow us to obtain a topology in Section 5, from which we will introduce another approach that has higher precision than the other approaches presented in soft rough set theory.

Definition 4.1. *Suppose that $S = (F, A)$ is a soft set over U, $A_S = (U, S)$ is a soft approximation space and X is a subset of U. The soft κ-lower aproximation and the soft κ-upper approximation of X are defined, respectively, as follows:*

$$\underline{S_\kappa}(X) = X \cap \overline{S}(\underline{S}(X)) \text{ and } \overline{S_\kappa}(X) = X \cup \overline{S}(X).$$

Throughout this work we will refer to $\underline{S_\kappa}(X)$ and $\overline{S_\kappa}(X)$ as the soft κ-rough approximations of X with respect to A_S.

Definition 4.2. *Let $A_S = (U, S)$ be a soft approximation space and X be a subset of U. The soft κ-negative region, the soft κ-positive region, the soft κ-boundary region and the soft κ-accuracy of X are defined, respectively, as follows:* $\mathrm{NEG}_\kappa(X) = U - \overline{S_\kappa}(X)$, $\mathrm{POS}_\kappa(X) = \underline{S_\kappa}(X)$, $\mathrm{BND}_\kappa(X) = \overline{S_\kappa}(X) - \underline{S_\kappa}(X)$ *and* $\mu_\kappa(X) = \dfrac{\left|\underline{S_\kappa}(X)\right|}{\left|\overline{S_\kappa}(X)\right|}$, *where* $\overline{S_\kappa}(X) \neq \emptyset$.

If $\underline{S_\kappa}(X) = \overline{S}_\kappa(X)$, then we say that X is a soft κ-exact or a soft κ-definable set. Otherwise, we say that X is a soft κ-rough set. Clearly, X is soft κ-exact if and only if $\mathrm{BND}_\kappa(X) = \emptyset$. Moreover, $\mu_\kappa(X) = 1$ implies that X is soft κ-exact.

In the following lemma we state the main properties of soft κ-rough approximations.

Lemma 4.3. *Let $S = (F, \Lambda)$ be a soft set over U, $A_S = (U, S)$ be a soft approximation space. For any subsets X and Y of U, the following statements hold:*

(1) $\underline{S_\kappa}(\emptyset) = \overline{S}_\kappa(\emptyset) = \emptyset$.

(2) $\underline{S_\kappa}(U) = \bigcup_{\lambda \in \Lambda} F(\lambda)$ *and* $\overline{S}_\kappa(U) = U$.

(3) If $X \subseteq Y$, *then* $\underline{S_\kappa}(X) \subseteq \underline{S_\kappa}(Y)$ *and* $\overline{S}_\kappa(X) \subseteq \overline{S}_\kappa(Y)$ *(monotony).*

(4) $\underline{S}_\kappa(X \cap Y) \subseteq \underline{S}_\kappa(X) \cap \underline{S}_\kappa(Y)$.
(5) $\underline{S}_\kappa(X) \cup \underline{S}_\kappa(Y) \subseteq \underline{S}_\kappa(X \cup Y)$.
(6) $\overline{S_\kappa}(X \cap Y) \subseteq \overline{S_\kappa}(X) \cap \overline{S_\kappa}(Y)$.
(7) $\overline{S_\kappa}(X \cup Y) = \overline{S_\kappa}(X) \cup \overline{S_\kappa}(Y)$.
(8) $\underline{S}(X) \subseteq \underline{S}_\kappa(X) \subseteq \underline{S}_p(X) \subseteq X \subseteq \overline{S_p}(X) \subseteq \overline{S_\kappa}(X)$ *and* $\overline{S}(X) \subseteq \overline{S_\kappa}(X)$.

Proof. (1) Obviously $\underline{S}_\kappa(\emptyset) = \emptyset \cap \overline{S}(\underline{S}(\emptyset)) = \emptyset$ and $\overline{S}_\kappa(\emptyset) = \emptyset \cup \overline{S}(\emptyset) = \emptyset \cup \emptyset = \emptyset$.

(2) By Proposition 2.4(2), we have $\overline{S}(U) = \underline{S}(U) = \bigcup_{\lambda\in\Lambda} F(\lambda) = F(\Lambda)$. Thus, by Proposition 2.6(2),

$$F(\Lambda) = \bigcup_{\lambda\in\Lambda} F(\lambda) = \underline{S}(U) \subseteq \overline{S}(\underline{S}(U)) = \overline{S}(F(\Lambda)) = F(\Lambda),$$

which implies that $\overline{S}(\underline{S}(U)) = \bigcup_{\lambda\in\Lambda} F(\lambda) \subseteq U$. Therefore,

$$\underline{S}_\kappa(U) = U \cap \overline{S}(\underline{S}(U) = \bigcup_{\lambda\in\Lambda} F(\lambda).$$

Also, we have

$$\overline{S_\kappa}(U) = U \cup \overline{S}(U) = U.$$

(3) By Proposition 2.4(3), we have $\underline{S}(X) \subseteq \underline{S}(Y)$ and $\overline{S}(X) \subseteq \overline{S}(Y)$, for any $X \subseteq Y$. Then,

$$\underline{S}_\kappa(X) = X \cap \overline{S}(\underline{S}(X)) \subseteq Y \cap \overline{S}(\underline{S}(Y)) = \underline{S}_\kappa(Y), \text{ for any } X \subseteq Y.$$

Also, we have

$$\overline{S_\kappa}(X) = X \cup \overline{S}(X) \subseteq Y \cup \overline{S}(Y) = \overline{S_\kappa}(Y), \text{ for any } X \subseteq Y.$$

(4) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, by monotony, it follows that $\underline{S}_\kappa(X \cap Y) \subseteq \underline{S}_\kappa(X)$ and $\underline{S}_\kappa(X \cap Y) \subseteq \underline{S}_\kappa(Y)$. Therefore, $\underline{S}_\kappa(X \cap Y) \subseteq \underline{S}_\kappa(X) \cap \underline{S}_\kappa(Y)$.

(5) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, by monotony the proof follows.

(6) Can be proved in a similar way to (4).

(7) The proof follows from the fact that $\overline{S}(X \cup Y) = \overline{S}(X) \cup \overline{S}(Y)$.

(8) By Proposition 2.6(2) and the monotony of $\overline{S}$ applied to the fact that $\underline{S}(X) \subseteq X$, we have $\underline{S}(X) \subseteq \overline{S}(\underline{S}(X)) \subseteq \overline{S}(X)$ for any $X \subseteq U$. Since $\underline{S}(X) \subseteq X$, it follows that

$$\underline{S}(X) \subseteq X \cap \overline{S}(\underline{S}(X)) \subseteq X \cap \overline{S}(X) \subseteq X \subseteq X \cup \overline{S}(\underline{S}(X)) \subseteq X \cup \overline{S}(X) \text{ for any } X \subseteq U.$$

This is, $\underline{S}(X) \subseteq \underline{S}_\kappa(X) \subseteq \underline{S}_p(X) \subseteq X \subseteq \overline{S_p}(X) \subseteq \overline{S_\kappa}(X)$. The inclusion $\overline{S}(X) \subseteq \overline{S_\kappa}(X)$ is obvious. □

Remark 4.4. *In Example 3.12, we have*

$$\overline{S_\kappa}(X) = X \cup \overline{S}(X) = \{x_3, x_4, x_5\} \cup \{x_1, x_2, x_3, x_5\} = \{x_1, x_2, x_3, x_4, x_5\} \not\subseteq \{x_1, x_2, x_3, x_5\} = \overline{S}(X).$$

Therefore, the inclusion $\overline{S_\kappa}(X) \subseteq \overline{S}(X)$*, in general, is not true.*

Corollary 4.5. *Let* $A_S = (U, S)$ *be a soft approximation space and* X *be a subset of* U*. Then, we have:*

(1) $\mathrm{BND}_p(X) \subseteq \mathrm{BND}_\kappa(X)$.
(2) If X is a soft pre-rough set, then it is a soft κ-rough set.
(3) If X is a soft κ-definable set, then it is a soft pre-definable set.
(4) $\mu_\kappa(X) \leq \mu_p(X)$.
(5) X is soft κ-definable if and only if $\underline{S}_\kappa(X) = \overline{S}_\kappa(X) = X$.

Proof. (1) By Lemma 4.3(8), we have $\underline{S}_\kappa(X) \subseteq \underline{S}_p(X) \subseteq \overline{S}_p(X) \subseteq \overline{S}_\kappa(X)$. Hence,

$$\mathrm{BND}_p(X) = \overline{S}_p(X) - \underline{S}_p(X) \subseteq \overline{S}_\kappa(X) - \underline{S}_\kappa(X) = \mathrm{BND}_\kappa(X).$$

(2) If X is a soft pre-rough set, then $\mathrm{BND}_p(X) \neq \emptyset$, and as $\mathrm{BND}_p(X) \subseteq \mathrm{BND}_\kappa(X)$, it follows that $\mathrm{BND}_\kappa(X) \neq \emptyset$. Therefore, X is a soft κ-rough set.

(3) If X is a soft κ-definable set, then $\mathrm{BND}_\kappa(X) = \emptyset$. Since $\mathrm{BND}_p(X) \subseteq \mathrm{BND}_\kappa(X)$, it follows that $\mathrm{BND}_p(X) = \emptyset$. Thus, X is a soft pre-definable set.

(4) Since $\underline{S}_\kappa(X) \subseteq \underline{S}_p(X) \subseteq \overline{S}_p(X) \subseteq \overline{S}_\kappa(X)$, we have $|\underline{S}_\kappa(X)| \leq |\underline{S}_p(X)| \leq |\overline{S}_p(X)| \leq |\overline{S}_\kappa(X)|$. Hence,

$$\mu_\kappa(X) = \frac{|\underline{S}_\kappa(X)|}{|\overline{S}_\kappa(X)|} \leq \frac{|\underline{S}_p(X)|}{|\overline{S}_p(X)|} = \mu_p(X).$$

(5) This is an immediate consequence of the inclusions $\underline{S}_\kappa(X) \subseteq X \subseteq \overline{S}_\kappa(X)$ and the definition of soft κ-definable set. □

Proposition 4.6. *Let $A_S = (U, S)$ be a soft approximation space. For any $X \subseteq U$, the following statements hold:*

(1) $\underline{S}_\kappa\left(\underline{S}_\kappa(X)\right) = \underline{S}_\kappa(X)$.
(2) $\overline{S}_\kappa(X) \subseteq \overline{S}_\kappa\left(\overline{S}_\kappa(X)\right)$.
(3) $\underline{S}_\kappa(X) \subseteq \overline{S}_\kappa\left(\underline{S}_\kappa(X)\right) \subseteq \overline{S}_\kappa(X)$.
(4) $\underline{S}_\kappa(X) \subseteq \underline{S}_\kappa\left(\overline{S}_\kappa(X)\right) \subseteq \overline{S}_\kappa(X)$.

Proof. (1) Since $\underline{S}_\kappa(X) \subseteq X$, we get that $\underline{S}_\kappa\left(\underline{S}_\kappa(X)\right) \subseteq \underline{S}_\kappa(X)$ by the monotony of $\underline{S}_\kappa$. To prove the opposite inclusion $\underline{S}_\kappa(X) \subseteq \underline{S}_\kappa(\underline{S}_\kappa(X))$, let us note that from Proposition 2.6, the inclusion $\underline{S}(X) \subseteq X$ and the monotony of $\underline{S}$ and $\overline{S}$, it follows that

$$\overline{S}(\underline{S}(X)) = \overline{S}(\underline{S}(\underline{S}(X))) = \overline{S}(\underline{S}(X \cap \underline{S}(X))) \subseteq \overline{S}(\underline{S}(X \cap \overline{S}(\underline{S}(X)))).$$

Thus,

$$\underline{S}_\kappa(X) = X \cap \overline{S}(\underline{S}(X)) \subseteq \overline{S}(\underline{S}(X)) \subseteq \overline{S}(\underline{S}(X \cap \overline{S}(\underline{S}(X)))) = \overline{S}(\underline{S}(\underline{S}_\kappa(X)))$$

and hence,

$$\underline{S}_\kappa(X) \subseteq \underline{S}_\kappa(X) \cap \overline{S}(\underline{S}(\underline{S}_\kappa(X))) = \underline{S}_\kappa(\underline{S}_\kappa(X)).$$

(2) Since $X \subseteq \overline{S}_\kappa(X)$, by monotony the proof follows.

(3) By Lemma 4.3(8), we have $Y \subseteq \overline{S}_\kappa(Y)$ for any $Y \subseteq U$. In particular, for $Y = \underline{S}_\kappa(X)$, we obtain that $\underline{S}_\kappa(X) \subseteq \overline{S}_\kappa(\underline{S}_\kappa(X))$. Also, since $\underline{S}_\kappa(X) \subseteq X$, by the monotony of $\overline{S}_\kappa$, it follows that $\overline{S}_\kappa(\underline{S}_\kappa(X)) \subseteq \overline{S}_\kappa(X)$.

(4) Since $X \subseteq \overline{S}_\kappa(X)$, by the monotony of $\underline{S}_\kappa$, it follows that $\underline{S}_\kappa(X) \subseteq \underline{S}_\kappa(\overline{S}_\kappa(X))$. Also, by Lemma 4.3(8), we have $\underline{S}_\kappa(Y) \subseteq Y$ for any $Y \subseteq U$. In particular, for $Y = \overline{S}_\kappa(X)$, we obtain that $\underline{S}_\kappa(\overline{S}_\kappa(X)) \subseteq \overline{S}_\kappa(X)$. □

Remark 4.7. *Observe that the inclusion relations in Proposition 4.6 may be strict, as shown in Examples 4.8 and 4.9.*

Example 4.8. *Consider the soft approximation space* $A_S = (U, S)$ *in Example 3.12 and let* $X = \{x_3, x_4, x_5\}$. *Then, we have* $\overline{S_\kappa}(X) = \{x_1, x_2, x_3, x_4, x_5\}$, *which implies that* $\overline{S_\kappa}(\overline{S_\kappa}(X)) = U$, *and hence* $\overline{S_\kappa}(\overline{S_\kappa}(X)) \neq \overline{S_\kappa}(X)$. *On the other hand, if* $Y = \{x_1, x_6\}$ *then* $\underline{S_\kappa}(Y) = Y$, $\overline{S_\kappa}(\underline{S_\kappa}(Y)) = \{x_1, x_2, x_5, x_6\}$ *and* $\underline{S_\kappa}(\overline{S_\kappa}(Y)) = \{x_1, x_2, x_5, x_6\}$. *Therefore,* $\overline{S_\kappa}(\underline{S_\kappa}(Y)) \neq \underline{S_\kappa}(Y)$ *and* $\underline{S_\kappa}(\overline{S_\kappa}(Y)) \neq \underline{S_\kappa}(Y)$.

Example 4.9. *Let* $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\Xi = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\}$ *and* $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. *Let* (F, Λ) *be a soft set over* U *given by*

$$F(\lambda_1) = \{x_1, x_6\}, F(\lambda_2) = \{x_3\}, F(\lambda_3) = \emptyset, F(\lambda_4) = \{x_1, x_2, x_5\}.$$

Let us consider the soft approximation space $A_S = (U, S)$ *and the subset* $X = \{x_1, x_4\}$. *Then, we have* $\overline{S_\kappa}(X) = \{x_1, x_2, x_4, x_5, x_6\}$, $\underline{S_\kappa}(X) = \emptyset$, $\overline{S_\kappa}(\underline{S_\kappa}(X)) = \emptyset$ *and* $\underline{S_\kappa}(\overline{S_\kappa}(X)) = \{x_1, x_2, x_5, x_6\}$. *Therefore,* $\overline{S_\kappa}(X) \neq \overline{S_\kappa}(\underline{S_\kappa}(X))$ *and* $\overline{S_\kappa}(X) \neq \underline{S_\kappa}(\overline{S_\kappa}(X))$. *Note that, in this case,* $\overline{S_\kappa}(\underline{S_\kappa}(X)) \neq \underline{S_\kappa}(\overline{S_\kappa}(X))$.

Proposition 4.10. *Let* $S = (F, \Lambda)$ *be a full soft set and* $A_S = (U, S)$ *be a soft approximation space. Then, we have:*

(1) $\underline{S_\kappa}(U) = U$.
(2) $\overline{S_\kappa}(X) = \overline{S}(X)$, *for any* $X \subseteq U$.
(3) $\underline{S_\kappa}\left(\overline{S_\kappa}(X)\right) = \overline{S_\kappa}(X)$, *for any* $X \subseteq U$.

Proof. (1) If $S = (F, A)$ is a full soft set, then by Proposition 2.8, we have

$$\underline{S_\kappa}(U) = U \cap \overline{S}(\underline{S}(U)) = U \cap \overline{S}(U) = U \cap U = U.$$

(2) Since $S = (F, \Lambda)$ is a full soft set, by Proposition 2.8, we have $X \subseteq \overline{S}(X)$ for any $X \subseteq U$. Therefore, $\overline{S_\kappa}(X) = X \cup \overline{S}(X) = \overline{S}(X)$.

(3) By part (2) and Proposition 2.6(4), we get that

$$\overline{S_\kappa}(X) = \overline{S}(X) = \underline{S}(\overline{S}(X)) \text{ for any } X \subseteq U.$$

Since $\underline{S}(Y) \subseteq \underline{S_\kappa}(Y) \subseteq Y$ for any $Y \subseteq U$, it follows that

$$\overline{S_\kappa}(X) = \underline{S}(\overline{S}(X)) \subseteq \underline{S_\kappa}(\overline{S}(X)) = \underline{S_\kappa}(\overline{S}(X)) = \underline{S_\kappa}(\overline{S_\kappa}(X)) \subseteq \overline{S_\kappa}(X).$$

This shows that $\underline{S_\kappa}\left(\overline{S_\kappa}(X)\right) = \overline{S_\kappa}(X)$, for any $X \subseteq U$. □

Motivated by the results of Lemma 4.3(8) and Proposition 4.10(2), we introduce the following classes of soft rough sets, which are related to the sets presented in Definition 2.10, as we will see later.

Definition 4.11. *Let* $S = (F, \Lambda)$ *be a soft set over* U *and* $A_S = (U, S)$ *be a soft approximation space. A subset* X *of* U *is said to be:*

(1) Roughly soft κ*-definable, if* $\underline{S_\kappa}(X) \neq \emptyset$ *and* $\overline{S_\kappa}(X) \neq U$.
(2) Internally soft κ*-indefinable, if* $\underline{S_\kappa}(X) = \emptyset$ *and* $\overline{S_\kappa}(X) \neq U$.

(3) *Externally soft κ-indefinable, if* $\underline{S_\kappa}(X) \neq \emptyset$ *and* $\overline{S_\kappa}(X) = U$.
(4) *Totally soft κ-indefinable, if* $\underline{S_\kappa}(X) = \emptyset$ *and* $\overline{S_\kappa}(X) = U$.

Theorem 4.12. *Let* $S = (F, \Lambda)$ *be a full soft set over* U, $A_S = (U, S)$ *be a soft approximation space and* X *be a subset of* U. *The following statements hold:*

(1) *If* X *is roughly soft* A_S*-definable, then it is roughly soft κ-definable.*
(2) *If* X *is internally soft κ-indefinable, then it is internally soft* A_S*-indefinable.*
(3) *If* X *is externally soft* A_S*-indefinable, then it is externally soft κ-indefinable.*
(4) *If* X *is totally soft κ-indefinable, then it is totally soft* A_S*-indefinable.*

Proof. The proof follows immediately from Proposition 4.10(2) and Definitions 2.10 and 4.11. □

5. A topology associated with the soft κ-upper approximation and related notions

The study of abstract spaces was initiated in the early years of the 20th century by Frechet, but it was Hausdorff who managed to abstract the basic properties of open sets by introducing a suitable notion to talk about these concepts and which is also independent of the idea of metrics. Thus, from the origin of the notion of topology on a set, it has influenced almost all other branches of mathematics. Following Singh's textbook [10], a *topology* on a set U is a collection τ of subsets of U such that the intersection of two members of τ is in τ; the union of any collection of members of τ is in τ; and the empty set $\emptyset$ and the entire set U are in τ. A set U endowed with a topology τ is called a *topological space*. The elements of U are called points, the members of τ are called τ-open sets and its complements are called τ-closed sets. In general, a topological space should be denoted as a pair (U, τ). Sometimes, if there is no danger of confusion, instead of saying "the space (U, τ)" one says "the space U", leaving the topology τ on U implied. The *topological interior* of a subset X of U, denoted by $Int(X)$, is defined as the union of all open sets contained in X and the *topological closure* of X, denoted by $Cl(X)$, is defined as the intersection of all closed sets containing X.

Čech closure spaces are extensions of topological spaces that form a field of interest for many authors, who have studied it extensively in algebra, topology, and computer theory. Recall that if $c : P(U) \to P(U)$ is a map satisfying (1) $c(\emptyset) = \emptyset$, (2) $X \in P(U) \Longrightarrow X \subseteq c(X)$ and (3) $X \in P(U), Y \in P(U) \Longrightarrow c(X \cup Y) = c(X) \cup c(Y)$; then c is called a *Čech closure operator* and the pair (U, c) is called a *Čech closure space*, or simply a closure space. If (U, c) is a closure space and $X \subseteq U$, then $c(X)$ is called the closure of X in (U, c). A subset X of U is said to be closed in (U, c), if $c(X) = X$ and is said to be open if its complement is closed, i.e. $c(U - X) = U - X$. With each closure space (U, c) there is an associated topology on U, which is denoted by $\tau(c)$ and is defined as $\tau(c) = \{O \subseteq U : c(U-O) = U-O\}$. Since the closure operator is not idempotent, in general, it is not the topological closure on the space $(U, \tau(c))$. For recent information on Čech closure space, see [7].

Now, as $\overline{S_\kappa} : P(U) \to P(U)$ satisfies all the required conditions for the map c (see Lemma 4.3), we have $c_\kappa(X) = \overline{S_\kappa}(X)$ is a Čech closure operator. We will denote by (U, c_κ) the Čech closure space generated by c_κ and, by τ_κ the topology on U associated with (U, c_κ), which we will call the *soft Čech κ-upper topology* on U. Note that

$$\begin{aligned} \tau_\kappa &= \{O \subseteq U : \overline{S_\kappa}(U - O) = U - O\} \\ &= \{O \subseteq U : U - \overline{S_\kappa}(U - O) = O\} \end{aligned}$$

$$= \{O \subseteq U : \mathrm{NEG}_\kappa(U - O) = O\}.$$

This tells us that the open sets in the topological space (U, τ_κ) are precisely those subsets of the universe U equal to the soft κ-negative region of their complements. We will denote by Int_κ and Cl_κ the topological interior operator and the topological closure operator on the space (U, τ_κ), respectively.

Remark 5.1. *It is important to note that, in general, the soft A_s-upper approximation $\overline{S}$ (resp. soft pre-upper approximation $\overline{S_p}$) does not induce a topology in the way that the soft κ-upper approximation $\overline{S_\kappa}$ does, because there is no guarantee that $\overline{S}$ (resp. $\overline{S_p}$) satisfies condition (2) (resp. condition (3)) of the definition of Čech closure operator.*

Proposition 5.2. *Let $S = (F, \Lambda)$ be a soft set and $A_S = (U, S)$ be a soft approximation space. Then, $\overline{S_\kappa}(X) \subseteq Cl_\kappa(X)$ for each $X \subseteq U$.*

Proof. Let X be any subset of U. If $\overline{S_\kappa}(X) = \emptyset$, then there is nothing to prove. Assume that there exists $x \in \overline{S_\kappa}(X)$ such that $x \notin Cl_\kappa(X)$. Then, $x \notin H$ for some subset H such that $X \subseteq H$ and $\overline{S_\kappa}(H) = H$, which implies that $x \notin \overline{S_\kappa}(H)$ and $X \subseteq H$. Due to monotony, we have $\overline{S_\kappa}(X) \subseteq \overline{S_\kappa}(H)$ and hence, $x \notin \overline{S_\kappa}(X)$, which is a contradiction. □

The following example shows that the inclusion of Proposition 5.2 can be strict.

Example 5.3. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\Lambda = \{\lambda_1, \lambda_2, \lambda_4\}$. Let $S = (F, \Lambda)$ a soft set over U given by*

$$F(\lambda_1) = \{x_1, x_4, x_7\}, F(\lambda_2) = \{x_5\}, F(\lambda_4) = \{x_2, x_4\}.$$

Consider the soft approximation space $A_S = (U, S)$ and the subset $X = \{x_1, x_5\}$ of U. Then,

$$\overline{S_\kappa}(X) = X \cup \overline{S}(X) = \{x_1, x_4, x_5, x_7\}.$$

Let (U, c_κ) be the Čech closure space induced by the operator $c_\kappa = \overline{S_\kappa}$ and let τ_κ be the soft Čech κ-upper topology on U. Since $Y = \{x_1, x_2, x_4, x_5, x_7\}$ is the smallest set (in the sense of inclusion) that contains X and satisfies that $\overline{S_\kappa}(Y) = Y$, we conclude that $Cl_\kappa(X) = Y = \{x_1, x_2, x_4, x_5, x_7\} \neq \{x_1, x_4, x_5, x_7\} = \overline{S_\kappa}(X)$.

By virtue of the previous results, we now introduce a new modification of the soft rough approximations using the topological interior operator of the space (U, τ_k) and the soft pre-rough approximations.

Definition 5.4. *Suppose that $S = (F, A)$ is a soft set over U, $A_S = (U, S)$ is a soft approximation space and X is a subset of U. Considering the space (U, τ_k) we define:*

(1) The soft pre-κ-lower aproximation as $\underline{S_{p\kappa}}(X) = \underline{S_p}(X) \cup Int_\kappa(X)$.
(2) The soft pre-κ-upper approximation as $\overline{S_{p\kappa}}(X) = X \cup Int_\kappa(\overline{S_p}(Int_\kappa(X)))$.
(3) The soft pre-κ-negative region as $\mathrm{NEG}_{p\kappa}(X) = U - \overline{S_{p\kappa}}(X)$.
(4) The soft pre-κ-positive region as $\mathrm{POS}_{p\kappa}(X) = \underline{S_{p\kappa}}(X)$.
(5) The soft pre-κ-boundary region as $\mathrm{BND}_{p\kappa}(X) = \overline{S_{p\kappa}}(X) - \underline{S_{p\kappa}}(X)$.

(6) The soft pre-κ-accuracy as $\mu_{p\kappa}(X) = \dfrac{\left|\underline{S}_{p\kappa}(X)\right|}{\left|\overline{S}_{p\kappa}(X)\right|}$, *where* $\overline{S}_{p\kappa}(X) \neq \emptyset$.

We will refer to $\underline{S}_{p\kappa}(X)$ and $\overline{S}_{p\kappa}(X)$ as the soft pre-κ-rough approximations of X with respect to A_S. If $\underline{S}_{p\kappa}(X) = \overline{S}_{p\kappa}(X)$, then X is said to be a soft pre-κ-exact or soft pre-κ-definable set. Otherwise, X is called a soft pre-κ-rough set. Obviously, X is soft pre-κ-exact if and only if $\mathrm{BND}_{p\kappa}(X) = \emptyset$. In addition, X is soft pre-κ-exact if $\mu_{p\kappa}(X) = 1$.

In the following proposition, the main properties of soft pre-κ-rough approximations are stated.

Proposition 5.5. *Let* $S = (F, \Lambda)$ *be a soft set over* U, $A_S = (U, S)$ *be a soft approximation space. For any subsets* X *and* Y *of* U, *the following statements hold:*

(1) $\underline{S}_{p\kappa}(\emptyset) = \overline{S}_{p\kappa}(\emptyset) = \emptyset$.
(2) $\underline{S}_{p\kappa}(U) = \overline{S}_{p\kappa}(U) = U$.
(3) If $X \subseteq Y$, *then* $\underline{S}_{p\kappa}(X) \subseteq \underline{S}_{p\kappa}(Y)$ *and* $\overline{S}_{p\kappa}(X) \subseteq \overline{S}_{p\kappa}(Y)$ *(monotony).*
(4) $\underline{S}_{p\kappa}(X \cap Y) \subseteq \underline{S}_{p\kappa}(X) \cap \underline{S}_{p\kappa}(Y)$.
(5) $\underline{S}_{p\kappa}(X) \cup \underline{S}_{p\kappa}(Y) \subseteq \underline{S}_{p\kappa}(X \cup Y)$.
(6) $\overline{S}_{p\kappa}(X \cap Y) \subseteq \overline{S}_{p\kappa}(X) \cap \overline{S}_{p\kappa}(Y)$.
(7) $\overline{S}_{p\kappa}(X) \cup \overline{S}_{p\kappa}(Y) \subseteq \overline{S}_{p\kappa}(X \cup Y)$.
(8) $\underline{S}(X) \subseteq \underline{S}_{\kappa}(X) \subseteq \underline{S}_{p}(X) \subseteq \underline{S}_{p\kappa}(X) \subseteq X \subseteq \overline{S}_{p\kappa}(X) \subseteq \overline{S}_{p}(X) \subseteq \overline{S}_{\kappa}(X)$.

Proof. (1) and (2) are clear from Definition 5.4.

(3) It is an immediate consequence of the monotony of $\underline{S}_p$, Int_κ and $\overline{S}_p$.

(4)–(7) Follow from part (3).

(8) By virtue of Lemma 4.3(8) and Definition 5.4, we only have to show that $\underline{S}_{p\kappa}(X) \subseteq X$ and $\overline{S}_{p\kappa}(X) \subseteq \overline{S}_p(X)$. Since $\underline{S}_p(X) \subseteq X$ and $Int_\kappa(X) \subseteq X$, it follows that

$$\underline{S}_{p\kappa}(X) = \underline{S}_p(X) \cup Int_\kappa(X) \subseteq X.$$

On the other hand, by the monotony of Int_κ and $\overline{S}_p$, we have

$$Int_\kappa(\overline{S}_p(Int_\kappa(X))) \subseteq Int_\kappa(\overline{S}_p(X)) \subseteq \overline{S}_p(X).$$

Since $X \subseteq \overline{S}_p(X)$, we get that $\overline{S}_{p\kappa}(X) = X \cup Int_\kappa(\overline{S}_p(Int_\kappa(X))) \subseteq \overline{S}_p(X)$. □

Corollary 5.6. *Let* $S = (F, \Lambda)$ *be a soft set over* U, $A_S = (U, S)$ *be a soft approximation space. For any* $X \subseteq U$, *the following statements hold:*

(1) $\mathrm{BND}_{p\kappa}(X) \subseteq \mathrm{BND}_p(X)$.
(2) If X is a soft pre-κ-rough set, then it is a soft pre-rough set.
(3) If X is a soft pre-exact set, then it is a soft pre-κ-exact set.
(4) $\mu_p(X) \leq \mu_{p\kappa}(X)$.
(5) X is soft pre-κ-definable if and only if $\underline{S}_{p\kappa}(X) = \overline{S}_{p\kappa}(X) = X \neq \emptyset$.

Proposition 5.7. *Let* $S = (F, \Lambda)$ *be a soft set over* U, $A_S = (U, S)$ *be a soft approximation space. For any* $X \subseteq U$, *the following statements hold:*

(1) $\underline{S_{p\kappa}}\left(\underline{S_{p\kappa}}(X)\right) \subseteq \underline{S_{p\kappa}}(X)$.
(2) $\overline{S_{p\kappa}}(X) \subseteq \overline{S_{p\kappa}}\left(\overline{S_{p\kappa}}(X)\right)$.
(3) $\underline{S_{p\kappa}}(X) \subseteq \overline{S_{p\kappa}}\left(\underline{S_{p\kappa}}(X)\right) \subseteq \overline{S_{p\kappa}}(X)$.
(4) $\underline{S_{p\kappa}}(X) \subseteq \underline{S_{p\kappa}}\left(\overline{S_{p\kappa}}(X)\right) \subseteq \overline{S_{p\kappa}}(X)$.
(5) $\overline{S_{p\kappa}}(Int_\kappa(\overline{X})) = Int_\kappa(X)$.
(6) $\overline{\overline{S_{p\kappa}}}(Int_\kappa(X)) = Int_\kappa(\overline{S_p}(Int_\kappa(X)))$.
(7) If $X \in \tau_\kappa$, then $\underline{S_{p\kappa}}(X) = X$ and $\overline{S_{p\kappa}}(X) = Int_\kappa(\overline{S_p}(X))$.
(8) If $S = (F, \Lambda)$ is a full soft set, then $\underline{S_{p\kappa}}\left(\overline{S_p}(X)\right) = \overline{S_p}(X)$.

The following theoretical example shows that the soft pre-κ-rough approximations approach is more efficient than all other approaches presented in this paper.

Example 5.8. *Suppose that*

$$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}, \quad E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}, \quad \Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}.$$

Let $S = (F, \Lambda)$ be a soft set over U given by

$$F(\lambda_1) = \{x_4\}, F(\lambda_2) = \{x_1, x_4\}, F(\lambda_3) = \{x_2, x_4\}, F(\lambda_4) = \{x_3, x_4\}.$$

Consider the soft approximation space $A_S = (U, S)$ and the subset $X = \{x_4, x_5\}$ of U. Then,

$$\underline{S}(X) = \{x_4\}, \quad \overline{S}(X) = \{x_1, x_2, x_3, x_4\}, \quad \underline{S_p}(X) = \{x_4\},$$
$$\overline{S_p}(X) = \{x_1, x_2, x_3, x_4, x_5\}, \quad \overline{S_\kappa}(X) = \{x_1, x_2, x_3, x_4, x_5\}.$$

Let (U, c_κ) be the Čech closure space induced by the operator $c_\kappa = \overline{S_\kappa}$ and let τ_κ be the topology consisting of the open sets in (U, c_κ), i.e. $\tau_\kappa = \{O \subseteq U : c_\kappa(U - O) = U - O\}$. Then, $Int_\kappa(X) = \{x_5\}$, which implies that $\underline{S_{p\kappa}}(X) = X$ and $\overline{S_{p\kappa}}(X) = X$. Note that $\mu_p(X) = \frac{1}{5} < \mu_{p\kappa}(X) = 1$. Also, $\mu_{A_S}(X) = \frac{1}{4} < \mu_{p\kappa}(X)$ and $\mu_\kappa(X) = \frac{1}{5} < \mu_{p\kappa}(X)$. On the other hand, for the set $Y = \{x_6, x_8\}$, we have $\underline{S}(Y) = \emptyset$, $\overline{S}(Y) = \emptyset$, $\underline{S_p}(Y) = \emptyset$, $\overline{S_p}(Y) = Y$, $\underline{S_\kappa}(Y) = \emptyset$, $\overline{S_\kappa}(Y) = Y$, $Int_\kappa(Y) = \{x_8\}$. According to this, $\underline{S_{p\kappa}}(Y) = \{x_8\}$ and $\overline{S_{p\kappa}}(Y) = Y$. Therefore, $\mu_p(Y) = 0 < \mu_{p\kappa}(Y) = \frac{1}{2}$. Moreover, $\mu_{A_S}(Y)$ is undefined and $\mu_\kappa(Y) = 0 < \mu_{p\kappa}(Y)$.

Next, we introduce the following classes of soft rough sets, which are related to the sets presented in the previous sections.

Definition 5.9. *Let $S = (F, \Lambda)$ be a soft set over U, $A_S = (U, S)$ be a soft approximation space. A subset X of U is said to be:*

(1) *Roughly soft pre-κ-definable, if $\underline{S_{p\kappa}}(X) \neq \emptyset$ and $\overline{S_{p\kappa}}(X) \neq U$.*
(2) *Internally soft pre-κ-indefinable, if $\underline{S_{p\kappa}}(X) = \emptyset$ and $\overline{S_{p\kappa}}(X) \neq U$.*
(3) *Externally soft pre-κ-indefinable, if $\underline{S_{p\kappa}}(X) \neq \emptyset$ and $\overline{S_{p\kappa}}(X) = U$.*
(4) *Totally soft pre-κ-indefinable, if $\underline{S_{p\kappa}}(X) = \emptyset$ and $\overline{S_{p\kappa}}(X) = U$.*

The intuitive interpretation of these classes is as follows:

- X is roughly soft pre-κ-definable means that we can decide for some elements of U that they belong to X, and meanwhile for some elements of U we can decide that they belong to X^c, by employing the available knowledge from the soft approximation space A_S.
- X is internally soft pre-κ-indefinable means that we can decide about some elements of U that they belong to X^c, but we cannot decide for any element of U that it belongs to X, by using A_S.
- X is externally soft pre-κ-indefinable means that we can decide for some elements of U that they belong to X, but we cannot decide, for any element of U that it belongs to X^c, by using A_S.
- X is totally soft pre-κ-indefinable means that we cannot decide for any element of U that it belongs to X or X^c, by employing A_S.

Theorem 5.10. *Let $S = (F, \Lambda)$ be a soft set over U, $A_S = (U, S)$ be a soft approximation space. For any subset X of U, the following statements hold:*

(1) If X is roughly soft pre-definable, then it is roughly soft pre-κ-definable.
(2) If X is totally soft pre-κ-indefinable, then it is totally soft pre-indefinable.

6. Diagnosis of COVID-19 in Colombia

At the end of 2019, a group of patients with pneumonia of unknown cause was detected in the city of Wuhan (China). Then, in January 2020, a new coronavirus responsible for severe acute respiratory syndrome was isolated, named by the International Committee on Taxonomy of Viruses, SARS-CoV-2. Later, in February 2020, the World Health Organization (WHO) named this contagious infectious disease "coronavirus disease 2019" (COVID-19) and a month later, it was declared a pandemic. Since the outbreak of the COVID-19 disease, it has caused great damage and brought great challenges to more than 200 countries and regions around the world [13, 14]. COVID-19 has a clinical spectrum that ranges from asymptomatic forms to severe forms [2]. Most symptomatic patients report fever, general symptoms, respiratory symptoms such as cough, dyspnea and, to a lesser extent, extrapulmonary manifestations . However, there is more and more evidence that many COVID-19 patients are asymptomatic or have only mild symptoms, but are capable of transmitting the virus to others [12]. The prevention and control of this disease has been difficult due to the complexity of investigating asymptomatic infections. The ability of these asymptomatic infections to spread the virus is high, and these patients are likely to cause a new round of outbreaks. Therefore, finding asymptomatic infections is the key point for early prevention and control of COVID-19 all over the world. It is at this point, where the implementation of classic mathematical tools and models, as well as their generalizations, can play a fundamental role in the detection of asymptomatic infections, since the potential of mathematics to face problems related to data from real-world situations has been proven.

With respect to Colombia, the Coronavirus pandemic required the National Government and the different territorial entities to work on the development and implementation of strategies to reduce the speed of infection of the virus and prevent its effects on the population. These strategies can be classified into three main sources, namely: 1) sanitary measures and sanitary emergency measures, 2) social, economic and ecological emergency measures, and 3) public order measures and other measures of an ordinary nature.

Among the sanitary and emergency health measures adopted, the Ministry of Health and Social Protection of the Republic of Colombia implemented the Sustainable Selective Testing, Screening and Isolation Program (abbreviated PRASS for its acronym in Spanish), which is largely based on contact tracing and isolation of probable or suspected cases of COVID-19, in an agile and timely manner; therefore, the objective is to increase the performance of COVID-19 diagnostic tests for epidemiological surveillance and early detection of cases among the population; to this end, free COVID-19 diagnostic test points were installed throughout the country.

Prior to the process of applying the diagnostic tests for Coronavirus, a poll was made to people to find out if they have had some of the most common symptoms of this disease, and it is worth noting that a significant percentage of respondents did not report any symptoms associated with COVID 19; however, in the corresponding diagnostic tests, they were positive for this disease (this type of people are called “asymptomatic”). Table 2 shows the data set of several persons who underwent the PCR diagnostic test for COVID-19, among whom it was determined that there was one asymptomatic person.

The columns of the Table 1 represent the attributes (the symptoms for Coronavirus) and the rows represent the objects (the patients). The entries in the table are the attribute values for each patients. Encoding the attribute values of Table 1, we obtain the Boolean-valued information system given in Table 2, where 0 and 1 represent “No” and “Yes”, respectively.

Table 1. A symptom information system.

Patients	Muscle pain	Headache	Nausea	Fever	Coronavirus
p_1	No	Yes	No	No	No
p_2	No	No	Yes	No	No
p_3	No	No	No	Yes	No
p_4	Yes	Yes	No	Yes	Yes
p_5	No	No	No	No	Yes
p_6	No	No	No	No	No
p_7	No	Yes	No	Yes	Yes
p_8	No	No	Yes	Yes	Yes

Table 2. A tabular representation of the soft set.

Patients	Muscle pain	Headache	Nausea	Fever	Coronavirus
p_1	0	1	0	0	0
p_2	0	0	1	0	0
p_3	0	0	0	1	0
p_4	1	1	0	1	1
p_5	0	0	0	0	1
p_6	0	0	0	0	0
p_7	0	1	0	1	1
p_8	0	0	1	1	1

The tabular representation in Table 2 describes the soft set $(F, \Lambda) = \{(\lambda_1, \{p_4\}), (\lambda_2, \{p_1, p_4, p_7\}), (\lambda_3, \{p_2, p_8\}), (\lambda_4, \{p_3, p_4, p_7, p_8\})\}$ over U, where $U = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$ is the universe of eight patients and $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is the set of decision parameters. The λ_i $(i = 1, 2, 3, 4)$ stands "joint pain", "headache", "nausea" and "fever". Let us note that the set of patients infected with Coronavirus is $X = \{p_4, p_5, p_7, p_8\}$. Thus, the approximations, the boundary, and the accuracy measure of X, by using soft rough approach, soft pre-rough approach and the current approach given in this paper are given respectively as follows:

Soft rough approach: By calculating, we have

$$\underline{S}(X) = \{p_4\}, \ \ \overline{S}(X) = \{p_1, p_2, p_3, p_4, p_7, p_8\},$$

$$\mathrm{BND}_{A_s}(X) = \{p_1, p_2, p_3, p_7, p_8\}, \ \ \mu_{A_S}(X) = \frac{1}{6}.$$

Note that X is not a subset of $\overline{S}(X)$. According to this approach, X is a soft A_S-rough set. Moreover, p_1, p_2 and p_3 which are not infected patients belongs to the boundary of X which consists of all those patients that cannot be classified uniquely as infected or uninfected, by using the available knowledge from the soft approximation space.

Soft pre-rough approach: By calculating, we get that

$$\underline{S_p}(X) = \{p_4, p_7, p_8\}, \ \ \overline{S_p}(X) = \{p_1, p_3, p_4, p_5, p_7, p_8\}$$

$$\mathrm{BND}_p(X) = \{p_1, p_3, p_5\}, \ \ \mu_p(X) = \frac{1}{2}.$$

According to this approach, X is a soft pre-rough set. Moreover, p_1 and p_3 which are not infected patients belongs to the boundary of X which consists of all those patients that cannot be classified uniquely as infected or uninfected, by using the available knowledge from the soft approximation space.

Current approach: By calculating, we have

$$\underline{S_{p\kappa}}(X) = X, \ \ \overline{S_{p\kappa}}(X) = X, \ \ \mathrm{BND}_{p\kappa}(X) = \emptyset, \ \ \mu_{p\kappa}(X) = 1.$$

According to this approach, X is a soft pre-κ-definable set, which means that by using the available knowledge, we can decide that the patients in X are the only ones infected with Coronavirus. Therefore, the proposed approach is more efficient than soft rough method and soft pre-rough method in decision making to extract information and help eliminate data vagueness in real-life problems.

To finalize this work, an algorithm is presented that can be used to compute the accuracy in an information system using soft pre-κ-rough approximations (see Algorithm 1).

Algorithm 1 Computation of accuracy using the soft pre-κ-rough approximations.

Input the soft set (F, Λ) and the target set X.
Obtain the indiscernibility relation for X.
Compute $\underline{S}_p(X)$ according to Definition 3.1.
Calculate $Int_\kappa(X)$ from the soft Čech κ-upper topology τ_κ.
Compute $\overline{S}_{p\kappa}(X)$ according to Definition 5.4, as $\overline{S}_{p\kappa}(X) = X \cup Int_\kappa(\overline{S}_p(Int_\kappa(X))$.
Determine $\underline{S}_{p\kappa}(X)$ according to Definition 5.4, as $\underline{S}_{p\kappa}(X) = \underline{S}_p(X) \cap Int_k(X)$.
Compute the accuracy of the approximation, $\mu_{p\kappa}(X)$, by using Definition 5.4.

Remark 6.1. In order to be able to perform data analysis with a larger universe and set of parameters, an algorithm has been programmed and its implementation was executed in *Octave* (see Algorithm 2). The structure of this program is based on the approaches presented throughout this work, where *Requel*, obtains the indiscernibility relation for X; *spinf* calculates the soft pre-lower aproximation; *Intk* finds the topological interior, Int_κ, of a subset X of U; *Union* calculates the union of sets; *spsup* compute the soft pre-upper aproximation, which are all predefined functions.

Algorithm 2 Computation of accuracy using the soft pre-κ-rough with GNU Octave programming

```
function acc_pk(U, soft_set, X)
    REQ = Requel(soft_set, X);
    Sp_inf = spinf(X, REQ);
    interior = Intk(U, X, REQ);
    Spk_inf = Union(Sp_inf, interior);
    Sp_sup_int = spsup(interior, REQ);
    int_aux = Intk(U, Sp_sup_int, REQ);
    Spk_sup = Union(X, int_aux)
    acc_pk = size(Spk_inf, 2)/size(Spk_sup, 2)
end function
```

7. Conclusions

In this paper, the theory of soft rough sets is used to make a review of the methods of soft pre-rough approximations and soft β-rough approximations and with the help of our Lemma 3.5 it is shown that these methods are the same. Moreover, examples are presented to show that the statements of part (ii) of Theorem 3.1 in [6] (resp. Theorem 3.9 in [6]) and part (viii) of Proposition 3.1 in [6] (resp. Proposition 3.3 in [6]) are false; i.e. the inclusion $\overline{S}_p(X) \subseteq \overline{S}(X)$ (resp. $\overline{S}_\beta(X) \subseteq \overline{S}(X)$) and the equality $\overline{S}_p(X \cup Y) = \overline{S}_p(X) \cup \overline{S}_p(X)$ (resp. $\overline{S}_\beta(X \cup Y) = \overline{S}_\beta(X) \cup \overline{S}_\beta(X)$) are not true. Considering the above, we first propose a new method, called soft κ-rough approximations, which satisfies many of the properties similar to the soft rough approximations, but it does not have higher accuracy than the soft pre-rough approximations (see Corollary 4.5). However, through some of the established properties and the theory of closure spaces, a topology is obtained that provides tools to formulate the method of pre-κ-rough soft approximations. With this last method the boundary region is considerably reduced and therefore the accuracy is improved with respect to the approaches mentioned in this section, as can be seen in Corollary 5.6, Example 5.8 and Section 6 of the Diagnosis of COVID-19 in Colombia. Therefore, in future research, new models of upper and lower approximations can be created by considering the topological results established here, to extend the field of application of soft rough sets to decision making problems in different contexts.

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Conflict of interest

The authors declare that they have no conflicts of interest in this article.

References

1. T. M. Al-Shami, Topological approach to generate new rough set models, *Complex Intell. Syst.*, **8** (2022), 4101–4113. https://doi.org/10.1007/s40747-022-00704-x
2. H. Chaolin, W. Yeming, L. Xingwang, R. Lili, Z. Jianping, Y. Hu, et al., Clinical features of patients infected with 2019 novel coronavirus in Wuhan, China, *Lancet*, **395** (2020), 497–506. https://doi.org/10.1016/S0140-6736(20)30183-5
3. X. Chen, B. Sun, X. Zhang, C. Qi, X. Chu, T. Wang, et al., An approach to multiple attribute decision making based on linguistic value soft rough set and VIKOR method, *J. Intell. Fuzzy Syst.*, **40** (2021), 9609–9626. https://doi.org/10.3233/JIFS-202085
4. M. K. El-Bably, A. A. El Atik, Soft β-rough sets and their application to determine COVID-19, *Turkish J. Math.*, **45** (2021), 1133–1148. https://doi.org/10.3906/mat-2008-93
5. M. A. El Safty, S. Al Zahrani, M. K. El-Bably, M. El Sayed, Soft ς-rough set and its applications in decision making of coronavirus, *Comput. Mater. Contin.*, **70** (2022), 267–285. https://doi.org/10.32604/cmc.2022.019345
6. M. El Sayed, A. Q. Al Qubati, M. K. El-Bably, Soft pre-rough sets and its applications in decision making, *Math. Biosci. Eng.*, **17** (2020), 6045–6063. https://doi.org/10.3934/mbe.2020321
7. A. Rieser, Čech closure spaces: a unified framework for discrete and continuous homotopy, *Topol. Appl.*, **296** (2021), 107613. https://doi.org/10.1016/j.topol.2021.107613
8. R. Sahu, S. R. Dash, S. Das, Career selection of students using hybridized distance measure based on picture fuzzy set and rough set theory, *Decis. Mak., Appl. Manag. Eng.*, **4** (2021), 104–126. https://doi.org/10.31181/dmame2104104s
9. H. K. Sharma, K. Kumari, S. Kar, Forecasting sugarcane yield of India based on rough sets combination approach, *Decis. Mak., Appl. Manag. Eng.*, **4** (2021), 163–177. https://doi.org/10.31181/dmame210402163s

10. T. B. Singh, *Introduction to topology*, Singapore: Springer, 2019. https://doi.org/10.1007/978-981-13-6954-4

11. B. Sun, C. Qi, W. Ma, T. Wang, L. Zhang, C. Jiang, Variable precision diversified attribute multigranulation fuzzy rough set-based multi-attribute group decision making problems, *Comput. Ind. Eng.*, **142** (2020), 106331. https://doi.org/10.1016/j.cie.2020.106331

12. J. Tan, S. Liu, L. Zhuang, L. Chen, M. Dong, J. Zhang, et al., Transmission and clinical characteristics of asymptomatic patients with SARS-CoV-2 infection, *Future Virology*, **15** (2020), 373–380. https://doi.org/10.2217/fvl-2020-0087

13. World Health Organization, Coronavirus disease 2019 (COVID-19), Situation Report-100, 2020. Available from: `https://www.who.int/emergencies/diseases/novel-coronavirus-2019/situation-reports`.

14. World Health Organization, Clinical management of severe acute respiratory infection (SARI) when COVID-19 disease is suspected, 2020. Available from: `https://apps.who.int/iris/handle/10665/331446`.

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