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Research article

Positivity analysis for mixed order sequential fractional difference operators

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Abstract: We consider the positivity of the discrete sequential fractional operators $\left({}_{a_0+1}^{\text{RL}}\nabla^{\nu_1}\ {}_{a_0}^{\text{RL}}\nabla^{\nu_2} f\right)(\tau)$ defined on the set $\mathscr{D}_1$ (see (1.1) and Figure 1) and $\left({}_{a_0+2}^{\text{RL}}\nabla^{\nu_1}\ {}_{a_0}^{\text{RL}}\nabla^{\nu_2} f\right)(\tau)$ of mixed order defined on the set $\mathscr{D}_2$ (see (1.2) and Figure 2) for $\tau \in \mathbb{N}_{a_0}$. By analysing the first sequential operator, we reach that $(\nabla f)(\tau) \geqq 0$, for each $\tau \in \mathbb{N}_{a_0+1}$. Besides, we obtain $(\nabla f)(3) \geqq 0$ by analysing the second sequential operator. Furthermore, some conditions to obtain the proposed monotonicity results are summarized. Finally, two practical applications are provided to illustrate the efficiency of the main theorems.

Keywords: discrete delta Riemann-Liouville fractional difference; negative lower bound; convexity analysis; analytical and numerical results
Mathematics Subject Classification: 26A48, 26A51, 33B10, 39A12, 39B62

1. Introduction

Discrete operators are the most essential branches of discrete fractional calculus that enable problems where the change of variables can be modeled in a numerical or theoretical continuum to derive, from it, the variation of these elements in specific kernels [1–4]. Besides, the discrete fractional difference/sum equations have important applications in areas like fluid dynamics [5–7], heat or mass transfer [8–10], chemical reaction processes [11–13], geometry [14, 15], ecology [16, 17], and contaminant transport [18–20].

The positivity and monotonicity analysis are basic parts of applied mathematics and mathematical analysis, and the development of discrete fractional calculus has enabled powerful mathematical tools for these areas. These kinds of problem can be solved either by analysing discrete operators or by using integrating by parts. By applying these techniques, it is possible to determine when the nabla operators are positive or the function is monotonically increasing or decreasing. Also, these have lead to additive (or splitting) schemes, but so far they are examined with various delta and nabla fractional difference operators in time space $\mathbb{N}_{a_0}$ (see previous works [21–26] and the references therein for more details).

In the literature of discrete fractional calculus mixed order sequential fractional difference operator has a form $\left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau)$, where ν_1 and ν_2 are two different orders. In addition, in most of the research on monotonicity and positivity analysis, discrete sequential fractional operators and mixed order fractional operators in discrete fractional environments are two of the most active research areas as you can see in previous studies. For this reason, there exists a wide literature about its reanalysis, numerically and analytically, see for example [27–31]. Therefore, it is of interest to analyse a discrete sequential fractional operator of mixed order correctly, provided that they allow a development of the applications and theory based on them successfully.

In view of the above discussion, this paper focuses on analysing discrete sequential fractional operator of mixed orders $\left({}^{\mathrm{RL}}_{a_0+1}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau)$ and $\left({}^{\mathrm{RL}}_{a_0+2}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau)$, and applying these to handle the positivity of $(\nabla f)(\tau)$ on the sets

$$\mathscr{D}_1 := \Big\{(\nu_2, \nu_1) \in (0,1) \times (1,2); \quad 1 < \nu_1 + \nu_2 < 2\Big\}, \tag{1.1}$$

and

$$\mathscr{D}_2 := \Big\{(\nu_2, \nu_1) \in (1,2) \times (0,1); \quad 1 < \nu_1 + \nu_2 < 2\Big\}, \tag{1.2}$$

respectively. The regions of these sets are plotted in the Figures 1 and 2 below.

The article set-up is structured as follows: in Section 2, we recall the basic discrete fractional operator tools and investigate the main lemmas and theorems concerning the designation of discrete sequential fractional operator of mixed orders. The sets $\mathscr{D}_1$ and $\mathscr{D}_2$, and the main theorems on these sets are given in Section 3. Section 4 is devoted to the study of practical applications with specificity of the order of the discrete sequential fractional operator. Finally, the conclusion and significance of the present article are elaborated in Section 5.

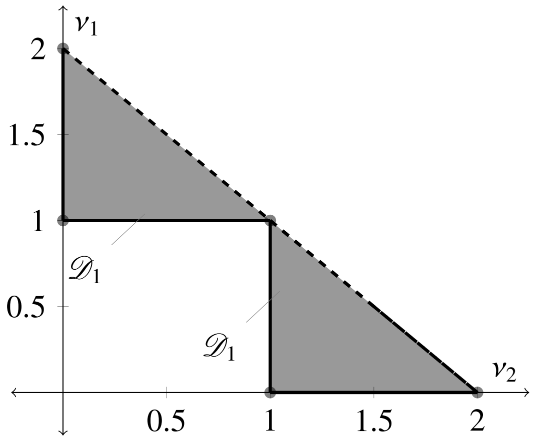

Figure 1. The regions of the set $\mathscr{D}_1$.

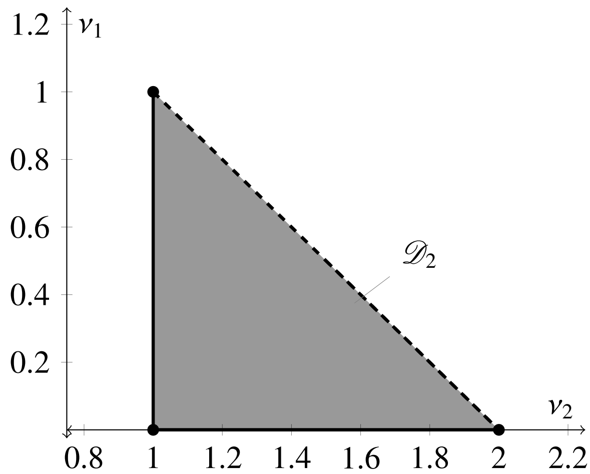

Figure 2. The region of the set $\mathscr{D}_2$.

2. Preliminaries

In this section, we will list some relevant preliminaries including the definition of discrete fractional difference and sum operators and their alternatives in the sense of Riemann-Liouville defined on the set $\mathbb{N}_{a_0}$, defined by $\mathbb{N}_{a_0} := \{a_0, a_0 + 1, a_0 + 2, \ldots\}$.

Definition 2.1. [32, Definition 3.58] Suppose that f is defined on $\mathbb{N}_{a_0}$ and $0 < \alpha$ is the order of the discrete fractional operator. Then the $\nabla-$fractional sum operator is given as follows

$$\left({}^{\mathrm{RL}}_{a_0}\nabla^{-\alpha} f\right)(\tau) = \sum_{\mathsf{s}=a_0+1}^{\tau} \frac{(\tau + 1 - \mathsf{s})^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(\mathsf{s}), \text{for } \tau \text{ in } \mathbb{N}_{a_0+1}, \tag{2.1}$$

where it is important to state that

$$\tau^{\overline{\alpha}} = \frac{\Gamma(\alpha + \tau)}{\Gamma(\tau)}, \; \nabla \tau^{\overline{\alpha}} = \alpha\, \tau^{\overline{\alpha-1}}, \tag{2.2}$$

where these lead to zero when the denominators are undefined but the numerators are well defined.

Definition 2.2. [26, Lemma 2.1] Suppose that f is defined on $\mathbb{N}_{a_0}$ and $\ell - 1 < \alpha < \ell$ is the order of the discrete fractional operator. Then the $\nabla-$fractional difference operator is given as follows

$$\begin{aligned} \left({}^{\mathrm{RL}}_{a_0}\nabla^{\alpha} f\right)(\tau) &= \frac{1}{\Gamma(-\alpha)} \sum_{\mathsf{s}=a_0+1}^{\tau} (\tau + 1 - \mathsf{s})^{\overline{-\alpha-1}} f(\mathsf{s}), \text{for } \tau \text{ in } \mathbb{N}_{a_0+\ell}, \\ (\nabla f)(\tau) &= f(\tau) - f(\tau-1), \text{for } \tau \in \mathbb{N}_{a_0+1}. \end{aligned} \tag{2.3}$$

Lemma 2.1. *Let $\nu_1 > 0$ and f be defined on $\mathbb{N}_{a_0}$. Then for $0 < \nu_2 < 1$, the following identity can be obtained*

$$\left({}^{\mathrm{RL}}_{a_0+1}\nabla^{-\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) = \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2-\nu_1} f\right)(\tau) - \frac{(\tau - a_0)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} f(a_0+1), \tag{2.4}$$

and for $1 < \nu_2 < 2$, the following identity can be obtained

$$\begin{aligned} \left({}^{\mathrm{RL}}_{a_0+2}\nabla^{-\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) = \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2-\nu_1} f\right)(\tau) &- \left[\frac{(\tau - a_0)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} - \nu_2 \frac{(\tau - a_0 - 1)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)}\right] f(a_0+1) \\ &- \frac{(\tau - a_0 - 1)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} f(a_0+2), \end{aligned} \tag{2.5}$$

for $\tau \in \mathbb{N}_{a_0+2}$.

Proof. Let $g(\tau) := \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau)$. Then by considering (2.1), we have

$$\begin{aligned} \left({}^{\mathrm{RL}}_{a_0+1}\nabla^{-\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) &= \left({}^{\mathrm{RL}}_{a_0+1}\nabla^{-\nu_1}\, g\right)(\tau) \\ &= \frac{1}{\Gamma(\nu_1)} \sum_{\mathsf{s}=a_0+2}^{\tau} (\tau - \mathsf{s} + 1)^{\overline{\nu_1-1}} g(\mathsf{s}) \\ &= \frac{1}{\Gamma(\nu_1)} \sum_{\mathsf{s}=a_0+1}^{\tau} (\tau - \mathsf{s} + 1)^{\overline{\nu_1-1}} g(\mathsf{s}) - \frac{(\tau - a_0)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} g(a_0+1) \\ &= \left({}^{\mathrm{RL}}_{a_0}\nabla^{-\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) - \frac{(\tau - a_0)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} f(a_0+1) \\ &= \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2-\nu_1} f\right)(\tau) - \frac{(\tau - a_0)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} f(a_0+1), \end{aligned}$$

where we have used

$$\begin{aligned} g(a_0+1) &= \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(a_0+1) \\ &= \frac{1}{\Gamma(-\nu_2)} \sum_{\mathsf{s}=a_0+1}^{a_0+1} (a_0 + 2 - \mathsf{s})^{\overline{-\nu_2-1}} f(\mathsf{s}) = f(a_0+1), \end{aligned}$$

which completes the proof of (2.4). Similarly, we can proceed

$$
\begin{aligned}
\left({}^{\mathrm{RL}}_{a_0+2}\nabla^{-\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) &= \left({}^{\mathrm{RL}}_{a_0+2}\nabla^{-\nu_1}\, g\right)(\tau) \\
&= \left({}^{\mathrm{RL}}_{a_0}\nabla^{-\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) - \frac{(\tau-a_0)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} g(a_0+1) \\
&- \frac{(\tau-a_0-1)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} g(a_0+2) \\
&= \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2-\nu_1} f\right)(\tau) - \left[\frac{(\tau-a_0)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} - \nu_2\,\frac{(\tau-a_0-1)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)}\right] f(a_0+1) \\
&- \frac{(\tau-a_0-1)^{\overline{\nu_1-1}}}{\Gamma(\nu_1)} f(a_0+2),
\end{aligned}
$$

where we have used

$$
g(a_0+2) = \frac{1}{\Gamma(-\nu_2)} \sum_{\mathsf{s}=a_0+1}^{a_0+2} (a_0+3-\mathsf{s})^{\overline{-\nu_2-1}} f(\mathsf{s}) = f(a_0+2) - \nu_2 f(a_0+1),
$$

which completes the proof of (2.5). Hence, the proof is done. □

Theorem 2.1. *Let f be defined on $\mathbb{N}_{a_0}$. Then for $0 < \nu_2 < 1$ and $1 < \nu_1 \leqq 2$, the following identity holds*

$$
\left({}^{\mathrm{RL}}_{a_0+1}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) = \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2+\nu_1}\, f\right)(\tau) - \frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+1), \tag{2.6}
$$

and for $1 < \nu_2 < 2$ and $0 < \nu_1 \leqq 1$, we have

$$
\begin{aligned}
\left({}^{\mathrm{RL}}_{a_0+2}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) &= \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2+\nu_1}\, f\right)(\tau) - \left[\frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \nu_2\,\frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) \\
&- \frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+2),
\end{aligned} \tag{2.7}
$$

for $\tau \in \mathbb{N}_{a_0+3}$.

Proof. With the help of Lemma 2.1, we have for $0 < \nu_2 < 1$ and $1 < \nu_1 \leqq 2$:

$$
\begin{aligned}
\left({}^{\mathrm{RL}}_{a_0+1}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) &= \nabla^2 \left({}^{\mathrm{RL}}_{a_0+1}\nabla^{-(2-\nu_1)}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) \\
&= \nabla^2 \left[\left({}^{\mathrm{RL}}_{a_0+1}\nabla^{\nu_2+\nu_1-2}\, f\right)(\tau) - \frac{(\tau-a_0)^{\overline{-\nu_1+1}}}{\Gamma(\nu_1)} f(a_0+1)\right] \\
&= \left({}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2+\nu_1}\, f\right)(\tau) - \frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+1),
\end{aligned}
$$

which completes the proof of (2.6). In similar way, we have for $1 < \nu_2 < 2$ and $0 < \nu_1 \leqq 1$:

$$
\begin{aligned}
\left({}_{a_0+2}^{\mathrm{RL}}\nabla^{\nu_1}\ {}_{a_0}^{\mathrm{RL}}\nabla^{\nu_2}f\right)(\tau) &= \nabla\left({}_{a_0+2}^{\mathrm{RL}}\nabla^{-(1-\nu_1)}\ {}_{a_0}^{\mathrm{RL}}\nabla^{\nu_2}f\right)(\tau)\\
&= \nabla\Bigg[\left({}_{a_0}^{\mathrm{RL}}\nabla^{\nu_2+\nu_1-1}f\right)(\tau)-\left(\frac{(\tau-a_0)^{\overline{-\nu_1}}}{\Gamma(1-\nu_1)}-\nu_2\frac{(\tau-a_0-1)^{\overline{-\nu_1}}}{\Gamma(1-\nu_1)}\right)f(a_0+1)\\
&\quad-\frac{(\tau-a_0-1)^{\overline{-\nu_1}}}{\Gamma(1-\nu_1)}f(a_0+2)\Bigg]\\
&= \left({}_{a_0}^{\mathrm{RL}}\nabla^{\nu_2+\nu_1}f\right)(\tau)-\left[\frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}-\nu_2\frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right]f(a_0+1)\\
&\quad-\frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}f(a_0+2),
\end{aligned}
$$

which completes the proof of (2.7), where in both we have used [32, Lemma 3.108] and [32, Theorem 3.57]. Thus, the proof is done. □

3. Monotonicity results

We start with the first result concerning the $\nabla-$fractional difference on the set $\mathscr{D}_1$.

Lemma 3.1. *Let f be defined on $\mathbb{N}_{a_0}$, $(\nu_2,\nu_1)\in\mathscr{D}_1$ and $\left({}_{a_0+1}^{\mathrm{RL}}\nabla^{\nu_1}\ {}_{a_0}^{\mathrm{RL}}\nabla^{\nu_2}f\right)(\tau)\geqq 0$, for $\tau\in\mathbb{N}_{a_0+3}$. Then the following inequality can be obtained:*

$$
\begin{aligned}
(\nabla f)(a_0+\mu) &\geqq \left[\frac{(\mu)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}-\frac{(\mu)^{\overline{-\nu_1+\nu_2}}}{\Gamma(1-\nu_1-\nu_2)}\right]f(a_0+1)\\
&\quad-\frac{1}{\Gamma(1-\nu_1-\nu_2)}\sum_{\jmath=0}^{\mu-3}(\mu-\jmath-1)^{\overline{-\nu_1-\nu_2}}(\nabla f)(a_0+\jmath+2),
\end{aligned}
\tag{3.1}
$$

for $\mu\in\mathbb{N}_3$. Furthermore,

$$
\frac{(\mu)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}-\frac{(\mu)^{\overline{-\nu_1+\nu_2}}}{\Gamma(1-\nu_1-\nu_2)}>0, \tag{3.2}
$$

and

$$
-\frac{1}{\Gamma(1-\nu_1-\nu_2)}(\mu-\jmath-1)^{\overline{-\nu_1-\nu_2}}>0, \tag{3.3}
$$

for $\jmath=0,1,\ldots,\mu-4$ and $\mu\in\mathbb{N}_4$.

Proof. The identity (2.6) and Definition (2.3) enable us to write

$$
\begin{aligned}
\left({}_{a_0+1}^{\mathrm{RL}}\nabla^{\nu_1}\ {}_{a_0}^{\mathrm{RL}}\nabla^{\nu_2}f\right)(\tau) &= \left({}_{a_0}^{\mathrm{RL}}\nabla^{\nu_2+\nu_1}f\right)(\tau)-\frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}f(a_0+1)\\
&= \frac{1}{\Gamma(-\nu_2-\nu_1)}\sum_{\mathsf{s}=a_0+1}^{\tau}(\tau-\mathsf{s}+1)^{\overline{-\nu_2-\nu_1-1}}f(\mathsf{s})-\frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}f(a_0+1)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{by}{\underset{(2.2)}{=}} \frac{1}{\Gamma(1-\nu_2-\nu_1)} \sum_{\mathsf{s}=a_0+1}^{\tau} \nabla_\tau (\tau-\mathsf{s}+1)^{\overline{-\nu_2-\nu_1}} f(\mathsf{s}) - \frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+1) \\
&= \frac{1}{\Gamma(1-\nu_2-\nu_1)}\Big[(\tau-a_0)^{\overline{-\nu_2-\nu_1}} f(a_0+1) + \Gamma(1-\nu_2-\nu_1)(\nabla f)(\tau) \\
&+ \sum_{\mathsf{s}=a_0+2}^{\tau-1} (\tau-\mathsf{s}+1)^{\overline{-\nu_2-\nu_1}} (\nabla f)(\mathsf{s})\Big] - \frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+1) \\
&= (\nabla f)(\tau) + \left[\frac{(\tau-a_0)^{\overline{-\nu_2-\nu_1}}}{\Gamma(1-\nu_2-\nu_1)} - \frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) \\
&+ \frac{1}{\Gamma(1-\nu_2-\nu_1)} \sum_{\mathsf{s}=a_0+2}^{\tau-1} (\tau-\mathsf{s}+1)^{\overline{-\nu_2-\nu_1}} (\nabla f)(\mathsf{s}), \qquad (3.4)
\end{aligned}
$$

where we have used that $(0)^{\overline{-\nu_2-\nu_1}} = 0$. By using the assumption that $\left({}^{\mathrm{RL}}_{a_0+1}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) \geqq 0$, it follows that

$$
(\nabla f)(\tau) \geqq \left[\frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \frac{(\tau-a_0)^{\overline{-\nu_2-\nu_1}}}{\Gamma(1-\nu_2-\nu_1)}\right] f(a_0+1) - \frac{1}{\Gamma(1-\nu_2-\nu_1)} \sum_{\mathsf{s}=a_0+2}^{\tau-1} (\tau-\mathsf{s}+1)^{\overline{-\nu_2-\nu_1}} (\nabla f)(\mathsf{s}).
$$

Changing the variable $\mu \coloneqq \tau - a_0$ gives the desired inequality (3.1).

The last part of the lemma is easy to be proved by considering the definition (2.2) as follows

$$
\frac{(\mu)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} = \frac{\overbrace{(-\nu_1)}^{<0}\overbrace{(1-\nu_1)}^{<0}\cdots(\mu-3-\nu_1)(\mu-2-\nu_1)}{(\mu-1)!} > 0,
$$

and

$$
\frac{(\mu)^{\overline{-\nu_1+\nu_2}}}{\Gamma(1-\nu_1-\nu_2)} = \frac{\overbrace{(1-\nu_1-\nu_2)}^{<0}(2-\nu_1-\nu_2)\cdots(\mu-3-\nu_1-\nu_2)(\mu-2-\nu_1-\nu_2)}{(\mu-1)!} < 0,
$$

for $\mu \in \mathbb{N}_3$, $1 < \nu_1 < 2$ and $1 < \nu_1 + \nu_2 < 2$, which rearranges to (3.2). And we can obtain (3.2) as follows

$$
\begin{aligned}
&- \frac{1}{\Gamma(1-\nu_1-\nu_2)} (\mu - \jmath - 1)^{\overline{-\nu_1-\nu_2}} \\
&- \frac{(\mu-\jmath-2-\nu_1-\nu_2)(\mu-\jmath-3-\nu_1-\nu_2)\cdots(2-\nu_1-\nu_2)\overbrace{(1-\nu_1-\nu_2)}^{<0}}{(\mu-\jmath-2)!} > 0,
\end{aligned}
$$

for $1 < \nu_1 + \nu_2 < 2$ and $\jmath = 0, 1, \ldots, \mu - 4$ with $\mu \in \mathbb{N}_4$. This ends the proof. □

Theorem 3.1. *Under the assumptions of Lemma 3.1 together with*

(1) $f(a_0 + 1) \geqq 0$*;*
(2) $(\nabla f)(a_0 + 1) \geqq 0$*;*

(3) $(\nabla f)(a_0+2) \geqq 0$,

one can have $(\nabla f)(\tau) \geqq 0$, *for* $\tau \in \mathbb{N}_{a_0+1}$.

Proof. From (3.1)–(3.3), the assumption (a) we have inductively that $(\nabla f)(\tau) \geqq 0$, for each $\tau \in \mathbb{N}_{a_0+3}$. Furthermore, from the assumptions (b) and (c) we get $(\nabla f)(\tau) \geqq 0$, for $\tau \in \mathbb{N}_{a_0+1}$. This concludes the proof. □

Our second result concerning the $\nabla-$fractional difference on the set $\mathscr{D}_2$.

Lemma 3.2. *Let f be defined on $\mathbb{N}_{a_0}$, $(\nu_2,\nu_1) \in \mathscr{D}_2$ and $\left({}^{\mathrm{RL}}_{a_0+2}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) \geqq 0$, for $\tau \in \mathbb{N}_{a_0+3}$. Then the following inequality holds*

$$
\begin{aligned}
(\nabla f)(a_0+\mu) \geqq & \left[\frac{(\mu)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \frac{(\mu)^{\overline{-\nu_1+\nu_2}}}{\Gamma(1-\nu_1-\nu_2)} - \nu_2\,\frac{(\mu-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) + \frac{(\mu-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+2) \\
& - \frac{1}{\Gamma(1-\nu_1-\nu_2)} \sum_{\jmath=0}^{\mu-3} (\mu-\jmath-1)^{\overline{-\nu_1-\nu_2}} (\nabla f)(a_0+\jmath+2),
\end{aligned}
\tag{3.5}
$$

for $\mu \in \mathbb{N}_3$.

Proof. In view of the identity (2.7) and (3.4), we have

$$
\begin{aligned}
\left({}^{\mathrm{RL}}_{a_0+2}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) = & \frac{1}{\Gamma(-\nu_2-\nu_1)} \sum_{\mathsf{s}=a_0+1}^{\tau} (\tau-\mathsf{s}+1)^{\overline{-\nu_2-\nu_1-1}} f(\mathsf{s}) \\
& - \left[\frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \nu_2\,\frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) - \frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+2) \\
= & (\nabla f)(\tau) - \left[\frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \frac{(\tau-a_0)^{\overline{-\nu_1+\nu_2}}}{\Gamma(1-\nu_1-\nu_2)} - \nu_2\,\frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) \\
& - \frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+2) + \frac{1}{\Gamma(1-\nu_2-\nu_1)} \sum_{\mathsf{s}=a_0+2}^{\tau-1} (\tau-\mathsf{s}+1)^{\overline{-\nu_2-\nu_1}} (\nabla f)(\mathsf{s}).
\end{aligned}
$$

By applying the assumption $\left({}^{\mathrm{RL}}_{a_0+2}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau) \geqq 0$ to the last identity, we get

$$
\begin{aligned}
(\nabla f)(\tau) \geqq & \left[\frac{(\tau-a_0)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \frac{(\tau-a_0)^{\overline{-\nu_1+\nu_2}}}{\Gamma(1-\nu_1-\nu_2)} - \nu_2\,\frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) \\
& + \frac{(\tau-a_0-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+2) - \frac{1}{\Gamma(1-\nu_2-\nu_1)} \sum_{\mathsf{s}=a_0+2}^{\tau-1} (\tau-\mathsf{s}+1)^{\overline{-\nu_2-\nu_1}} (\nabla f)(\mathsf{s}),
\end{aligned}
$$

for $\tau \in \mathbb{N}_{a_0+3}$. The last inequality together with changing the variable $\mu := \tau - a_0$ rearrange the desired inequality (3.5). □

Theorem 3.2. *Let the assumptions of Lemma 3.2 be fulfilled with $\tau = a_0 + 3$. Suppose that*

(1) $f(a_0+1) \geqq 0$*;*
(2) $(\nabla f)(a_0+2) \geqq 0$*;*
(3) $\delta f(a_0+1) \geqq f(a_0+2) \geqq 0$*, for some* $1 \leqq \delta$*.*

Then one can have $(\nabla f)(a_0+3) \geqq 0$ *such that* $\delta \leqq 1 + \frac{3\nu_2-\nu_2^2-2}{2\nu_1}$*.*

Proof. Rewriting (3.5) at $\mu = 3$ to get

$$(\nabla f)(a_0+3) \geqq \left[\frac{(3)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \frac{(3)^{\overline{-\nu_1+\nu_2}}}{\Gamma(1-\nu_1-\nu_2)} - \nu_2\frac{(2)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) + \frac{(2)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+2) - \frac{(2)^{\overline{-\nu_1-\nu_2}}}{\Gamma(1-\nu_1-\nu_2)}(\nabla f)(a_0+2). \tag{3.6}$$

We know that $-\frac{(2)^{\overline{-\nu_1-\nu_2}}}{\Gamma(1-\nu_1-\nu_2)} = -(1-\nu_1-\nu_2) > 0$ by $1 < \nu_1+\nu_2 < 2$, and $(\nabla f)(a_0+2) \geqq 0$ by condition (b), so (3.6) becomes

$$\begin{aligned}(\nabla f)(a_0+3) &\geqq \left[\frac{(3)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \frac{(3)^{\overline{-\nu_1+\nu_2}}}{\Gamma(1-\nu_1-\nu_2)} - \nu_2\frac{(2)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) + \frac{(2)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+2)\\ &= \left[\frac{-1}{2}\nu_1(1-\nu_1) - \frac{1}{2}(2-\nu_1-\nu_2)(1-\nu_1-\nu_2) + \nu_2\,\nu_1\right] f(a_0+1) - \nu_1\, f(a_0+2)\\ &\underset{\text{condition (c)}}{\overset{by}{\geqq}} \left[\frac{-1}{2}\nu_1(1-\nu_1) - \frac{1}{2}(2-\nu_1-\nu_2)(1-\nu_1-\nu_2) + \nu_2\,\nu_1\right] f(a_0+1) - \delta\,\nu_1\, f(a_0+1)\\ &= \frac{2\nu_1+3\nu_2-2\delta\,\nu_1-\nu_2^2-2}{2} f(a_0+1),\end{aligned}$$

which is $\geqq 0$ by condition (a) provided that $\frac{2\nu_1+3\nu_2-2\delta\,\nu_1-\nu_2^2-2}{2} \geqq 0$, or equivalently, $\delta \leqq 1 + \frac{3\nu_2-\nu_2^2-2}{2\nu_1}$. Hence, the proof is finished. □

Remark 3.1. *It is worth mentioning that in Theorem 3.2 the quantity* $\frac{3\nu_2-\nu_2^2-2}{2\nu_1}$ *is positive for* $(\nu_2,\nu_1) \in \mathscr{D}_2$*.*

4. Application

Here we provide two numerical examples in infinite time set $\mathbb{N}_{a_0}$ to demonstrate the performance of Theorems 3.1 and 3.2. Furthermore, we have performed all implementations using Matlab 2018-b, installed on laptop with Intel(R) Core(TM) i7–2600 CPU@2.30GHz and 16.00 Gb-RAM running on Windows 10 operating system.

Example 4.1. Let f be a function defined by

$$f(\tau) = 2^{\tau-a_0}, \quad \text{for } \tau \in \mathbb{N}_{a_0}.$$

From the proof of Lemma 3.1, we have for $\tau := a_0+\mu$:

$$\left({}^{\mathrm{RL}}_{a_0+1}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(a_0+\mu) = \frac{1}{\Gamma(-\nu_2-\nu_1)}\sum_{\jmath=0}^{\mu-1}(\mu-\jmath)^{\overline{-\nu_2-\nu_1-1}} f(\jmath+a_0+1) - \frac{(\mu)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+1),$$

for $\mu \in \mathbb{N}_3$. Letting $a_0 = 0$, it follows that

$$\left({}^{\mathrm{RL}}_{1}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{0}\nabla^{\nu_2} f\right)(\mu) = \frac{1}{\Gamma(-\nu_2-\nu_1)}\sum_{\jmath=0}^{\mu-1}(\mu-\jmath)^{\overline{-\nu_2-\nu_1-1}} f(\jmath+1) - \frac{(\mu)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(1)$$
$$= \frac{1}{\Gamma(-\nu_2-\nu_1)}\sum_{\jmath=0}^{\mu-1}\frac{\Gamma(\mu-\jmath-\nu_2-\nu_1-1)}{\Gamma(\mu-\jmath)}2^{\jmath+1} - 2\frac{\Gamma(\mu-\nu_1-1)}{\Gamma(-\nu_1)\Gamma(\mu)}, \tag{4.1}$$

for $\mu \in \mathbb{N}_3$. Computing (4.1) at $\nu_1 = 1.2$, $\nu_2 = 0.3$, and some values of μ, we get

$$\left({}^{\mathrm{RL}}_{1}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{0}\nabla^{\nu_2} f\right)(\mu) = \frac{51}{100}, \text{for } \mu = 3,$$
$$= \frac{5561}{1000}, \text{for } \mu = 4,$$
$$= \frac{3741}{332}, \text{for } \mu = 5,$$

and so on, we get $\left({}^{\mathrm{RL}}_{1}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{0}\nabla^{\nu_2} f\right)(\mu) \geqq 0$, for each $\mu \in \mathbb{N}_3$. Furthermore, we have

$$f(1) = 2, \quad (\nabla f)(1) = 1, (\nabla f)(2) = 2.$$

Hence, Theorem 3.1 confirms that $(\nabla f)(\mu) \geqq 0$, for each $\mu \in \mathbb{N}_1$.

Example 4.2. In this example, let us define f by

$$f(\tau) = \left(\frac{3}{2}\right)^{\tau-a_0}, \text{for } \tau \in \mathbb{N}_{a_0}.$$

In view of the proof of Lemma 3.2, we have for $\tau := a_0 + \mu$:

$$\left({}^{\mathrm{RL}}_{a_0+2}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{a_0}\nabla^{\nu_2} f\right)(a_0+\mu) = \frac{1}{\Gamma(-\nu_2-\nu_1)}\sum_{\jmath=0}^{\mu-1}(\mu-\jmath)^{\overline{-\nu_2-\nu_1-1}} f(\jmath+a_0+1)$$
$$-\left[\frac{(\mu)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} - \nu_2\frac{(\mu-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)}\right] f(a_0+1) - \frac{(\mu-1)^{\overline{-\nu_1-1}}}{\Gamma(-\nu_1)} f(a_0+2),$$

for $\mu \in \mathbb{N}_3$. Putting $a_0 = 0$, it follows that

$$\left({}^{\mathrm{RL}}_{2}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{0}\nabla^{\nu_2} f\right)(\mu) = \frac{1}{\Gamma(-\nu_2-\nu_1)}\sum_{\jmath=0}^{\mu-1}\frac{\Gamma(\mu-\jmath-\nu_2-\nu_1-1)}{\Gamma(\mu-\jmath)}\left(\frac{3}{2}\right)^{\jmath+1}$$
$$-\frac{3}{2}\left[\frac{\Gamma(\mu-\nu_1-1)}{\Gamma(-\nu_1)\Gamma(\mu)} - \nu_2\frac{\Gamma(\mu-\nu_1-2)}{\Gamma(-\nu_1)\Gamma(\mu-1)}\right] - \frac{9}{4}\frac{\Gamma(\mu-\nu_1-2)}{\Gamma(-\nu_1)\Gamma(\mu-1)}, \tag{4.2}$$

for $\mu \in \mathbb{N}_3$. Calculating (4.2) at $\nu_2 = 1.1$, $\nu_1 = 0.05$, and $\mu = 3$ to obtain

$$\left({}^{\mathrm{RL}}_{2}\nabla^{\nu_1}\ {}^{\mathrm{RL}}_{0}\nabla^{\nu_2} f\right)(3) = \frac{393}{400} \geqq 0.$$

Besides, by choosing $1 \leqq \delta = 1.6 \leqq 1 + \frac{3\nu_2-\nu_2^2-2}{2\nu_1} = 1.9$, we have

$$f(1) = 1.5, f(2) = 2.25, (\nabla f)(2) = 0.75, \text{and} f(2) - \delta f(1) = -0.15.$$

Then $(\nabla f)(3) \geqq 0$, which confirms the conclusion of Theorem 3.2.

5. Conclusions

The investigation of positivity analysis development for discrete sequential fractional operator of mixed order was explored in this research article based on the time set $\mathbb{N}_{a_0}$. In general, our results can be summarized as follows:

(1) The standardized discrete sequential operators $\left({}^{\text{RL}}_{a_0+1}\nabla^{\nu_1}\ {}^{\text{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau)$ and $\left({}^{\text{RL}}_{a_0+2}\nabla^{\nu_1}\ {}^{\text{RL}}_{a_0}\nabla^{\nu_2} f\right)(\tau)$ are firstly formulated in (2.6) and (2.7), respectively.
(2) The above formulations are defined on the sets $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively.
(3) Based on the first formulation (2.6), the positivity of nabla is discussed in details for each $\tau \in \mathbb{N}_{a_0}$.
(4) Although it was difficult to examine the positivity of nabla at each time step $\tau \in \mathbb{N}_{a_0}$, we have found the positivity of nabla at $\tau = a_0 + 3$ with an extra condition (see condition (c) in Theorem 3.2) based on the formulation (2.7).
(5) We have demonstrated the accuracy and efficiency of the main results using two examples. In the first example, we have found that $f(\tau) = 2^{\tau - a_0}$ is increasing (i.e. $(\nabla f)(\tau) \geqq 0$) for each $\tau \in \mathbb{N}_{a_0+1}$ based on Theorem 3.1. In the second example, Theorem 3.2 confirmed that $f(\tau) = \left(\frac{3}{2}\right)^{\tau - a_0}$ is increasing at $\tau = \{a_0 + 1, a_0 + 2, a_0 + 3\}$.

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Conflict of interest

The authors declare that they have no conflicts interests.

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