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Research article

A study of Wiener-Hopf dynamical systems for variational inequalities in the setting of fractional calculus

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Abstract: In this paper, we consider a new fractional dynamical system for variational inequalities using the Wiener Hopf equations technique. We show that the fractional Wiener-Hopf dynamical system is exponentially stable and converges to its unique equilibrium point under some suitable conditions . We also discuss some special cases, which can be obtained from our main results.

Keywords: dynamical systems; fractional derivative; convergence; variational inequalities
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1. Introduction

In modern analysis theory of inequalities has played a significant role due to its wide range of applications both in pure and applied sciences. Variational inequalities are one of the most significant and intensively studied inequalities. Recently these inequalities have appeared as an interesting and dynamic field of pure and applied sciences. Variational techniques are being used to study a wide class of problems with ‘applications in industry, structural engineering, mathematical finance, economics, optimization, transportation, and optimization problems, see [1, 9–11, 18, 21, 23–25, 30] and references therein. This has motivated researchers to introduce and study several classes of variational inequalities. Stampacchia [30] has studied that the minimum of differentiable convex functions on a convex set can be characterized by variational inequalities.

Using the fixed-point formulation of the variational inequalities, Dupuis and Nagurney [4] introduced and considered the projected dynamical systems in which the right hand side of the ordinary differential equation is a projection operator associated with variational inequalities. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problem. Hence, equilibrium and nonlinear problems arising in various branches in pure and applied sciences, which can be formulated in the setting of the variational inequalities, can be studied in the more general setting of dynamical systems. It has been shown [5, 6, 10, 11, 14, 16, 17, 32] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. In recent years, much attention has been given to study the globally asymptotic stability of these projected dynamical systems. Xia and Wang [32] have shown that the projected dynamical systems can be used effectively in designing neural network for solving variational inequalities.

Zeng et al. [31] have investigated the fractional dynamical systems associated with linear variational inequalities. They have investigated the criteria for the asymtotically stability of the equilibrium points. Noor et al. [22] have also suggested and analyzed fractional implicit dynamical systems for linear quasi variational inequalities by extending and modifying their techniques. In this paper, we propose and consider a fractional Wiener-Hopf dynamical system associated with the variational inequalities. We show that the fractional Wiener-Hopf dynamical system is exponentially stable and converges to its unique equilibrium point. Some special cases are discussed. Our results are more general than the results of Zeng et al. [31]. The ideas and techniques of this paper may inspire the interested readers for further research in this area.

2. Formulation and basic results

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be any closed and convex set in H.

For given operator $\mathcal{T} : H \to H$, consider a problem of finding $u \in K$ such that

$$\langle \mathcal{T}u, v-u\rangle \geq 0, \quad \forall v \in K. \tag{2.1}$$

The inequality of type (2.1) is called the variational inequality. This problem was introduced and studied by Stampacchia [30]. For the recent applications, numerical algorithms, sensitivity analysis, dynamical systems and formulations of variational inequalities, see [1–29, 33] and the references therein.

Definition 2.1. *A nonlinear operator $\mathcal{T} : H \to H$ is said to be monotone, if*

$$\langle \mathcal{T}u - \mathcal{T}v, u-v\rangle \geq 0, \quad \forall u, v \in H.$$

Definition 2.2. *A nonlinear operator $\mathcal{T} : H \to H$ is said to be pseudomonotone, if*

$$\langle \mathcal{T}u, v-u\rangle \geq 0, \quad \textit{implies} \quad \langle \mathcal{T}v, v-u\rangle \geq 0, \quad \forall u, v \in H.$$

Definition 2.3. *A nonlinear operator $\mathcal{T} : H \to H$ is said to be Lipschitz continuous, if there exists a constant $\beta > 0$ such that*

$$\|\mathcal{T}u - \mathcal{T}v\| \leq \beta \|u-v\|, \quad \forall u, v \in H.$$

It is well known that monotonicity implies pseudomonotonicity, but not conversely.

Lemma 2.4. *[16] Let K be a closed and convex set in H. Then for a given $z \in H$, $u \in K$ satisfies*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.2}$$

if and only if

$$u = P_K[z],$$

where P_K is the projection of H onto a closed and convex set K in H.

It is well known that the projection operator P_K is non-expensive, that is

$$\|P_K[u] - P_K[v]\| \leq \|u - v\|, \quad \forall u, v \in H.$$

Using Lemma 2.4, one can show that Problem (2.1) is equivalent to the fixed point problem.

Lemma 2.5. *[16] Let K be a closed and convex set in a real Hilbert space H. The function $u \in K$ is a solution of Problem* (2.1)*, if and only if, $u \in K$ satisfies the relation*

$$u = P_K[u - \rho \mathcal{T} u], \tag{2.3}$$

where $\rho > 0$ is a constant.

From Lemma 2.5, it follows that Problem (2.1) is equivalent to a fixed point Problem (2.3). This equivalent formulation plays an important role in developing several iterative methods, see [17, 19, 20].

We now define a residue vector $\mathcal{R}(u)$ as:

$$\mathcal{R}(u) = u - P_K[u - \rho \mathcal{T} u]. \tag{2.4}$$

It is clear from Lemma 2.5 that Problem (2.1) has a solution $u \in K$, if and only if, $u \in K$ is a zero of the equation

$$\mathcal{R}(u) = 0.$$

It is well known that the variational inequalities are also equivalent to Wiener Hopf equations. The Wiener Hopf equations technique has been used to develop some efficient and powerful iterative methods for solving variational inequalities and complementarity problems. We now use the Wiener Hopf equations technique to suggest and analyze another dynamical system.

Let $Q_K = I - P_K$, where I is the identity operator and P_K is the projection of H onto the closed and convex set K. For given nonlinear operator $\mathcal{T} : H \rightarrow H$, consider a problem of finding $z \in H$ such that

$$\mathcal{T} P_K[z] + \rho^{-1} Q_K[z] = 0. \tag{2.5}$$

Problem (2.5) is called Wiener Hopf equations associated with the variational inequalities (2.1). This problem was introduced and studied by Shi [29]. It is well known that Problems (2.1) and (2.5) are equivalent. For the sake of completeness, we recall this result without proof.

Lemma 2.6. *[16] The Problem* (2.1) *has a solution* $u \in K$, *if and only if, Problem* (2.5) *has a solution* $z \in H$*, where*

$$u = P_K[z], \tag{2.6}$$

$$z = u - \rho\mathcal{T}u, \tag{2.7}$$

where $\rho > 0$ *is a constant.*

Using Lemma 2.6, the Wiener-Hopf Eq (2.5) can be written as

$$u - \rho\mathcal{T}u - P_K[u - \rho\mathcal{T}u] + \rho\mathcal{T}P_K[u - \rho\mathcal{T}u] = 0, \tag{2.8}$$

which is equivalent to

$$\mathcal{R}(u) - \rho\mathcal{T}u + \rho\mathcal{T}P_K[u - \rho\mathcal{T}u] = 0. \tag{2.9}$$

Thus it is clear from Lemma 2.6 that $u \in K$ is a solution of Problem (2.1) , if and only if, $u \in K$ satisfies the Eq (2.9).

We now suggest a new dynamical system:

$$D_\omega^\alpha u(\omega) = \gamma\{-\mathcal{R}(u) - \rho\mathcal{T}P_K[u - \rho\mathcal{T}u] + \rho\mathcal{T}u\}, \quad u(\omega_0) = u_0 \in K, \tag{2.10}$$

where $0 < \alpha < 1$ and γ is a constant, associated with Problem (2.1). The dynamical system of type (2.10) is called fractional Wiener-Hopf dynamical system related to the Problem (2.1).

We now discuss some special cases of Problem (2.10).

(1) If $\alpha = 1$, then the fractional Wiener-Hopf dynamical System (2.10) reduces to following dynamical system:

$$\begin{aligned} \frac{du}{dt} &= \gamma\{P_K[u - \rho\mathcal{T}u] - \rho\mathcal{T}P_K[u - \rho\mathcal{T}u] + \rho\mathcal{T}u - u\}, \\ u(\omega_0) &= u_0 \in K, \end{aligned} \tag{2.11}$$

which was introduced and considered by Noor [16].

(2) In the affine case, that is, if $\mathcal{T}u = Au + b$, where $A = a_{ij}$ is a real $n \times n$ matrix and $b = b_j$ is an n-dimensional vector, then the System (2.10) reduces to:

$$\begin{aligned} D_\omega^\alpha u(\omega) &= \gamma\{P_K[u - \rho Au - \rho b] - \rho AP_K[u - \rho Au - \rho b] + \rho Au - u\}, \\ u(\omega_0) &= u_0 \in K, \end{aligned} \tag{2.12}$$

which is called linear fractional Wiener-Hopf dynamical system.

We also need the following well-known fundamental results and concepts.

Definition 2.7. *[8] The fractional integral (Riemann-Liouville integral) with order* $\alpha \in \mathcal{R}_+$ *of continuous function* $u(\omega)$ *is defined as:*

$$I_{\omega_0}^\alpha u(\omega) = \frac{1}{\Gamma(\alpha)}\int_{\omega_0}^{\omega}(\omega - \tau)^{\alpha-1}u(\tau)\,d\tau, \quad \omega > \omega_0,$$

where Γ *denotes the Euler gamma function.*

Definition 2.8. *[8] The Caputo derivative with order* $\alpha \in \mathcal{R}_+$ *of continuous function* $u(\omega) \in C^n([\omega_0, +\infty], \mathcal{R})$ *is defined as:*

$$D_{\omega_0}^{\alpha} u(\omega) = I_{\omega_0}^{n-\alpha} u^{(n)}(\omega) = \frac{1}{\Gamma(\alpha)} \int_{\omega_0}^{\omega} (\omega - \tau)^{n-\alpha-1} u^{(n)}(\tau)\, d\tau, \quad \omega > \omega_0,$$

where n *is positive integer such that* $n - 1 < \alpha < n$ *and* Γ *denotes the Euler gamma function.*

Definition 2.9. *[31] The point* u^* *is said to be an equilibrium point of the fractional projected dynamical System* (2.4)*, if* u^* *satisfies the following*

$$P_K[u^*(\omega) - \rho \mathcal{T} u^*(\omega)] - u^*(\omega) = 0.$$

Definition 2.10. *[33] The dynamical system* (2.4) *is said to be* α*-exponentially stable with degree* λ *if, for any two solutions* $u(\omega)$ *and* $v(\omega)$ *of* (2.4) *with different initial values by* u_0 *and* v_0 *satisfies*

$$\|u(\omega) - v(\omega)\| \le \eta \|u_0 - v_0\| e^{-\lambda \omega^{\alpha}}, \quad \forall \omega \ge \omega_0,$$

where $\eta > 0$ *is a constant.*

Lemma 2.11. *(Gronwall's Lemma [17]) Let* u *and* v *be real valued non-negative continuous functions with domain* $\{\omega : \omega \ge \omega_0\}$ *and let* $\alpha(\omega) = \alpha_0 |\omega - \omega_0|$*, where* α_0 *is a monotone increasing function. If, for* $\omega \ge \omega_0$*,*

$$u(\omega) \le \alpha(\omega) + \int_{\omega_0}^{\omega} u(s) v(s)\, ds,$$

then

$$u(\omega) \le \alpha(\omega) \cdot \exp\left(\int_{\omega_0}^{\omega} v(s)\, ds \right).$$

Lemma 2.12. *[8] Let* n *is a positive integer such that* $n - 1 < \alpha < n$*. If* $u(\omega) \in C^n[a, b]$*, then*

$$I_{\omega}^{\alpha} D_{\omega}^{\alpha} u(\omega) = u(\omega) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (\omega - a)^k.$$

In particular, if $0 < \alpha \le 1$ *and* $u(\omega) \in C^1[a, b]$

$$I_{\omega}^{\alpha} D_{\omega}^{\alpha} u(\omega) = u(\omega) - u(a). \tag{2.13}$$

Lemma 2.13. *[33] Let* $u(\omega)$ *be a continuous function on* $[0, +\infty)$ *and satisfies*

$$D_{\omega}^{\alpha} u(\omega) \le \theta . u(\omega), \tag{2.14}$$

where $0 < \alpha < 1$ *and* θ *is a constant. Then*

$$u(\omega) \le u(0) . \exp\left(\frac{\theta . \omega^{\alpha}}{\Gamma(\alpha + 1)} \right).$$

Proof. For a nonnegative continuous function $h(\omega)$, relation (2.14) can be written as:

$$D_{\omega}^{\alpha}u(\omega) + h(\omega) = \theta.u(\omega). \tag{2.15}$$

By taking the fractional integral of order α of (2.15), we have

$$I_{\omega}^{\alpha} D_{\omega}^{\alpha}u(\omega) + I^{\alpha}h(\omega) = I^{\alpha}\theta.u(\omega). \tag{2.16}$$

Using Lemma 2.12, we have

$$I_{\omega}^{\alpha} D_{\omega}^{\alpha}u(\omega) = u(\omega) - u(0). \tag{2.17}$$

Since $h(\omega)$ is a nonnegative continuous function, therefore by the definition of fractional integral we have

$$I_{\omega}^{\alpha}h(\omega) = \frac{1}{\Gamma(\alpha)}\int_{0}^{\omega}(\omega-\tau)^{\alpha-1}h(\tau)\,d\tau \geq 0, \quad \omega > 0, \tag{2.18}$$

and

$$I_{\omega}^{\alpha}\theta.u(\omega) = \frac{\theta}{\Gamma(\alpha)}\int_{0}^{\omega}(\omega-\tau)^{\alpha-1}u(\tau)\,d\tau, \quad \omega > 0. \tag{2.19}$$

Combining (2.16)–(2.19), we have

$$u(\omega) - u(0) + 0 \leq \frac{\theta}{\Gamma(\alpha)}\int_{0}^{\omega}(\omega-\tau)^{\alpha-1}u(\tau)\,d\tau, \quad \omega > 0,$$

which implies

$$\begin{aligned} u(\omega) &\leq u(0) + \frac{\theta}{\Gamma(\alpha)}\int_{0}^{\omega}(\omega-\tau)^{\alpha-1}u(\tau)\,d\tau, \quad \omega > 0 \\ &\leq u(0).\exp\left(\frac{\theta}{\Gamma(\alpha)}\int_{0}^{\omega}(\omega-\tau)^{\alpha-1}\,d\tau\right) \\ &= u(0).\exp\left(\frac{\theta.\omega^{\alpha}}{\Gamma(\alpha+1)}\right), \end{aligned}$$

where we have used Lemma 2.11. This is the desired result. □

Lemma 2.14. *[12, 31] Consider the system*

$$D_{\omega}^{\alpha}u(\omega) = g(\omega, u(\omega)), \quad \omega > \omega_0, \tag{2.20}$$

with initial condition $u(\omega_0)$, where $0 < \alpha \leq 1$ and $g : [\omega_0, \infty) \times \Omega \to H$, $\Omega \subset H$. If $g(\omega, u(\omega))$ satisfies the locally Lipschitz condition with respect to u, then there exists a unique solution of (2.20) *on* $[\omega_0, \infty) \times \Omega$.

Lemma 2.15. *[12] For the real valued continuous function $g(\omega, u(\omega))$, defined in* (2.20)*, we have $\left\| I_{\omega}^{\alpha}g(\omega, u(\omega)) \right\| \leq I_{\omega}^{\alpha}\left\| g(\omega, u(\omega)) \right\|$, where $\alpha \geq 0$ and $\|\cdot\|$ denotes an arbitrary norm.*

3. Main results

In this section we study the main properties of the dynamical system (2.10) and analyze the global stability of the systems.

Theorem 3.1. *Let the operator $\mathcal{T}$ is Lipschitz continuous with constant $\beta > 0$. If $\gamma > 0$, then there exists a unique solution $u(\omega) \in H$ of Problem* (2.10) *with $u(\omega_0) = u_0$, that is defined for all $\omega \in [\omega_0, \infty)$.*

Proof. Let

$$G(u(\omega)) = \gamma \{P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - \rho\mathcal{T}P_K[u(\omega) - \rho\mathcal{T}u(\omega)] + \rho\mathcal{T}u(\omega) - u(\omega)\}.$$

To prove that $G(u(\omega))$ is Lipschitz continuous for all $u(\omega), v(\omega) \in H$, we have to consider

$$\begin{aligned} &\|G(u(\omega)) - G(v(\omega))\| \\ = {}& \gamma \left\|P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - \rho\mathcal{T}P_K[u(\omega) - \rho\mathcal{T}u(\omega)] + \rho\mathcal{T}u(\omega) - u(\omega) \right. \\ & \left. - P_K[v(\omega) - \rho\mathcal{T}v(\omega)] + \rho\mathcal{T}P_K[v(\omega) - \rho\mathcal{T}v(\omega)] - \rho\mathcal{T}v(\omega) + v(\omega)\right\| \\ \leq {}& \gamma \left\|P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - P_K[v(\omega) - \rho\mathcal{T}v(\omega)]\right\| + \gamma\rho\|\mathcal{T}u(\omega) - \mathcal{T}v(\omega)\| \\ & + \gamma\rho\left\|\mathcal{T}P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - \mathcal{T}P_K[v(\omega) - \rho\mathcal{T}v(\omega)]\right\| + \gamma\|u(\omega) - v(\omega)\| \\ \leq {}& \gamma\|u(\omega) - v(\omega) - \rho(\mathcal{T}u(\omega) - \mathcal{T}v(\omega))\| + \gamma\rho\beta\|u(\omega) - v(\omega)\| \\ & + \gamma\rho\beta\|u(\omega) - v(\omega) - \rho(\mathcal{T}u(\omega) - \mathcal{T}v(\omega))\| + \gamma\|u(\omega) - v(\omega)\| \\ \leq {}& 2\gamma\|u(\omega) - v(\omega)\| + 3\gamma\rho\beta\|u(\omega) - v(\omega)\| + \gamma\rho^2\beta^2\|u(\omega) - v(\omega)\| \\ = {}& \gamma\left(2 + 3\rho\beta + \rho^2\beta^2\right)\|u(\omega) - v(\omega)\|, \end{aligned}$$

where we have used Lipschitz continuity of operator $\mathcal{T}$ with constant $\beta > 0$. This implies that operator $G(u)$ is a Lipschitz continuous in H. Thus from Lemma 2.14, it is clear that there exists a unique solution $u(\omega)$ of Problem (2.10). Let $[\omega_0, \infty)$ be its maximal interval of existence; we show that $\mathcal{T}_1 = \infty$. Consider

$$\begin{aligned} \left\|D^\alpha_\omega u(\omega)\right\| = {}& \|G(u(\omega))\| \\ = {}& \gamma\left\|P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - \rho\mathcal{T}P_K[u(\omega) - \rho\mathcal{T}u(\omega)] + \rho\mathcal{T}u(\omega) - u(\omega)\right\| \\ \leq {}& \gamma\left\|P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - u(\omega)\right\| + \gamma\rho\beta\left\|P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - u(\omega)\right\| \\ = {}& \gamma(1 + \rho\beta)\left\|P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - P_K[u(\omega)] \right. \\ & \left. + P_K[u(\omega)] - P_K[u^*] + P_K[u^*] - u(\omega)\right\| \\ \leq {}& \gamma(1 + \rho\beta)\left\{\left\|P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - P_K[u(\omega)]\right\| \right. \\ & \left. + \|P_K[u(\omega)] - P_K[u^*]\| + \|P_K[u^*] - u(\omega)\|\right\} \\ \leq {}& \gamma(1 + \rho\beta)\{\|u(\omega) - \rho\mathcal{T}u(\omega) - u(\omega)\| + \|u(\omega) - u^*\| \\ & + \|P_K[u^*]\| + \|u(\omega)\|\} \\ \leq {}& \gamma(1 + \rho\beta)\{(2 + \rho\beta)\|u(\omega)\| + \|u^*\| + \|P_K[u^*]\|\} \\ = {}& \gamma(1 + \rho\beta)\{\|u^*\| + \|P_K[u^*]\|\} + \gamma(1 + \rho\beta)(2 + \rho\beta)\|u(\omega)\| \\ = {}& k_1 + k_2\|u(\omega)\|, \end{aligned} \tag{3.1}$$

where we have used the Lipschitz continuity of operator $\mathcal{T}$, with constant $\beta > 0$.

Taking the fractional integral of (3.1), we have

$$\begin{aligned} I_{\omega}^{\alpha}\left\|D_{\omega}^{\alpha}u(\omega)\right\| &\leq I_{\omega}^{\alpha}\{k_1 + k_2\|u(\omega)\|\} \\ &= \frac{k_1}{\Gamma(\alpha)}\int_{\omega_0}^{\omega}(\omega-\tau)^{\alpha-1}d\tau + \frac{k_2}{\Gamma(\alpha)}\int_{\omega_0}^{\omega}(\omega-\tau)^{\alpha-1}\|u(\tau)\|d\tau \\ &= \frac{k_1(\omega-\omega_0)^{\alpha}}{\Gamma(\alpha+1)} + \frac{k_2}{\Gamma(\alpha)}\int_{\omega_0}^{\omega}(\omega-\tau)^{\alpha-1}\|u(\tau)\|d\tau, \end{aligned}$$

using Lemma 2.12 and Lemma 2.15, we have

$$\|u(\omega) - u(\omega_0)\| \leq \frac{k_1(\omega-\omega_0)^{\alpha}}{\Gamma(\alpha+1)} + \frac{k_2}{\Gamma(\alpha)}\int_{\omega_0}^{\omega}(\omega-\tau)^{\alpha-1}\|u(\tau)\|d\tau.$$

From which by using Lemma 2.11, we obtain

$$\begin{aligned} \|u(\omega)\| &\leq \left\{\|u(\omega_0)\| + \frac{k_1(\omega-\omega_0)^{\alpha}}{\Gamma(\alpha+1)}\right\} + \frac{k_2}{\Gamma(\alpha)}\int_{\omega_0}^{\omega}(\omega-\tau)^{\alpha-1}\|u(\tau)\|d\tau \\ &\leq \left\{\|u(\omega_0)\| + \frac{k_1(\omega-\omega_0)^{\alpha}}{\Gamma(\alpha+1)}\right\}\exp\left\{\frac{k_2(\omega-\omega_0)^{\alpha}}{\Gamma(\alpha+1)}\right\}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} k_1 &= \gamma(1+\rho\beta)\{\|u^*\| + \|P_K[u^*]\|\} \\ k_2 &= \gamma(1+\rho\beta)(2+\rho\beta) > 0. \end{aligned}$$

Thus from (3.2), it follows that the solution is bounded on $[\omega_0, \infty)$. □

Theorem 3.2. *Let the operator $\mathcal{T}$ be pseudomonotone and Lipschitz continuous. If $\gamma > 0$, then the dynamical system* (2.10) *is stable in the sense of Lyapunov and globally converges to the solution of variational inequality* (2.1).

Proof. Since the operator is Lipschitz continuous, it follows from Theorem 3.1 that the dynamical system (2.10) has a unique continuous solution $u(\omega)$ over $[\omega, \mathcal{T}_1)$ for any fixed $u_0 \in K$. Let $u(\omega, \omega_0; u_0)$ be he solution of the initial value Problem (2.1). For a given $u^* \in K$, consider the following Lyapunov function

$$L(u(\omega)) = \|u(\omega) - u^*\|^2, \quad u \in K. \tag{3.3}$$

It is clear that $\lim_{n\to\infty} L(u_n) = \infty$, whenever the sequence $\{u_n\} \subset K$ and $\lim_{n\to\infty} u_n = \infty$. Consequently, we conclude that the level sets of L are bounded. Let $u^* \in K$ be a solution of the variational inequality (2.1). Then

$$\langle \mathcal{T}u^*, v - u^*\rangle \geq 0, \quad \forall v \in K,$$

which implies, using pseudomonotonicity of operator $\mathcal{T}$,

$$\langle \mathcal{T} v, v - u^* \rangle \geq 0, \quad \forall v \in K. \tag{3.4}$$

Setting $v = P_K [u - \rho \mathcal{T} u]$ in (3.4), we have

$$\langle \mathcal{T} P_K [u - \rho \mathcal{T} u], P_K [u - \rho \mathcal{T} u] - u^* \rangle \geq 0. \tag{3.5}$$

Now, setting $v = u^*$, $u = P_K [u - \rho \mathcal{T} u]$ and $z = u - \rho \mathcal{T} u$ in (2.2), we have

$$\langle P_K [u - \rho \mathcal{T} u] - u + \rho \mathcal{T} u, u^* - P_K [u - \rho \mathcal{T} u] \rangle \geq 0. \tag{3.6}$$

Adding (3.5) and (3.6), we have

$$\langle P_K [u - \rho \mathcal{T} u] - u + \rho \mathcal{T} u - \rho \mathcal{T} P_K [u - \rho \mathcal{T} u], u^* - P_K [u - \rho \mathcal{T} u] \rangle \geq 0, \tag{3.7}$$

using (2.4), (3.7) can be written as

$$\langle -\mathcal{R}(u) + \rho \mathcal{T} u - \rho \mathcal{T} P_K [u - \rho \mathcal{T} u], u^* - u + \mathcal{R}(u) \rangle \geq 0,$$

which implies that

$$\begin{aligned} \langle u - u^*, \mathcal{R}(u) - \rho \mathcal{T} u + \rho \mathcal{T} P_K [u - \rho \mathcal{T} u] \rangle & \\ \geq\ & \|\mathcal{R}(u)\|^2 - \rho \langle \mathcal{R}(u), \mathcal{T} u - \mathcal{T} P_K [u - \rho \mathcal{T} u] \rangle \\ \geq\ & \|\mathcal{R}(u)\|^2 - \rho \|\mathcal{R}(u)\| \left\| \mathcal{T} u - \mathcal{T} P_K [u - \rho \mathcal{T} u] \right\| \\ \geq\ & \|\mathcal{R}(u)\|^2 - \rho\delta \|\mathcal{R}(u)\| \left\| u - P_K [u - \rho \mathcal{T} u] \right\| \\ =\ & (1 - \rho\delta) \|\mathcal{R}(u)\|^2, \end{aligned} \tag{3.8}$$

where we have used the fact that the operator $\mathcal{T}$ is Lipschitz continuous with constant $\delta > 0$.

Thus from (2.10), (3.3) and (3.8), we have

$$\begin{aligned} D_\omega^\alpha L(u(\omega)) &= (D_u^\alpha L(u(\omega)))(D_\omega^\alpha u(\omega)) \\ &= \frac{\Gamma(3)}{\Gamma(3-\alpha)} \langle u - u^*, D_\omega^\alpha u \rangle \\ &= \frac{2\gamma}{\Gamma(3-\alpha)} \langle u - u^*, -\mathcal{R}(u) + \rho \mathcal{T} u - \rho \mathcal{T} P_K [u - \rho \mathcal{T} u] \rangle \\ &\leq \frac{-2\gamma(1-\rho\delta)}{\Gamma(3-\alpha)} \|\mathcal{R}(u)\|^2 \leq 0. \end{aligned}$$

This implies that $L(u)$ is a global Lyapunov function for the System (2.10) and the system is stable in the sense of Lyapunov. Since $\{u(\omega) : \omega \geq \omega_0\} \subset K_0$, where $K_0 = \{u \in K : L(u) \leq L(u_0)\}$ and the function is differentiable on the bounded and closed set K, then it follows from LaSalle's invariance principle that the trajectory will converge to Ω, the largest invariant subset of the following subset:

$$E = \{u \in K : D_\omega^\alpha L(u(\omega)) = 0\}.$$

Note that, if $D_\omega^\alpha L(u(\omega)) = 0$, then

$$\left\| u - P_K [u - \rho \mathcal{T} u] \right\| = 0,$$

and hence u is the equilibrium point of the dynamical system (2.4), that is,

$$D_{\omega}^{\alpha}u(\omega) = 0.$$

Conversely, if $D_{\omega}^{\alpha}u(\omega) = 0$, then it follows that $D_{\omega}^{\alpha}L(u(\omega)) = 0$. Thus, we conclude that

$$E = \{u \in K : D_{\omega}^{\alpha}u(\omega) = 0\} = K_0 \cap K^*,$$

which is nonempty, convex and invariant set containing the solution set K^*. So

$$\lim_{\omega\to\infty} \text{dist}(u(\omega), E) = 0.$$

Therefore the dynamical system (2.10) converges globally to the solution set of the variational inequalities (2.1). In particular, if the set $E = \{u^*\}$, then

$$\lim_{\omega\to\infty} u(\omega) = u^*.$$

Hence the dynamical system (2.10) is globally asymptotically stable. □

We now discuss the stability and existence of the equilibrium point for the dynamical System (2.10) under some suitable conditions.

Theorem 3.3. *Let the operator $\mathcal{T}$ be Lipschitz continuous with constant $\beta > 0$. If $\gamma < 0$, then fractional dynamical System* (2.10) *is α-exponentially stable.*

Proof. Let $u(\omega)$ and $v(\omega)$ be any two solutions of dynamical System (2.10) with initial values $u(0) = u_0$ and $v(0) = v_0$ respectively.

Let

$$e(\omega) = u(\omega) - v(\omega),$$

then $e(0) \neq 0$ and taking the fractional derivative of above equation, we have

$$\begin{aligned} D_{\omega}^{\alpha}e(\omega) &= D_{\omega}^{\alpha}u(\omega) - D_{\omega}^{\alpha}v(\omega) \\ &= \gamma\{P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - \rho\mathcal{T}P_K[u(\omega) - \rho\mathcal{T}u(\omega)] + \rho\mathcal{T}u(\omega) - u(\omega)\} \\ &\quad -\gamma\{P_K[v(\omega) - \rho\mathcal{T}v(\omega)] - \rho\mathcal{T}P_K[v(\omega) - \rho\mathcal{T}v(\omega)] + \rho\mathcal{T}v(\omega) - v(\omega)\} \\ &= \gamma\{P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - P_K[v(\omega) - \rho\mathcal{T}v(\omega)]\} + \gamma\rho\{\mathcal{T}u(\omega) - \mathcal{T}v(\omega)\} \\ &\quad -\gamma\rho\{\mathcal{T}P_K[u(\omega) - \rho\mathcal{T}u(\omega)] - \mathcal{T}P_K[v(\omega) - \rho\mathcal{T}v(\omega)]\} - \gamma e(\omega), \end{aligned} \tag{3.9}$$

where $0 < \alpha < 1$, $\omega \geq 0$. From Theorem 3.1, $u(\omega)$ and $v(\omega)$ are uniquely determined solutions. Therefore $e(\omega)$ is the uniquely determined solution of error System (3.9) with initial value $e(0) = e_0$.

We claim that if $e(0) > 0$, then $e(\omega) \geq 0$ for $\omega \geq 0$, if $e(0) < 0$, then $e(\omega) \leq 0$ for $\omega \geq 0$. In fact, if $e(0) > 0$, there exists ω_1, such that $e(\omega) < 0$ for $\omega \geq \omega_1$, so there must be $0 < \omega_0 < \omega_1$ such that $e(\omega_0) = 0$. It means that dynamical System (2.10) has two different solutions with initial value ω_0 to $e(\omega_0) \leq 0$ for $\omega \geq \omega_0$, which contradicts to Theorem 3.1. In a similar way, we can prove that if $e(0) < 0$, then $e(\omega_0) \leq 0$ for $\omega \geq 0$.

So, we can have

$$\begin{aligned}
D_{\omega}^{\alpha}G(\omega) &= D_{\omega}^{\alpha}\|e(\omega)\| \\
&= \operatorname{sgn}(e(\omega))\left(D_{\omega}^{\alpha}e(\omega)\right) \\
&= \gamma\operatorname{sgn}(e(\omega))\{P_K[u(\omega)-\rho\mathcal{T}u(\omega)]-P_K[v(\omega)-\rho\mathcal{T}v(\omega)]\} \\
&\quad -\gamma\rho\operatorname{sgn}(e(\omega))\{\mathcal{T}P_K[u(\omega)-\rho\mathcal{T}u(\omega)]-\mathcal{T}P_K[v(\omega)-\rho\mathcal{T}v(\omega)]\} \\
&\quad +\gamma\rho\operatorname{sgn}(e(\omega))\{\mathcal{T}u(\omega)-\mathcal{T}v(\omega)\}-\gamma\operatorname{sgn}(e(\omega))\,e(\omega) \\
&\le \gamma\left\|P_K[u(\omega)-\rho\mathcal{T}u(\omega)]-P_K[v(\omega)-\rho\mathcal{T}v(\omega)]\right\| \\
&\quad +\gamma\rho\left\|\mathcal{T}P_K[u(\omega)-\rho\mathcal{T}u(\omega)]-\mathcal{T}P_K[v(\omega)-\rho\mathcal{T}v(\omega)]\right\| \\
&\quad +\gamma\rho\|\mathcal{T}u(\omega)-\mathcal{T}v(\omega)\|+\gamma\|e(\omega)\| \\
&\le \gamma(1+\rho\beta)\|u(\omega)-v(\omega)-\rho(\mathcal{T}u(\omega)-\mathcal{T}v(\omega))\| \\
&\quad +\gamma\rho\beta\|u(\omega)-v(\omega)\|+\gamma\|e(\omega)\| \\
&\le \gamma(1+\rho\beta)\|u(\omega)-v(\omega)\|+\gamma\rho\beta(1+\rho\beta)\|u(\omega)-v(\omega)\| \\
&\quad +\gamma(1+\rho\beta)\|e(\omega)\| \\
&= \gamma(1+\rho\beta)(2+\rho\beta)\|e(\omega)\|,
\end{aligned}$$

which implies

$$\begin{aligned}
D_{\omega}^{\alpha}G(\omega) &\le \gamma(1+\rho\beta)(2+\rho\beta)G(\omega) \\
&= \theta_1 G(\omega),
\end{aligned}$$

thus by using Lemma 2.13, we have

$$G(\omega)\le G(0)\exp\left(\frac{\theta_1\omega^{\alpha}}{\Gamma(\alpha+1)}\right),$$

where

$$\theta_1=\gamma(1+\rho\beta)(2+\rho\beta)<0.$$

Let $\theta_1=-\theta_2$, where θ_2 is a positive constant. Then

$$G(\omega)\le G(0)\exp\left(\frac{-\theta_2\omega^{\alpha}}{\Gamma(\alpha+1)}\right),$$

which shows that the dynamical system (2.10) is α-exponentially stable. □

Remark 3.4. *We would like to mention that, for* $\alpha = 1$, *Theorems 3.1–3.3 are also valid under the same conditions. This gives us a new technique to study dynamical systems associated with variational inequalities.*

4. Conclutions

In this paper, we have introduced and studied fractional Wiener-Hopf dynamical systems for classical variational inequalities. These fractional dynamical systems related to the variational inequalities are proposed and analyzed using the projection technique. We have proved that these dynamical systems converge exponentially to the unique solution of variational inequality problems under some suitable conditions. The suggested dynamical systems can be used in designing recurrent neural networks for solving variational inequalities and related optimization problems.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. A. Baiocchi, A. Capelo, *Variational and quasi-variational inequalities*, New York: John Wiley and Sons, 1984.
2. J. Dong, D. Zhang, A. Nagurney, A projected dynamical systems model of general financial equilibrium with stability analysis, *Math. Comput. Model.*, **24** (1996), 35–44. https://doi.org/10.1016/0895-7177(96)00088-X
3. T. S. Du, J. G. Liao, L. Z. Chen, M. U. Awan, Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized (α, m)-preinvex functions, *J. Inequal. Appl.*, **306** (2016). https://doi.org/10.1186/s13660-016-1251-5
4. P. Dupuis, A. Nagurney, Dynamical systems and variational inequalities, *Ann. Oper. Res.*, **44** (1993), 7–42. https://doi.org/10.1007/BF02073589
5. T. L. Friesz, D. Bernstein, N. J. Mehta, R. L. Tobin, S. Ganjalizadeh, Day-to-day dynamic network disequilibria and idealized traveler information systems, *Oper. Res.*, **42** (1994), 1120–1136. https://doi.org/10.1287/opre.42.6.1120
6. T. L. Friesz, D. Bernstein, R. Stough, Dynamic systems, variational inequalities and control theoretic models for predicting time-varying urban network flows, *Transport. Sci.*, **30** (1996), 14–31. https://doi.org/10.1287/trsc.30.1.14
7. A. A. Khan, M. Sama, Optimal control of multivalued quasi variational inequalities, *Nonlinear Anal.-Theo.*, **75** (2012), 1419–1428. https://doi.org/10.1016/j.na.2011.08.005
8. A. A. Kilbas, M. H. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
9. A. S. Kravchuk, P. J. Neittaanmki, *Variational and quasi-variational inequalities in mechanics*, Berlin: Springer, 2007.
10. Q. Liu, J. Cao, A recurrent neural network based on projection operator for extended general variational inequalities, *IEEE. T. Syst. Man Cy. B*, **40** (2010), 928–938. https://doi.org/10.1109/TSMCB.2009.2033565
11. Q. Liu, Y. Yang, Global exponential system of projection neural networks for system of generalized variational inequalities and related nonlinear minimax problems, *Neurocomputing*, **73** (2010), 2069–2076. https://doi.org/10.1016/j.neucom.2010.03.009

12. Y. Li, Y. Q. Chen, I. Podlubny, Mittag-Leffler stability of fractional order nonlinear dynamic systems, *Automatica*, **45** (2009), 1965–1969. https://doi.org/10.1016/j.automatica.2009.04.003

13. Y. Li, Y. Q. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct nethod and generalized Mittag-Leffler stability, *Comput. Math. Appl.*, **59** (2010), 1810–1821. https://doi.org/10.1016/j.camwa.2009.08.019

14. A. Nagurney, A. D. Zhang, *Projected dynamical systems and variational inequalities with applications*, Boston: Kluwer Academic, 1996.

15. C. Niculescu, L. E. Persson, *Convex functions and their applications: a contemporary approach*, Berlin: Springer, 2006.

16. M. A. Noor, A Wiener-Hopf dynamical system for variational inequalities, *New Zealand J. Math.*, **31** (2002), 173–182.

17. M. A. Noor, Implicit dynamical systems and quasi variational inequalities, *Appl. Math. Comput.*, **134** (2003), 69–81. https://doi.org/10.1016/S0096-3003(01)00269-7

18. M. A. Noor, Auxiliary principle technique for extended general variational inequalities, *Banach J. Math. Anal.*, **2** (2008), 33–39.

19. M. A. Noor, K. I. Noor, Some new classes of quasi split feasibility problems, *Appl. Math. Inform. Sci.*, **7** (2013), 1547–1552. http://dx.doi.org/10.12785/amis/070439

20. M. A. Noor, K. I. Noor, A. G. Khan, Some iterative schemes for solving extended general quasi variational inequalities, *Appl. Math. Inform. Sci.*, **7** (2013), 917–925.

21. M. A. Noor, K. I. Noor, A. G. Khan, Dynamical systems for quasi variational inequalities, *Ann. Funct. Anal.*, **6** (2015), 193–209.

22. M. A. Noor, K. I. Noor and A. G. Khan, Fractional projected dynamical system for quasi variational inequalities, *U. Polithe. Buch. Ser. A*, **80** (2018), 99–112.

23. M. A. Noor, K. I. Noor, B. B. Mohsen, M. T. Rassias, A. Raigorodskii, General preinvex functions and variational-like inequalities, In: *Approximation and computation in science and engineering*, Berlin: Springer, 2022.

24. M. A. Noor, K. I. Noor, M. T. Rassias, New trends in general variational inequalities, *Acta Appl. Math.*, **170** (2020), 981–1064. https://doi.org/10.1007/s10440-020-00366-2

25. M. A. Noor, K. I. Noor, M. T. Rassias, Strongly biconvex functions and bivariational inequalities, In: *Mathematical analysis, optimization, approximation and applications*, Singapore: World Scientific Publishing Company, 2021.

26. I. Petras, *Fractioanl-order nonlinear systems: modeling, analysis and simulation*, New York: Springer, 2011.

27. I. Podlubny, *Fractional differential equations*, San Siego: Academic Press, 1999.

28. Y. Shehu, Iterative methods for fixed points and equilibrium problems, *Ann. Funct. Anal.*, **1** (2010), 121–132. https://doi.org/10.15352/afa/1399900594

29. P. Shi, Equivalence of variational inequalities with Wiener-Hopf equations, *Proc. Amer. Math. Soc.*, **111** (1991), 339–346.

30. G. Stampacchia, Formes bilineaires coercivites sur les ensembles convexes, *C. R. Acad. Scz. Paris*, **258** (1964), 4413–4416.

31. Z. B. Wu, Y. Z. Zou, Global fractional-order projective dynamical systems, *Commun. Nonlinear Sci.*, **19** (2014), 2811–2819. https://doi.org/10.1016/j.cnsns.2014.01.007

32. Y. Xia, J. Wang, On the stability of globally projected dynamical systems, *J. Optim. Theory Appl.*, **106** (2000), 129–150.

33. J. Yu, C. Hu, H. J. Jiang, α-stability and α-synchronization for fractional-order neural networks, *Neural Networks*, **35** (2012), 82–87. https://doi.org/10.1016/j.neunet.2012.07.009