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Research article

Existence, uniqueness and Hyers-Ulam stability of random impulsive stochastic integro-differential equations with nonlocal conditions

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Abstract: In this article, we study the existence and stability results of mild solutions for random impulsive stochastic integro-differential equations (RISIDEs) with noncompact semigroups and resolvent operators in Hilbert spaces. Initially, we prove the existence of mild solutions using Hausdorff measures of noncompactness and Mönch fixed point theorem. Then, we explore the stability results which includes continuous dependence of initial conditions, Hyers-Ulam stability and mean-square stability of the system by developing some new analysis techniques and establishing an improved inequality. Finally, we propose an example to validate the obtained results.

Keywords: stochastic differential equations; random impulse; Hyers-Ulam stability; mean-square exponential stability
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1. Introduction

In general, impulsive effects are a widespread natural phenomena caused by instantaneous perturbations at certain moments, such as various biological models involving thresholds, bursting explosive models in medicine biological theory, the optimal control model in economics and so on [1–3]. In the past few decades, differential equations (DEs) with impulses are utilized to model the processes subjected to abrupt changes at discrete moments and the dynamics of impulsive DEs have attracted the attention of a large number of scholars, see [4–8]. Furthermore, since real-world systems and natural phenomena are almost inevitably affected by stochastic perturbations, mathematical models cannot ignore the stochastic factors due to a combination of uncertainties and complexities. (Partial) DEs driven by stochastic processes or equations with random impulses provide a natural and effective method for explicating a variety of impulsive occurrences in order to take them into consideration. The study of stochastic differential equations have been considered by many researchers, such as [9–15], and the references therein. In particular, Wu and his teams first investigated random impulsive DEs, in [16]. Then, they applied the random impulsive stochastic differential equations (SDEs) to study the stock pricing models [17] combining Brownian motions through the empirical analysis of the historical data. Wu et al. [18] investigated the existence and uniqueness of SDEs with random impulses. Zhou and Wu [19] developed the existence and uniqueness of solutions with random impulses under Lipschitz conditions. Yang et al. [20] existence and stability results of mild solutions for random impulsive stochastic PDEs with noncomact semigroups in Hilbert spaces using M$\ddot{o}$nch fixed point theorem.

However, there are not many papers considering the Hyers-Ulam stability problem of stochastic impulsive differential equations. Recently, Li et al. [21] investigated the existence and Hyers-Ulam stability of mild solutions for random impulsive stochastic functional ordinary differential equations using Krasnoselskii's fixed point theorem. Anguraj et al. [22] established the stability of random impulsive stochastic functional differential equations driven by Poisson jumps with finite delays by using Banach fixed point theorem. Many authors have studied the various kinds of integro-differential equations with random impulses [23,24]. Recently, there have been massive studies covering existence and stability of solutions to partial differential equations (PDEs) with impulses or randomness. Chan [25] established a new impulsive integral inequality to obtain the sufficient conditions in order to prove the existence and stability of mild solutions for impulsive stochastic PDEs with delays. Gao and Li [26] developed a new criterion in proving the mean-square exponential stability of the proposed existence of mild solutions for the impulsive stochastic PDEs with noncompact semigroup.

To the best of our knowledge, no notable work has yet been published that conducts research on RISIDEs with nonlocal conditions combining the Mönch fixed point theorem and the Hausdorff measure of noncompactness via resolvent operator. In this paper, we consider the following RISIDEs with nonlocal conditions of the form:

$$\begin{aligned} d\vartheta(\mathrm{t}) &= \left[\mathfrak{A}\vartheta(\mathrm{t}) + \int_0^{\mathrm{t}} \Theta(\mathrm{t}-s)\vartheta(s)ds + \mathfrak{f}(\mathrm{t},\vartheta(\mathrm{t}))\right]d\mathrm{t} + \mathfrak{g}(\mathrm{t},\vartheta(\mathrm{t}))d\omega(\mathrm{t}), \quad \mathrm{t} \geq \mathrm{t}_0, \quad \mathrm{t} \neq \xi_{\mathrm{k}}, \\ \vartheta(\xi_{\mathrm{k}}) &= \mathfrak{b}_{\mathrm{k}}(\delta_{\mathrm{k}})\vartheta(\xi_{\mathrm{k}}^-), \quad \mathrm{k} = 1,2,\cdots, \\ \vartheta(0) &= \mathfrak{h}(\vartheta) + \vartheta_0, \end{aligned} \tag{1.1}$$

where $\mathfrak{A}$ is the infinitesimal generator of a strongly continuous semigroup $(\mathfrak{S}(\mathrm{t}))_{\mathrm{t}\geq 0}$ on $\mathbb{X}$ with domain $\mathscr{D}(\mathfrak{A})$. $(\Theta(t))_{t\geq 0}$ is a closed linear operators on $\mathbb{X}$ with $\mathscr{D}(\Theta(t)) \supset \mathscr{D}(\mathfrak{A})$ which is independent of t and $\omega(t)$ is a standard Wiener process on $\mathbb{X}$. Let $\mathbb{Y}$ be another separable Hilbert space. We may denote $\mathcal{L}^2(\Omega,\mathbb{X})$ the collection of all strongly measurable, square-integrable $\mathbb{X}$-valued random variables. Then $\|\vartheta(.)\|_{\mathcal{L}^2} = \left(\mathbb{E}\,\|\vartheta(.,\omega)\|^2\right)^{1/2}$ is a Banach space.

Throughout this work, we may consider the subspace of $\mathcal{L}^2(\Omega,\mathbb{X})$ given by $\mathcal{L}_2^0(\Omega,\mathbb{X}) = \{\vartheta \in \mathcal{L}_2^0(\Omega,\mathbb{X}) \mid \vartheta \text{ is } \mathcal{F}_0\text{-measurable }\}$. We denote $\mathscr{C}([\mathrm{t}_0,\infty];\mathcal{L}^2(\Omega,\mathbb{X}))$ the space of all continuous

$\mathcal{F}_t-$adapted measurable processes from $[t_0, \infty]$ to $\mathcal{L}^2(\Omega, \mathbb{X})$ satisfying $\sup_{t\in[t_0,\infty]} \mathbb{E}\|\vartheta(t)\|^2 < \infty$. Then, it is obvious that $\mathscr{C}([t_0, \infty]; \mathcal{L}^2(\Omega, \mathbb{X}))$ is a Banach space equipped with the subnorm

$$\|\vartheta\|_{\mathscr{C}} = \left(\sup_{t\in[t_0,\infty]} \mathbb{E}\|\vartheta(t)\|^2\right)^{1/2}.$$

The map $\mathfrak{f} : [t_0, +\infty] \times \mathbb{X} \to \mathbb{X}$, $\mathfrak{g} : [t_0, +\infty] \times \mathbb{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$, $\mathfrak{h} : \mathscr{C}([t_0, +\infty]; \mathcal{L}^2(\Omega, \mathbb{X})) \to \mathbb{X}$ are Borel measurable functions. Let δ_k be a random variable from Ω to $\mathscr{D}_k := (0, \mathfrak{d}_k)$ with $0 < \mathfrak{d}_k < +\infty$ for $k = 1, 2, \cdots$ and suppose that δ_i and δ_j are independent of each other as $i \neq j$ for $i, j = 1, 2, \cdots$. Here $\mathfrak{b}_k : \mathscr{D}_k \to \mathbb{X}$, and $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \delta_k$ for $k = 1, 2, \cdots$, where $t_0 \in [\delta, +\infty]$ is an arbitrary given non-negative number. It is obvious that

$$t_0 = \xi_0 < \xi_1 < \cdots < \lim_{k\to\infty} \xi_k = +\infty,$$

then, $\{\xi_k\}$ is a process with independent increments. Denoting $\vartheta(\xi_k^-) := \lim_{\vartheta\to\xi_k^-} \vartheta(t)$, the norm

$$\|\vartheta\|_t := \sup_{t-\delta\leq s\leq t} \|\vartheta\|_{\mathbb{X}},$$

with the jump

$$\Delta\vartheta(\xi_k) := [\mathfrak{b}_k(\delta_k) - 1]\, \vartheta(\xi_k^-),$$

represents the random impulsive effect in the state ϑ at time ξ_k. The preliminary data $\varphi : [-\delta, 0] \to \mathbb{X}$ is a function with respect to ϑ when $t = t_0$. Let us suppose, $\{\mathscr{N}(t), t \geq 0\}$ is a simple counting process generated by $\{\xi_k\}$, $\mathfrak{I}_t^{(1)}$ be the $\sigma-$algebra generated by $\{\mathscr{N}(t), t \geq 0\}$ and $\mathfrak{I}_t^{(2)}$ indicates the $\sigma-$algebra generated by $\{\omega(t) : t \geq 0\}$ where $\mathfrak{I}_\infty^{(1)}, \mathfrak{I}_\infty^{(2)}$ and ξ being mutually independent.

The novelties of this paper are the following aspects:

- Motivated by the previously mentioned literatures [25–27], we add the random impulses into the system. We discover how stochastic integro-differential equations with nonlocal conditions driven by Brownian motions interact with random impulses in the proof of existence of mild solutions by using Hausdorff measure of noncompactness and Mönch fixed point theorem via resolvent operator.

- Under the influence of both white noises and random impulses, we investigate stability with continuous dependence of initial conditions, Hyers-Ulam stability and mean-square exponential stability of mild solution for the RISIDEs with nonlocal conditions.

2. Preliminaries and notations

Let $\mathbb{X}$ and $\mathbb{Y}$ be real separable Hilbert space with norms $\|.\|$,$\|.\|_{\mathbb{Y}}$ and $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ denotes the space of bounded linear operators from $\mathbb{Y}$ to $\mathbb{X}$. $(\Omega, \mathfrak{F}, \mathcal{P})$ be a complete filtered probability space provided the filtration $\mathfrak{I}_t^{(1)} \vee \mathfrak{I}_t^{(2)}(t \geq 0)$ satisfies the usual notation. In the probability space $(\Omega, \mathfrak{F}, \mathcal{P})$, $\{\beta_n(t), t \geq 0\}$ represents a real-valued one dimensional standard Brownian motion being mutually independent. For the probability measure $\mathcal{P}$, $\mathcal{L}^2(\Omega)$ be the space of square-integrable random variables . Let $Q \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$

be a positive trace class operator on $\mathcal{L}^2(\mathbb{X})$ and $(\lambda_n, e_n)_n$ symbolizes its spectral elements. The Weiner process $\omega(\mathrm{t})$ is exhibited as:

$$\omega(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda_n}\beta_n(\mathrm{t})e_n,$$

with $trQ = \sum_{n=1}^{+\infty} \lambda_n < +\infty$. Then, the $\mathbb{Y}-$valued stochastic process $\omega(\mathrm{t})$ is called a $Q-$Weiner process.

Definition 2.1. *[20] Let $\Xi \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$, we define*

$$\|\Xi\|^2_{\mathcal{L}_2^0} := tr(\Xi Q \Xi^*) = \{\sum_{n=1}^{+\infty} \| \sqrt{\lambda_n}\Xi e_n\|^2\}.$$

If $\|\Xi\|^2_{\mathcal{L}_2^0} < +\infty$, then Ξ is called a $Q-$Hilbert-Schmidt operator and $\mathcal{L}_2^0$ is the space of all $Q-$Schmidt operators $\Xi : \mathbb{Y} \to \mathbb{X}$.

Partial integro-differential equations: Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces such that

$$|y|_{\mathbb{Y}} := |\mathfrak{A}y| + |y| \quad \text{for} \quad y \in \mathbb{Y}.$$

$\mathfrak{A}$ and $\Theta(t)$ are closed linear operator on $\mathbb{X}$.

Let $\mathscr{C}([0, +\infty); \mathbb{Y})$, stand for the space of all continuous functions from $[0, +\infty)$ into $\mathbb{Y}$, the set of all bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$. For further purposes, let us consider the following system

$$\begin{aligned} v'(t) &= \mathfrak{A}v(t) + \int_0^t \Theta(t-s)v(s)ds, \quad t \geq 0, \\ v(0) &= v_0 \in \mathbb{X}. \end{aligned} \tag{2.1}$$

Definition 2.2. *[28] A resolvent for Eq (2.1) is a bounded linear operator valued function $\Re(t) \in \mathcal{L}(\mathbb{X})$ for $t \geq 0$, having the following properties:*
(i) $\Re(0) = \mathcal{I}$ and $\|\Re(t)\| \leq Me^{\lambda t}$ for some constants M and λ.
(ii) For each $v \in \mathbb{X}$, $\Re(t)v$ is continuous for $t \geq 0$.
(iii) $\Re(t) \in \mathcal{L}(\mathbb{Y})$, for $t \geq 0$. For $v \in \mathbb{Y}, \Re(\cdot)v \in \mathscr{C}^1([0, +\infty); \mathbb{X}) \cap \mathscr{C}([0, +\infty); \mathbb{Y})$ and

$$\begin{aligned} \Re'(t)v &= \mathfrak{A}\Re(t)v + \int_0^t \Theta(t-s)\Re(s)vds \\ &= \Re(t)\mathfrak{A}v + \int_0^t \Re(t-s)\Theta(s)vds, \; t \geq 0. \end{aligned}$$

The resolvent operator is supposed to be exponentially stable as Definition 2.2 (i) holds for $\lambda > 0$. The following constrains acquired from Grimmer [28] are enough to ensure the existence of solutions for (2.1).

(H1) The operator $\mathfrak{A}$ is an infinitesimal generator of a strongly continuous semigroup on $\mathbb{X}$.
(H2) For all $t \geq 0$, $\Theta(t)$ represents a closed continuous linear operator from $\mathscr{D}(\mathfrak{A})$ to $\mathbb{X}$ and $\Theta(t) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$. For any $\mathfrak{y} \in \mathbb{Y}$, the map $t \mapsto \Theta(t)\mathfrak{y}$ is bounded, differentiable and its derivative $t \mapsto \Theta'(t)\mathfrak{y}$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.1. *[28] Assume that (H1)–(H2) hold. Then, there exists a unique resolvent operator for the Cauchy problem 2.1.*

Now, consider the conditions that ensure the existence of solutions to the deterministic integro-differential equation.

$$\begin{aligned} v'(t) &= \mathfrak{A}v(t) + \int_0^t \Theta(t-s)v(s)ds + m(t), \quad t \geq 0, \\ v(0) &= v_0 \in \mathbb{X}, \end{aligned} \tag{2.2}$$

where $m : [0, +\infty) \to \mathbb{X}$ is a continuous function.

Lemma 2.1. *[28] If v is a strict solution of [2.2], then*

$$v(t) = \Re(t)v_0 + \int_0^t \Re(t-s)m(s)ds, \quad t \geq 0. \tag{2.3}$$

Lemma 2.2. *[28] Assuming (H1), (H2) holds, the resolvent operator $\Re(t)$ is continuous for $t \geq 0$ on the operator norm, namely for $t_0 \geq 0$,*

$$\lim_{\tau \to 0} \|\Re(t_0 + \tau) - \Re(t_0)\| = 0.$$

Lemma 2.3. *[28] Assume (H1), (H2) gets satisfied, then $\exists\, \mathscr{G} > 0 \ni$*

$$\|\Re(t + \epsilon) - \Re(\epsilon)\Re(t)\| \leq \mathscr{G}\epsilon.$$

Lemma 2.4. *[20] For any $\mathfrak{p} \geq 0$ and for arbitrary $\mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$-valued predictable process $\Psi(\cdot)$, we have*

$$\sup_{s \in [0,\mathrm{t}]} \left\| \int_0^s \Psi(s) d\omega(s) \right\|^{\mathfrak{p}} \leq \mathscr{C}'_{\mathfrak{p}} \left(\int_0^{\mathrm{t}} \left(\mathbb{E}\|\Psi(s)\|^{\mathfrak{p}}_{\mathcal{L}_2^0} \right)^{\frac{2}{\mathfrak{p}}} ds \right)^{\frac{2}{\mathfrak{p}}}, \ \mathrm{t} \in l,$$

where $\mathscr{C}'_{\mathfrak{p}} = \left(\frac{\mathfrak{p}(\mathfrak{p}-1)}{2} \right)^{\frac{p}{2}}$.

The Hausdorff measure of noncompactness $\alpha(.)$ defined on a bounded subset $\mathscr{E}$ of a Banach space $\mathbb{X}$ by

$$\alpha(\mathscr{E}) = \inf\{\varepsilon > 0 : \mathscr{E} \text{ has a finite } \varepsilon - \text{net in } \mathbb{X}\}.$$

Lemma 2.5. *[20] Let $\mathbb{X}$ be a real Banach space and $\mathscr{M}, \mathscr{N} \subset \mathbb{X}$ be bounded. Then we have the following properties:*
(1) *$\mathscr{M}$ is precompact if and only if $\alpha(\mathscr{M}) = 0$.*
(2) *$\alpha(\mathscr{M}) = \alpha(\overline{\mathscr{M}}) = \alpha(conv\mathscr{M})$, where $\overline{\mathscr{M}}$ and $conv\mathscr{M}$ are the closure and the convex hull.*
(3) *$\alpha(\mathscr{M}) \leq \alpha(\mathscr{N})$ while $\mathscr{M} \subset \mathscr{N}$.*
(4) *$\alpha(\mathscr{M} + \mathscr{N}) \leq \alpha(\mathscr{M}) + \alpha(\mathscr{N})$, wherever $\mathscr{M} + \mathscr{N} = \{\vartheta + \varpi : \vartheta \in \mathscr{M}, \varpi \in \mathscr{N}\}$.*
(5) *$\alpha(\mathscr{M} \cup \mathscr{N}) \leq \max\{\alpha(\mathscr{M}), \alpha(\mathscr{N})\}$.*
(6) *$\alpha(\lambda\mathscr{M}) \leq |\lambda|\alpha(\mathscr{N})$ for any $\lambda \in \Re$.*
(7) *If $\mathscr{K} \subset \mathscr{C}([0, \mathrm{T}])$ is bounded, then*

$$\alpha(\mathscr{K}(\mathrm{t})) \leq \alpha(\mathscr{K}) \ \ \forall \ \ \mathrm{t} \in [0, \mathrm{T}],$$

where $\mathscr{K}(\mathrm{t}) = \{m(\mathrm{t}) : m \in \mathscr{K} \subset \mathbb{X}\}$*. Further, if* $\mathscr{K}$ *is equicontinuous on* $[0, \mathrm{T}]$*, then* $\mathrm{t} \to \mathscr{K}(\mathrm{t})$ *is continuous on* $[0, \mathrm{T}]$*, and* $\alpha(\mathscr{K}) = \sup\{\mathscr{K}(\mathrm{t}) : \mathrm{t} \in [0, \mathrm{T}]\}$.
(8) *If* $\mathscr{K} \subset \mathscr{C}([0, \mathrm{T}], \mathbb{X})$ *is bounded and equicontinuous, then* $\mathrm{t} \to \alpha(\mathscr{K}(\mathrm{t}))$ *is continuous on* $[0, \mathrm{T}]$ *and* $\alpha\left(\int_0^{\mathrm{t}} \mathscr{K}(s)ds\right) \leq \int_0^{\mathrm{t}} \alpha(\mathscr{K}(s))ds \ \forall\ \mathrm{t} \in [0, \mathrm{T}]$ *where* $\int_0^{\mathrm{t}} \mathscr{K}(s)ds = \{\int_0^{\mathrm{t}} m(s)ds : m \in \mathscr{K}\}$.
(9) *Let* $\{m_n\}_{n=1}^{\infty}$ *be a sequence of Bochner integrable functions from* $[0, \mathrm{T}]$ *to* $\mathbb{X}$ *with* $\|m_n(\mathrm{t})\| \leq \hat{u}(\mathrm{t})\ \forall\ \mathrm{t} \in [0, \mathrm{T}]$ *and* $n \geq 1$*, where* $\hat{u}(\mathrm{t}) \in \mathcal{L}([0, \mathrm{T}], \mathscr{R}^+)$*, then* $\Psi(\mathrm{t}) = \alpha(\{m_n(\mathrm{t})\}_{n=1}) \in \mathcal{L}([0, \mathrm{T}], \mathscr{R}^+)$ *and satisfies*

$$\alpha\left(\{\int_0^{\mathrm{t}} m_n(s)ds : n \geq 1\}\right) \leq 2\int_0^{\mathrm{t}} \Psi(s)ds.$$

Lemma 2.6. *[20] If* $\mathscr{K} \subset \mathscr{C}([0, \mathrm{T}], \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X}))$ *and* ω *being a Weiner process,*

$$\alpha\left(\int_0^{\mathrm{t}} \mathscr{K}(s)d\omega(s)\right) \leq \sqrt{\mathrm{T}}\alpha(\mathscr{K}(\mathrm{t})),$$

where,

$$\int_0^{\mathrm{t}} \mathscr{K}(s)d\omega(s) = \left\{\int_0^{\mathrm{t}} m(s)d\omega(s) :\ \forall\, m \in \mathscr{K},\ \mathrm{t} \in [0, \mathrm{T}]\right\}.$$

Lemma 2.7. *[20] Let* $\mathbb{D}$ *be a closed convex subset of* $\mathbb{X}$ *with* $0 \in \mathbb{D}$*. Suppose* $\Psi : \mathbb{D} \to \mathbb{D}$ *is a continuous map of Mönch type which satisfies:*

$$\mathcal{M} \subset \mathbb{D} \text{ countable and } \mathcal{M} \subset \overline{co}(\{0\} \cup \Psi(\mathcal{M})) \text{ implies that } \mathcal{M} \text{ is relatively compact},$$

then, Ψ *has a fixed point in* $\mathbb{D}$.

3. Existence results

Definition 3.1. *For* $\mathrm{T} \in (\mathrm{t}_0, +\infty)$*, an* $\mathbb{X}-$*valued stochastic process* $\{\vartheta(\mathrm{t}), \mathrm{t} \in [\mathrm{t}_0, \mathrm{T}]\}$ *is said to be a mild solution of [1.1] provided,*
(i) $\vartheta(\mathrm{t})$ *is an* $\mathfrak{I}_{\mathrm{t}}-$*adapted process for* $\mathrm{t} \geq \mathrm{t}_0$*;*
(ii) $\vartheta(\mathrm{t}) \in \mathbb{X}$ *contains cadlag path on* $\mathrm{t} \in [\mathrm{t}_0, \mathrm{T}]$ *almost surely,*
(iii) $\vartheta(\mathrm{t}) = \varphi$*,* $\forall\ \mathrm{t} \in [\mathrm{t}_0, \mathrm{T}]$,

$$\begin{aligned}\vartheta(\mathrm{t}) \quad = \quad & \sum_{\mathrm{k}=0}^{+\infty}\Big[\prod_{i=1}^{\mathrm{k}} \mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)[\varphi(0) + \mathfrak{h}(\vartheta)] + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s, \vartheta(s))ds \\ + & \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s, \vartheta(s))ds + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s, \vartheta(s))d\omega(s) \\ + & \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\Big]\mathcal{I}_{[\xi_{\mathrm{k}}, \xi_{\mathrm{k}+1})}(\mathrm{t}),\end{aligned}$$

where $\prod_{j=i}^{\mathrm{k}}(.) = 1$ *as* $i > \mathrm{k}$, $\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j) = \mathfrak{b}_k(\delta_k)\mathfrak{b}_{\mathrm{k}-1}(\delta_{\mathrm{k}-1})\cdots\mathfrak{b}_i(\delta_i)$*,* $\mathcal{I}_{\mathfrak{A}}(.)$ *be the indicator function expressed as,*

$$\mathcal{I}_{\mathfrak{A}}(\mathrm{t}) = \begin{cases} 1, & if\ \ \mathrm{t} \in \mathfrak{A}, \\ 0, & if\ \ \mathrm{t} \notin \mathfrak{A}. \end{cases}$$

We may take into consideration the following hypotheses

(A1) There exists a positive constant $\mathscr{H}$, such that for all $\mathrm{t} \geq 0$, $\|\Re(\mathrm{t})\| \leq \mathscr{H}$,

(A2) The map $\mathfrak{f} : [\mathrm{t}_0, \mathrm{T}] \times \mathbb{X} \to \mathbb{X}$ satisfies

(i) For $\vartheta \in \mathbb{X}$, $\mathfrak{f}(., \vartheta) : [\mathrm{t}_0, , \mathrm{T}] \to \mathbb{X}$ is measurable and $\mathfrak{f}(\mathrm{t}, .) : \mathbb{X} \to \mathbb{X}$ being continuous for $\mathrm{t} \in [\mathrm{t}_0, \mathrm{T}]$.

(ii) As in there, a continuous $\nu_{\mathfrak{f}}(\mathrm{t}) : [\mathrm{t}_0, \mathrm{T}] \to \mathscr{R}^+$ and a continuous non-decreasing function $\Gamma_{\mathfrak{f}} : \mathscr{R}^+ \to \mathscr{R}^+$ and $\|\vartheta\|^2 \leq \mathfrak{r}$ such that

$$\|\mathfrak{f}(\mathrm{t}, \vartheta)\|^2 \leq \nu_{\mathfrak{f}}(\mathrm{t})\Gamma_{\mathfrak{f}}(\|\vartheta\|^2) \leq \nu_{\mathfrak{f}}(\mathrm{t})\Gamma_{\mathfrak{f}}(\mathfrak{r}).$$

(iii) There exists a positive function $\mathscr{C}_{\mathfrak{f}}(\mathrm{t}) \in \mathcal{L}^1([\mathrm{t}_0, \mathrm{T}], \mathscr{R}^+)$ such that, for any bounded subsets $\beta_1 \subset \mathbb{X}$, we have

$$\alpha(\mathfrak{f}(\mathrm{t}, \vartheta)) \leq \mathscr{C}_{\mathfrak{f}}(\mathrm{t}) \sup_{\theta \in (-\delta, 0]} \alpha(\beta_1(\theta)).$$

(A3) The function $\mathfrak{g} : [\mathrm{t}_0, \mathrm{T}] \times \mathbb{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ satisfies

(i) $\mathfrak{g}(., \vartheta) : [\mathrm{t}_0, \mathrm{T}] \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ is measurable for $\vartheta \in \mathbb{X}$ and $\mathfrak{g}(\mathrm{t}, .) : \mathbb{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ be continuous for $\mathrm{t} \in [\mathrm{t}_0, \mathrm{T}]$.

(ii) There appears a continuous function $\nu_{\mathfrak{g}}(\mathrm{t}) : [\mathrm{t}_0, \mathrm{T}] \to \mathscr{R}^+$ and a continuous non-decreasing function $\Gamma_{\mathfrak{g}} : \mathscr{R}^+ \to \mathscr{R}^+$ such that

$$\|\mathfrak{g}(\mathrm{t}, \vartheta)\|^2 \leq \nu_{\mathfrak{g}}(\mathrm{t})\Gamma_{\mathfrak{g}}(\|\vartheta\|^2) \leq \nu_{\mathfrak{g}}(\mathrm{t})\Gamma_{\mathfrak{g}}(\mathfrak{r}).$$

(iii) There exists a positive function $\mathscr{C}_{\mathfrak{g}}(\mathrm{t}) \in \mathcal{L}^1([\mathrm{t}_0, \mathrm{T}], \mathscr{R}^+)$ such that, for any bounded subsets $\beta_2 \subset \mathbb{X}$, we have

$$\alpha(\mathfrak{g}(\mathrm{t}, \vartheta)) \leq \mathscr{C}_{\mathfrak{g}}(\mathrm{t}) \sup_{\theta \in (-\delta, 0]} \alpha(\beta_2(\theta)).$$

(A4) (i) The nonlocal function $\mathfrak{h} : \mathscr{C}([\mathrm{t}_0, \mathrm{T}], \mathcal{L}^2(\Omega, \mathscr{X})) \to \mathbb{X}$ is continuous and compact.

(ii) $\exists$ a constant $\mathfrak{m} > 0$ and a nondecreasing continuous function $\Gamma_{\mathfrak{h}} : \mathscr{R}^+ \to \mathscr{R}^+ \ni$

$$\mathbb{E}\|\mathfrak{h}(\vartheta)\|^2 \leq \Gamma_{\mathfrak{h}}(\mathfrak{r}) \ \ \textit{and} \ \ \lim_{\mathfrak{r} \to +\infty} \inf \frac{\Gamma_{\mathfrak{h}}(\mathfrak{r})}{\mathfrak{r}} := \mathfrak{m} < +\infty.$$

(A5) $\mathbb{E}\left[\max_{i,k}\left\{\prod_{j=i}^{k} \|\mathfrak{b}_j(\delta_j)\|\right\}\right] < +\infty$. In other words, there exists a constant $\mathscr{B} > 0$ such that

$$\mathbb{E}\left(\max_{i,k}\left\{\prod_{j=i}^{k} \|\mathfrak{b}_j(\delta_j)\|\right\}\right) \leq \mathscr{B} \ \text{ for all } \ \delta_j \in \mathscr{D}_j, \ j \in \mathbb{N}.$$

(A6)

$$4\mathscr{B}^2\mathscr{H}^2\mathfrak{m} + 4\max\{1, \mathscr{B}^2\}(\mathrm{T} - \mathrm{t}_0)\mathscr{H}^2\left[\lim_{\mathfrak{r} \to +\infty} \frac{\Gamma_{\mathfrak{f}}(\mathfrak{r})}{\mathfrak{r}} \int_{\mathrm{t}_0}^{\mathrm{t}} \nu_{\mathfrak{f}}(s)ds + \lim_{\mathfrak{r} \to +\infty} \frac{\Gamma_{\mathfrak{g}}(\mathfrak{r})}{\mathfrak{r}} \int_{\mathrm{t}_0}^{\mathrm{t}} \nu_{\mathfrak{g}}(s)ds\right] \leq 1.$$

Theorem 3.1. *Assume the conditions (A1)–(A6) holds, then there exist at least one mild solution for [1.1] provided:*

$$\mathscr{B}^2\mathscr{H}^2\mathfrak{m} + \max\{1, \mathscr{B}^2\}\mathscr{H}^2(\mathrm{T} - \mathrm{t}_0)\|\mathscr{C}_{\mathfrak{f}}\|_{\mathcal{L}^1([\mathrm{t}_0,\mathrm{T}],\mathscr{R}^+)} + \max\{1, \mathscr{B}^2\}\mathscr{H}^2(\mathrm{T} - \mathrm{t}_0)^{\frac{1}{2}}\|\mathscr{C}_{\mathfrak{g}}\|_{\mathcal{L}^2([\mathrm{t}_0,\mathrm{T}],\mathscr{R}^+)} < 1. \quad (3.1)$$

Proof. Let us initiate the set $\Upsilon_{\mathtt{T}} : PC\Big([\mathrm{t}_0-\delta,\mathtt{T}],\mathcal{L}^2(\Omega,\mathbb{X})\Big)$ endowed with the norm

$$\|\vartheta\|^2_{\Upsilon_{\mathtt{T}}} = \sup_{\mathrm{t}\in[\mathrm{t}_0,\mathtt{T}]}\mathbb{E}\|\vartheta\|^2_{\mathrm{t}} = \sup_{\mathrm{t}\in[\mathrm{t}_0,\mathtt{T}]}\mathbb{E}\left(\sup_{\mathrm{t}-\delta\le s\le \mathrm{t}}\|\vartheta(s)\|^2\right).$$

It is clear that $\Upsilon_{\mathtt{T}}$ is a Banach space and, furthermore, we consider the closed subset of $\Upsilon_{\mathtt{T}}$ defined by

$$\overline{\Upsilon}_{\mathtt{T}} = \{\vartheta\in\Upsilon_{\mathtt{T}} : \vartheta(s)=\varphi(s),\ \ \text{for}\ \ s\in[-\delta,0]\}$$

with the norm $\|\vartheta\|^2_{\Upsilon_{\mathtt{T}}}$. Thus, [1.1] can be modified to a fixed point problem. Define an operator $\Theta : \overline{\Upsilon}_{\mathtt{T}}\to\overline{\Upsilon}_{\mathtt{T}}$ by

$$\begin{aligned}
(\Theta\vartheta)(\mathrm{t}) &= \sum_{\mathrm{k}=0}^{+\infty}\Big[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\left[\varphi(0)+\mathfrak{h}(\vartheta)\right]+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s)))ds\\
&+\int_{\xi_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s)))ds+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_k}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s))d\omega(s)\\
&+\int_{\xi_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s))d\omega(s)\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})(\mathrm{t})},\ \ \mathrm{t}\in[\mathrm{t}_0,\mathtt{T}],\\
\text{and}&\\
(\Theta\vartheta) &= \varphi(\theta),\quad \mathrm{t}\in[-\delta,0].
\end{aligned}$$

Let us split the proof into four steps.

Step 1: Initially, we need to verify that the operator Θ satisfies the property $\mathcal{N}(\mathbb{B}_{\mathrm{r}})\subset\mathbb{B}_{\mathrm{r}}$, where $\mathbb{B}_{\mathrm{r}}=\{\vartheta\in\Upsilon_{\mathtt{T}} : \|\vartheta\|^2_{\Upsilon_{\mathtt{T}}}\le\mathrm{r}\}$. If the result contradicts, for $\vartheta\in\mathbb{B}_{\mathrm{r}}$, $\mathcal{N}(\mathbb{B}_{\mathrm{r}})\not\subseteq\mathbb{B}_{\mathrm{r}}$. Thus, we may find $\mathrm{t}\in[\mathrm{t}_0,\mathtt{T}]$ satisfying $\mathbb{E}\|(\Theta\vartheta)(\mathrm{t})\|^2>\mathrm{r}$. By the aforementioned assumptions, we have

$$\begin{aligned}
\mathbb{E}\|(\Theta\vartheta)(\mathrm{t})\|^2 &= \mathbb{E}\Big\|\sum_{\mathrm{k}=0}^{+\infty}\Big[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)[\varphi(0)+\mathfrak{h}(\vartheta)]+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s))ds\\
&+\int_{\xi_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s))ds+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s))d\omega(s)\\
&+\int_{\xi_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s))d\omega(s)\Big]\Big\|^2\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})(\mathrm{t})},\\
&\le 4\mathbb{E}\Bigg(\bigg(\max_{\mathrm{k}}\{\prod_{i=1}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|\}\bigg)^2\Bigg)\|\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\|^2\mathbb{E}\|\varphi(0)+\mathfrak{h}(\vartheta)\|^2+4\mathbb{E}\bigg(\max_{i,\mathrm{k}}\{\prod_{j=i}^{\mathrm{k}}\|\mathfrak{b}_j(\delta_j)\|,1\}\bigg)^2\\
&\times\ \mathbb{E}\left(\|\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s))ds\|^2\right)+4\mathbb{E}\Bigg(\bigg(\max_{i,\mathrm{k}}\{\prod_{j=i}^{\mathrm{k}}\|\mathfrak{b}_j(\delta_j)\|,1\}\bigg)^2\Bigg)\\
&\times\ \mathbb{E}\left(\|\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s))d\omega(s)\|^2\right)\\
&\le 4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\|\varphi(0)\|^2+4\mathscr{B}^2\mathscr{H}^2\Gamma_{\mathfrak{h}}(\mathrm{r})+4\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathtt{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\nu\mathfrak{f}(s)\Gamma_{\mathfrak{f}}(\mathrm{r})ds
\end{aligned}$$

$$+ \quad 4\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathtt{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{g}}(s)\Gamma_{\mathfrak{g}}(\mathfrak{r})ds$$

Dividing the above inequality by $\mathfrak{r}$, and letting $\mathfrak{r}\to+\infty$, we have

$$4\mathscr{B}^2\mathscr{H}^2\mathfrak{m}+4\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathtt{T}-\mathrm{t}_0)\left(\lim_{\mathfrak{r}\to+\infty}\frac{\Gamma_{\mathfrak{f}}(\mathfrak{r})}{\mathfrak{r}}\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{f}}(s)ds+\lim_{\mathfrak{r}\to+\infty}\frac{\Gamma_{\mathfrak{g}}(\mathfrak{r})}{\mathfrak{r}}\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{g}}(s)ds\right)>1,$$

which contradicts our assumption (A4). Thus, there exist some $\vartheta\in\mathbb{B}_{\mathfrak{r}}$ such that $\mathcal{N}(\mathbb{B}_{\mathfrak{r}})\subset\mathbb{B}_{\mathfrak{r}}$.

Step 2: In order to prove the continuity of the operator Θ in $\mathbb{B}_{\mathfrak{r}}$, let $\vartheta,\vartheta_n\in\mathbb{B}_{\mathfrak{r}}$ and $\vartheta_n\to\vartheta$ as $n\to+\infty$. By condition (ii) of (A1),(A2), we get

$$\begin{aligned}\mathfrak{f}(\mathrm{t},\vartheta_n)\to\mathfrak{f}(\mathrm{t},\vartheta),\ n\to+\infty,\ \|\mathfrak{f}(\mathrm{t},\vartheta_n)-\mathfrak{f}(\mathrm{t},\vartheta)\|^2&\le 2\nu_{\mathfrak{f}}(\mathrm{t})\Gamma_{\mathfrak{f}}(\mathrm{t}),\\ \mathfrak{g}(\mathrm{t},\vartheta_n)\to\mathfrak{g}(\mathrm{t},\vartheta),\ n\to+\infty,\ \|\mathfrak{g}(\mathrm{t},\vartheta_n)-\mathfrak{g}(\mathrm{t},\vartheta)\|^2&\le 2\nu_{\mathfrak{g}}(\mathrm{t})\Gamma_{\mathfrak{g}}(\mathrm{t}),\\ \mathfrak{h}(\vartheta_n)\to\mathfrak{h}(\vartheta),\ n\to+\infty,\ \|\mathfrak{h}(\vartheta_n)-\mathfrak{h}(\vartheta)\|^2&\le 2\Gamma_{\mathfrak{h}}.\end{aligned}$$

Using Dominated Convergence theorem and (A3), we may deduce that

$$\begin{aligned}&\mathbb{E}\,\|(\Theta\vartheta_n)(\mathrm{t})-(\Theta\vartheta)(\mathrm{t})\|^2\\ &\le 4\mathbb{E}\left\|\sum_{\mathrm{k}=0}^{+\infty}\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\Re(\mathrm{t}-\mathrm{t}_0)(\vartheta_n(0)-\vartheta(0))\right\|^2+4\mathbb{E}\left\|\sum_{\mathrm{k}=0}^{+\infty}\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\Re(\mathrm{t}-\mathrm{t}_0)(\mathfrak{h}(\vartheta_n)-\mathfrak{h}(\vartheta))\right\|^2\\ &\quad+4\mathbb{E}\Big\|\sum_{\mathrm{k}=0}^{+\infty}\Big(\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}\Re(\mathrm{t}-s)[\mathfrak{f}(s,\vartheta_n(s))-\mathfrak{f}(s,\vartheta(s))]ds\\ &\quad+\int_{\xi_{\mathrm{k}}}^{\mathrm{t}}\Re(\mathrm{t}-s)[\mathfrak{f}(s,(\vartheta_n(s))-\mathfrak{f}(s,\vartheta(s))]ds\Big)\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}\Big\|^2\\ &\quad+4\mathbb{E}\Big\|\sum_{\mathrm{k}=0}^{+\infty}\Big(\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}\Re(\mathrm{t}-s)[\mathfrak{g}(s,\vartheta_n(s))-\mathfrak{g}(s,\vartheta(s))]d\omega(s)\\ &\quad+\int_{\xi_{\mathrm{k}}}^{\mathrm{t}}\Re(\mathrm{t}-s)[\mathfrak{g}(s,\vartheta_n(s))-\mathfrak{g}(s,\vartheta(s))]d\omega(s)\Big)\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}\Big\|^2\\ &\le 4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\,\|\vartheta_n(0)-\vartheta(0)\|^2+4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\,\|\mathfrak{h}(\vartheta_n)-\mathfrak{h}(\vartheta)\|^2\\ &\quad+4\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{t}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\mathbb{E}\,\|\mathfrak{f}(s,\vartheta_n(s))-\mathfrak{f}(s,\vartheta(s))\|^2ds\\ &\quad+4\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{t}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\mathbb{E}\,\|\mathfrak{g}(s,\vartheta_n(s))-\mathfrak{g}(s,\vartheta(s))\|^2_{\mathcal{L}_2^0}ds\\ &\to 0\ \ as\ \ n\to+\infty.\end{aligned}$$

Therefore, Θ is continuous on $\mathbb{B}_{\mathfrak{r}}$.

Step 3: To prove Θ is equicontinuous on $[\mathrm{t}_0,\mathtt{T}]$, for $\mathrm{t}_0<\mathrm{t}_1<\mathrm{t}_2<\mathtt{T}$ and $\vartheta\in\mathbb{B}_{\mathfrak{r}}$, we have
$(\Theta\vartheta)(\mathrm{t}_2)-(\Theta\vartheta)(\mathrm{t}_1)$

$$=\quad\sum_{\mathrm{k}=0}^{+\infty}\Big[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\Re(\mathrm{t}_2-\mathrm{t}_0)[\varphi(0)+\mathfrak{h}(\vartheta)]+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}\Re(\mathrm{t}_2-s)\mathfrak{f}(s,\vartheta(s))ds$$

$$
\begin{aligned}
&+ \int_{\xi_k}^{t_2} \Re(t_2 - s)\mathfrak{f}(s, \vartheta(s))ds + \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} \Re(t_2 - s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\\
&+ \int_{\xi_k}^{t_2} \Re(t_2 - s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\Big]\mathcal{I}_{[\xi_k,\xi_{k+1})}(t_2)\\
&- \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} \mathfrak{b}_i(\delta_i)\Re(t_1 - t_0)[\varphi(0) + \mathfrak{h}(\vartheta)] + \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} \Re(t_1 - s)\mathfrak{f}(s, \vartheta(s))ds\\
&+ \int_{\xi_k}^{t_1} \Re(t_1 - s)\mathfrak{f}(s, \vartheta(s))ds + \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} \Re(t_1 - s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\\
&+ \int_{\xi_k}^{t_1} \Re(t_1 - s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\Big]\mathcal{I}_{[\xi_k,\xi_{k+1})}(t_1)\\
=& \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} \mathfrak{b}_i(\delta_i)\Re(t_2 - t_0)[\varphi(0) + \mathfrak{h}(\vartheta)] + \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} \Re(t_2 - s)\mathfrak{f}(s, \vartheta(s))ds\\
&+ \int_{\xi_k}^{t_2} \Re(t_2 - s)\mathfrak{f}(s, \vartheta(s))ds + \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} \Re(t_2 - s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\\
&+ \int_{\xi_k}^{t_2} \Re(t_2 - s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\Big]\Big(\mathcal{I}_{[\xi_k,\xi_{k+1})}(t_2) - \mathcal{I}_{[\xi_k,\xi_{k+1})}(t_1)\Big)\\
&+ \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} \mathfrak{b}_i(\delta_i)(\Re(t_2 - t_0) - \Re(t_2 - t_1))[\varphi(0) + \mathfrak{h}(\vartheta)]\\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} (\Re(t_2 - s) - \Re(t_1 - s))\mathfrak{f}(s, \vartheta(s))ds + \int_{\xi_k}^{t_2} (\Re(t_2 - s) - \Re(t_1 - s))\mathfrak{f}(s, \vartheta(s))ds\\
&+ \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} (\Re(t_2 - s) - \Re(t_1 - s))\mathfrak{g}(s, \vartheta(s))d\omega(s) + \int_{\xi_k}^{t_2} (\Re(t_2 - s) - \Re(t_1 - s))\\
&\times \mathfrak{g}(s, \vartheta(s))d\omega(s)\Big]\mathcal{I}_{[\xi_k,\xi_{k+1})(t_g)}\\
=& 2\mathbb{E}\left\|\mathscr{J}_1\right\|^2 + 2\mathbb{E}\left\|\mathscr{J}_2\right\|^2.
\end{aligned}
$$

Where,

$$
\begin{aligned}
\mathbb{E}\left\|\mathscr{J}_1\right\|^2 =& \mathbb{E}\Big\|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} \mathfrak{b}_i(\delta_i)\Re(t_2 - t_0)[\varphi(0) + \mathfrak{h}(\vartheta)] + \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} \Re(t_2 - s)\mathfrak{f}(s, \vartheta(s))ds\\
&+ \int_{\xi_k}^{t_2} \Re(t_2 - s)\mathfrak{f}(s, \vartheta(s))ds + \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j)\int_{\xi_{k-1}}^{\xi_k} \Re(t_2 - s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\\
&+ \int_{\xi_k}^{t_2} \Re(t_2 - s)\mathfrak{g}(s, \vartheta(s))d\omega(s)\Big]\Big(\mathcal{I}_{[\xi_k,\xi_{k+1})}(t_2) - \mathcal{I}_{[\xi_k,\xi_{k+1})}(t_1)\Big)\Big\|^2
\end{aligned}
$$

$$\begin{aligned}
\mathbb{E}\left\|\mathscr{J}_2\right\|^2 &= \mathbb{E}\Big\|\sum_{\mathrm{k}=0}^{+\infty}\Big[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)(\Re(\mathrm{t}_2-\mathrm{t}_0)-\Re(\mathrm{t}_2-\mathrm{t}_1))[\varphi(0)+\mathfrak{h}(\vartheta)] \\
&+ \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}(\Re(\mathrm{t}_2-s)-\Re(\mathrm{t}_1-s))\mathfrak{f}(s,\vartheta(s))ds \\
&+ \int_{\xi_{\mathrm{k}}}^{\mathrm{t}_2}(\Re(\mathrm{t}_2-s)-\Re(\mathrm{t}_1-s))\mathfrak{f}(s,\vartheta(s))ds \\
&+ \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}(\Re(\mathrm{t}_2-s)-\Re(\mathrm{t}_1-s))\mathfrak{g}(s,\vartheta(s))d\omega(s) \\
&+ \int_{\xi_{\mathrm{k}}}^{\mathrm{t}_2}(\Re(\mathrm{t}_2-s)-\Re(\mathrm{t}_1-s))\mathfrak{g}(s,\vartheta(s))d\omega(s)\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})(\mathrm{t}_g)}\Big\|^2.
\end{aligned}$$

By treating each term separately,

$$\begin{aligned}
\mathbb{E}\|\mathscr{J}_1\|^2 &\leq 4\mathbb{E}\left(\max_{\mathrm{k}}\{\prod_{i=1}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|^2\}\right)\|\Re(\mathrm{t}_2-\mathrm{t}_0)\|^2\,\mathbb{E}\|\varphi(0)\|^2\left(\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t}_2)-\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t}_1)\right)^2 \\
&+ 4\mathbb{E}\left(\max_{\mathrm{k}}\{\prod_{i=1}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|^2\}\right)\|\Re(\mathrm{t}_2-\mathrm{t}_0)\|^2\,\mathbb{E}\|\mathfrak{h}(\vartheta)\|^2\left(\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t}_2)-\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t}_1)\right)^2 \\
&+ 4\mathbb{E}\left(\max_{i,\mathrm{k}}\{\prod_{j=i}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|,1\}\right)^2\,\mathbb{E}\left(\sum_{\mathrm{k}=0}^{+\infty}\int_{\mathrm{t}_0}^{\mathrm{t}_2}\|\Re(\mathrm{t}_2-s)\|^2\,\|\mathfrak{f}(s,\vartheta(s))\|^2\,ds\right) \\
&\times \left(\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t}_2)-\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t}_1)\right)^2+4\mathbb{E}\left(\max_{i,\mathrm{k}}\{\prod_{j=i}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|,1\}\right)^2 \\
&\times \mathbb{E}\left(\sum_{\mathrm{k}=0}^{+\infty}\int_{\mathrm{t}_0}^{\mathrm{t}_2}\|\Re(\mathrm{t}_2-s)\|^2\,\|\mathfrak{g}(s,\vartheta(s))\|^2\,d\omega(s)\right)\left(\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t}_2)-\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t}_1)\right)^2 \\
&\to 0\ \ as\ \ \mathrm{t}_2\to\mathrm{t}_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\|\mathscr{J}_2\|^2 &\leq 6\mathscr{B}^2\,\|\Re(\mathrm{t}_2-\mathrm{t}_0)-\Re(\mathrm{t}_1-\mathrm{t}_0)\|^2\,\mathbb{E}\|\varphi(0)\|^2+6\mathscr{B}^2\,\|\Re(\mathrm{t}_2-\mathrm{t}_0)-\Re(\mathrm{t}_1-\mathrm{t}_0)\|^2\,\mathbb{E}\|\mathfrak{h}(\vartheta)\|^2 \\
&+ 6\max\{1,\mathscr{B}^2\}(\mathrm{t}_1-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}_1}\|\Re(\mathrm{t}_2-s)-\Re(\mathrm{t}_1-s)\|^2\,\mathbb{E}\,\|\mathfrak{f}(s,\vartheta(s))\|^2\,ds \\
&+ 6(\mathrm{t}_2-\mathrm{t}_1)\int_{\mathrm{t}_1}^{\mathrm{t}_2}\|\Re(\mathrm{t}_2-s)\|^2\int_{\mathrm{t}_1}^{\mathrm{t}_2}\|\Re(\mathrm{t}_2-s)\|^2\mathbb{E}\,\|\mathfrak{f}(s,\vartheta(s))\|^2\,ds \\
&+ 6\max\{1,\mathscr{B}^2\}(\mathrm{t}_1-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}_1}\|\Re(\mathrm{t}_2-s)-\Re(\mathrm{t}_1-s)\|^2\,\mathbb{E}\,\|\mathfrak{g}(s,\vartheta(s))\|^2\,ds \\
&+ 6(\mathrm{t}_2-\mathrm{t}_1)\int_{\mathrm{t}_1}^{\mathrm{t}_2}\|\Re(\mathrm{t}_2-s)\|^2\mathbb{E}\,\|\mathfrak{g}(s,\vartheta(s))\|^2\,ds \\
&\to 0\ \ as\ \ \mathrm{t}_2\to\mathrm{t}_1.
\end{aligned}$$

Thus, we have

$$\mathbb{E}\|(\Theta\vartheta)(\mathrm{t}_2)-(\Theta\vartheta)(\mathrm{t}_1)\|^2 \to 0 \ \ as \ \ \mathrm{t}_2 \to \mathrm{t}_1,$$

which implies Θ is equicontinuous on $[\mathrm{t}_0, \mathtt{T}]$.

Step 4: Now to establish M*ö*nch condition, let $\gamma \subset \Upsilon_{\mathtt{T}}$ be a nonempty set and $\vartheta_1, \vartheta_2 \in \gamma$, by probability 1,

$$d(\Theta\vartheta_1(\mathrm{t}), \Theta\vartheta_2(\mathrm{t})) = d(\overline{\Theta}\vartheta_1(\mathrm{t}), \overline{\Theta}\vartheta_2(\mathrm{t})),$$

where

$$\begin{aligned}(\overline{\Theta}\vartheta)(\mathrm{t}) &= \mathscr{B}\mathfrak{h}(\vartheta) + \max\{1,\mathscr{B}\}\sum_{\mathrm{k}=0}^{+\infty}\Big[\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}\Re(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s))ds + \int_{\xi_{\mathrm{k}}}^{\mathrm{t}}\Re(\mathrm{t}-s)\mathfrak{f}(s,\vartheta(s))ds\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t})\\ &+ \max\{1,\mathscr{B}\}\sum_{\mathrm{k}=0}^{+\infty}\Big[\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}}\Re(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s))ds + \int_{\xi_{\mathrm{k}}}^{\mathrm{t}}\Re(\mathrm{t}-s)\mathfrak{g}(s,\vartheta(s))ds\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t})\\ &= \overline{\Theta}_1 + \overline{\Theta}_2.\end{aligned}$$

By the similar procedure used in Lemma 2.3,

$$\alpha((\Theta\vartheta)(\mathrm{t})) = \alpha((\overline{\Theta})(\mathrm{t})).$$

Let $\Delta \subset \mathbb{B}_{\mathfrak{r}}$ be countable and $\Delta \subset \overline{co}(\{0\} \cup \Theta(\Delta))$. by proving $\alpha(\Delta) = 0$ the Mönch condition is then verified. Set $\Delta = \{\vartheta^n\}_{n=1}^{\infty}$, then it is well defined $\Delta \subset \overline{co}(\{0\} \cup \Theta(\Delta))$ is equicontinuous on $[\mathrm{t}_0, \mathtt{T}]$ by step 3.

By Lemmas 2.2 and 2.3,

$$\begin{aligned}\alpha(\{\overline{\Theta}_1\vartheta^n(\mathrm{t})\}_{n=1}^{\infty}) &\le \max\{1,\mathscr{B}\}\mathscr{H}(\mathtt{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\mathscr{C}_{\mathfrak{f}}(\mathrm{t})\sup_{\theta\in(-\delta,0]}\alpha(\{\vartheta^n(\theta-\mu(\theta))\}_{n=1}^{\infty})ds\\ &\le \max\{1,\mathscr{B}\}\mathscr{H}(\mathtt{T}-\mathrm{t}_0)\left\|\mathscr{C}_{\mathfrak{f}}\right\|_{\mathcal{L}^1([\mathrm{t}_0,\mathtt{T}],\Re^+)}\sup_{\mathrm{t}\in[\mathrm{t}_0,\mathtt{T}]}\alpha(\{\vartheta^n(\mathrm{t})\}_{n=1}^{\infty}),\\ \alpha(\{\overline{\Theta}_2\vartheta^n(\mathrm{t})\}_{n=1}^{\infty}) &\le \max\{1,\mathscr{B}\}\mathscr{H}(\mathtt{T}-\mathrm{t}_0)^{\frac{1}{2}}\left\|\mathscr{C}_{\mathfrak{g}}\right\|_{\mathcal{L}^2([\mathrm{t}_0,\mathtt{T}],\Re^+)}\sup_{\mathrm{t}\in[\mathrm{t}_0,\mathtt{T}]}\alpha(\{\vartheta^n(\mathrm{t})\}_{n=1}^{\infty}).\end{aligned}$$

By using Lemma 2.3,

$$\begin{aligned}\alpha(\{\Theta_1\vartheta^n(\mathrm{t})\}_{n=1}^{\infty}) &= \alpha(\{\overline{\Theta}_1\vartheta^n(\mathrm{t})\}_{n=1}^{\infty})\\ &\le \alpha(\{\overline{\Theta}_1\vartheta^n(\mathrm{t})\}_{n=1}^{\infty}) + \alpha(\{\overline{\Theta}_2\vartheta^n(\mathrm{t})\}_{n=1}^{\infty})\\ &\le \Big[\max\{1,\mathscr{B}\}\mathscr{H}(\mathtt{T}-\mathrm{t}_0)\left\|\mathscr{C}_{\mathfrak{f}}\right\|_{\mathcal{L}^1([\mathrm{t}_0,\mathtt{T}],\Re^+)} + \max\{1,\mathscr{B}\}\mathscr{H}(\mathtt{T}-\mathrm{t}_0)^{\frac{1}{2}}\\ &\times \left\|\mathscr{C}_{\mathfrak{g}}\right\|_{\mathcal{L}^2([\mathrm{t}_0,\mathtt{T}],\Re^+)}\Big]\alpha(\{\vartheta^n(\mathrm{t})\}_{n=1}^{\infty}).\end{aligned}$$

It follows that

$$\alpha(\Delta) \le \alpha(\overline{co}(\{0\} \cup \Theta(\Delta))) = \alpha(\Theta(\Delta)) \le \alpha(\Delta),$$

implying $\alpha(\Delta) = 0$ and then Δ is relatively compact set. Thus in Δ, Θ has a fixed point which is the mild solution of [1.1]. This completes the proof. □

4. Stability

4.1. Continuous dependence of solutions on initial conditions

(A7) There exists a constants $\mathscr{C}_1, \mathscr{C}_2$ such that

$$\|\mathfrak{f}(\mathrm{t}, \vartheta) - \mathfrak{f}(\mathrm{t}, \varpi)\| \leq \mathscr{C}_1 \|\vartheta - \varpi\|, \quad \|\mathfrak{g}(\mathrm{t}, \vartheta) - \mathfrak{g}(\mathrm{t}, \varpi)\|_{\mathcal{L}_2^0} \leq \mathscr{C}_2 \|\vartheta - \varpi\|_{\mathcal{L}_2^0}.$$

Theorem 4.1. *Let $\vartheta(\mathrm{t})$ and $\overline{\vartheta}(\mathrm{t})$ be mild solutions for [1.1] with $\varphi(0)$ and $\overline{\varphi}(0)$ as initial values. Assuming (A3), (A7) holds, the mild solution of [1.1] is stable in the mean square.*

Proof. $\mathbb{E}\left\|\vartheta - \overline{\vartheta}\right\|_{\mathrm{t}}^2$

$$\begin{aligned}
&\leq 4\mathbb{E}\left\|\sum_{k=0}^{+\infty}\prod_{i=1}^{\mathrm{k}} \mathfrak{b}_i(\delta_i)\right\|^2 \left\|\Re(\mathrm{t}-\mathrm{t}_0)(\varphi(0) - \overline{\varphi(0)})\right\|^2 + 4\mathbb{E}\left\|\sum_{k=0}^{+\infty}\prod_{i=1}^{\mathrm{k}} \mathfrak{b}_i(\delta_i)\right\|^2 \left\|\Re(\mathrm{t}-\mathrm{t}_0)(\mathfrak{h}(\vartheta) - \overline{\mathfrak{h}(\vartheta)})\right\|^2 \\
&+ 4\mathbb{E}\Bigg\|\sum_{\mathrm{k}=0}^{+\infty}\Big[\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s)\Big(\mathfrak{f}(s,\vartheta(s)) - \mathfrak{f}(s,\overline{\vartheta(s)})\Big)ds \\
&+ \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \Re(\mathrm{t}-s)\left(\mathfrak{f}(s,\vartheta(s)) - \mathfrak{f}(s,\overline{\vartheta(s)})\right)ds\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t})\Bigg\|^2 \\
&+ 4\mathbb{E}\Bigg\|\sum_{\mathrm{k}=0}^{+\infty}\Big[\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s)\left(\mathfrak{g}(s,\vartheta(s)) - \mathfrak{g}(s,\overline{\vartheta(s)})\right)ds \\
&+ \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \Re(\mathrm{t}-s)\left(\mathfrak{g}(s,\vartheta(s)) - \mathfrak{g}(s,\overline{\vartheta(s)})\right)ds\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t})\Bigg\|^2 \\
&\leq 4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\left\|\varphi(0) - \overline{\varphi}(0)\right\|^2 + 4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\left\|\mathfrak{h}(\vartheta) - \overline{\mathfrak{h}(\vartheta)}\right\|^2 + 4\max\{1,\mathscr{B}^2\}(\mathrm{T}-\mathrm{t}_0) \\
&\times \Big[\int_{\mathrm{t}_0}^{\mathrm{t}} \mathbb{E}\left\|\mathfrak{f}(s,\vartheta(s)) - \mathfrak{f}(s,\overline{\vartheta(s)})\right\|^2 ds + \int_{\mathrm{t}_0}^{\mathrm{t}} \mathbb{E}\left\|\mathfrak{g}(s,\vartheta(s)) - \mathfrak{g}(s,\overline{\vartheta(s)})\right\|^2 ds\Big],
\end{aligned}$$

which implies

$$\begin{aligned}
\sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]}\mathbb{E}\left\|\vartheta - \overline{\vartheta}\right\|_{\mathrm{t}}^2 &\leq 4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\left\|\varphi(0) - \overline{\varphi}(0)\right\|^2 + 4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\left\|\mathfrak{h}(\vartheta) - \overline{\mathfrak{h}(\vartheta)}\right\|^2 + 4\max\{1,\mathscr{B}^2\}\mathscr{H}^2 \\
&\times (\mathrm{T}-\mathrm{t}_0)(\mathscr{C}_1 + \mathscr{C}_2)\int_{\mathrm{t}_0}^{\mathrm{t}} \sup_{s\in[\mathrm{t}_0,\mathrm{t}]}\mathbb{E}\left\|\vartheta - \overline{\vartheta}\right\|_s^2 ds.
\end{aligned}$$

By Gronwall's inequality

$$\sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]}\mathbb{E}\left\|\vartheta - \overline{\vartheta}\right\|_{\mathrm{t}}^2 \leq 4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\left\|[\varphi(0) - \overline{\varphi}(0)] + \left[\mathfrak{h}(\vartheta) - \overline{\mathfrak{h}(\vartheta)}\right]\right\|^2 \exp\left\{4\mathscr{H}^2\max\{1,\mathscr{B}^2\}(\mathrm{T}-\mathrm{t}_0)(\mathscr{C}_1 + \mathscr{C}_2)\right\}.$$

For $\epsilon > 0$, there exist a positive number

$$\tau = \frac{\epsilon}{4\mathscr{B}^2\mathscr{H}^2\exp\{4\mathscr{H}^2\max\{1,\mathscr{B}^2\}(\mathrm{T}-\mathrm{t}_0)(\mathscr{C}_1 + \mathscr{C}_2)\}} > 0.$$

$\ni \mathbb{E}\,\|\varphi(0) - \overline{\varphi}(0)\|^2 < \tau$, subsequently

$$\sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \mathbb{E}\left\|\vartheta - \overline{\vartheta}\right\|_{\mathrm{t}}^2 \leq \epsilon.$$

This completes the proof. □

4.2. Hyers-Ulam stability

Definition 4.1. *Suppose $\varpi(\mathrm{t})$ is a $\mathbb{Y}-$valued stochastic process and if there exists a real number $\mathscr{C} > 0$ such that, for arbitrary $\epsilon > 0$, satisfying*

$$\begin{aligned}\mathbb{E}\Big\|\varpi(\mathrm{t}) - \sum_{\mathrm{k}=0}^{+\infty}\Big[\prod_{i=1}^{\mathrm{k}} \mathfrak{b}_i(\delta_i)\Re(\mathrm{t}-\mathrm{t}_0)[\varphi(0)+\mathfrak{h}(\vartheta)] + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s)\mathfrak{f}(s,\varpi(s))ds \\ + \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \Re(\mathrm{t}-s)\mathfrak{f}(s,\varpi(s))ds + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s)\mathfrak{g}(s,\varpi(s))d\omega(s) \\ + \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \Re(\mathrm{t}-s)\mathfrak{g}(s,\varpi(s))d\omega(s)\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t})\Big\|^2 \leq \epsilon, \qquad \forall\ \mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]. \qquad (4.1)\end{aligned}$$

For each solution $\varpi(\mathrm{t})$ with the initial value $\varpi_{\mathrm{t}_0} = \vartheta_{\mathrm{t}_0} = \varphi$, if $\exists$ a solution $\vartheta(\mathrm{t})$ of [1.1] with $\mathbb{E}\,\|\varpi(\mathrm{t}) - \vartheta(\mathrm{t})\|^2 \leq \mathscr{C}\epsilon$, for $\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]$. Then [1.1] has Hyers-Ulam stability.

Theorem 4.2. *Assume conditions (A3) and (A5) gets satisfied, then [1.1] has the Hyers-Ulam stability.*

Proof. Let $\vartheta(\mathrm{t})$ be a mild solution of [1.1] and $\varpi(\mathrm{t})$ a $\mathbb{Y}-$valued stochastic process assuring [4.1]. Obviously, $\mathbb{E}\,\|\varpi(\mathrm{t}) - \vartheta(\mathrm{t})\|^2 = 0$ for $\mathrm{t}\in[-\delta,0]$. Moreover, as $\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]$, we posses

$$\begin{aligned}\mathbb{E}\,\|\varpi - \vartheta\|_{\mathrm{t}}^2 \leq\ & 2\mathbb{E}\Big\|\varpi(\mathrm{t}) - \sum_{\mathrm{k}=0}^{+\infty}\Big[\prod_{i=1}^{\mathrm{k}} \mathfrak{b}_i(\delta_i)\Re(\mathrm{t}-\mathrm{t}_0)[\varphi(0)+\mathfrak{h}(\vartheta)] + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s)\mathfrak{f}(s,\varpi(s))ds \\ & + \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \Re(\mathrm{t}-s)\mathfrak{f}(s,\varpi(s))ds + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s)\mathfrak{g}(s,\varpi(s))d\omega(s) \\ & + \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \Re(\mathrm{t}-s)\mathfrak{g}(s,\varpi(s))d\omega(s)\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t})\Big\|^2 + 2\mathbb{E}\Big\|\sum_{\mathrm{k}=0}^{+\infty}\Big[\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s) \\ & \times\ (\mathfrak{f}(s,\varpi(s)) - \mathfrak{f}(s,\vartheta(s)))\,ds + \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \Re(\mathrm{t}-s)\,(\mathfrak{f}(s,\varpi(s)) - \mathfrak{f}(s,\vartheta(s)))\,ds \\ & + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s)\,(\mathfrak{g}(s,\varpi(s)) - \mathfrak{g}(s,\vartheta(s)))\,ds + \int_{\xi_{\mathrm{k}}}^{\mathrm{t}} \Re(\mathrm{t}-s) \\ & \times\ (\mathfrak{g}(s,\varpi(s)) - \mathfrak{g}(s,\vartheta(s)))\,ds\Big]\mathcal{I}_{[\xi_{\mathrm{k}},\xi_{\mathrm{k}+1})}(\mathrm{t})\Big\|^2 \\ \leq\ & 2\epsilon + 2\mathbb{E}\|\mathscr{J}\|^2,\end{aligned}$$

Now, we consider

$$\mathbb{E}\|\mathscr{J}\|^2 = 2\mathbb{E}\Big\|\sum_{\mathrm{k}=0}^{+\infty}\Big[\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\xi_{\mathrm{k}-1}}^{\xi_{\mathrm{k}}} \Re(\mathrm{t}-s)\,(\mathfrak{f}(s,\varpi(s)) - \mathfrak{f}(s,\vartheta(s)))\,ds$$

$$
\begin{aligned}
&+ \int_{\xi_k}^{t} \Re(t-s)\left(\mathfrak{f}(s,\varpi(s)) - \mathfrak{f}(s,\vartheta(s))\right) ds + \sum_{i=1}^{k}\prod_{j=i}^{k} \mathfrak{b}_j(\delta_j) \int_{\xi_{k-1}}^{\xi_k} \Re(t-s) \\
&\times \left(\mathfrak{g}(s,\varpi(s)) - \mathfrak{g}(s,\vartheta(s))\right) ds + \int_{\xi_k}^{t} \Re(t-s)\Big(\mathfrak{g}(s,\varpi(s)) - \mathfrak{g}(s,\vartheta(s))\Big) ds\Big] \mathcal{I}_{[\xi_k,\xi_{k+1})}(t)\Big\|^2 \\
&\leq 2\max\{1,\mathscr{B}^2\}\mathscr{H}^2(T-t_0)\int_{t_0}^{t} \mathbb{E}\,\|\mathfrak{f}(s,\varpi(s)) - \mathfrak{f}(s,\vartheta(s))\|^2\, ds \\
&+ 2\max\{1,\mathscr{B}^2\}\mathscr{H}^2 \int_{t_0}^{t} \mathbb{E}\,\|\mathfrak{g}(s,\varpi(s)) - \mathfrak{g}(s,\vartheta(s))\|^2\, ds.
\end{aligned}
$$

Taking supremum on both sides and using (A5),

$$
\begin{aligned}
\sup_{t\in[t_0,T]} \mathbb{E}\,\|\varpi-\vartheta\|_t^2 \quad &\leq \quad 2\epsilon + 4\max\{1,\mathscr{B}^2\}\mathscr{H}^2(T-t_0)\mathscr{C}_1 \int_{t_0}^{t} \sup_{t\in[t_0,T]} \mathbb{E}\,\|\varpi-\vartheta\|_s^2\, ds \\
&+ \quad 4\max\{1,\mathscr{B}^2\}\mathscr{H}^2\mathscr{C}_2 \int_{t_0}^{t} \sup_{t\in[t_0,T]} \mathbb{E}\,\|\varpi-\vartheta\|_s^2\, ds.
\end{aligned}
$$

By following Gronwall's inequality, there occurs a constant

$$
\mathscr{C} := 2\exp\{\max\{1,\mathscr{B}^2\}\mathscr{H}^2[(T-t_0)\mathscr{C}_1 + \mathscr{C}_2]\} > 0.
$$

This implies that

$$
\sup_{t\in[t_0,T]} \mathbb{E}\,\|\varpi-\vartheta\|_t^2 \leq \mathscr{C}\epsilon,
$$

This signifies that [1.1] is Hyers-Ulam stable. As a direct consequence, the proof is complete. □

4.3. Mean-square exponential stability

Now, we will analyze the exponential stability in the mean square moment for the mild solution to system 1.1. We need to impose some additional assumption and lemma:

(A8) The resolvent operator $\Re(t)_{t\geq 0}$ satisfies the further condition: There exist a constant $\mathscr{H} > 0$ and a real number $\varsigma > 0$ such that $\|\Re(t)\| \leq \mathscr{H} e^{\varsigma t}$, $t \geq 0$.

In order to prove the theorem we may take into consideration the following lemma

Lemma 4.1. *[20] For $\varsigma > 0$, $\exists$ some positive constants $\upsilon, \upsilon' > 0 \ni$ if $\upsilon' < \varsigma$, the following inequality*

$$
\varpi(t) = \begin{cases} \upsilon e^{-\varsigma(t-t_0)}, & t \in [-\delta, 0] \\ \upsilon e^{-\varsigma(t-t_0)} + \upsilon' \displaystyle\int_{t_0}^{t} e^{-\varsigma(t-s)} \sup_{\theta\in(-\delta,0]} \varpi(s+\theta) ds, & t \geq t_0 \end{cases}
$$

holds. We have $\varpi(t) \leq \mathscr{F} e^{-\tau(t-t_0)}$, where $\tau > 0$ satisfying

$$
\frac{\upsilon'}{\varsigma-\tau} e^{\tau(\delta+t_0)} = 1
$$

and

$$
\mathscr{F} = \max\{\frac{\upsilon}{\upsilon'}(\varsigma-\tau)e^{-\tau\delta}, \varsigma\}.
$$

Theorem 4.3. *Assume (A3), (A8) gets satisfied, then the mild solution of [1.1] are mean-square exponentially stable.*

Proof. Together with the assumed hypotheses and Holder's inequality,

$$
\begin{aligned}
\mathbb{E}\|\vartheta(\mathrm{t})\|^2 \quad \leq \quad & 4\mathbb{E}\left(\max_{\mathrm{k}}\{\prod_{i=1}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|^2\}\right)^2 \|\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\|^2\,\mathbb{E}\|\varphi(0)\|^2 \\
+ \quad & 4\mathbb{E}\left(\max_{\mathrm{k}}\{\prod_{i=1}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|^2\}\right)^2 \|\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\|^2\,\mathbb{E}\|[\mathfrak{h}(\vartheta)]\|^2 \\
+ \quad & 4\mathbb{E}\left(\max_{i,\mathrm{k}}\{\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\},1\right)^2 \mathbb{E}\left(\int_{\mathrm{t}_0}^{\mathrm{t}}\|\mathfrak{R}(\mathrm{t}-s)\|\,\|\mathfrak{f}(s,\vartheta(s))\|\,ds\right)^2 \\
+ \quad & 4\mathbb{E}\left(\max_{i,\mathrm{k}}\{\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\},1\right)^2 \mathbb{E}\left(\int_{\mathrm{t}_0}^{\mathrm{t}}\|\mathfrak{R}(\mathrm{t}-s)\|\,\|\mathfrak{g}(s,\vartheta(s))\|\,d\omega(s)\right)^2 \\
\leq \quad & 4\mathscr{B}^2\mathscr{H}^2e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\varphi(0)\|^2 + 4\mathscr{B}^2\mathscr{H}^2e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\mathfrak{h}(\vartheta)\|^2 \\
+ \quad & 4\max\{1,\mathscr{B}^2\}\mathscr{H}^2\int_{\mathrm{t}}^{\mathrm{t}_0}e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\mathfrak{f}(s,\vartheta(s))\|^2\,ds\int_{\mathrm{t}_0}^{\mathrm{t}}e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}ds \\
+ \quad & 4\max\{1,\mathscr{B}^2\}\mathscr{H}^2\int_{\mathrm{t}_0}^{\mathrm{t}}e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}ds\int_{\mathrm{t}}^{\mathrm{t}_0}e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\mathfrak{g}(s,\vartheta(s))\|^2\,ds \\
\leq \quad & 4\mathscr{B}^2\mathscr{H}^2e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\varphi(0)\|^2 + 4\mathscr{B}^2\mathscr{H}^2e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)}\mathbb{E}\,\|\mathfrak{h}(\vartheta)\|^2 \\
+ \quad & 4\max\{1,\mathscr{B}^2\}\frac{\mathscr{H}^2(\mathscr{C}_1+\mathscr{C}_2)}{\varsigma}\int_{\mathrm{t}_0}^{\mathrm{t}}\sup_{\theta\in[-\delta,0]}\mathbb{E}\,\|\vartheta(s+\theta)\|^2\,ds \\
\leq \quad & \mathscr{F}e^{-\varsigma(\mathrm{t}-\mathrm{t}_0)},\quad \forall\ \mathrm{t}\in[-\delta,0],
\end{aligned}
$$

where $\mathscr{F} = \max\{4\mathscr{B}^2\mathscr{H}^2\mathbb{E}\|[\varphi(0)+\mathfrak{h}(\vartheta)]\|^2, \sup_{\theta\in[-\delta,0]}\mathbb{E}\|[\varphi+\mathfrak{h}]\|^2\}$.

Thus, by lemma 4.1, $\forall\ \mathrm{t}\in[\mathrm{t}_0-\delta,+\infty]$,

$$\mathbb{E}\|\vartheta(\mathrm{t})\|^2 \leq \mathscr{F}e^{-\tau\mathrm{t}}.$$

This completes the proof. □

5. Illustration

We may take into account the domain $\Omega \subset \mathscr{R}^n$ with the boundary $\partial\Omega$:

$$
\begin{aligned}
[\frac{dz(\mathrm{t},\vartheta)}{\partial\mathrm{t}}] \quad = \quad & \frac{\partial^2}{\partial\vartheta^2}z(\mathrm{t},\vartheta) + \int_0^{\mathrm{t}}\alpha(\mathrm{t}-s)\frac{\partial^2}{\partial\vartheta^2}z(s,\vartheta)ds + \int_{-\mathrm{r}}^{\mathrm{t}}[\kappa_1(\theta)z(\mathrm{t}+\theta)d\theta] \\
+ \quad & \int_{-\mathrm{r}}^{\mathrm{t}}[\kappa_2(\theta)z(\mathrm{t}+\theta)d\theta]\,d\omega(\mathrm{t}), \qquad \mathrm{t}\geq\delta,\quad \mathrm{t}\neq\xi_{\mathrm{k}}, \\
z(\xi_{\mathrm{k}},\vartheta) \quad = \quad & h(\mathrm{k})\delta_{\mathrm{k}}z(\xi_{\mathrm{k}}^-,\vartheta), \qquad\qquad \vartheta\in\Omega \\
z(\mathrm{t}_0,\vartheta) \quad = \quad & \varphi(\theta,\vartheta) + \int_0^1\int_0^{\mathrm{t}}\Lambda(\mathrm{t},\vartheta)\log(1+|z(\mathrm{t},\mathrm{r})|^{1/2})d\mathrm{t}d\mathrm{r},
\end{aligned}
$$

$$\begin{array}{rcll} & = & \{\varphi(\theta) \leq \theta < 0\} & \vartheta \in \Omega, \theta \in [-\delta, 0] \\ z(\mathrm{t}, \vartheta) & = & 0, & \vartheta \in \partial\Omega. \end{array} \tag{5.1}$$

Let $\mathscr{X} = \mathcal{L}^2(\Omega)$, $\alpha : \mathscr{R}_+ \to \mathscr{R}_+$. κ_1, κ_2 be positive functions from $[-\delta, 0]$ to $\mathscr{R}$. Assume δ_k is a random variable described on $\mathscr{D}_\mathrm{k} = (0, \mathfrak{d}_\mathrm{k})$ with $0 < \mathfrak{d}_\mathrm{k} < +\infty$ for $\mathrm{k} = 1, 2, \cdots$. Without loss of generality, we suppose $\{\delta_\mathrm{k}\}$ follows Erlang distribution. δ_i, δ_j being mutually independent with $i \neq j$ for $i, j = 1, 2, \cdots$. $\xi_\mathrm{k} = \xi_{\mathrm{k}-1} + \delta_\mathrm{k}$ where $\{\xi_\mathrm{k}\}$ is a strictly increasing process with independently increasing increments and $\mathrm{t}_0 \in [0, \mathtt{T}]$ be an arbitrary real number.
Let $\mathfrak{A}$ be an operator on $\mathscr{X}$ by $\mathfrak{A}z = \frac{\partial^2 z}{\partial \vartheta}$ provided,

$$\mathscr{D}(\mathfrak{A}) = \{z \in \mathscr{X} : z \ and\ z_\vartheta \text{ are absolutely continuous}, \ \ z_{\vartheta\vartheta} \in \mathscr{X}, \ z = 0 \ on\ \partial\Omega\}.$$

Also, let the map $\mathfrak{B} : \mathscr{D}(\mathfrak{A}) \subset \mathscr{X} \to \mathscr{X}$ be the operator interpreted as

$$\mathfrak{B}(\mathrm{t})(z) = \alpha(\mathrm{t})\mathfrak{A}z \ \text{ for } \ \mathrm{t} \geq 0 \ \text{ and } \ z \in \mathscr{D}\mathfrak{A}.$$

The operator $\mathfrak{A}$ can be exhibited as

$$\mathfrak{A}z = \sum_{n=1}^{\infty} \mathfrak{n}^2 \langle z, z_\mathfrak{n} \rangle z_\mathfrak{n}, \quad z \in \mathscr{D}\mathfrak{A},$$

where $\mathfrak{A}$ is provided with the eigenvectors $z_\mathfrak{n}(\varpi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}$. It is evident that $z_\mathfrak{n}(\varpi)$ forms an orthonormal system in $\mathbb{X}$. Moreover, for the analytic semigroup $(\mathfrak{R}(\mathrm{t}))_{\mathrm{t}\geq 0}$ in $\mathbb{X}$, $\mathfrak{A}$ is the infinitesimal generator satisfying:

$$\|\mathfrak{R}(\mathrm{t})\| \leq \exp\{-\pi^2(\mathrm{t} - \mathrm{t}_0)\}, \quad \mathrm{t} \geq \mathrm{t}_0.$$

Additionally, there are the following conditions:

(i) $\displaystyle\int_{-\delta}^{0} \kappa_1(\theta)^2 d\theta < \infty, \int_{-\delta}^{0} \kappa_2(\theta)^2 d\theta < \infty,$

(ii) $\displaystyle\mathbb{E}\left(\max_{i,\mathrm{k}}\{\prod_{j=i}^{\mathrm{k}} \left\|h(j)(\delta_j)\right\|\}^2\right) < \infty.$

Using the aforementioned conditions, [5.1] can be represented by the abstract random impulsive stochastic differential equation of the form [1.1],

$$\begin{array}{rcl} \mathfrak{f}(\mathrm{t}, z(\mathrm{t})) & = & \displaystyle\int_{-\mathrm{r}}^{\mathrm{t}} \kappa_1(\theta) z(\mathrm{t} + \theta) d\theta, \\ \mathfrak{g}(\mathrm{t}, z(\mathrm{t})) & = & \displaystyle\int_{-\mathrm{r}}^{\mathrm{t}} \kappa_2(\theta) z(\mathrm{t} + \theta) d\theta, \\ \mathfrak{h}(\vartheta) & = & \displaystyle\int_0^1 \int_0^{\mathrm{t}} \Lambda(\mathrm{t}, \vartheta) \log(1 + |z(\mathrm{t}, \mathrm{r})|^{1/2}) d\mathrm{t} d\mathrm{r}, \\ \mathfrak{b}_\mathrm{k}(\delta_\mathrm{k}) & = & h(\mathrm{k})\delta_\mathrm{k}. \end{array}$$

Condition (i) implies (A6) holds with

$$\mathscr{C}_i = \int_{\mathrm{r}}^{0} \kappa_i^2(\theta) d\theta, \quad for \ \ i = 1, 2,$$

alongwith condition (ii) implying (A4). This depicts that [5.1] has a mild solution. Moreover, achieving continuous dependence of solution on initial conditions and Hyers Ulam Stability as in Section 4. Finally, if $\lambda' \leq \tau$, (i.e.,)

$$4\max\{1, \mathscr{B}^2\}(\mathscr{C}_1 + \mathscr{C}_2)/(\pi^2) \leq \pi^2,$$

then [5.1] is mean square exponentially stable under the assumptions (A3) and (A7).

6. Conclusions

In this paper, we have obtained the existence and various types of stability results for the RISIDEs with nonlocal conditions by means of functional analysis and the stochastic analysis method. In addition, it is of great interest for future research to study RISIDEs including more complicated stochastic factors, such as the stochastics processes driven by fractional Brownian motions, Rosenblatt process and Poisson jumps, which describe some stochastic phenomena more precisely, see [13, 22] for more details.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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