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Research article

Generalized Ξ-metric-like space and new fixed point results with an application

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Abstract: This paper is devoted to generalizing Ξ-metric spaces and b-metric-like spaces to present the structure of generalized Ξ-metric-like spaces. The topological properties of this space and examples to support it are being investigated. Moreover, as demonstrated in the previous literature, the concept of Lipschitz mappings is presented more generally and some results of fixed points are derived in the aforementioned space. Finally, some theoretical results have been implicated in the discussion of the existence and uniqueness of the solution to the Fredholm integral equation.

Keywords: generalized Ξ-metric-like space; fixed point style; Lipschitz mapping; Fredholm integral equation
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1. Introduction

The fixed point (FP) theory beautifully combines analysis, topology, and geometry. In the past few decades, it has been clear that the theory of FPs is a very effective and significant instrument for the investigation of nonlinear processes. Particularly in the areas of biology, chemistry, economics, engineering, game theory, physics, and logic programming, fixed point theory has been utilized. The FP method became more effective and attractive to scientists after Banach presented his principle [1] that states: Every contraction mapping defined on a complete metric space owns a unique FP.

A cone metric space is a concept that Huang and Zhang [2] developed in 2007 which considerably generalizes metric spaces. Additionally, they obtained FP theorems for contractions of the Banach, Kannan, and Chatterjea types. Following that, a significant number of FP outcomes in cone metric spaces were reported, see [3–7]. In 2012, Rawashdeh et al. [8] established the existence of the ordered space, known as an $\Xi-$metric space, and demonstrated that the convergent sequence in this space is a Cauchy sequence.

The FP theorems derived by Cevik and Altun [9], Critescu [10], Matkowski [11], and Wegrzyk [12] were subsequently generalized by Pales and Petre [13] in 2013, who also introduced the idea of stringent positivity in Riesz spaces. In order to find the Hardy-Rogers type FP theorems in Ξ-metric spaces devoid of solid cones, Wang et al. [14] examined the topological features pertaining to semi-interior points in those spaces.

The study of FP theorems in Ξ-metric spaces has yielded few research findings to date. In this manuscript, we build a new space and call it a generalized Ξ-metric-like space ($G\Xi ML$-space, for short), which is a combination of results of Ξ-metric-like spaces and b-metric-like spaces. Moreover, we suggest that FPs for Cirić type contraction [15] in $G\Xi ML$-spaces exist and are unique. We also provide the existence and uniqueness of the FP for the η-$\wp$-type contraction in a $G\Xi ML$-space. Our findings are fresh enough in our own eyes because no FP findings for Cirić type contraction in $G\Xi ML$-spaces have been reported so far. Furthermore, as is well known, metric-like spaces, cone metric-like spaces, Ξ-metric spaces, and several other spaces, are all considerably generalized by $G\Xi ML$-spaces. From this perspective, the relevance of our FP results from $G\Xi ML$-spaces is profound and far-reaching. Ultimately, the existence and uniqueness of the Fredholm integral equation solution have been provided by some theoretical findings.

2. Basic concepts

In this part, we present some basic definitions and theorems introduced earlier in order to assist the reader in understanding our manuscript.

Throughout this paper, $(\Xi^+)^{SI}$ represents the set of all semi-interior points of Ξ^+ and $\lll$ refers to a partial order on Ξ^+ and it is defined as

$$\ell_1, \ell_2 \in \Xi^+,\ \ell_1 \lll \ell_2 \Leftrightarrow \ell_2 - \ell_1 \in (\Xi^+)^{SI}.$$

Definition 2.1. [2] Let Ξ be a normed space, ϑ_Ξ be a zero element of Ξ and $P \neq \emptyset$ be a closed subset of Ξ. P is called a cone if it satisfies

(a) $P \neq \{\vartheta_\Xi\}$;
(b) $\zeta_1, \zeta_2 \in [0, +\infty)$ implies $\zeta_1 P + \zeta_2 P \subseteq P$;
(c) $\{\vartheta_\Xi\} = P \cap (-P)$.

P is called a solid cone if $P^\circ \neq \emptyset$, where P° is the set of all interior points of P.

Note, from here to the rest of the paper the symbols $\preceq$ and $\lll$ refer to the partial orders in Ξ and defined as

$$\ell_1, \ell_2 \in \Xi \text{ and } \ell_1 \preceq \ell_2 \text{ iff } \ell_2 - \ell_1 \in P,$$

and

$$\ell_1, \ell_2 \in \Xi \text{ and } \ell_1 \lll \ell_2 \text{ iff } \ell_2 - \ell_1 \in P^\circ.$$

Definition 2.2. [2] Let Ξ be a normed space, ϑ_Ξ be a zero element of Ξ and $\Xi^+ \neq \emptyset$ be a closed convex subset of Ξ. Then Ξ^+ is called a positive cone iff the two assertions below hold

(1) $\rho \in \Xi^+$, $\zeta_1 \geq 0$ implies $\zeta_1\rho \in \Xi^+$,
(2) $\rho \in \Xi^+$, $-\rho \in \Xi^+$ implies $\rho = \{\vartheta_\Xi\}$.

Assume that $\rho_0 \in \Xi^+$, if there is $\zeta_1 > 0$ so that $\rho_0 - \zeta_1 W_+ \subseteq \Xi^+$, ρ_0 is called a semi-interior (SI) point in Ξ^+ [16], where W_+ is the positive part of W so that $W_+ = W \cap \Xi^+$ and W is closed unit ball of E.

Definition 2.3. [8] Let Ω be a non-empty set defined on a real normed space Ξ. The mapping $d_\Xi : \Omega^2 \to [0, +\infty)$ is called an Ξ-metric if the hypotheses below hold for $\ell_1, \ell_2, \ell_3 \in \Omega$,

(i) $\vartheta_\Xi \preceq d_\Xi(\ell_1, \ell_2)$ and $d_\Xi(\ell_1, \ell_2) = \vartheta_\Xi \Leftrightarrow \ell_1 = \ell_2$,
(ii) $d_\Xi(\ell_1, \ell_2) = d_\Xi(\ell_2, \ell_1)$,
(iii) $d_\Xi(\ell_1, \ell_2) \preceq d_\Xi(\ell_1, \ell_3) + d_\Xi(\ell_3, \ell_2)$.

Then, (Ω, d_Ξ) is called a Ξ-metric space.

The topological properties of this space, which includes convergence, Cauchy sequences, examples, some facts on e-sequence theory and others were studied in detail by [17–19].

Definition 2.4. [18] If for each $\vartheta_\Xi \lll e$, there is $k^* \in \widetilde{\mathbb{N}} = \mathbb{N} \cup \{0\}$ so that $\ell_k \lll e$ for all $k > k^*$, then the sequence $\{\ell_k\}$ in Ξ^+ is called an e-sequence.

Lemma 2.5. *[18] Assume that $\{\ell_k\}$ and $\{\rho_k\}$ are two sequences in Ξ so that*

$$\ell_k \preceq \rho_k \text{ and } \rho_k \to \vartheta_\Xi \text{ as } k \to \infty.$$

Then $\{\ell_k\}$ is an e-sequence.

Lemma 2.6. *[18] The collection $\{u\ell_k+v\rho_k\}$ is an e-sequence provided that $\{\ell_k\}$ and $\{\rho_k\}$ are e-sequences and $u, v \geq 0$.*

Lemma 2.7. *[18] Suppose that $\xi_1, \xi_2, \xi_3 \in \Xi$ and $\xi_1 \preceq \xi_2 \lll \xi_3$, then $\xi_1 \lll \xi_3$.*

Lemma 2.8. *[18] If $\vartheta_\Xi \preceq v \lll e$ for any $\vartheta_\Xi \lll e$, then $v = \vartheta_\Xi$.*

Lemma 2.9. *[19] If $\vartheta_\Xi \preceq v \preceq \alpha v$, then $v = \vartheta_\Xi$, where $\alpha \in [0, 1)$.*

Lemma 2.10. *[18] Let $\ell, \rho \in \Xi$ and $\ell \lll \rho + e$ for all $\vartheta_\Xi \lll e$, then $\ell \lll \rho$.*

Theorem 2.11. *[18] Assume that (Ω, d_Ξ) is an e-complete Ξ-metric space $(\Xi^+)^{SI} \neq \emptyset$. Let $\Upsilon : \Omega \to \Omega$ be a mapping verifying*

$$d_\Xi(\Upsilon\ell_1, \Upsilon\ell_2) \preceq \vartheta d_\Xi(\ell_1, \ell_2),\ \forall \ell_1, \ell_2 \in \Omega,$$

where $\vartheta \in [0, 1)$. Then Υ owns a FP.

The idea of $\eta-$admissible function is defined in [20] as follows:

Definition 2.12. [20] For a set $\Omega \neq \emptyset$, let $\eta : \Omega^2 \to [0, +\infty)$ be a function and Υ be a self-mapping on Ω. Then Υ is called an $\eta-$admissible function if

$$\eta(\ell_1, \ell_2) \geq 1 \text{ implies } \eta(\Upsilon\ell_1, \Upsilon\ell_2) \geq 1,\ \forall \ell_1, \ell_2 \in \Omega.$$

Definition 2.13. [21] For a set $\Omega \neq \emptyset$, let $\eta : \Omega^2 \to [0, +\infty)$ be a function, $\rho \in \Omega$ and $\{\rho_k\}$ be a sequence in Ω. Then Ω is called an $\eta-$regular if for any $k \in \widetilde{\mathbb{N}}$,

$$\eta(\rho_k, \rho_{k+1}) \geq 1 \text{ and } \lim_{k\to\infty} \rho_k = \rho \text{ implies } \eta(\rho_k, \rho) \geq 1.$$

Alghamdi et al. [22] introduced the idea of a $b-$metric-like as a generalization of a $b-$metric as follows:

Definition 2.14. [22,23] A $b-$metric-like on the set $\Omega \neq \emptyset$ is a function $\varpi : \Omega^2 \to [0, +\infty)$ so that for all $\ell_1, \ell_2, \ell_3 \in \Omega$, the assertions below are true

(*i*) $d_\Xi(\ell_1, \ell_2) = 0 \Rightarrow \ell_1 = \ell_2$,
(*ii*) $d_\Xi(\ell_1, \ell_2) = d_\Xi(\ell_2, \ell_1)$,
(*iii*) $d_\Xi(\ell_1, \ell_2) \leq s\,[d_\Xi(\ell_1, \ell_3) + d_\Xi(\ell_3, \ell_2)]$.

Here, the pair (Ω, ϖ) is called a $b-$metric-like space with a constant $s \geq 1$.

In relation to this space, they analyzed the topological structure and discovered some relevant FP consequences. Numerous findings have been made on the fixed points of mappings under specific contractive conditions in the aforementioned spaces, for example, see [24–29].

3. Generalized Ξ-metric-like space

By combining the results of Ξ-metric and b-metric-like spaces, we introduce a generalized Ξ-metric-like space as follows:

Definition 3.1. Let Ω be a non-empty subset of a normed space Ξ. We say that the mapping $d_\Xi : \Omega^2 \to [0, +\infty)$ is a $G\Xi ML$-space, if for each $\ell_1, \ell_2, \ell_3 \in \Omega$, the conditions below hold
(GM_1) $\vartheta_\Xi \preceq d_\Xi(\ell_1, \ell_2)$ and $d_\Xi(\ell_1, \ell_2) = \vartheta_\Xi \Rightarrow \ell_1 = \ell_2$,
(GM_2) $d_\Xi(\ell_1, \ell_2) = d_\Xi(\ell_2, \ell_1)$,
(GM_3) $d_\Xi(\ell_1, \ell_2) \preceq \varpi(\ell_1, \ell_2)\,[d_\Xi(\ell_1, \ell_3) + d_\Xi(\ell_3, \ell_2)]$,
where $\varpi : \Omega^2 \to [1, +\infty)$ is a mapping. Then the pair (Ω, d_Ξ) is called a $G\Xi ML$-space.

Remark 3.2. A $G\Xi ML$-space generalizes several known metric structures such that for all $\ell_1, \ell_2 \in \Omega$,

(i) If $\varpi(\ell_1, \ell_2) = 1$, then a $G\Xi ML$-space reduces to an Ξ-metric-like space,
(ii) If $\varpi(\ell_1, \ell_2) = s > 1$, then a $G\Xi ML$-space reduces to a b-metric-like space over the normed space Ξ.

Example 3.3. Consider $\Omega = \{0\} \cup \mathbb{N}$ and q is a positive even integer. Describe a mapping $\varpi : \Omega^2 \to [1, +\infty)$ as

$$\varpi(\ell_1, \ell_2) = \begin{cases} 1 + \ell_1\ell_2, & \text{if } \ell_1 \neq \ell_2, \\ 1, & \text{if } \ell_1 = \ell_2, \end{cases} \quad \forall \ell_1, \ell_2 \in \Omega.$$

Let $d_\Xi : \Omega^2 \to [0, +\infty)$ be defined by $d_\Xi(\ell_1, \ell_2) = (\ell_1 + \ell_2)^q\, e^{2\tau}$, for all $\ell_1, \ell_2 \in \Omega$ and for $\tau \in [0, 1]$. Then, (Ω, d_Ξ) is a $G\Xi ML$-space. Stipulation (GM_1) and (GM_2) are clearly verified. Now, we satisfy the stipulation (GM_3). For this, let $\ell_1 \in \Omega$ be an arbitrary, then we obtain

- The axiom (GM_3) is clear, if $\ell_1 = \ell_2$.

- If $\ell_1 \neq \ell_2$ and $\ell_1 = \ell_3$, then, we have

$$\begin{aligned}
\varpi(\ell_1, \ell_2)\,[d_\Xi(\ell_1, \ell_3) + d_\Xi(\ell_3, \ell_2)]\,(\tau) &= (1 + \ell_1\ell_2)\,[(\ell_1 + \ell_3)^q + (\ell_3 + \ell_2)^q]\,e^{2\tau} \\
&\geq (1 + \ell_1\ell_2)\,(\ell_1 + \ell_2)^q\,e^{2\tau} \\
&\geq (\ell_1 + \ell_2)^q\,e^{2\tau} = d_\Xi(\ell_1, \ell_2)\,(\tau)\,.
\end{aligned}$$

- If $\ell_1 \neq \ell_2$, $\ell_2 \neq \ell_3$ and $\ell_3 \neq \ell_1$, then, we get

$$\begin{aligned}
\varpi(\ell_1, \ell_2)\,[d_\Xi(\ell_1, \ell_3) + d_\Xi(\ell_3, \ell_2)]\,(\tau) &= (1 + \ell_1\ell_2)\,[(\ell_1 + \ell_3)^q + (\ell_3 + \ell_2)^q]\,e^{2\tau} \\
&\geq \left(\frac{1 + \ell_1\ell_2}{3}\right)^q (\ell_1 + \ell_3 + \ell_3 + \ell_2)^q\,e^{2\tau} \\
&= \left(\frac{1 + \ell_1\ell_2}{3}\right)^q (\ell_1 + 2\ell_3 + \ell_2)^q\,e^{2\tau} \\
&\geq (\ell_1 + 2\ell_3 + \ell_2)^q\,e^{2\tau} \\
&\geq (\ell_1 + \ell_2)^q\,e^{2\tau} = d_\Xi(\ell_1, \ell_2)\,(\tau)\,.
\end{aligned}$$

Example 3.4. Consider $\Omega = \{0\} \cup \mathbb{N}$ and q is a positive even integer. Describe a mapping $\varpi : \Omega^2 \to [1, +\infty)$ as

$$\varpi(\ell_1, \ell_2) = \begin{cases} 1 + \ell_2 + \ell_2, & \text{if } \ell_1 \neq \ell_2, \\ 1, & \text{if } \ell_1 = \ell_2, \end{cases} \quad \forall \ell_1, \ell_2 \in \Omega.$$

Let $d_\Xi : \Omega^2 \to [0, +\infty)$ be defined by $d_\Xi(\ell_1, \ell_2) = \left(\ell_1^q + \ell_2^q\right)^q e^\tau$, for all $\ell_1, \ell_2 \in \Omega$ and for $\tau \in [0, 1]$. Then, (Ω, d_Ξ) is a $G\Xi ML$-space. Stipulation (GM_1) and (GM_2) are clearly fulfilled. Only, we verify the axiom (GM_3). For this regard, choose $\ell_1 \in \Omega$ as arbitrary, then we find that the cases below:

- The axiom (GM_3) is clear, if $\ell_1 = \ell_2$.
- If $\ell_1 \neq \ell_2$ and $\ell_1 = \ell_3$, then

$$\begin{aligned}
\varpi(\ell_1, \ell_2)\,[d_\Xi(\ell_1, \ell_3) + d_\Xi(\ell_3, \ell_2)]\,(\tau) &= (1 + \ell_1 + \ell_2)\left[\left(\ell_1^q + \ell_3^q\right)^q + \left(\ell_3^q + \ell_2^q\right)^q\right] e^\tau \\
&\geq (1 + \ell_1 + \ell_2)\left|\left(\ell_1^q + \ell_2^q\right)\right|^q e^\tau \\
&\geq \left(\ell_1^q + \ell_2^q\right)^q e^{2\tau} = d_\Xi(\ell_1, \ell_2)\,(\tau)\,.
\end{aligned}$$

If $\ell_1 \neq \ell_2$, $\ell_2 \neq \ell_3$and $\ell_3 \neq \ell_1$, then

$$\begin{aligned}
\varpi(\ell_1, \ell_2)\,[d_\Xi(\ell_1, \ell_3) + d_\Xi(\ell_3, \ell_2)]\,(\tau) &= (1 + \ell_1 + \ell_2)\left[\left(\ell_1^q + \ell_3^q\right)^q + \left(\ell_3^q + \ell_2^q\right)^q\right] e^\tau \\
&\geq \left(\frac{1 + \ell_1 + \ell_2}{3}\right)^q \left(\ell_1^q + \ell_3^q + \ell_3^q + \ell_2^q\right)^q e^\tau \\
&= \left(\frac{1 + \ell_1 + \ell_2}{3}\right)^q \left(\ell_1^q + 2\ell_3^q + \ell_2^q\right)^q e^\tau \\
&\geq \left(\ell_1^q + \ell_2^q\right)^q e^\tau = d_\Xi(\ell_1, \ell_2)\,(\tau)\,.
\end{aligned}$$

Now, we define a topology on a $G\Xi ML$-space.

Definition 3.5. Let (Ω, d_Ξ) be a $G\Xi ML$-space, $\ell \in \Omega$ and $\varrho > \vartheta_\Xi$. We define a d_Ξ-ball with radius $\varrho > \vartheta_\Xi$ and center ℓ as

$$\mathfrak{R}_{d_\Xi}(\ell, \varrho) = \{\ell_1 \in \Omega : |d_\Xi(\ell, \ell_1) - d_\Xi(\ell_1, \ell_1)| < \varrho\},$$

and take

$$\Theta = \{\mathfrak{R}_{d_\Xi}(\ell, \varrho) : \ell \in \Omega \text{ and } \vartheta_\Xi \lll \varrho\}.$$

Theorem 3.6. *The family* $\Theta = \{\mathfrak{R}_{d_\Xi}(\ell, \varrho) : \ell \in \Omega \text{ and } \vartheta_\Xi \lll \varrho\}$ *of all open balls is a basis for the topology* $\mathfrak{I}_{d_\Xi}$.

Proof. (i) Assume that $\ell \in \Omega$. It is obvious that $\ell \in \mathfrak{R}_{d_\Xi}(\ell, \varrho)$ for $\varrho > \vartheta_\Xi$. This yields

$$\varrho \in \mathfrak{R}_{d_\Xi}(\ell, \varrho) \subseteq \cup_{\ell \in \Omega, \varrho > \vartheta_\Xi} \mathfrak{R}_{d_\Xi}(\ell, \varrho).$$

(ii) Let $r \in \mathfrak{R}_{d_\Xi}(\ell, \varrho_1) \cap \mathfrak{R}_{d_\Xi}(\ell, \varrho_2)$. Then there exists $\varrho > \vartheta_\Xi$ such that $\mathfrak{R}_{d_\Xi}(r, \varrho) \subseteq \mathfrak{R}_{d_\Xi}(\ell, \varrho_1)$ and $\mathfrak{R}_{d_\Xi}(r, \varrho) \subseteq \mathfrak{R}_{d_\Xi}(\ell, \varrho_2)$. Assume also $w \in \mathfrak{R}_{d_\Xi}(r, \varrho)$, then we have $d_\Xi(r, w) - d_\Xi(r, r) \lll \varrho$. Hence,

$$\mathfrak{R}_{d_\Xi}(r, \varrho) \subseteq \mathfrak{R}_{d_\Xi}(\ell, \varrho_1) \cap \mathfrak{R}_{d_\Xi}(\ell, \varrho_2).$$

Therefore, a collection Θ is a basis for the topology $\mathfrak{I}_{d_\Xi}$.

□

Definition 3.7. Let (Ω, d_Ξ) be a $G\Xi ML$-space, $\{\ell_k\}$ be a sequence in Ω and $\ell \in \Omega$, $(\Xi^+)^{SI} \neq \emptyset$. We say that

(1) $\{\ell_k\}$ is e-convergent to ℓ iff

$$\lim_{k\to\infty} d_\Xi(\ell_k, \ell) = d_\Xi(\ell, \ell),$$

on the other words, if for any $\vartheta_\Xi \lll e$, there is $k^* \in \widetilde{\mathbb{N}}$ so that $d_\Xi(\ell_k, \ell) \lll e$ for all $k > k^*$.

(2) $\{\ell_k\}$ is e-Cauchy sequence, if for any $\vartheta_\Xi \lll e$, there is $k^* \in \widetilde{\mathbb{N}}$ so that $d_\Xi(\ell_k, \ell_l) \lll e$ for all $k, l > k^*$, or equivalently

$$\lim_{k,j\to\infty} d_\Xi(\ell_k, \ell_j) = \lim_{k,j\to\infty} d_\Xi(\ell_k, \ell) = d_\Xi(\ell, \ell) = \vartheta_\Xi.$$

(3) (Ω, d_Ξ) is e-complete, if every e-Cauchy sequence is e-convergent to some point in Ω.

4. Lipschitz mappings in the wider sense

Here, we generalize the Lipschitz mappings on a $G\Xi ML$-space and we present some of the results found in the previous literature that can be generalized in the space under study.

Definition 4.1. Let (Ω, d_Ξ) be a $G\Xi ML$-space. A self-mapping $Z : \Omega \to \Omega$ is called an extended Lipschitz mapping if there is a constant $\delta < 1$ and for each $\ell_1, \ell_2 \in \Omega$, we get

$$d_\Xi(Z\ell_1, Z\ell_2) \leq \delta d_\Xi(\ell_1, \ell_2).$$

Example 4.2. Suppose that Ω and ϖ are as in Example 3.4. Describe a mapping $d_\Xi : \Omega^2 \to [0, +\infty)$ as

$$d_\Xi(\ell_1, \ell_2) = (\ell_1 + \ell_2)^2 e^{2\tau}, \ \forall \ell_1, \ell_2 \in \Omega, \ \tau \in (0,1).$$

Then (Ω, d_Ξ) is a $G\Xi ML$-space with $\varpi(\ell_1, \ell_2) = 2^{q-1}$. Define the mapping $Z : \Omega \to \Omega$ by $Z(\ell_1) = \frac{\ell_1}{3}$ for all $\ell_1 \in \Omega$. Then, we have

$$\begin{aligned} d_\Xi(Z\ell_1, Z\ell_2)(\tau) &= (Z\ell_1 + Z\ell_2)^2 e^{2\tau} = \left(\frac{\ell_1}{3} + \frac{\ell_2}{3}\right)^2 e^{2\tau} \\ &\leq \frac{1}{9}(\ell_1 + \ell_2)^2 e^{2\tau} = \delta(u) d_\Xi(\ell_1, \ell_2), \end{aligned}$$

where $\delta(u) = \frac{1}{9} < 1$. It follows that Z is an extended Lipschitz mapping.

The following lemma is very important in the sequel.

Lemma 4.3. *Let (Ω, d_Ξ) be a $G\Xi ML$-space with a normal cone, $\ell, \rho \in \Omega$ and $\{\ell_k\}$ and $\{\rho_k\}$ be sequences in Ω so that $\ell_k \to \ell$ and $\rho_k \to \rho$ as $k \to \infty$. Then $d_\Xi(\ell_k, \rho_k) \to d_\Xi(\ell, \rho)$ as $k \to \infty$.*

Proof. For every $\vartheta_\Xi \lll \epsilon$. Choose $e \in \Xi$ with $\varpi(\ell, \rho) \lll 1 + e$ and $\|e\| < \frac{\epsilon}{4G+2}$, where G is a normal constant. Since $\ell_k \to \ell$ and $\rho_k \to \rho$ as $k \to \infty$, then there is $k^* \in \widetilde{\mathbb{N}}$ so that $d_\Xi(\ell_k, \ell) \lll e$ and $d_\Xi(\rho_k, \rho) \lll e$ for all $k > k^*$, by Axiom (GM_3) of Definition 3.1, we get

$$\begin{aligned} d_\Xi(\ell_k, \rho_k) &\leq \varpi(\ell_k, \rho_k)\left[d_\Xi(\ell_k, \ell) + d_\Xi(\ell, \rho_k)\right] \\ &\leq \varpi(\ell_k, \rho_k) d_\Xi(\ell_k, \ell) + \varpi(\ell_k, \rho_k)\varpi(\ell, \rho_k)\left[d_\Xi(\ell, \rho) + d_\Xi(\rho, \rho_k)\right] \\ &\leq \varpi(\ell_k, \rho_k) d_\Xi(\ell_k, \ell) + \varpi(\ell_k, \rho_k)\varpi(\ell, \rho_k) d_\Xi(\ell, \rho) + \varpi(\ell_k, \rho_k)\varpi(\ell, \rho_k) d_\Xi(\rho, \rho_k) \\ &\leq (1+e)e + (1+e)^2 d_\Xi(\ell, \rho) + (1+e)^2 e, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} d_\Xi(\ell, \rho) &\leq \varpi(\ell, \rho)\left[d_\Xi(\ell, \ell_k) + d_\Xi(\ell_k, \rho)\right] \\ &\leq \varpi(\ell, \rho) d_\Xi(\ell_k, \ell) + \varpi(\ell, \rho)\varpi(\ell_k, \rho)\left[d_\Xi(\ell_k, \rho_k) + d_\Xi(\rho_k, \rho)\right] \\ &\leq \varpi(\ell, \rho) d_\Xi(\ell_k, \ell) + \varpi(\ell, \rho)\varpi(\ell_k, \rho) d_\Xi(\ell_k, \rho_k) + \varpi(\ell, \rho)\varpi(\ell_k, \rho) d_\Xi(\rho_k, \rho) \\ &\leq (1+e)e + (1+e)^2 d_\Xi(\ell_k, \rho_k) + (1+e)^2 e. \end{aligned} \tag{4.2}$$

From (4.2), we have

$$\vartheta_\Xi \leq d_\Xi(\ell, \rho) - (1+e)^2 d_\Xi(\ell_k, \rho_k) \leq 2(1+e)^2 e,$$

which implies that

$$\vartheta_\Xi \leq \frac{d_\Xi(\ell, \rho)}{(1+e)^2} - d_\Xi(\ell_k, \rho_k) + 2e \leq 4e.$$

Or, equivalently

$$\vartheta_\Xi \leq \frac{d_\Xi(\ell, \rho)}{(1+e)^2} - d_\Xi(\ell_k, \rho_k) + 2e \leq d_\Xi(\ell, \rho) + 2e - d_\Xi(\ell_k, \rho_k) \leq 4e. \tag{4.3}$$

Similarly, from (4.1) and using (4.3), we can write

$$\|d_\Xi(\ell_k, \rho_k) - d_\Xi(\ell, \rho)\| \leq \|d_\Xi(\ell, \rho) + 2e - d_\Xi(\ell_k, \rho_k)\| + \|2e\| \leq (4G+2)\|e\| \leq \epsilon.$$

Therefore, $d_\Xi(\ell_k, \rho_k) \to d_\Xi(\ell, \rho)$ as $k \to \infty$. □

It should be noted that this lemma is not satisfied on $b-$metric spaces [30].

5. Main results

This part is devoted to obtaining some FP results in a $G\Xi ML$ space (Ω, d_Ξ) if it meets the criterion given below:

$$d_\Xi(\Upsilon\ell, \Upsilon\rho) \leq \delta^2 G(\ell,\rho),\ \forall \ell,\rho \in \Omega,$$

where

$$G(\ell,\rho) \in \max\left\{d_\Xi(\ell,\rho), d_\Xi(\ell,\Upsilon\ell), d_\Xi(\rho,\Upsilon\rho), \frac{d_\Xi(\ell,\Upsilon\rho)+d_\Xi(\rho,\Upsilon\ell)}{2}\right\}. \tag{5.1}$$

Theorem 5.1. *Let (Ω, d_Ξ) be an e-complete $G\Xi ML$ space, $(\Xi^+)^{SI} \neq \emptyset$ and P be a cone on Ξ. Assume that the mapping $\Upsilon : \Omega \to \Omega$ fulfills a generalized Cirić contractive condition*

$$d_\Xi(\Upsilon\ell, \Upsilon\rho) \leq \delta^2 G(\ell,\rho),\ \forall \ell,\rho \in \Omega, \tag{5.2}$$

where $\delta \in [0, \frac{1}{2})$ and $G(\ell_1, \ell_2)$ is given as (5.1). If $\lim_{k,j\to+\infty} \varpi(\ell_k, \ell_j) < \frac{1}{\delta}$ and $\{\ell_k\} = \{\Upsilon^k \ell_0\}$ is the Picard iteration sequence produced by $\ell_0 \in \Omega$. Then Υ owns a unique FP in Ω.

Proof. Let $\ell_0 \in \Omega$ and create the iterative Picard's sequence $\{\ell_k\}$ by assuming $\ell_1 = \Upsilon\ell_0$, $\ell_2 = \Upsilon\ell_1$, ..., $\ell_k = \Upsilon\ell_{k-1}$, ... If there is $k_0 \in \mathbb{N}$ so that $\ell_{k_0+1} = \Upsilon\ell_{k_0} = \ell_{k_0}$, then ℓ_{k_0} is a FP of Υ and nothing proof. Let's assume, without losing the wider context, $\ell_k \neq \ell_{k+1}$ for all $k \in \mathbb{N}$. By utilizing (5.2), we find that

$$d_\Xi(\ell_{k+1}, \ell_{k+2}) = d_\Xi(\Upsilon\ell_k, \Upsilon\ell_{k+1}) \leq \delta^2 G(\ell_k, \ell_{k+1}), \tag{5.3}$$

where

$$\begin{aligned} G(\ell_k, \ell_{k+1}) &\in \left\{d_\Xi(\ell_k,\ell_{k+1}), d_\Xi(\ell_k,\Upsilon\ell_k), d_\Xi(\ell_{k+1},\Upsilon\ell_{k+1}), \frac{d_\Xi(\ell_k,\Upsilon\ell_{k+1})+d_\Xi(\ell_{k+1},\Upsilon\ell_k)}{2}\right\} \\ &= \left\{d_\Xi(\ell_k,\ell_{k+1}), d_\Xi(\ell_{k+1},\ell_{k+2}), \frac{d_\Xi(\ell_k,\ell_{k+2})+\vartheta_\Xi}{2}\right\} \\ &\leq \left\{d_\Xi(\ell_k,\ell_{k+1}), d_\Xi(\ell_{k+1},\ell_{k+2}), d_\Xi(\ell_k,\ell_{k+2})\right\}. \end{aligned}$$

Now, we consider the cases below for (5.3):

(C_1) If $G(\ell_k, \ell_{k+1}) = d_\Xi(\ell_k, \ell_{k+1})$, we have

$$d_\Xi(\ell_{k+1}, \ell_{k+2}) \leq \delta^2 d_\Xi(\ell_k, \ell_{k+1}).$$

Additionally

$$d_\Xi(\ell_k, \ell_{k+1}) \leq \delta^2 d_\Xi(\ell_{k-1}, \ell_k) \leq \delta^4 d_\Xi(\ell_{k-2}, \ell_{k-1}) \leq \cdots \leq \delta^{2k} d_\Xi(\ell_0, \ell_1). \tag{5.4}$$

Hence, based on the axiom (GM_3) of Definition 3.1 and (5.4), for each $k \in \mathbb{N}$ and for any $r = 1, 2, ...$, we have

$$\begin{aligned} & d_\Xi(\ell_k, \ell_{k+r}) \\ \leq\ & \varpi(\ell_k, \ell_{k+r})\left[d_\Xi(\ell_k, \ell_{k+1}) + d_\Xi(\ell_{k+1}, \ell_{k+r})\right] \\ =\ & \varpi(\ell_k, \ell_{k+r}) d_\Xi(\ell_k, \ell_{k+1}) + \varpi(\ell_k, \ell_{k+r}) d_\Xi(\ell_{k+1}, \ell_{k+r}) \end{aligned}$$

$$
\begin{aligned}
&\leq \varpi(\ell_k,\ell_{k+r})\, d_\Xi(\ell_k,\ell_{k+1}) + \varpi(\ell_k,\ell_{k+r})\,\varpi(\ell_{k+1},\ell_{k+r})\,[d_\Xi(\ell_{k+1},\ell_{k+2}) + d_\Xi(\ell_{k+2},\ell_{k+r})]\\
&= \varpi(\ell_k,\ell_{k+r})\, d_\Xi(\ell_k,\ell_{k+1}) + \varpi(\ell_k,\ell_{k+r})\,\varpi(\ell_{k+1},\ell_{k+r})\, d_\Xi(\ell_{k+1},\ell_{k+2})\\
&\quad +\varpi(\ell_k,\ell_{k+r})\,\varpi(\ell_{k+1},\ell_{k+r})\, d_\Xi(\ell_{k+2},\ell_{k+r})\\
&\leq \cdots\\
&\leq \varpi(\ell_k,\ell_{k+r})\, d_\Xi(\ell_k,\ell_{k+1}) + \varpi(\ell_k,\ell_{k+r})\,\varpi(\ell_{k+1},\ell_{k+r})\, d_\Xi(\ell_{k+1},\ell_{k+2}) + ...\\
&\quad +\varpi(\ell_k,\ell_{k+r})\,\varpi(\ell_{k+1},\ell_{k+r}) ...\varpi(\ell_{k+r-2},\ell_{k+r})\,[d_\Xi(\ell_{k+r-2},\ell_{k+r-1}) + d_\Xi(\ell_{k+r-1},\ell_{k+r})]\\
&\leq \varpi(\ell_k,\ell_{k+r})\,\delta^{2k} d_\Xi(\ell_0,\ell_1) + \varpi(\ell_k,\ell_{k+r})\,\varpi(\ell_{k+1},\ell_{k+r})\,\delta^{2(k+1)} d_\Xi(\ell_0,\ell_1) + ...\\
&\quad +\varpi(\ell_k,\ell_{k+r})\,\varpi(\ell_{k+1},\ell_{k+r}) ...\varpi(\ell_{k+r-2},\ell_{k+r})\,\delta^{2(k+r-1)} d_\Xi(\ell_0,\ell_1)\\
&\leq \left[\sum_{j=k}^{k+p-1} \delta^{2j} \prod_{s=1}^{j} \varpi(\ell_s,\ell_{s+r})\right] d_\Xi(\ell_0,\ell_1) \to \vartheta_\Xi \text{ as } k\to\infty. \qquad (5.5)
\end{aligned}
$$

It follows from Lemma 2.5 and (5.5) that the sequence $\{\ell_k\}$ is an e-Cauchy sequence in Ω.

(C_2) If $G(\ell_k,\ell_{k+1}) = d_\Xi(\ell_{k+1},\ell_{k+2})$, we get

$$
d_\Xi(\ell_{k+1},\ell_{k+2}) \preceq \delta^2 d_\Xi(\ell_{k+1},\ell_{k+2}),
$$

which implies that

$$
\left(1-\delta^2\right) d_\Xi(\ell_{k+1},\ell_{k+2}) \preceq \vartheta_\Xi.
$$

Since $\delta \in [0,\frac{1}{2})$, then $d_\Xi(\ell_{k+1},\ell_{k+2}) = \vartheta_\Xi$. The consequence is inconsistent with our assumption.

(C_3) If $G(\ell_k,\ell_{k+1}) = d_\Xi(\ell_k,\ell_{k+2})$, we have

$$
\begin{aligned}
d_\Xi(\ell_{k+1},\ell_{k+2}) &\preceq \delta^2 d_\Xi(\ell_k,\ell_{k+2})\\
&\preceq \delta^2 \varpi(\ell_k,\ell_{k+2})\,[d_\Xi(\ell_k,\ell_{k+1}) + d_\Xi(\ell_{k+1},\ell_{k+2})]\\
&\preceq \delta\,[d_\Xi(\ell_k,\ell_{k+1}) + d_\Xi(\ell_{k+1},\ell_{k+2})],
\end{aligned}
$$

which leads to

$$
d_\Xi(\ell_{k+1},\ell_{k+2}) \preceq \frac{\delta}{1-\delta} d_\Xi(\ell_k,\ell_{k+1}) = \xi d_\Xi(\ell_k,\ell_{k+1}),
$$

where $0 \leq \frac{\delta}{1-\delta} = \xi < 1$. Furthermore

$$
d_\Xi(\ell_k,\ell_{k+1}) \preceq \xi d_\Xi(\ell_{k-1},\ell_k) \preceq \xi^2 d_\Xi(\ell_{k-2},\ell_{k-1}) \preceq \cdots \preceq \xi^k d_\Xi(\ell_0,\ell_1).
$$

Again, using the axiom (GM_3) of Definition 3.1 and follows the same steps of (5.5), we conclude that

$$
d_\Xi(\ell_k,\ell_{k+1}) \preceq \left[\sum_{j=k}^{k+p-1} \delta^{j} \prod_{s=1}^{j} \varpi(\ell_s,\ell_{s+r})\right] d_\Xi(\ell_0,\ell_1) \to \vartheta_\Xi \text{ as } k\to\infty.
$$

Hence, the sequence $\{\ell_k\}$ is an e-Cauchy sequence in Ω. From the above cases, we obtain that $\{\ell_k\}$ is an e-Cauchy sequence in Ω.

The completeness of Ω leads to there is an element $\ell \in \Omega$ so that

$$
\lim_{k,j\to\infty} d_\Xi\left(\ell_k,\ell_j\right) = \lim_{k\to\infty} d_\Xi(\ell_k,\ell) = d_\Xi(\ell,\ell) = \vartheta_\Xi,
$$

that is, $\{d_\Xi(\ell_k,\ell)\}$ and $\{d_\Xi\left(\ell_k,\ell_j\right)\}$ are e-sequences in Ξ. Now, we prove that Υ has a FP. Using the axiom (GM_3) with the inequality (5.2), one can write

$$\begin{aligned} d_\Xi(\Upsilon\ell,\ell) &\preceq \varpi(\Upsilon\ell,\ell)\left[d_\Xi(\Upsilon\ell,\ell_k)+d_\Xi(\ell,\ell_k)\right] \\ &\preceq \delta d_\Xi(\Upsilon\ell,\Upsilon\ell_{k-1})+\delta d_\Xi(\ell,\ell_k) \\ &\preceq \delta^2 G(\ell,\ell_{k-1})+\delta d_\Xi(\ell,\ell_k) \\ &\preceq \delta G(\ell,\ell_{k-1})+d_\Xi(\ell,\ell_k), \end{aligned} \tag{5.6}$$

where

$$G(\ell,\ell_{k-1})\in\max\left\{d_\Xi(\ell,\ell_{k-1}),d_\Xi(\ell,\Upsilon\ell),d_\Xi(\ell_{k-1},\ell_k),\frac{d_\Xi(\ell,\ell_k)+d_\Xi(\ell_{k-1},\Upsilon\ell)}{2}\right\}.$$

We shall categorize it into four cases in the following:

- If $G(\ell,\ell_{k-1})=d_\Xi(\ell,\ell_{k-1})$, then by (5.6), we get

$$d_\Xi(\Upsilon\ell,\ell)\preceq\delta d_\Xi(\ell,\ell_{k-1})+d_\Xi(\ell,\ell_k).$$

  Using Lemma 2.6 and the fact $\{d_\Xi(\ell_k,\ell)\}$ is an e-sequence, we have $\{\delta d_\Xi(\ell,\ell_{k-1})+d_\Xi(\ell,\ell_k)\}$ is an e-sequence. Based on Lemmas 2.8 and 2.9, cleraly $d_\Xi(\Upsilon\ell,\ell)=\vartheta_\Xi$, i.e., $\Upsilon\ell=\ell$, that is, ℓ is a FP of Υ.
- If $G(\ell,\ell_{k-1})=d_\Xi(\ell,\Upsilon\ell)$, then by (5.6), we have

$$d_\Xi(\Upsilon\ell,\ell)\preceq\delta d_\Xi(\ell,\Upsilon\ell)+d_\Xi(\ell,\ell_k),$$

  which implies that

$$(1-\delta)\,d_\Xi(\Upsilon\ell,\ell)\preceq d_\Xi(\ell,\ell_k).$$

  Because $\{d_\Xi(\ell_k,\ell)\}$ is an e-sequence, then by Lemmas 2.8 and 2.9, we obtain $(1-\delta)\,d_\Xi(\Upsilon\ell,\ell)=\vartheta_\Xi$. Hence $d_\Xi(\Upsilon\ell,\ell)=\vartheta_\Xi$, which leads to $\Upsilon\ell=\ell$.
- If $G(\ell,\ell_{k-1})=d_\Xi(\ell_{k-1},\ell_k)$, then by (5.6), one has

$$d_\Xi(\Upsilon\ell,\ell)\preceq\delta d_\Xi(\ell_{k-1},\ell_k)+d_\Xi(\ell,\ell_k).$$

  It should be noted that $\{\ell_k\}$ is an e-sequence, then $\{d_\Xi(\ell_{k-1},\ell_k)\}$ is an e-sequence. Because $\{d_\Xi(\ell_k,\ell)\}$ is an e-sequence, then by Lemma 2.6, we conclude that $\{\delta d_\Xi(\ell_{k-1},\ell_k)+d_\Xi(\ell,\ell_k)\}$ is an e-sequence. Via Lemmas 2.8 and 2.9, $d_\Xi(\Upsilon\ell,\ell)=0$ that is $\Upsilon\ell=\ell$.
- If $G(\ell,\ell_{k-1})=\frac{d_\Xi(\ell,\ell_k)+d_\Xi(\ell_{k-1},\Upsilon\ell)}{2}$, then by (5.6), one can obtain

$$\begin{aligned} d_\Xi(\Upsilon\ell,\ell) &\preceq \delta\frac{d_\Xi(\ell,\ell_k)+d_\Xi(\ell_{k-1},\Upsilon\ell)}{2}+d_\Xi(\ell,\ell_k) \\ &= \left(1+\frac{\delta}{2}\right)d_\Xi(\ell,\ell_k)+\frac{\delta}{2}d_\Xi(\ell_{k-1},\Upsilon\ell) \\ &\preceq \left(1+\frac{\delta}{2}\right)d_\Xi(\ell,\ell_k)+\frac{\delta}{2}\varpi(\ell_{k-1},\Upsilon\ell)\left[d_\Xi(\ell_{k-1},\ell)+d_\Xi(\ell,\Upsilon\ell)\right] \\ &\preceq \left(1+\frac{\delta}{2}\right)d_\Xi(\ell,\ell_k)+\frac{\delta}{2}.\frac{1}{\delta}\left[d_\Xi(\ell_{k-1},\ell)+d_\Xi(\ell,\Upsilon\ell)\right] \end{aligned}$$

$$\leq \frac{\delta}{2}d_{\Xi}(\ell,\ell_k)+\frac{1}{2}d_{\Xi}(\ell_{k-1},\ell)+\frac{1}{2}d_{\Xi}(\ell,\Upsilon\ell),$$

which implies that

$$d_{\Xi}(\Upsilon\ell,\ell)\leq\delta d_{\Xi}(\ell,\ell_k)+d_{\Xi}(\ell_{k-1},\ell).$$

As $\{d_{\Xi}(\ell_k,\ell)\}$ is an e-sequence, then by Lemma 2.6, we can write $\{\delta d_{\Xi}(\ell,\ell_k)+d_{\Xi}(\ell_{k-1},\ell)\}$ is an e-sequence. Thanks to Lemmas 2.8 and 2.9 for the conclusion that $d_{\Xi}(\Upsilon\ell,\ell)=0$, that is ℓ is a FP of Υ.

For the uniqueness, assume that ℓ^* is another FP of Υ so that $\ell\neq\ell^*$. Based on (5.2), we get

$$d_{\Xi}(\ell,\ell^*)=d_{\Xi}(\Upsilon\ell,\Upsilon\ell^*)\leq\delta^2G(\ell,\ell^*),\ \forall\ell,\ell^*\in\Omega, \tag{5.7}$$

where

$$\begin{aligned}G(\ell,\ell^*)&\in\left\{d_{\Xi}(\ell,\ell^*),d_{\Xi}(\ell,\Upsilon\ell),d_{\Xi}(\ell^*,\Upsilon\ell^*),\frac{d_{\Xi}(\ell,\Upsilon\ell^*)+d_{\Xi}(\ell^*,\Upsilon\ell)}{2}\right\}\\&=\left\{d_{\Xi}(\ell,\ell^*),d_{\Xi}(\ell,\ell),d_{\Xi}(\ell^*,\ell^*),\frac{d_{\Xi}(\ell,\ell^*)+d_{\Xi}(\ell^*,\ell)}{2}\right\}\\&=\{d_{\Xi}(\ell,\ell^*),\vartheta_{\Xi}\}.\end{aligned}$$

Confer two cases about (5.7) as follows:

- If $G(\ell,\ell^*)=d_{\Xi}(\ell,\ell^*)$, then, we have

$$d_{\Xi}(\ell,\ell^*)\leq\delta^2d_{\Xi}(\ell,\ell^*).$$

Since $\delta\in[0,\frac{1}{2})$, then by Lemma 2.10, we obtain that $d_{\Xi}(\ell,\ell^*)=\vartheta_{\Xi}$, that is $\ell=\ell^*$.

- If $G(\ell,\ell^*)=\vartheta_{\Xi}$, then, we get

$$d_{\Xi}(\ell,\ell^*)\leq\vartheta_{\Xi}.$$

From the axiom (GM_1) in Definition 3.1, we conclude that $d_{\Xi}(\ell,\ell^*)=\vartheta_{\Xi}$. Thus, $\ell=\ell^*$.

□

The result below follows immediately from Theorem 5.1.

Corollary 5.2. *Let* (Ω,d_{Ξ}) *be an e-complete* $G\Xi ML$ *space,* $(\Xi^+)^{SI}\neq\emptyset$ *and* P *be a cone on* Ξ. *Suppose that* $\Upsilon:\Omega\to\Omega$ *is a mapping so that*

$$d_{\Xi}(\Upsilon\ell,\Upsilon\rho)\leq\delta^2G^*(\ell,\rho),\ \forall\ell,\rho\in\Omega,$$

where $\delta\in[0,\frac{1}{2})$ *and*

$$G^*(\ell,\rho)\in\max\{d_{\Xi}(\ell,\rho),d_{\Xi}(\ell,\Upsilon\ell),d_{\Xi}(\rho,\Upsilon\rho)\}.$$

If $\lim_{k,j\to+\infty}\varpi(\ell_k,\ell_j)<\frac{1}{\delta}$. *Then* Υ *possess a unique FP in* Ω.

The following examples support Theorem 5.1.

Example 5.3. Let Ω_k be a subset of $\mathbb{R}^2$ endowed that point-wise partial order including the unit disk and $P_n \in \mathbb{R}^2$ is a polygon with the vertices

$$(-1,0),(0,-1),(k,-k),(1,0),(0,1),(-k,k).$$

Define the norm $\|.\|_k$ by

$$\|(\ell,\rho)\|_k(\tau) = e^{2\tau} \times \begin{cases} |\ell_1+\ell_2|^2, & \text{if } \ell\rho \geq 0, \\ \left(\max\{|\ell_1|^2,|\ell_2|^2\} - \frac{k-1}{k}\min\{|\ell_1|^2,|\ell_2|^2\}\right), & \text{if } \ell\rho < 0. \end{cases}$$

Choose a sequence $\mathbf{L} = \{\ell_k\}_{k\in\mathbb{N}}$ in Ξ, where

$$\ell_k = \left(\ell_k^1, \ell_k^2\right) \in \Omega_k, \ \ \|\mathbf{L}\|_k \leq s_{\mathbf{L}}, \ \forall k \in \mathbb{N}.$$

and $s_{\mathbf{L}} > 0$, which depends on $\mathbf{L}$. Assume also Ξ is an ordered space. The cone P can be described by

$$P = \left\{\mathbf{L} = \{\ell_k\} \in \Xi : \ell_k \in \mathbb{R}^+, \ k \in \widetilde{\mathbb{N}}\right\},$$

endowed with the norm

$$\|\mathbf{L}\|_\infty = \sup_{k\in\mathbb{N}} \|\ell_k\|_k.$$

Suppose that $\Omega = P$ is a subspace of Ξ, $d_\Xi : \Omega^2 \to [0,+\infty)$ and $\varpi : \Omega^2 \to [1,+\infty)$ are mappings described as

$$d_\Xi(\mathbf{L},\mathbf{C}) = (\|\mathbf{L}+\mathbf{C}\|_\infty, \|\mathbf{L}+\mathbf{C}\|_\infty) \text{ and } \varpi(\mathbf{L},\mathbf{C}) = 1 + \|\mathbf{L}\|_\infty + \|\mathbf{C}\|_\infty.$$

Putting $\Upsilon\mathbf{L} = \frac{1}{9}\mathbf{L}$, we get

$$d_\Xi(\Upsilon\mathbf{L},\Upsilon\mathbf{C}) = d_\Xi\left(\frac{\mathbf{L}}{9},\frac{\mathbf{C}}{9}\right) = \frac{1}{9}(\|\mathbf{L}+\mathbf{C}\|_\infty, \|\mathbf{L}+\mathbf{C}\|_\infty) = \frac{1}{9}d_\Xi(\mathbf{L},\mathbf{C}).$$

Since

$$\begin{aligned} G(\mathbf{L},\mathbf{C}) &\in \max\left\{d_\Xi(\mathbf{L},\mathbf{C}), d_\Xi(\mathbf{L},\Upsilon\mathbf{L}), d_\Xi(\mathbf{C},\Upsilon\mathbf{C}), \frac{d_\Xi(\mathbf{L},\Upsilon\mathbf{C}) + d_\Xi(\mathbf{C},\Upsilon\mathbf{L})}{2}\right\} \\ &= \max\left\{d_\Xi(\mathbf{L},\mathbf{C}), d_\Xi\left(\mathbf{L},\frac{\mathbf{L}}{9}\right), d_\Xi\left(\mathbf{C},\frac{\mathbf{C}}{9}\right), \frac{d_\Xi\left(\mathbf{L},\frac{\mathbf{C}}{9}\right) + d_\Xi\left(\mathbf{C},\frac{\mathbf{L}}{9}\right)}{2}\right\} \\ &\preceq \max\left\{d_\Xi(\mathbf{L},\mathbf{C}), d_\Xi\left(\mathbf{L},\frac{\mathbf{L}}{9}\right), d_\Xi\left(\mathbf{C},\frac{\mathbf{C}}{9}\right), \frac{d_\Xi(\mathbf{L},\mathbf{C}) + d_\Xi(\mathbf{C},\mathbf{L})}{2}\right\} \\ &= \max\left\{d_\Xi(\mathbf{L},\mathbf{C}), d_\Xi\left(\mathbf{L},\frac{\mathbf{L}}{9}\right), d_\Xi\left(\mathbf{C},\frac{\mathbf{C}}{9}\right)\right\}, \end{aligned}$$

we take $G(\mathbf{L},\mathbf{C}) = d_\Xi(\mathbf{L},\mathbf{C})$, then

$$d_\Xi(\Upsilon\mathbf{L},\Upsilon\mathbf{C}) \preceq \frac{1}{9}d_\Xi(\mathbf{L},\mathbf{C}).$$

Hence, Υ fulfills the stipulation (5.2) of Theorem 5.1 with $\delta = \frac{1}{3} < \frac{1}{2}$, so Υ owns a unique FP.

Example 5.4. Let $\Xi = C([0,1],\mathbb{R})$ be a normed space under the norm $\|\ell\|_{\Xi} = \|\ell\|_{\infty} = \sup_{k\in\mathbb{N}} \|\ell_k\|$. Define a cone $P = \{\ell \in \Xi : \ell(\tau) \geq 0,\ \forall\tau \in [0,1]\}$. Consider $\Omega = \{0,1,2\}$ and describe the mappings $d_{\Xi} : \Omega^2 \to [0,+\infty)$ and $\varpi : \Omega^2 \to [1,+\infty)$ as $d_{\Xi}(0,0)(\tau) = d_{\Xi}(1,1)(\tau) = d_{\Xi}(2,2)(\tau) = \vartheta_{\Xi}$, $d_{\Xi}(0,1)(\tau) = d_{\Xi}(1,0)(\tau) = e^{2\tau}$, $d_{\Xi}(1,2)(\tau) = d_{\Xi}(2,1)(\tau) = 4e^{2\tau}$, $d_{\Xi}(0,2)(\tau) = d_{\Xi}(2,0)(\tau) = 8e^{2\tau}$, for all $\tau \in [0,1]$ and $\varpi(\ell,\rho) = \frac{3}{2} + \ell + \rho$. Then (Ω, d_{Ξ}) is an e-complete $G\Xi ML$ space but not a cone Ξ-metric-like space. Define a mapping $\Upsilon : \Omega \to \Omega$ by $\Upsilon 0 = \Upsilon 1 = 1$, $\Upsilon 2 = 0$. To verify the stipulation (5.2) of Theorem 5.1, the cases below hold:

(i) If $(\ell,\rho) = (0,1)$ or $(1,0)$, we have

$$d_{\Xi}(\Upsilon\ell,\Upsilon\rho) = d_{\Xi}(\Upsilon 0,\Upsilon 1) = d_{\Xi}(1,1) = \vartheta_{\Xi} \preceq \delta^2 G(\ell,\rho),$$

the above inequality is true for any value of δ and $G(\ell,\rho)$.

(ii) If $(\ell,\rho) = (0,2)$ or $(2,0)$, we get

$$\begin{aligned}
d_{\Xi}(\Upsilon\ell,\Upsilon\rho) &= d_{\Xi}(\Upsilon 0,\Upsilon 2) = d_{\Xi}(1,0) = e^{2\tau} \\
&\preceq \frac{1}{5}8e^{2\tau} = \delta^2 \max\left\{8e^{2\tau}, e^{2\tau}, \frac{\vartheta_{\Xi} + 4e^{2\tau}}{2}\right\} \\
&= \delta^2 = \max\left\{d_{\Xi}(0,2), d_{\Xi}(0,1), d_{\Xi}(2,0), \frac{d_{\Xi}(0,0) + d_{\Xi}(2,1)}{2}\right\} \\
&= \max\left\{d_{\Xi}(0,2), d_{\Xi}(0,\Upsilon 0), d_{\Xi}(2,\Upsilon 2), \frac{d_{\Xi}(0,\Upsilon 2) + d_{\Xi}(2,\Upsilon 0)}{2}\right\} \\
&= \delta^2 \max\left\{d_{\Xi}(\ell,\rho), d_{\Xi}(\ell,\Upsilon\ell), d_{\Xi}(\rho,\Upsilon\rho), \frac{d_{\Xi}(\ell,\Upsilon\rho) + d_{\Xi}(\rho,\Upsilon\ell)}{2}\right\}.
\end{aligned}$$

Hence, the condition (5.2) is fulfilled with $\delta = \frac{1}{\sqrt{5}} < 0.5$.

(iii) If $(\ell,\rho) = (1,2)$ or $(2,1)$, one has

$$\begin{aligned}
d_{\Xi}(\Upsilon\ell,\Upsilon\rho) &= d_{\Xi}(\Upsilon 1,\Upsilon 2) = d_{\Xi}(1,0) = e^{2\tau} \\
&\preceq \frac{1}{5}8e^{2\tau} = \delta^2 \max\left\{4e^{2\tau}, \vartheta_{\Xi}, 8e^{2\tau}, e^{2\tau}, \frac{4e^{2\tau} + 8e^{2\tau}}{2}\right\} \\
&= \delta^2 = \max\left\{d_{\Xi}(1,2), d_{\Xi}(1,1), d_{\Xi}(2,0), \frac{d_{\Xi}(1,0) + d_{\Xi}(2,1)}{2}\right\} \\
&= \max\left\{d_{\Xi}(1,2), d_{\Xi}(1,\Upsilon 1), d_{\Xi}(2,\Upsilon 2), \frac{d_{\Xi}(1,\Upsilon 2) + d_{\Xi}(2,\Upsilon 1)}{2}\right\} \\
&= \delta^2 \max\left\{d_{\Xi}(\ell,\rho), d_{\Xi}(\ell,\Upsilon\ell), d_{\Xi}(\rho,\Upsilon\rho), \frac{d_{\Xi}(\ell,\Upsilon\rho) + d_{\Xi}(\rho,\Upsilon\ell)}{2}\right\}.
\end{aligned}$$

Also, the condition (5.2) is fulfilled with $\delta = \frac{1}{\sqrt{5}} < 0.5$.

From the above cases, we conclude that all requirements of Theorem 5.1 are fulfilled and $1 \in M$ is a unique FP of Υ.

According to the notion of $\eta-$admissible functions, we present the following theorem:

Theorem 5.5. *Let (Ω, d_Ξ) be an e-complete GΞML space, $(\Xi^+)^{S1} \neq \emptyset$ and P be a normal cone on Ξ. Assume that $\xi : \mathbb{R}_+ \to [0,1)$ is a function and $\wp : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function. Let $\Upsilon : \Omega \to \Omega$ be an $\eta-$admissible function satisfying*

$$\eta(\ell,\rho)\,\wp(\|d_\Xi(\Upsilon\ell,\Upsilon\rho)\|) \leq \xi\left(\wp\left(\widetilde{G}(\ell,\rho)\right)\right)\wp\left(\alpha^2\widetilde{G}(\ell,\rho)\right),\ \forall \ell,\ \rho \in \Omega, \tag{5.8}$$

where $\widetilde{G}(\ell,\rho) \in \max\{\|d_\Xi(\ell,\rho)\|, \|d_\Xi(\ell,\Upsilon\ell)\|, \|d_\Xi(\rho,\Upsilon\rho)\|\}$ and $\alpha \in [0,1)$. If $\lim_{k,j\to+\infty}\varpi(\ell_k,\ell_j) < \frac{1}{\alpha}$, there is $\ell_0 \in \Omega$ so that $\eta(\ell_0,\Upsilon\ell_0) \geq 1$, and one of the assertions below hold:

(1) Υ is continuous,
(2) Ω is $\eta-$regular,
then Υ admits a FP. Furthermore, this point is a unique if the following axiom is true
(3) For all $\ell,\rho \in \Omega$ there is a $\varkappa \in \Omega$ so that $\eta(\ell,\varkappa) \geq 1$ and $\eta(\rho,\varkappa) \geq 1$.

Proof. According to our hypothesis of the theorem $\ell_0 \in \Omega$ so that $\eta(\ell_0,\Upsilon\ell_0) \geq 1$. We build the sequence $\{\ell_n\}$ as follows: $\ell_1 = \Upsilon\ell_0$, $\ell_2 = \Upsilon\ell_1, ..., \ell_k = \Upsilon\ell_{k-1}$. Because $\eta(\ell_0,\ell_1) = \eta(\ell_0,\Upsilon\ell_0) \geq 1$ and the mapping Υ is an $\eta-$admissible, one has $\eta(\ell_1,\ell_2) = \eta(\Upsilon\ell_0,\Upsilon\ell_1) \geq 1$. In the same scenario, we conclude that $\eta(\ell_k,\ell_{k+1}) \geq 1$. Now, if $\ell_{k_0+1} = \Upsilon\ell_{k_0} = \ell_{k_0}$ for any $k_0 \in \mathbb{N}$, then ℓ_{k_0} is a FP of Υ and the proof stops here. So, we assume that for each $k \in \mathbb{N}$, $\ell_k \neq \ell_{k+1}$. Utilizing (5.8), we get

$$\begin{aligned}
\wp(\|d_\Xi(\ell_{k+1},\ell_{k+2})\|) &\leq \eta(\ell_k,\ell_{k+1})\,\wp(\|d_\Xi(\Upsilon\ell_k,\Upsilon\ell_{k+1})\|) \\
&\leq \xi\left(\wp\left(\widetilde{G}(\ell_k,\ell_{k+1})\right)\right)\wp\left(\alpha^2\widetilde{G}(\ell_k,\ell_{k+1})\right) \\
&\leq \wp\left(\alpha^2\widetilde{G}(\ell_k,\ell_{k+1})\right),
\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}
\widetilde{G}(\ell_k,\ell_{k+1}) &\in \max\{\|d_\Xi(\ell_k,\ell_{k+1})\|, \|d_\Xi(\ell_k,\Upsilon\ell_k)\|, \|d_\Xi(\ell_{k+1},\Upsilon\ell_{k+1})\|\} \\
&= \max\{\|d_\Xi(\ell_k,\ell_{k+1})\|, \|d_\Xi(\ell_{k+1},\ell_{k+2})\|\}.
\end{aligned}$$

For (5.9), we consider two cases below:

- If $\widetilde{G}(\ell_k,\ell_{k+1}) = \|d_\Xi(\ell_k,\ell_{k+1})\|$, then

  $$\wp(\|d_\Xi(\ell_{k+1},\ell_{k+2})\|) \leq \wp\left(\alpha^2\|d_\Xi(\ell_k,\ell_{k+1})\|\right),$$

  the non-decreasing property of $\wp$ implies that

  $$\|d_\Xi(\ell_{k+1},\ell_{k+2})\| \leq \alpha^2\|d_\Xi(\ell_k,\ell_{k+1})\|,$$

  which yields that

  $$\begin{aligned}
  \|d_\Xi(\ell_k,\ell_{k+1})\| &\leq \alpha^2\|d_\Xi(\ell_{k-1},\ell_k)\| \leq \alpha^4\|d_\Xi(\ell_{k-2},\ell_{k-1})\| \\
  &\leq \cdots \leq \alpha^{2k}\|d_\Xi(\ell_0,\ell_1)\|.
  \end{aligned}$$

  Proving the sequence $\{\ell_k\}$ is an e-Cauchy follows immediately from Case (C_1) of the proof of Theorem 5.1. The completeness of Ω implies that there is an element $\ell \in \Omega$ so that

  $$\lim_{k,j\to\infty} d_\Xi\left(\ell_k,\ell_j\right) = \lim_{k\to\infty} d_\Xi(\ell_k,\ell) = d_\Xi(\ell,\ell) = \vartheta_\Xi,$$

  that is, $\{d_\Xi(\ell_k,\ell)\}$ and $\{d_\Xi\left(\ell_k,\ell_j\right)\}$ are e-sequences in Ξ.

- If $\widetilde{G}(\ell_k, \ell_{k+1}) = \|d_\Xi(\ell_{k+1}, \ell_{k+2})\|$, then

$$\wp(\|d_\Xi(\ell_{k+1}, \ell_{k+2})\|) \leq \wp\left(\alpha^2 \|d_\Xi(\ell_{k+1}, \ell_{k+2})\|\right).$$

Since $\wp$ is non-decreasing, then we obtain

$$\|d_\Xi(\ell_{k+1}, \ell_{k+2})\| \leq \alpha^2 \|d_\Xi(\ell_{k+1}, \ell_{k+2})\|,$$

which implies that

$$(1-\alpha^2)\|d_\Xi(\ell_{k+1}, \ell_{k+2})\| \leq \vartheta_\Xi.$$

As $\alpha \in [0,1)$, then $d_\Xi(\ell_{k+1}, \ell_{k+2}) = \vartheta_\Xi$. Clearly $\ell_{k+1} = \ell_{k+2}$, which contradicts our assumption $(\ell_k \neq \ell_{k+1})$.

Now, we shall discuss the existence of the FP for Υ.

(1) If Υ is continuous, then

$$\ell = \lim_{k\to\infty} \ell_{k+1} = \lim_{k\to\infty} \Upsilon\ell_k = \Upsilon\left(\lim_{k\to\infty} \ell_k\right) = \Upsilon\ell,$$

i.e., ℓ is a FP of Υ.

(2) Ω is $\eta-$regular, from (5.8), we can write

$$\begin{aligned} \wp(\|d_\Xi(\Upsilon\ell_k, \Upsilon\ell)\|) &\leq \eta(\ell_k, \ell)\,\wp(\|d_\Xi(\Upsilon\ell_k, \Upsilon\ell)\|) \\ &\leq \xi\left(\wp\left(\widetilde{G}(\ell_k, \ell)\right)\right)\wp\left(\alpha^2\widetilde{G}(\ell_k, \ell)\right) \\ &\leq \wp\left(\alpha^2\widetilde{G}(\ell_k, \ell)\right). \end{aligned}$$

Since $\wp$ is non-decreasing, we get

$$\|d_\Xi(\Upsilon\ell_k, \Upsilon\ell)\| \leq \alpha^2\widetilde{G}(\ell_k, \ell),$$

where

$$\begin{aligned} \widetilde{G}(\ell_k, \ell) &\in \max\{\|d_\Xi(\ell_k, \ell)\|, \|d_\Xi(\ell_k, \Upsilon\ell_k)\|, \|d_\Xi(\ell, \Upsilon\ell)\|\} \\ &= \max\{\|d_\Xi(\ell_k, \ell)\|, \|d_\Xi(\ell_k, \ell_{k+1})\|, \|d_\Xi(\ell, \Upsilon\ell)\|\}. \end{aligned}$$

Now, we discuss the following cases:

(i) If $\widetilde{G}(\ell_k, \ell) = \|d_\Xi(\ell_k, \ell)\|$, we have

$$\|d_\Xi(\ell_{k+1}, \Upsilon\ell)\| = \|d_\Xi(\Upsilon\ell_k, \Upsilon\ell)\| \leq \alpha^2 \|d_\Xi(\ell_k, \ell)\|. \tag{5.10}$$

Passing $k \to \infty$ in (5.10), using Lemma 4.3 and P is a normal cone on Ξ, we have $\|d_\Xi(\ell, \Upsilon\ell)\| = \vartheta_\Xi$, that is, $\ell = \Upsilon\ell$.

(ii) If $\widetilde{G}(\ell_k,\ell) = \|d_\Xi(\ell_k,\Upsilon\ell_k)\|$, we get

$$\|d_\Xi(\ell_{k+1},\Upsilon\ell)\| \le \alpha^2 \|d_\Xi(\ell_k,\ell_{k+1})\|.$$

From the axiom (GM_3) of Definition 3.1, one can write

$$\begin{aligned}\|d_\Xi(\ell_{k+1},\Upsilon\ell)\| &\le \alpha^2 \|d_\Xi(\ell_k,\ell_{k+1})\| \\ &\le \alpha^2 \varpi(\ell_k,\ell_{k+1})[d_\Xi(\ell_k,\ell) + d_\Xi(\ell,\ell_{k+1})]. \end{aligned} \tag{5.11}$$

Letting $k\to\infty$ in (5.11), P is a normal cone on Ξ, using $\lim_{k,j\to+\infty}\varpi(\ell_k,\ell_j) < \frac{1}{\alpha}$ and Lemma 4.3, we get $\|d_\Xi(\ell,\Upsilon\ell)\| = \vartheta_\Xi$, that is, $\ell = \Upsilon\ell$.

(iii) If $\widetilde{G}(\ell_k,\ell) = \|d_\Xi(\ell,\Upsilon\ell)\|$, we obtain

$$\|d_\Xi(\ell_{k+1},\Upsilon\ell)\| \le \alpha^2 \|d_\Xi(\ell,\Upsilon\ell)\|. \tag{5.12}$$

Taking $k\to\infty$ in (5.12), P is a normal cone on Ξ and using Lemma 4.3, we have

$$\|d_\Xi(\ell,\Upsilon\ell)\| \le \alpha^2 \|d_\Xi(\ell,\Upsilon\ell)\|,$$

which implies that

$$(1-\alpha^2)\|d_\Xi(\ell,\Upsilon\ell)\| \le \vartheta_\Xi.$$

Since $\alpha\in[0,1)$, then, we must write $\|d_\Xi(\ell,\Upsilon\ell)\| = \vartheta_\Xi$, that is, $\ell = \Upsilon\ell$.

Based on the three cases above, we conclude that Υ possess a FP $\ell\in\Omega$.

For the uniqueness, assume that the hypothesis (3) of Theorem 5.5 is true and Υ has two distinct FP $\ell,\rho\in\Omega$. From this hypothesis, there is a $\varkappa\in\Omega$ so that

$$\eta(\ell,\varkappa)\ge 1 \text{ and } \eta(\rho,\varkappa)\ge 1. \tag{5.13}$$

As Υ is an $\eta-$admissible, then by (5.13), one can deduce

$$\eta\left(\ell,\Upsilon^k\varkappa\right)\ge 1 \text{ and } \eta\left(\rho,\Upsilon^k\varkappa\right)\ge 1. \tag{5.14}$$

It follows from (5.8) and (5.14) that

$$\begin{aligned}\wp\left(\left\|d_\Xi\left(\Upsilon^{k+1}\varkappa,\Upsilon\ell\right)\right\|\right) &\le \eta\left(\Upsilon^k\varkappa,\ell\right)\wp\left(\left\|d_\Xi\left(\Upsilon^{k+1}\varkappa,\Upsilon\ell\right)\right\|\right) \\ &\le \xi\left(\wp\left(\widetilde{G}\left(\Upsilon^k\varkappa,\ell\right)\right)\right)\wp\left(\alpha^2\widetilde{G}\left(\Upsilon^k\varkappa,\ell\right)\right) \\ &\le \wp\left(\alpha^2\widetilde{G}\left(\Upsilon^k\varkappa,\ell\right)\right). \end{aligned} \tag{5.15}$$

Because $\wp$ is non-decreasing, the inequality (5.14) reduces to

$$\left\|d_\Xi\left(\Upsilon^{k+1}\varkappa,\Upsilon\ell\right)\right\| \le \alpha^2\widetilde{G}\left(\Upsilon^k\varkappa,\ell\right),$$

where

$$\begin{aligned}\widetilde{G}\left(\Upsilon^k\varkappa,\ell\right) &\in \max\left\{\left\|d_\Xi\left(\Upsilon^k\varkappa,\ell\right)\right\|, \left\|d_\Xi\left(\Upsilon^k\varkappa,\Upsilon^{k+1}\varkappa\right)\right\|, \|d_\Xi(\ell,\Upsilon\ell)\|\right\} \\ &= \max\{\|d_\Xi(\varkappa_k,\ell)\|, \|d_\Xi(\varkappa_k,\varkappa_{k+1})\|, \vartheta_\Xi\}.\end{aligned}$$

The proof ends, if we can prove that

$$\lim_{k\to\infty}\varkappa_k = \ell. \tag{5.16}$$

For this regards, we discuss the following cases:

(i) If $\widetilde{G}\left(\Upsilon^k \varkappa, \ell\right) = \|d_\Xi(\varkappa_k, \ell)\|$, we have

$$\|d_\Xi(\varkappa_{k+1}, \ell)\| \le \alpha^2 \|d_\Xi(\varkappa_k, \ell)\| \le \left(\alpha^2\right)^2 \|d_\Xi(\varkappa_{k-1}, \ell)\| \le \cdots \le \left(\alpha^2\right)^k \|d_\Xi(\varkappa, \ell)\|.$$

Passing $k \to \infty$ in the above inequality and since $\alpha \in [0, 1)$, we have (5.16).

(ii) If $\widetilde{G}\left(\Upsilon^k \varkappa, \ell\right) = \|d_\Xi(\varkappa_k, \varkappa_{k+1})\|$, we get

$$\|d_\Xi(\varkappa_{k+1}, \ell)\| \le \alpha^2 \|d_\Xi(\varkappa_k, \varkappa_{k+1})\|. \tag{5.17}$$

It is easy to find that $\{\varkappa_k\}$ (similar to case (c_1) of the proof of Theorem 5.1) is an e-Cauchy sequence. So $\lim_{k\to\infty} \|d_\Xi(\varkappa_{k+1}, \varkappa_k)\| = \vartheta_\Xi$. Thus, by (5.17), one has (5.16).

(iii) If $\widetilde{G}\left(\Upsilon^k \varkappa, \ell\right) = \vartheta_\Xi$, then

$$\|d_\Xi(\varkappa_{k+1}, \ell)\| \le \vartheta_\Xi,$$

which implies (5.16).

In the same method, from (5.8) and (5.14), we obtain

$$\lim_{k\to\infty} \varkappa_k = \rho. \tag{5.18}$$

Combining (5.16) and (5.18), we claim that $\rho = \ell$ and this finishes the proof. □

6. Application to Fredholm integral equation

In this part, we attempt to apply Corollary 5.2 to examine the existence of solution to the following Fredholm integral equation:

$$\ell(\tau) = \int_0^1 R(\tau, z, \ell(z))\, dz, \ \text{for all } \tau, z \in [0, 1], \tag{6.1}$$

where $\ell : [0, 1] \to \mathbb{R}$ and $R : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $\Omega = C^1[0, 1]$ be the set of all continuous functions on $[0, 1]$ equipped with the norm $\|\ell\| = \|\ell\|_\infty + \|\ell'\|_\infty$. Set $P = \{\ell \in \Xi : \ell \ge 0\}$, then $(\Xi^+)^{SI} \ne \emptyset$. Define the mapping $d_\Xi : \Omega^2 \to [0, +\infty)$ and $\varpi : \Omega^2 \to [1, +\infty)$ as

$$d_\Xi(\ell, \rho) = \sup_{\tau\in[0,1]} \left\{ \frac{e^\tau}{2} |\ell(\tau) - \rho(\tau)| \right\} \text{ and } \varpi(\ell, \rho) = 1 + |\ell| + |\rho|, \ \forall \ell, \rho \in \Omega,$$

respectively. Then, (Ω, d_Ξ) is an e-complete $G\Xi ML$ space.

Now, we present and prove our theorem in this part as follows:

Theorem 6.1. *Suppose that for $\ell, \rho \in C[0, 1]$*

$$|R(\tau, z, \ell(z)) - R(\tau, z, \rho(z))| \le \left(\frac{|\ell(\tau) - \rho(\tau)|}{4} \right), \ \forall \tau, z \in [0, 1].$$

Then, the Fredholm integral equation (6.1) has a unique solution on Ω.

Proof. Define the mapping $\Upsilon : \Omega \to \Omega$ by

$$\Upsilon\ell(\tau) = \int_0^1 R(\tau, z, \ell(z))\, dz, \ \forall \tau, z \in [0, 1].$$

Clearly, a unique FP of Υ is equivalent to a unique solution to integral equation (6.1).

Consider

$$\begin{aligned}
\frac{e^\tau}{2} |\Upsilon\ell(\tau) - \Upsilon\rho(\tau)| &= \frac{e^\tau}{2} \left| \int_0^1 (R(\tau, z, \ell(z)) - R(\tau, z, \rho(z)))\, dz \right| \\
&\leq \frac{e^\tau}{2} \int_0^1 |R(\tau, z, \ell(z)) - R(\tau, z, \rho(z))|\, dz \\
&\leq \frac{e^\tau}{2} \int_0^1 \left(\frac{|\ell(\tau) - \rho(\tau)|}{4} \right) dz \\
&= \frac{e^\tau}{4} \left\{ \frac{|\ell(\tau) - \rho(\tau)|}{2} \right\},
\end{aligned}$$

taking the suprimum in the both sides, we have

$$\begin{aligned}
d_\Xi(\Upsilon\ell, \Upsilon\rho)(\tau) &= \sup_{\tau \in [0,1]} \left\{ \frac{e^\tau}{2} |\Upsilon\ell(\tau) - \Upsilon\rho(\tau)| \right\} \\
&\leq \frac{1}{4} \sup_{\tau \in [0,1]} \left\{ \frac{e^\tau}{2} |\ell(\tau) - \rho(\tau)| \right\} \\
&= \delta^2 d_\Xi(\ell, \rho) \\
&\leq \delta^2 G^*(\ell, \rho),
\end{aligned}$$

where $\delta = \frac{1}{2} < 1$. Hence the requirements of Corollary 5.2 are satisfied. Then the considered problem (6.1) has a unique solution on Ω. □

7. Conclusions and open problems

The fixed point technique has assumed a prominent position in non-linear analysis, where it enters into a variety of intriguing and fascinating applications. In order to generalize their findings, many researchers adopted a variety of techniques, either by changing the contractive condition or by extending the scope of the study. So, in this manuscript, a new space was introduced called a $G\Xi ML$ space, which is a mixture of $\Xi-$metric spaces and $b-$metric-like spaces. Topological properties and examples to support it are also presented. As usual, after the space is ready, a mapping is defined under suitable contractive conditions, and then some new results related to the FPs are obtained. Finally, some of the results obtained were applied to the existence of the solution to the Fredholm integral equation as an application. In future work, we will tackle the following problems:

- What would the proofs of theorems look like if $\lim_{k,j\to+\infty} \varpi(\ell_k, \ell_j) < +\infty$?
- What if the definition of mapping in Hausdorff space was changed from single-valued to multi-valued?

- Can the regularity condition be replaced by an equivalent condition?
- Can we define the space under consideration using the Banach algebra?
- Produce comparable results for Kannan, Chatterjee, Reich, Ciric, and Hardy-Rogers contractions.
- Replace the current application in integrodifferential equations, functional eqintegrodifferential equations, and matrix equations with another.

Conflicts of interest

The authors declare that they have no conflicts of interest.

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