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Research article

Statistical inference of the stress-strength reliability for inverse Weibull distribution under an adaptive progressive type-II censored sample

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Abstract: In this paper, we investigate classical and Bayesian estimation of stress-strength reliability $\delta = P(X > Y)$ under an adaptive progressive type-II censored sample. Assume that X and Y are independent random variables that follow inverse Weibull distribution with the same shape but different scale parameters. In classical estimation, the maximum likelihood estimator and asymptotic confidence interval are deduced. An approximate maximum likelihood estimator approach is used to obtain the explicit form. In Bayesian estimation, the Bayesian estimators are derived based on symmetric entropy loss function and LINEX loss function. Due to the complexity of integrals, we proposed Lindley's approximation to get the approximate Bayesian estimates. To compare the different estimators, we performed Monte Carlo simulations. Under gamma prior, the approximate maximum likelihood estimator performs better than Bayesian estimators. Under non-informative prior, the approximate maximum likelihood estimator has the same behavior as Bayesian estimators. In the end, two data sets are used to prove the effectiveness of the proposed methods.

Keywords: approximate maximum likelihood estimation; Bayesian estimation; inverse Weibull distribution; stress-strength reliability
Mathematics Subject Classification: 62F10, 62F15

1. Introduction

The stress-strength model has an essential role in lifetime study and engineering application. In terms of reliability, stress-strength reliability is an interesting topic, which is defined as $\delta = P(X > Y)$, X denotes the strength of a system or unit with stress Y. The system or unit works normally when $X > Y$. Aziz and Chassapis [1] considered the performance $\delta = P(X > Y)$ of a gearing system, which denotes the stress on the gear tooth and X denotes the strength of the tooth root. Dong et al. [2] studied the biomechanical performance $\delta = P(X > Y)$ of the healthy and reconstructed pelvic model, which denotes the strength of the pelvic model and Y indicates daily activities such as knee bending, standing up, stair descent and stair ascent. Zhou et al. [3] studied the effect of the stress-strength ratio and fiber length on the creep properties of polypropylene fiber-reinforced alkali-activated slag concrete.

Since the application of stress-strength reliability is wide, its statistical inference has attracted the concern of many researchers. Mehdi and Mehrdad [4] assumed that strength X has the Pareto distribution within outliers but stress Y follows an unsullied Pareto distribution, and considered the stress-strength reliability estimation. They found that maximum likelihood estimation and the modified maximum likelihood estimation perform better than the method of moments and least squares. Mohamed and Reda [5] proposed a stress-strength model with a type-II censored sample and studied in odd generalized exponential-exponential distribution. They observed that the performance of Bayesian estimation is better than maximum likelihood estimation in terms of mean square error. Based on progressive first failure censored samples, Shi and Shi [6] derived the estimators of stress-strength reliability for beta log Weibull distribution. It can be shown that the Bayesian estimation is better than the maximum likelihood estimation in terms of average absolute bias and mean squared error. For more research on stress-strength reliability, please refer to [7–17].

Inverse Weibull distribution (IWD) is a lifetime distribution commonly employed in reliability analysis, and its application fields include engineering, medicine and so on. Aljeddani and Mohammed [18] proposed that IWD is an effective tool for modeling wind speed characteristics, offering a deep understanding of the density function and cumulative distribution function of wind speed. IWD can also be used for statistical process control. Baklizi and Ghannam [19] proposed a control chart based on the case that the product lifetime obeys the IWD and extended the applicability of the control chart method to the case involving censored lifetime tests. The probability density function (PDF) and cumulative density function (CDF) of IWD are given by Eqs (1) and (2), respectively.

$$f(x;\zeta,\sigma) = \zeta\sigma x^{-\sigma-1}\exp(-\zeta x^{-\sigma}) \;; \quad x > 0, \tag{1}$$

$$F(x;\zeta,\sigma) = \exp(-\zeta x^{-\sigma}) \;; \quad x > 0, \tag{2}$$

where $\zeta > 0$ is the scale parameter and $\sigma > 0$ is the shape parameter. For convenience, denote IWD with PDF (1) as $\mathrm{IW}(\zeta,\sigma)$. In practical production, some hazard rate functions are often non-monotonic. As shown in Figure 1, the hazard rate function (hrf) of IWD exhibits an inverted bathtub shape, making it highly suitable for modeling such data. In the degradation process of diesel engine mechanical parts, Keller and Kanath [20] pointed out that IWD is more suitable for fitting failure data of pistons, crankshafts and main bearings compared to the exponential distribution and Weibull distribution.

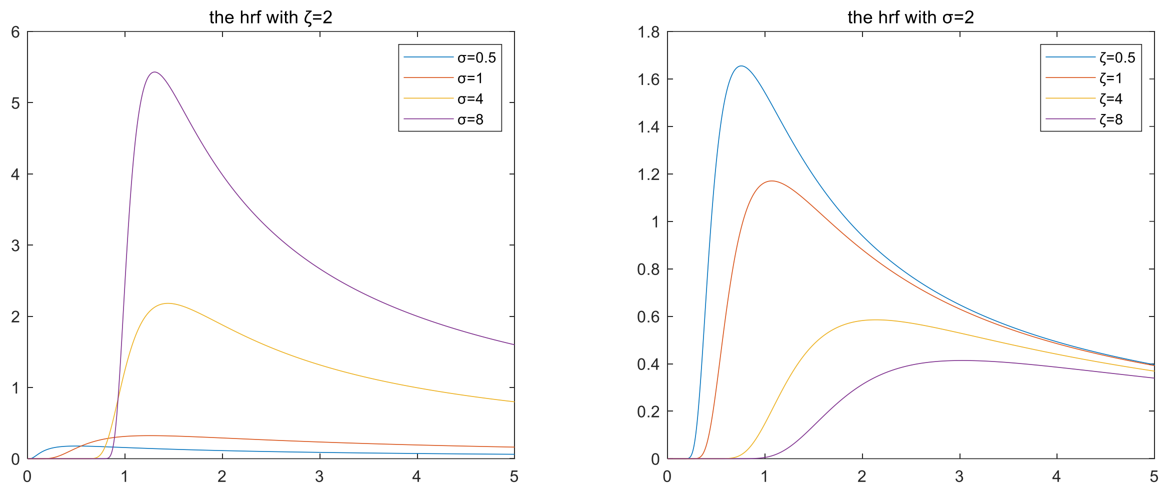

Figure 1. The hazard rate functions of IWD.

In recent years, the statistical inference of IWD has attracted many authors. Alam and Nassar [21] considered the estimation of entropy for IWD based on improved adaptive progressive type-II censored data. Lin et al. [22] considered the estimation of parameters and percentiles for Marshall Olkin extended IWD based on progressive type-II censored data. They found that the least-squares estimation, maximum likelihood estimation and percentiles estimation are not stable. Therefore, Bayesian estimation is focused. Nassar and Ahmed [23] studied the constant stress partial accelerated life test using adaptive progressive type-I censored samples. The research assumes that the life of the product under normal use conditions obeys IWD. The maximum likelihood estimation, the maximum product of the interval process and Bayesian estimation were used to estimate the point and interval estimation of model parameters and acceleration factors. Amany [24] proposed different predictive and reconstructive pivotal quantities for IWD based on dual generalized order statistics. Based on complete samples, Hassan [25] obtained a modified maximum likelihood estimator and confidence intervals of stress-strength reliabilities for IWD by ranked set sampling. Jia et al. [26] discussed the maximum likelihood and Bayesian estimation of the stress-strength model $P(X>Y)$ under the first-failure progressive unified hybrid censored sample, which X and Y were independent random variables from IWD. Based on complete samples, Bi and Gui [27] considered the classical and Bayesian estimation of stress-strength reliability of IWD. Under the adaptive progressive type-II (APT-II) censored samples, Alslman and Helu [28] obtained the maximum likelihood and maximum product of spacing estimators of the stress-strength reliability for IWD. Yadav et al. [29] derived the maximum likelihood estimator and Bayesian estimator of stress-strength reliability for IWD under progressively type-II censoring data.

In the available references, it is still not comprehensive enough in terms of the censored scheme and estimation method. Therefore, we consider the estimation of stress-strength reliability $\delta = P(X>Y)$ under APT-II censored samples, where X and Y are two independent random variables from IWD with the same shape parameter but different scale parameters. The rest of this paper is organized as follows: Section 2 introduces the APT-II censored scheme. Section 3 derives the maximum likelihood estimator (MLE) and asymptotic distribution of δ. Approximate maximum likelihood estimator (AMLE) and asymptotic confidence interval (ACI) are constructed. Section 4 derives the Bayesian estimators (BEs) of δ and approximates them using Lindley's approximation. Section 5 presents the Monte Carlo simulation. In Section 6, the application of the mentioned methods is illustrated by two real datasets. Section 7 contains the conclusions.

2. Adaptive progressive type-II censored scheme

In situations where products have long life spans, obtaining failure time data can be time-consuming and costly. To address this issue, experimenters often employ censored schemes. Two commonly used censored schemes are the progressive type-I censored scheme and progressive type-II censored scheme. The progressive type-I censored scheme involves ending the test at a predetermined time. However, it may result in a small number of observed failures when the product life is long. This can limit the accuracy and efficiency of statistical inference. The progressive type-II censored scheme ends the test after a predetermined number of failures occur. While this scheme ensures a sufficient number of failures are observed, it can lead to prolonged test times, which can be costly and impractical in some cases. Ng et al. [30] developed APT-II censored scheme to address these limitations. In this scheme, the experimenter can not only ensure to observe enough numbers of failures, but also speed up the test process, which greatly improves the efficiency of statistical inference.

Assume that n units are put into the lifetime test. Only m failure units can be observed. A censored scheme $Q=(Q_1,Q_2,...,Q_m)$ satisfies $Q_1+Q_2+...+Q_m+m=n$. Denote the lifetime of the observed failure units by X_i ($i=1,2,...,m$). When the first failure X_1 is observed, Q_1 units are randomly removed from the residual $n-1$ units that have not failed. Similarly, Q_2 units are randomly removed from the remaining $n-Q_1-2$ units at the time of the second failure X_2. When the m time of failure X_m is observed, all the remaining Q_m units are removed. Then, $(X_1,X_2,...,X_m)$ is a set of progressive type-II censored samples.

The APT-II censored scheme is essentially a hybrid of the type-I censored scheme and type-II progressive censored scheme, as detailed in Figures 2 and 3. A desired total test time T is given, but the actual test time is allowed to exceed T as well. If the number of failure units has reached m before time T, the test will be stopped before T. On the contrary, if the test time exceeds T and the failure units observed are less than m, the testers would like to terminate the test as soon as possible. To fulfill this expectation, the testers will make some changes during the test. Ensure that there is enough time to observe m failure units without the actual test time exceeding T too much. Therefore, to terminate the test as soon as possible without changing m, it is necessary to retain more surviving units in the test. The specific situations of the APT-II censored scheme are shown below.

(1) If m failure units have been observed before T, the censored scheme is $Q=(Q_1,Q_2,...,Q_m)$.

(2) Suppose that J $(J<m)$ failure units are observed before time T, that is, $X_J<T<X_{J+1}$. To retain more surviving units in the test, the testers set $Q_{J+1}=Q_{J+2}=...=Q_{m-1}=0$ and $Q_m=n-m-Q_1-Q_2-...-Q_J$.

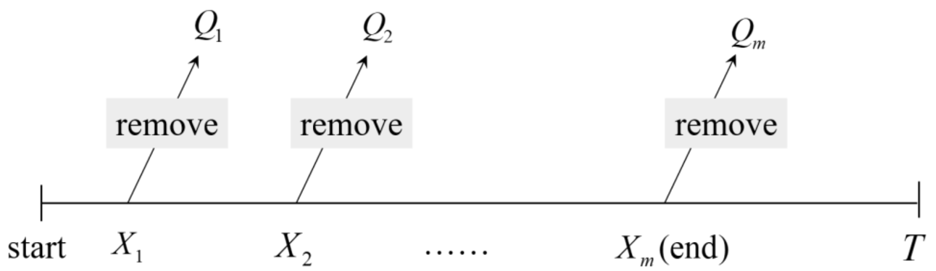

Figure 2. The APT-II censored scheme with the situation (1).

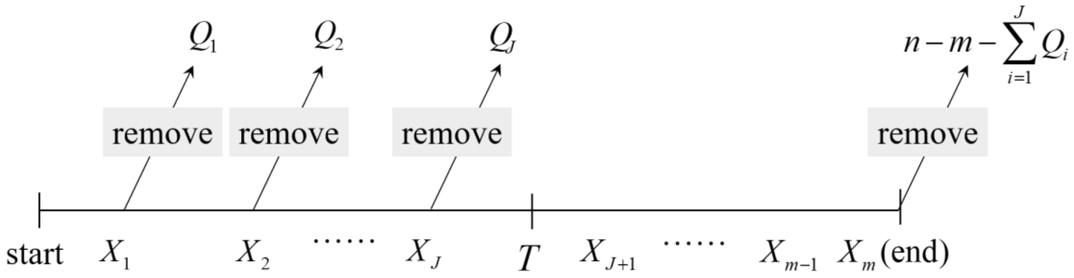

Figure 3. The APT-II censored scheme with the situation (2).

3. Maximum likelihood estimation

3.1. Maximum likelihood estimator

Suppose that X and Y are two independent random variables, where $X \sim \mathrm{IW}(\zeta_1, \sigma)$ and $Y \sim \mathrm{IW}(\zeta_2, \sigma)$. The stress-strength reliability $\delta = P(X > Y)$ is given by

$$\begin{aligned}\delta &= P(X > Y) \\ &= \int_0^{+\infty} f(x;\zeta_1,\sigma) P(Y \le x) dx \\ &= \int_0^{+\infty} f(x;\zeta_1,\sigma) F(x;\zeta_2,\sigma) dx \\ &= \frac{\zeta_1}{\zeta_1+\zeta_2}\end{aligned} \tag{3}$$

Let $X = (X_1, X_2, \ldots, X_m)$ be an APT-II censored sample from $\mathrm{IW}(\zeta_1, \sigma)$ with $X_1 < X_2 < \ldots < X_m$ under censored scheme $Q = (Q_1, \ldots, Q_J, 0, \ldots, 0, Q_m = n_1 - m - \sum_{i=1}^{J} Q_i)$ such that $X_J < T_1 < X_{J+1}$. Let $Y = (Y_1, Y_2, \ldots, Y_t)$ be an APT-II censored sample from $\mathrm{IW}(\zeta_2, \sigma)$ with $Y_1 < Y_2 < \ldots < Y_t$ under censored scheme $R = (R_1, \ldots, R_K, 0, \ldots, 0, R_t = n_2 - t - \sum_{i=1}^{K} R_i)$ such that $Y_K < T_2 < Y_{K+1}$. Denote $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_t)$ as the observation of X and Y, respectively. The joint likelihood function can be written as

$$\begin{aligned}l(\zeta_1,\zeta_2,\sigma;x,y) &= A_1 A_2 [\prod_{i=1}^{m} f_1(x_i)] \prod_{i=1}^{J} [1-F_1(x_i)]^{Q_i} [1-F_1(x_m)]^{Q_m} [\prod_{i=1}^{t} f_2(y_i)] \\ &\quad \prod_{i=1}^{K} [1-F_2(y_i)]^{R_i} [1-F_2(y_t)]^{R_t} \\ &= A_1 A_2 \zeta_1^m \zeta_2^t \sigma^{m+t} \prod_{i=1}^{m} x_i^{-\sigma-1} e^{-\zeta_1 x_i^{-\sigma}} [\prod_{i=1}^{J} (1-e^{-\zeta_1 x_i^{-\sigma}})^{Q_i}] (1-e^{-\zeta_1 x_m^{-\sigma}})^{Q_m} \\ &\quad \prod_{i=1}^{t} y_i^{-\sigma-1} e^{-\zeta_2 y_i^{-\sigma}} [\prod_{i=1}^{K} (1-e^{-\zeta_2 y_i^{-\sigma}})^{R_i}] (1-e^{-\zeta_2 y_t^{-\sigma}})^{R_t}\end{aligned} \tag{4}$$

where

$$A_1 = n_1(n_1 - 1 - Q_1)(n_1 - 2 - Q_1 - Q_2)....(n_1 - m + 1 - \sum_{i=1}^{m-1} Q_i),$$

$$A_2 = n_2(n_2 - 1 - R_1)(n_2 - 2 - R_1 - R_2)....(n_2 - t + 1 - \sum_{i=1}^{t-1} R_i),$$

$$f_1(x) = \zeta_1 \sigma x^{-\sigma-1} e^{-\zeta_1 x^{-\sigma}},$$

$$f_2(y) = \zeta_2 \sigma y^{-\sigma-1} e^{-\zeta_2 y^{-\sigma}},$$

$$F_1(x) = e^{-\zeta_1 x^{-\sigma}},$$

$$F_2(y) = e^{-\zeta_2 y^{-\sigma}}.$$

The log-likelihood function is

$$\begin{aligned} L(\zeta_1, \zeta_2, \sigma; x, y) &= \ln l(\zeta_1, \zeta_2, \sigma; x, y) \\ &= \ln A_1 A_2 + m \ln \zeta_1 + t \ln \zeta_2 + (m+t) \ln \sigma - (\sigma+1) \sum_{i=1}^{m} \ln x_i - \zeta_1 \sum_{i=1}^{m} x_i^{-\sigma} \\ &+ \sum_{i=1}^{J} Q_i \ln(1 - e^{-\zeta_1 x_i^{-\sigma}}) + Q_m \ln(1 - e^{-\zeta_1 x_m^{-\sigma}}) - (\sigma+1) \sum_{i=1}^{t} \ln y_i - \zeta_2 \sum_{i=1}^{t} y_i^{-\sigma} \\ &+ \sum_{i=1}^{K} R_i \ln(1 - e^{-\zeta_2 y_i^{-\sigma}}) + R_t \ln(1 - e^{-\zeta_2 y_t^{-\sigma}}) \end{aligned} \quad (5)$$

The partial derivatives of the log-likelihood function $L(\zeta_1, \zeta_2, \sigma; x, y)$ with respect to ζ_1, ζ_2 and σ are given by

$$\frac{\partial L(\zeta_1, \zeta_2, \sigma; x, y)}{\partial \zeta_1} = \frac{m}{\zeta_1} - \sum_{i=1}^{m} x_i^{-\sigma} + \sum_{i=1}^{J} \frac{Q_i x_i^{-\sigma} \exp(-\zeta_1 x_i^{-\sigma})}{1 - \exp(-\zeta_1 x_i^{-\sigma})} + \frac{Q_m x_m^{-\sigma} \exp(-\zeta_1 x_m^{-\sigma})}{1 - \exp(-\zeta_1 x_m^{-\sigma})}, \quad (6)$$

$$\frac{\partial L(\zeta_1, \zeta_2, \sigma; x, y)}{\partial \zeta_2} = \frac{t}{\zeta_2} - \sum_{i=1}^{t} y_i^{-\sigma} + \sum_{i=1}^{K} \frac{R_i y_i^{-\sigma} \exp(-\zeta_2 y_i^{-\sigma})}{1 - \exp(-\zeta_2 y_i^{-\sigma})} + \frac{R_t y_t^{-\sigma} \exp(-\zeta_2 y_t^{-\sigma})}{1 - \exp(-\zeta_2 y_t^{-\sigma})}, \quad (7)$$

$$\begin{aligned} \frac{\partial L(\zeta_1, \zeta_2, \sigma; x, y)}{\partial \sigma} &= \frac{m+t}{\sigma} + \sum_{i=1}^{m} (\zeta_1 x_i^{-\sigma} \ln x_i - \ln x_i) + \sum_{i=1}^{t} (\zeta_2 y_i^{-\sigma} \ln y_i - \ln y_i) \\ &- \zeta_1 \sum_{i=1}^{J} \frac{Q_i x_i^{-\sigma} \exp(-\zeta_1 x_i^{-\sigma}) \ln x_i}{1 - \exp(-\zeta_1 x_i^{-\sigma})} - \zeta_2 \sum_{i=1}^{K} \frac{R_i y_i^{-\sigma} \exp(-\zeta_2 y_i^{-\sigma}) \ln y_i}{1 - \exp(-\zeta_2 y_i^{-\sigma})}, \\ &- \frac{\zeta_1 Q_m x_m^{-\sigma} \exp(-\zeta_1 x_m^{-\sigma}) \ln x_m}{1 - \exp(-\zeta_1 x_m^{-\sigma})} - \frac{\zeta_2 R_t y_t^{-\sigma} \exp(-\zeta_2 y_t^{-\sigma}) \ln y_t}{1 - \exp(-\zeta_2 y_t^{-\sigma})} \end{aligned} \quad (8)$$

$$\begin{cases} \dfrac{\partial L(\zeta_1,\zeta_2,\sigma;x,y)}{\partial \zeta_1}=0 \\ \dfrac{\partial L(\zeta_1,\zeta_2,\sigma;x,y)}{\partial \zeta_2}=0 \\ \dfrac{\partial L(\zeta_1,\zeta_2,\sigma;x,y)}{\partial \sigma}=0 \end{cases}. \tag{9}$$

The MLEs $\hat{\zeta}_{1,ML}$, $\hat{\zeta}_{2,ML}$ and $\hat{\sigma}_{ML}$ are the solutions of likelihood Eq (9). Considering the nonlinearity, we propose an iteration method to obtain the approximate solutions. Because of the invariance of maximum likelihood estimation, the MLE $\hat{\delta}_{ML}$ of δ can be written as

$$\hat{\delta}_{ML}=\frac{\hat{\zeta}_{1,ML}}{\hat{\zeta}_{1,ML}+\hat{\zeta}_{2,ML}}. \tag{10}$$

3.3. *Approximate maximum likelihood estimator*

Since the explicit form of $\hat{\delta}_{ML}$ cannot be obtained in Section 3.1, we consider the approximate maximum likelihood estimation now.

Let $W=-\ln X$ and $V=-\ln Y$. The CDFs of W and V can be obtained easily.

$$F_W(w)=P(W\le w)=1-P(X\le e^{-w})=1-\exp(-\zeta_1 e^{-\sigma w}) \tag{11}$$

$$F_V(v)=P(V\le v)=1-P(Y\le e^{-v})=1-\exp(-\zeta_2 e^{-\sigma v}) \tag{12}$$

Let $\sigma=-\beta^{-1}$, $\zeta_1=e^{\sigma\alpha_1}$ and $\zeta_2=e^{\sigma\alpha_2}$. The CDFs (11) and (12) can be rewritten as

$$F_W(w)=1-\exp[-\exp(\frac{w-\alpha_1}{\beta})]\ ,\quad w>0\ , \tag{13}$$

$$F_V(v)=1-\exp[-\exp(\frac{v-\alpha_2}{\beta})]\ ,\quad v>0\ . \tag{14}$$

It's obvious that W and V follow the extreme value distribution. Denote that $W\sim \mathrm{EV}(\alpha_1,\beta)$ and $V\sim \mathrm{EV}(\alpha_2,\beta)$. We assume that $w_j=-\ln x_j$ ($j=1,2,...,m$) and $v_k=-\ln y_k$ ($k=1,2,...,t$).

Given the observations $w_1,w_2,...,w_m$ and $v_1,v_2,...,v_t$, the log-likelihood function of α_1, α_2 and β is

$$L_{WV}=A_3-(m+t)\ln\beta+\sum_{j=1}^{m}\omega_j-\sum_{j=1}^{m}e^{\omega_j}-\sum_{j=1}^{J}Q_j e^{\omega_j}-Q_m e^{\omega_m}+\sum_{k=1}^{t}\upsilon_k-\sum_{k=1}^{t}e^{\upsilon_k}-\sum_{k=1}^{K}R_k e^{\upsilon_k}-R_t e^{\upsilon_t}\ , \tag{15}$$

where $\omega_j=\beta^{-1}(w_j-\alpha_1)$, $\upsilon_k=\beta^{-1}(v_k-\alpha_2)$ and A_3 is a constant.

Next, expending the function e^{ω_j} and e^{υ_k} at $\omega_j^0=\ln[-\ln(1-p_j)]$ and $\upsilon_k^0=\ln[-\ln(1-q_k)]$, respectively, and retaining the first derivative.

$$e^{\omega_j}\simeq a_{x,j}+b_{x,j}\omega_j, \tag{16}$$

$$e^{\upsilon_k} \simeq a_{y,k} + b_{y,k}\upsilon_k, \tag{17}$$

where

$$p_j = 1 - \prod_{i=m-j+1}^{m} \frac{i + Q_{m-i+1} + \ldots + Q_m}{i + 1 + Q_{m-i+1} + \ldots + Q_m}\ ,\ \ a_{x,j} = e^{\omega_j^0}(1-\omega_j^0)\ ,\ \ b_{x,j} = e^{\omega_j^0}$$

$$q_k = 1 - \prod_{i=t-k+1}^{t} \frac{i + R_{t-i+1} + \ldots + R_t}{i + 1 + R_{t-i+1} + \ldots + R_t}\ ,\ \ a_{y,k} = e^{\upsilon_k^0}(1-\upsilon_k^0)\ ,\ \ b_{y,k} = e^{\upsilon_k^0}.$$

Thus,

$$\frac{\partial L_{WV}}{\partial \alpha_1} \simeq -\frac{1}{\beta}[m - \sum_{j=1}^{m}(a_{x,j} + b_{x,j}\omega_j) - \sum_{j=1}^{J} Q_j(a_{x,j} + b_{x,j}\omega_j) - Q_m(a_{x,m} + b_{x,m}\omega_m)], \tag{18}$$

$$\frac{\partial L_{WV}}{\partial \alpha_2} \simeq -\frac{1}{\beta}[t - \sum_{k=1}^{t}(a_{y,k} + b_{y,k}\upsilon_k) - \sum_{k=1}^{K} R_k(a_{y,k} + b_{y,k}\upsilon_k) - R_t(a_{y,t} + b_{y,t}\upsilon_t)], \tag{19}$$

$$\begin{aligned}\frac{\partial L_{WV}}{\partial \beta} \simeq -\frac{1}{\beta}[m + t + \sum_{j=1}^{m}\omega_j + \sum_{k=1}^{t}\upsilon_k - \sum_{j=1}^{m}\omega_j(a_{x,j} + b_{x,j}\omega_j) - \sum_{k=1}^{t}\upsilon_k(a_{y,k} + b_{y,k}\upsilon_k) \\ -\sum_{j=1}^{J} Q_j\omega_j(a_{x,j} + b_{x,j}\omega_j) - \sum_{k=1}^{K} R_k\upsilon_k(a_{y,k} + b_{y,k}\upsilon_k) - Q_m\omega_m(a_{x,m} + b_{x,m}\omega_m) \\ -R_t\upsilon_t(a_{y,t} + b_{y,t}\upsilon_t)]\end{aligned}, \tag{20}$$

$$\begin{cases} \dfrac{\partial L_{WV}}{\partial \alpha_1} = 0 \\ \dfrac{\partial L_{WV}}{\partial \alpha_2} = 0 \\ \dfrac{\partial L_{WV}}{\partial \beta} = 0 \end{cases}. \tag{21}$$

The solutions of likelihood Eq (21) are

$$\begin{cases} \hat{\alpha}_1 = (B_x - A_x\hat{\beta})C_x^{-1} \\ \hat{\alpha}_2 = (B_y - A_y\hat{\beta})C_y^{-1} \\ \hat{\beta} = [\sqrt{(DC_x^2 - A_xB_xC_x)^2 - 4mC_x^2(B_x^2C_x - EC_x^2)} + A_xB_xC_x - DC_x^2](2mC_x^2)^{-1} \end{cases}, \tag{22}$$

where

$$A_x = m - \sum_{j=1}^{m} a_{x,j} - \sum_{j=1}^{J} Q_j a_{x,j} - Q_m a_{x,m},$$

$$A_y = t - \sum_{k=1}^{t} a_{y,k} - \sum_{k=1}^{K} R_k a_{y,k} - R_t a_{y,t},$$

$$B_x = \sum_{j=1}^{m} b_{x,j} w_j + \sum_{j=1}^{J} Q_j b_{x,j} w_j + Q_m b_{x,m} w_m,$$

$$B_y = \sum_{k=1}^{t} b_{y,k} v_k + \sum_{k=1}^{K} R_k b_{y,k} v_k + R_t b_{y,t} v_t ,$$

$$C_x = \sum_{j=1}^{m} b_{x,j} + \sum_{j=1}^{J} Q_j b_{x,j} + Q_m b_{x,m} ,$$

$$C_y = \sum_{k=1}^{t} b_{y,k} + \sum_{k=1}^{K} R_k b_{y,k} + R_t b_{y,t} ,$$

$$D = \sum_{j=1}^{m} w_j - \sum_{j=1}^{m} a_{x,j} w_j - \sum_{j=1}^{J} Q_j a_{x,j} w_j - Q_m a_{x,m} w_m ,$$

$$E = \sum_{j=1}^{m} b_{x,j} w_j^2 + \sum_{j=1}^{J} Q_j b_{x,j} w_j^2 + Q_m b_{x,m} w_m^2 .$$

Hence, the AMLE of δ is given by

$$\hat{\delta}_{AML} = \frac{\hat{\zeta}_{1,AML}}{\hat{\zeta}_{1,AML} + \hat{\zeta}_{2,AML}} , \tag{23}$$

where

$$\hat{\sigma}_{AML} = -\hat{\beta} , \quad \hat{\zeta}_{1,AML} = \exp(\hat{\sigma}_{AML} \hat{\alpha}_1) , \quad \hat{\zeta}_{2,AML} = \exp(\hat{\sigma}_{AML} \hat{\alpha}_2) .$$

3.3. Asymptotic confidence interval

It can be seen from Section 3.1 that the MLE of δ cannot be given in an explicit form. Therefore, we cannot construct the exact confidence interval. Based on the asymptotically normal property of maximum likelihood estimation, we construct the ACI of δ in this subsection.

Denote $\theta = (\zeta_1, \zeta_2, \sigma)$ and $\hat{\theta}_{ML} = (\hat{\zeta}_{1,ML}, \hat{\zeta}_{2,ML}, \hat{\sigma}_{ML})$. The observed Fisher information matrix can be expressed as

$$\mathbf{H} = \begin{bmatrix} -H_{11}(\hat{\theta}_{ML}) & -H_{12}(\hat{\theta}_{ML}) & -H_{13}(\hat{\theta}_{ML}) \\ -H_{21}(\hat{\theta}_{ML}) & -H_{22}(\hat{\theta}_{ML}) & -H_{23}(\hat{\theta}_{ML}) \\ -H_{31}(\hat{\theta}_{ML}) & -H_{32}(\hat{\theta}_{ML}) & -H_{33}(\hat{\theta}_{ML}) \end{bmatrix} . \tag{24}$$

Here,

$$H_{11}(\theta) = \frac{\partial^2 L(\zeta_1, \zeta_2, \sigma; x, y)}{\partial \zeta_1^2} = -\frac{m}{\zeta_1} - \sum_{i=1}^{J} \frac{Q_i x_i^{-2\sigma} e^{-\zeta_1 x_i^{-\sigma}}}{(1 - e^{-\zeta_1 x_i^{-\sigma}})^2} - \frac{Q_m x_m^{-2\sigma} e^{-\zeta_1 x_m^{-\sigma}}}{(1 - e^{-\zeta_1 x_m^{-\sigma}})^2} ,$$

$$H_{22}(\theta) = \frac{\partial^2 L(\zeta_1, \zeta_2, \sigma; x, y)}{\partial \zeta_2^2} = -\frac{t}{\zeta_2} - \sum_{i=1}^{K} \frac{R_i y_i^{-2\sigma} e^{-\zeta_2 y_i^{-\sigma}}}{(1 - e^{-\zeta_2 y_i^{-\sigma}})^2} - \frac{R_t y_t^{-2\sigma} e^{-\zeta_2 y_t^{-\sigma}}}{(1 - e^{-\zeta_2 y_t^{-\sigma}})^2} ,$$

$$H_{13}(\theta) = \frac{\partial^2 L(\zeta_1, \zeta_2, \sigma; x, y)}{\partial \zeta_1 \partial \sigma} = \sum_{i=1}^{m} x_i^{-\sigma} \ln x_i + \zeta_1 \sum_{i=1}^{J} \frac{Q_i x_i^{-\sigma} e^{-\zeta_1 x_i^{-\sigma}} \ln x_i}{(1 - e^{-\zeta_1 x_i^{-\sigma}})^2} + \frac{\zeta_1 Q_m x_m^{-\sigma} e^{-\zeta_1 x_m^{-\sigma}} \ln x_m}{(1 - e^{-\zeta_1 x_m^{-\sigma}})^2} ,$$

$$H_{23}(\theta)=\frac{\partial^2 L(\zeta_1,\zeta_2,\sigma;x,y)}{\partial\zeta_2\partial\sigma}=\sum_{i=1}^{t} y_i^{-\sigma}\ln y_i+\zeta_2\sum_{i=1}^{K}\frac{R_i y_i^{-\sigma}e^{-\zeta_2 y_i^{-\sigma}}\ln y_i}{(1-e^{-\zeta_2 y_i^{-\sigma}})^2}+\frac{\zeta_2 R_t y_t^{-\sigma}e^{-\zeta_2 y_t^{-\sigma}}\ln y_t}{(1-e^{-\zeta_2 y_t^{-\sigma}})^2},$$

$$\begin{aligned}H_{33}(\theta)&=\frac{\partial^2 L(\zeta_1,\zeta_2,\sigma;x,y)}{\partial\sigma^2}\\&=-\frac{m+t}{\sigma^2}-\zeta_1\sum_{i=1}^{m}x_i^{-\sigma}(\ln x_i)^2-\zeta_2\sum_{i=1}^{t}y_i^{-\sigma}(\ln y_i)^2+\zeta_1\sum_{i=1}^{J}Q_i x_i^{-\sigma}e^{-\zeta_1 x_i^{-\sigma}}(\ln x_i)^2\\&+\zeta_2\sum_{i=1}^{K}R_i y_i^{-\sigma}e^{-\zeta_2 y_i^{-\sigma}}(\ln y_i)^2+\zeta_1 Q_m x_m^{-\sigma}e^{-\zeta_1 x_m^{-\sigma}}(\ln x_m)^2+\zeta_2 R_t y_t^{-\sigma}e^{-\zeta_2 y_t^{-\sigma}}(\ln y_t)^2,\\&-\zeta_1^2\sum_{i=1}^{J}\frac{Q_i x_i^{-2\sigma}e^{-\zeta_1 x_i^{-\sigma}}(\ln x_i)^2(1+e^{-\zeta_1 x_i^{-\sigma}})}{1-e^{-\zeta_1 x_i^{-\sigma}}}-\frac{\zeta_1^2 Q_m x_m^{-2\sigma}e^{-\zeta_1 x_m^{-\sigma}}(\ln x_m)^2(1+e^{-\zeta_1 x_m^{-\sigma}})}{1-e^{-\zeta_1 x_m^{-\sigma}}}\\&-\zeta_2^2\sum_{i=1}^{K}\frac{R_i y_i^{-2\sigma}e^{-\zeta_2 y_i^{-\sigma}}(\ln y_i)^2(1+e^{-\zeta_2 y_i^{-\sigma}})}{1-e^{-\zeta_2 y_i^{-\sigma}}}-\frac{\zeta_2^2 R_t y_t^{-2\sigma}e^{-\zeta_2 y_t^{-\sigma}}(\ln y_t)^2(1+e^{-\zeta_2 y_t^{-\sigma}})}{1-e^{-\zeta_2 y_t^{-\sigma}}}\end{aligned}$$

$$H_{12}(\theta)=H_{21}(\theta)=0,\quad H_{31}(\theta)=H_{13}(\theta),\quad H_{32}(\theta)=H_{23}(\theta).$$

Next, the Delta method is used to derive the ACI of δ. Let $\phi=(\phi_1(\hat{\theta}_{ML}),\phi_2(\hat{\theta}_{ML}),\phi_3(\hat{\theta}_{ML}))^{\mathrm{T}}$, and

$$\phi_1(\theta)=\frac{\partial\delta}{\partial\zeta_1}=\frac{\zeta_2}{(\zeta_1+\zeta_2)^2},\quad \phi_2(\theta)=\frac{\partial\delta}{\partial\zeta_2}=\frac{-\zeta_1}{(\zeta_1+\zeta_2)^2},\quad \phi_3(\theta)=\frac{\partial\delta}{\partial\sigma}=0.$$

According to the Delta method, the estimate of variance $\mathrm{Var}(\hat{\delta}_{ML})$ is approximated by Eq (25), where $\mathbf{H}^{-1}$ is the inverse matrix of Fisher information matrix $\mathbf{H}$.

$$\mathrm{Var}(\hat{\delta}_{ML})=\phi^{\mathrm{T}}\mathbf{H}^{-1}\phi. \tag{25}$$

Then, the $100(1-\lambda)\%$ ACI of δ is present by Eq (26), where $z_{\frac{\lambda}{2}}$ is the upper $\frac{\lambda}{2}$ th quantile of the standardized normal distribution.

$$\left(\hat{\delta}_{ML}-z_{\frac{\lambda}{2}}\sqrt{\mathrm{Var}(\hat{\delta}_{ML})},\ \hat{\delta}_{ML}+z_{\frac{\lambda}{2}}\sqrt{\mathrm{Var}(\hat{\delta}_{ML})}\right). \tag{26}$$

4. Bayesian estimation

In this section, we assume that ζ_1 and ζ_2 are independent random variables and follow gamma priors. The BEs of ζ_1 and ζ_2 are derived under symmetric entropy loss function and LINEX loss function.

In Bayesian estimation, selecting prior distribution for unknown parameter is a significant matter. First, the gamma prior is versatile for adjusting different shapes of the distribution density function.

Second, the gamma prior is relatively simple, and there will not be too complicated computational issues. Its advantage is to provide conjugacy and mathematical ease. As a result, we investigate the gamma prior. Then, the prior distributions of ζ_1 and ζ_2 are given as

$$\pi(\zeta_1) \propto \zeta_1^{a_1-1}\exp(-b_1\zeta_1), \quad a_1, b_1 > 0, \tag{27}$$

$$\pi(\zeta_2) \propto \zeta_2^{a_2-1}\exp(-b_2\zeta_2), \quad a_2, b_2 > 0. \tag{28}$$

Denote that $\zeta_1 \sim G(a_1, b_1)$ and $\zeta_2 \sim G(a_2, b_2)$. The joint prior is

$$\pi(\zeta_1, \zeta_2) \propto \zeta_1^{a_1-1}\zeta_2^{a_2-1}\exp(-b_1\zeta_1 - b_2\zeta_2). \tag{29}$$

Therefore, the joint posterior distribution given observation data is

$$\begin{aligned}\pi(\zeta_1, \zeta_2, \sigma \mid x, y) = A_4\zeta_1^{m+a_1-1}\zeta_2^{t+a_2-1}\sigma^{m+t}e^{-b_1\zeta_1-b_2\zeta_2}\prod_{i=1}^{m}x_i^{-\sigma-1}e^{-\zeta_1 x_i^{-\sigma}}\prod_{i=1}^{t}y_i^{-\sigma-1}e^{-\zeta_2 y_i^{-\sigma}} \\ \prod_{i=1}^{J}(1-e^{-\zeta_1 x_i^{-\sigma}})^{Q_i}[\prod_{i=1}^{K}(1-e^{-\zeta_2 y_i^{-\sigma}})^{R_i}](1-e^{-\zeta_1 x_m^{-\sigma}})^{Q_m}(1-e^{-\zeta_2 y_t^{-\sigma}})^{R_t}\end{aligned}, \tag{30}$$

and $A_4^{-1} = \iiint \pi(\zeta_1, \zeta_2, \sigma \mid x, y) l(\zeta_1, \zeta_2, \sigma; x, y) d\zeta_1 d\zeta_2 d\sigma$.

Let $\hat{\rho}$ be the estimator of ρ. The symmetric entropy loss function (Xu et al. [31]) and LINEX loss function (Varian [32]) are defined as

$$L_S(\rho, \hat{\rho}) = \frac{\hat{\rho}}{\rho} + \frac{\rho}{\hat{\rho}} - 2, \tag{31}$$

$$L_E(\rho, \hat{\rho}) = \exp[d(\hat{\rho} - \rho)] - d(\hat{\rho} - \rho) - 1, \tag{32}$$

where d is the hype-parameter of LINEX loss function. Given observations x and y, the BEs of ρ under symmetric entropy loss function and LINEX loss function are presented by Eqs (33) and (34), where $\mathrm{E}(\cdot \mid x, y)$ denotes the posterior expectation.

$$\hat{\rho}_S = [\frac{\mathrm{E}(\rho \mid x, y)}{\mathrm{E}(\rho^{-1} \mid x, y)}]^{\frac{1}{2}} \tag{33}$$

$$\hat{\rho}_E = -\frac{1}{d}\ln[\mathrm{E}(e^{-d\rho} \mid x, y)] \tag{34}$$

Thus, based on APT-II censored samples, the BE $\hat{\delta}_S$ of δ under symmetric entropy loss function is given by

$$\hat{\delta}_S = [\frac{\mathrm{E}(\delta \mid x, y)}{\mathrm{E}(\delta^{-1} \mid x, y)}]^{\frac{1}{2}}$$
$$= [\frac{\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \delta \pi(\zeta_1, \zeta_2, \sigma \mid x, y) d\zeta_1 d\zeta_2 d\sigma}{\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \delta^{-1} \pi(\zeta_1, \zeta_2, \sigma \mid x, y) d\zeta_1 d\zeta_2 d\sigma}]^{\frac{1}{2}}$$
$$= \{(\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \zeta_1^{m+a_1} \zeta_2^{t+a_2-1} (\zeta_1+\zeta_2)^{-1} \sigma^{m+t} e^{-b_1\zeta_1 - b_2\zeta_2} \prod_{i=1}^{m} x_i^{-\sigma-1} e^{-\zeta_1 x_i^{-\sigma}} \prod_{i=1}^{t} y_i^{-\sigma-1} e^{-\zeta_2 y_i^{-\sigma}} \quad (35)$$
$$\prod_{i=1}^{J} (1-e^{-\zeta_1 x_i^{-\sigma}})^{Q_i} [\prod_{i=1}^{K} (1-e^{-\zeta_2 y_i^{-\sigma}})^{R_i}] (1-e^{-\zeta_1 x_m^{-\sigma}})^{Q_m} (1-e^{-\zeta_2 y_t^{-\sigma}})^{R_t} d\zeta_1 d\zeta_2 d\sigma)$$
$$[\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \zeta_1^{m+a_1-2} \zeta_2^{t+a_2-1} (\zeta_1+\zeta_2) \sigma^{m+t} e^{-b_1\zeta_1 - b_2\zeta_2} \prod_{i=1}^{m} x_i^{-\sigma-1} e^{-\zeta_1 x_i^{-\sigma}} \prod_{i=1}^{t} y_i^{-\sigma-1} e^{-\zeta_2 y_i^{-\sigma}}$$
$$\prod_{i=1}^{J} (1-e^{-\zeta_1 x_i^{-\sigma}})^{Q_i} [\prod_{i=1}^{K} (1-e^{-\zeta_2 y_i^{-\sigma}})^{R_i}] (1-e^{-\zeta_1 x_m^{-\sigma}})^{Q_m} (1-e^{-\zeta_2 y_t^{-\sigma}})^{R_t} d\zeta_1 d\zeta_2 d\sigma]^{-1}\}^{\frac{1}{2}}$$

The BE $\hat{\delta}_E$ of δ under LINEX loss function is given by

$$\hat{\delta}_E = -\frac{1}{d} \ln[\mathrm{E}(e^{-d\delta} \mid x, y)]$$
$$= -\frac{1}{d} \ln[A_4 \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \zeta_1^{m+a_1-1} \zeta_2^{t+a_2-1} \sigma^{m+t} e^{-(b_1+d)\zeta_1 - b_2\zeta_2 - d(\zeta_1+\zeta_2)^{-1}} \prod_{i=1}^{m} x_i^{-\sigma-1} e^{-\zeta_1 x_i^{-\sigma}} \quad (36)$$
$$\prod_{i=1}^{t} y_i^{-\sigma-1} e^{-\zeta_2 y_i^{-\sigma}} \prod_{i=1}^{J} (1-e^{-\zeta_1 x_i^{-\sigma}})^{Q_i} [\prod_{i=1}^{K} (1-e^{-\zeta_2 y_i^{-\sigma}})^{R_i}] (1-e^{-\zeta_1 x_m^{-\sigma}})^{Q_m} (1-e^{-\zeta_2 y_t^{-\sigma}})^{R_t} d\zeta_1 d\zeta_2 d\sigma]$$

It can be seen that both Eqs (35) and (36) involve the ratio of two integrals, and the form of integral is complex. Hence, we use Lindley's approximation (Lindley [33]) to compute the approximate Bayesian estimates. Lindley's approximation provides a method to obtain an approximation of the posterior expectation like the following form.

$$\mathrm{E}[\eta(\zeta_1, \zeta_2, \sigma) \mid x, y] = \frac{\int \eta(\zeta_1, \zeta_2, \sigma) e^{L(\zeta_1, \zeta_2, \sigma; x, y) + \pi^*(\zeta_1, \zeta_2, \sigma)} d(\zeta_1, \zeta_2, \sigma)}{\int e^{L(\zeta_1, \zeta_2, \sigma; x, y) + \pi^*(\zeta_1, \zeta_2, \sigma)} d(\zeta_1, \zeta_2, \sigma)}. \quad (37)$$

In Eq (37), $\eta(\zeta_1, \zeta_2, \sigma)$ is a function of ζ_1, ζ_2 and σ, and $\pi^*(\zeta_1, \zeta_2, \sigma) = \ln \pi(\zeta_1, \zeta_2, \sigma)$. According to Lindley's approximation, the form of posterior expectation (37) can be rewritten as

$$\mathrm{E}[\eta(\zeta_1, \zeta_2, \sigma) \mid x, y] = \eta + \frac{1}{2}[(\eta_{11} + 2\eta_1\pi_1^*)\varphi_{11} + (\eta_{21} + 2\eta_2\pi_1^*)\hat{\varphi}_{21} + (\eta_{12} + 2\eta_1\pi_2^*)\varphi_{12}$$
$$+(\eta_{22} + 2\eta_2\pi_2^*)\varphi_{22} + (\eta_1\varphi_{11} + \eta_2\varphi_{12})(L_{111}\varphi_{11} + L_{121}\varphi_{12} + L_{211}\varphi_{21} + L_{221}\varphi_{22}), \quad (38)$$
$$+(\eta_1\varphi_{21} + \eta_2\varphi_{22})(L_{112}\varphi_{11} + L_{122}\varphi_{12} + L_{212}\varphi_{21} + L_{222}\varphi_{22})]$$

where

$$L_{111} = \frac{\partial^3 L(\zeta_1, \zeta_2, \sigma; x, y)}{\partial \zeta_1^3} = \frac{2m}{\zeta_1^3} + \sum_{i=1}^{J} \frac{Q_i x_i^{-3\sigma} e^{-\zeta_1 x_i^{-\sigma}}}{(1-e^{-\zeta_1 x_i^{-\sigma}})^3} + \frac{Q_m x_m^{-3\sigma} e^{-\zeta_1 x_m^{-\sigma}}}{(1-e^{-\zeta_1 x_m^{-\sigma}})^3},$$

$$L_{222}=\frac{\partial^3 L(\zeta_1,\zeta_2,\sigma;x,y)}{\partial\zeta_2^3}=\frac{2mt}{\zeta_2^3}+\sum_{i=1}^{K}\frac{R_i y_i^{-3\sigma}e^{-\zeta_2 y_i^{-\sigma}}}{(1-e^{-\zeta_2 y_i^{-\sigma}})^3}+\frac{R_t y_t^{-3\sigma}e^{-\zeta_2 y_t^{-\sigma}}}{(1-e^{-\zeta_2 y_t^{-\sigma}})^3},$$

$$\pi_1^*=\frac{a_1-1}{\zeta_1}-b_1\ ,\quad \pi_2^*=\frac{a_2-1}{\zeta_2}-b_2\ ,$$

$$L_{121}=L_{211}=L_{221}=L_{112}=L_{122}=L_{212}=0\ ,$$

$$\varphi=\begin{bmatrix}-\frac{\partial^2 L(\zeta_1,\zeta_2,\sigma;x,y)}{\partial\zeta_1^2} & 0\\ 0 & -\frac{\partial^2 L(\zeta_1,\zeta_2,\sigma;x,y)}{\partial\zeta_2^2}\end{bmatrix}^{-1},$$

and φ_{ij} $(i,j=1,2)$ is the element of φ.

Under symmetric entropy loss function, we need to approximate $\mathrm{E}(\delta\,|\,x,y)$ and $\mathrm{E}(\delta^{-1}\,|\,x,y)$ referring Eq (38). Let $\eta=\eta(\zeta_1,\zeta_2)$ be a function of ζ_1 and ζ_2, and we denote

$$\eta_1=\frac{\partial\eta}{\partial\zeta_1},\eta_2=\frac{\partial\eta}{\partial\zeta_2},\eta_{11}=\frac{\partial^2\eta}{\partial\zeta_1^2},\eta_{22}=\frac{\partial^2\eta}{\partial\zeta_2^2},\eta_{12}=\frac{\partial^2\eta}{\partial\zeta_1\partial\zeta_2}\ ,\eta_{21}=\frac{\partial^2\eta}{\partial\zeta_2\partial\zeta_1}.$$

When the function mentioned in Eq (37) is $\eta=\zeta_1(\zeta_1+\zeta_2)^{-1}$, the partial derivatives are

$$\eta_1=\frac{\zeta_2}{(\zeta_1+\zeta_2)^2}\ ,\quad \eta_2=\frac{-\zeta_1}{(\zeta_1+\zeta_2)^2},$$
$$\eta_{11}=\frac{-2\zeta_2}{(\zeta_1+\zeta_2)^3}\ ,\quad \eta_{12}=\frac{\zeta_1-\zeta_2}{(\zeta_1+\zeta_2)^3}\ ,\quad \eta_{22}=\frac{2\zeta_1}{(\zeta_1+\zeta_2)^3}\ ,\quad \eta_{21}=\eta_{12}. \tag{39}$$

Therefore,

$$\mathrm{E}(\delta\,|\,x,y)=\frac{\zeta_1}{\zeta_1+\zeta_2}+[\frac{-\zeta_2}{(\zeta_1+\zeta_2)^3}+\frac{\zeta_2\pi_1^*}{(\zeta_1+\zeta_2)^2}]\varphi_{11}+[\frac{\zeta_1}{(\zeta_1+\zeta_2)^3}-\frac{\zeta_1\pi_2^*}{(\zeta_1+\zeta_2)^2}]\varphi_{22}$$
$$+\frac{1}{2}[(\frac{\zeta_2}{(\zeta_1+\zeta_2)^2}\varphi_{11}^2 L_{111}-\frac{\zeta_1}{(\zeta_1+\zeta_2)^2}\varphi_{22}^2 L_{222}]\ . \tag{40}$$

When the function mentioned in Eq (37) is $\eta=\zeta_1^{-1}(\zeta_1+\zeta_2)$, the partial derivatives are

$$\eta_1=\frac{-\zeta_2}{\zeta_1^2}\ ,\ \eta_2=\frac{1}{\zeta_1}\ ,\ \eta_{11}=\frac{2\zeta_2}{\zeta_1^3}\ ,\ \eta_{12}=\frac{-1}{\zeta_1^2}\ ,\ \eta_{22}=0\ ,\ \eta_{21}=\eta_{12}\,. \tag{41}$$

Therefore,

$$\mathrm{E}(\delta^{-1}\,|\,x,y)=1+\frac{\zeta_2}{\zeta_1}+(\frac{\zeta_2}{\zeta_1^3}-\frac{\zeta_2}{\zeta_1^2}\pi_1^*)\varphi_{11}+\frac{1}{\zeta_1}\pi_2^*\varphi_{22}+\frac{1}{2}(\frac{1}{\zeta_1}L_{222}\varphi_{22}^2-\frac{\zeta_2}{\zeta_1^2}\varphi_{11}^2 L_{111})\,. \tag{42}$$

The BE $\hat{\delta}_S$ is given by Eq (43).

$$\hat{\delta}_S = [\frac{\mathrm{E}(\delta \mid x, y)}{\mathrm{E}(\delta^{-1} \mid x, y)}]^{\frac{1}{2}} |_{(\zeta_1, \zeta_2, \sigma)=(\hat{\zeta}_{1,ML}, \hat{\zeta}_{2,ML}, \hat{\sigma}_{ML})} . \tag{43}$$

Under the LINEX loss function, we only need to approximate $\mathrm{E}(e^{-d\delta} \mid x, y)$. When $\eta = \exp(\frac{-d\zeta_1}{\zeta_1 + \zeta_2})$, the partial derivatives are

$$\begin{aligned} &\eta_1 = \frac{-d\zeta_2}{(\zeta_1+\zeta_2)^2}\exp(\frac{-d\zeta_1}{\zeta_1+\zeta_2}) , \quad \eta_2 = \frac{d\zeta_1}{(\zeta_1+\zeta_2)^2}\exp(\frac{-d\zeta_1}{\zeta_1+\zeta_2}) , \\ &\eta_{11} = [\frac{2d\zeta_2}{(\zeta_1+\zeta_2)^3} + \frac{d^2\zeta_2^2}{(\zeta_1+\zeta_2)^4}]\exp(\frac{-d\zeta_1}{\zeta_1+\zeta_2}) , \\ &\eta_{22} = [\frac{-2d\zeta_1}{(\zeta_1+\zeta_2)^3} + \frac{d^2\zeta_1^2}{(\zeta_1+\zeta_2)^4}]\exp(\frac{-d\zeta_1}{\zeta_1+\zeta_2}) , \\ &\eta_{12} = [\frac{d\zeta_2 - d\zeta_1}{(\zeta_1+\zeta_2)^3} - \frac{d^2\zeta_1\zeta_2}{(\zeta_1+\zeta_2)^4}]\exp(\frac{-d\zeta_1}{\zeta_1+\zeta_2}) , \quad \eta_{21} = \eta_{12} . \end{aligned} \tag{44}$$

Thus,

$$\begin{aligned} \mathrm{E}(e^{-d\delta} \mid x, y) = &\exp(\frac{-d\zeta_1}{\zeta_1+\zeta_2}) + \frac{1}{2}\exp(\frac{-d\zeta_1}{\zeta_1+\zeta_2})\{[\frac{2d\zeta_2}{(\zeta_1+\zeta_2)^3} + \frac{d^2\zeta_2^2}{(\zeta_1+\zeta_2)^4} - \frac{2d\zeta_2}{(\zeta_1+\zeta_2)^2}\pi_1^*]\varphi_{11} \\ &+[\frac{-2d\zeta_1}{(\zeta_1+\zeta_2)^3} + \frac{d^2\zeta_1^2}{(\zeta_1+\zeta_2)^4} + \frac{2d\zeta_1}{(\zeta_1+\zeta_2)^2}\pi_2^*]\varphi_{22} + \frac{-d\zeta_2}{(\zeta_1+\zeta_2)^2}\varphi_{11}^2 L_{111} + \frac{d\zeta_1}{(\zeta_1+\zeta_2)^2}L_{222}\varphi_{22}^2\} \end{aligned} . \tag{45}$$

The BE $\hat{\delta}_E$ is given by Eq (46)

$$\hat{\delta}_E = -\frac{1}{d}\ln[\mathrm{E}(e^{-d\delta} \mid x, y)] |_{(\zeta_1, \zeta_2, \sigma)=(\hat{\zeta}_{1,ML}, \hat{\zeta}_{2,ML}, \hat{\sigma}_{ML})} . \tag{46}$$

5. Monte Carlo simulation

In this section, Monte Carlo simulation is used to evaluate the behavior of different estimators under different APT-II censored schemes. We take the true values are $(\zeta_{1,real}, \zeta_{2,real}, \sigma_{real}) = (2,3,5)$. Hence, the true value δ_{real} is 0.4000. Consider two priors, namely, Priors 1 and 2. The hyper-parameters of Prior 1 are $(a_1, b_1) = (5,2)$ and $(a_2, b_2) = (3,6)$. Prior 2 is non-informative prior, that is, $a_1 = a_2 = b_1 = b_2 = 0$. Without loss of generality, let $T_1 = x_{m/2}$ and $T_2 = y_{t/5}$. On this basis, the trails are N at 10,000 times. We consider two cases with different censored schemes, which are detailed in Table 1. The point estimates are compared by average bias (AB) and mean squared error (MSE). The performance of confidence interval is represented by the average width (AW) and coverage probability (CP). All the results are displayed in Tables 2–8. It is necessary to select initial values using iteration method, so we take AMLE $\hat{\delta}_{AML}$ to substitute for MLE $\hat{\delta}_{ML}$. The algorithm of generating APT-II censored data is shown in Algorithm 1. Finally, the AB, MSE and AW are calculated by the following formulas:

$$\text{AB} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\delta}_i - \delta_{real}),\ \ \text{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\delta}_i - \delta_{real})^2 \ \ \text{and}\ \ \text{AW} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\delta}_{i,up} - \hat{\delta}_{i,low}).$$

Algorithm 1.

(1) Generate two sets of random numbers $(w_{x,1}, w_{x,2}, ..., w_{x,m})$ and $(w_{y,1}, w_{y,2}, ..., w_{y,t})$ from $\text{U}(0,1)$.

(2) Let $v_{x,i} = w_{x,i}^{(i+Q_m+Q_{m-1}+...+Q_{m-i+1})^{-1}}$ ($i = 1, 2, ..., m$) and $v_{y,j} = w_{y,j}^{(j+R_t+R_{t-1}+...+R_{t-j+1})^{-1}}$ ($j = 1, 2, ..., t$). Set $u_{x,i} = 1 - v_{x,m}v_{x,m-1}...v_{x,m-i+1}$ and $u_{y,j} = 1 - v_{y,t}v_{y,t-1}...v_{y,t-j+1}$.

(3) Let $x_i = F^{-1}(u_{x,i}; \zeta_{1,real}, \sigma_{real})$ and $y_j = F^{-1}(u_{y,j}; \zeta_{2,real}, \sigma_{real})$, where F is the CDF of IWD. Then, $(x_1, x_2, ..., x_m)$ is the progressive type-II censored data from $\text{IW}(\zeta_{1,real}, \sigma_{real})$ with censored scheme $(Q_1, Q_2, ..., Q_m)$ and $(y_1, y_2, ..., y_t)$ is the progressive type-II censored data from $\text{IW}(\zeta_{2,real}, \sigma_{real})$ with censored scheme $(R_1, R_2, ..., R_t)$.

(4) Determine J and K such that $x_J < T_1 < x_{J+1}$ and $y_K < T_2 < y_{K+1}$. Remove $x_{J+2}, x_{J+3}, ..., x_m$ and $y_{K+2}, y_{K+3}, ..., y_t$.

(5) Generate the first $m - J - 1$ order statistics from the truncated distribution $\frac{f(x; \zeta_{1,real}, \sigma_{real})}{1 - F(x_{J+1}; \zeta_{1,real}, \sigma_{real})}$ and denote them as $x_{J+2}, x_{J+3}, ..., x_m$. Then, the censored scheme changes to $(Q_1, ..., Q_J, 0, ..., 0, Q_m = n_1 - m - \sum_{i=1}^{J} Q_i)$. Similarly, generate the first $t - K + 1$ order statistics from $\frac{f(y; \zeta_{2,real}, \sigma_{real})}{1 - F(y_{K+1}; \zeta_{2,real}, \sigma_{real})}$ as $y_{K+2}, y_{K+3}, ..., y_t$. Then, the censored scheme changes to $(R_1, ..., R_K, 0, ..., 0, R_t = n_2 - t - \sum_{i=1}^{K} R_i)$.

Table 1. The censored schemes.

	(n_1, m)	Q	(n_2, t)	R
Case 1	$(30,10)$	$Q1 = (0*8, 10*2)$	$(40, 20)$	$R1 = (2*10, 0*10)$
		$Q2 = (20*1, 0*9)$		$R2 = (0*19, 20*1)$
		$Q3 = ((0,5)*5)$		$R3 = ((0,0,0,0,5)*4)$
Case 2	$(50, 20)$	$Q1 = (10*1, 0*18, 20*1)$	$(50,30)$	$R1 = (5*2, 0*13, 5*2, 0*13)$
		$Q2 = (0*19, 30*1)$		$R2 = (0*20, 2*10)$
		$Q3 = (0*10, 2*15)$		$R3 = (10*1, 0*28, 10*1)$
Case 3	$(100, 50)$	$Q1 = (10*5, 0*45)$	$(150, 70)$	$R1 = (2*40, 0*30)$
		$Q2 = (20*1, 0*24, 30*1, 0*24)$		$R2 = (30*1, 0*30, 50*1, 0*38)$
		$Q3 = (0*20, 50*1, 0*29)$		$R3 = (0*45, 80*1, 0*24)$

Table 2. The MSEs and ABs of δ under Prior 1 based on Case 1.

Censored scheme	MSE				AB			
	$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$		$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$	
			$d=3$	$d=-3$			$d=3$	$d=-3$
$Q1, R1$	0.0165	0.0495	0.0406	0.0397	0.1129	0.2181	0.2003	0.1957
$Q1, R2$	0.0064	0.0044	0.0868	0.0291	-0.0152	0.3182	0.3529	0.1627
$Q1, R3$	0.0060	0.0031	0.0663	0.0331	0.0400	0.2613	0.2569	0.1765
$Q2, R1$	0.0008	0.0034	0.0031	0.0030	-0.0140	0.0543	0.0514	0.0492
$Q2, R2$	0.0032	0.0025	0.0034	0.0019	-0.0481	0.0571	0.0532	0.0373
$Q2, R3$	0.0021	0.0029	0.0025	0.0018	-0.0374	0.0473	0.0447	0.0368
$Q3, R1$	0.0032	0.0151	0.0137	0.0135	0.0422	0.1189	0.1139	0.1114
$Q3, R2$	0.0027	0.0078	0.0167	0.0107	-0.0011	0.1285	0.1247	0.0968
$Q3, R3$	0.0021	0.0185	0.0130	0.0106	0.0130	0.1155	0.1099	0.0973

Table 3. The MSEs and ABs of δ under Prior 1 based on Case 2.

Censored scheme	MSE				AB			
	$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$		$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$	
			$d=3$	$d=-3$			$d=3$	$d=-3$
$Q1, R1$	0.0021	0.0096	0.0090	0.0090	0.0385	0.0955	0.0923	0.0922
$Q1, R2$	0.0008	0.0076	0.0072	0.0061	0.0047	0.0850	0.0824	0.0751
$Q1, R3$	0.0017	0.0094	0.0088	0.0086	0.0331	0.0944	0.0913	0.0899
$Q2, R1$	0.0136	0.0320	0.0288	0.0298	0.1113	0.1775	0.1683	0.1709
$Q2, R2$	0.0029	0.0349	0.0273	0.0222	0.0530	0.1707	0.1640	0.1464
$Q2, R3$	0.0113	0.0311	0.0281	0.0284	0.1008	0.1749	0.1662	0.1665
$Q3, R1$	0.0129	0.0318	0.0286	0.0292	0.1068	0.1767	0.1676	0.1685
$Q3, R2$	0.0035	0.0297	0.0281	0.0203	0.0377	0.1712	0.1661	0.1390
$Q3, R3$	0.0102	0.0306	0.0276	0.0272	0.0941	0.1731	0.1646	0.1625

Table 4. The MSEs and ABs of δ under Prior 1 based on Case 3.

Censored scheme	MSE				AB			
	$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$		$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$	
			$d=3$	$d=-3$			$d=3$	$d=-3$
$Q1, R1$	3.61E-4	0.0012	0.0011	0.0012	0.0137	0.0318	0.0304	0.0326
$Q1, R2$	7.16E-4	0.0018	0.0017	0.0019	0.0229	0.0403	0.0389	0.0412
$Q1, R3$	6.24E-4	1.60E-4	1.56E-4	1.65E-4	-0.0208	0.0013	2.01E-4	0.0019
$Q2, R1$	0.0052	0.0082	0.0078	0.0083	0.0708	0.0893	0.0872	0.0899
$Q2, R2$	0.0065	0.0096	0.0092	0.0097	0.0793	0.0972	0.0950	0.0977
$Q2, R3$	0.0016	0.0038	0.0035	0.0038	0.0373	0.0597	0.0579	0.0601
$Q3, R1$	0.0024	0.0046	0.0040	0.0046	0.0421	0.0638	0.0620	0.0641
$Q3, R2$	0.0039	0.0066	0.0063	0.0067	0.0574	0.0775	0.0757	0.0780
$Q3, R3$	9.39E-4	9.27E-4	8.63 E-4	9.09E-4	-0.0144	0.0205	0.0192	0.0193

Table 5. The MSEs and ABs of δ under Prior 2 based on Case 1.

Censored scheme	MSE				AB			
	$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$		$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$	
			$d=3$	$d=-3$			$d=3$	$d=-3$
$Q1, R1$	0.0165	0.0179	0.0129	0.0175	0.0385	0.1173	0.0967	0.1177
$Q1, R2$	0.0064	0.0061	0.0059	0.0059	0.0047	-0.0063	-0.0235	-0.0015
$Q1, R3$	0.0060	0.0066	0.0044	0.0066	0.0331	0.0447	0.0240	0.0483
$Q2, R1$	0.0008	0.0009	0.0010	0.0007	0.1113	-0.0162	-0.0204	-0.0100
$Q2, R2$	0.0032	0.0035	0.0038	0.0027	0.0530	-0.0504	-0.0538	-0.0427
$Q2, R3$	0.0021	0.0023	0.0025	0.0017	0.1008	-0.0397	-0.0434	-0.0327
$Q3, R1$	0.0032	0.0032	0.0026	0.0037	0.1068	0.0424	0.0353	0.0480
$Q3, R2$	0.0027	0.0028	0.0027	0.0027	0.0377	-0.0019	-0.0082	0.0059
$Q3, R3$	0.0021	0.0022	0.0019	0.0023	0.0941	0.0134	0.0063	0.0201

Table 6. The MSEs and ABs of δ under Prior 2 based on Case 2.

Censored scheme	MSE				AB			
	$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$		$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$	
			$d=3$	$d=-3$			$d=3$	$d=-3$
$Q1, R1$	0.0021	0.0021	0.0018	0.0025	0.0385	0.0386	0.0342	0.0433
$Q1, R2$	0.0008	0.0008	0.0007	0.0009	0.0047	0.0048	0.0009	0.0101
$Q1, R3$	0.0017	0.0018	0.0015	0.0021	0.0331	0.0340	0.0295	0.0380
$Q2, R1$	0.0136	0.0141	0.0118	0.0145	0.1113	0.1139	0.1036	0.1161
$Q2, R2$	0.0029	0.0049	0.0037	0.0053	0.0530	0.0565	0.0465	0.0605
$Q2, R3$	0.0113	0.0118	0.0097	0.0123	0.1008	0.1026	0.0923	0.1052
$Q3, R1$	0.0129	0.0134	0.0110	0.0136	0.1068	0.1097	0.0987	0.1111
$Q3, R2$	0.0035	0.0037	0.0029	0.0040	0.0377	0.0408	0.0313	0.0451
$Q3, R3$	0.0102	0.0108	0.0087	0.0111	0.0941	0.0964	0.0855	0.0988

Table 7. The MSEs and ABs of δ under Prior 2 based on Case 3.

Censored scheme	MSE				AB			
	$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$		$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$	
			$d=3$	$d=-3$			$d=3$	$d=-3$
$Q1, R1$	3.61E-4	3.50E-4	3.11E-4	3.95E-4	0.0137	0.0132	0.0117	0.0149
$Q1, R2$	7.16E-4	6.92E-4	6.24E-4	7.66E-4	0.0229	0.0223	0.0208	0.0239
$Q1, R3$	6.24E-4	6.54E-4	7.08E-4	5.76E-4	-0.0208	-0.0215	-0.0228	-0.0196
$Q2, R1$	0.0052	0.0053	0.0049	0.0055	0.0708	0.0710	0.0688	0.0724
$Q2, R2$	0.0065	0.0066	0.0062	0.0068	0.0793	0.0798	0.0775	0.0811
$Q2, R3$	0.0016	0.0016	0.0015	0.0017	0.0373	0.0371	0.0350	0.0387
$Q3, R1$	0.0024	0.0024	0.0022	0.0025	0.0421	0.0426	0.0407	0.0442
$Q3, R2$	0.0039	0.0039	0.0037	0.0041	0.0574	0.0574	0.0554	0.0589
$Q3, R3$	9.39E-4	9.52E-4	9.92E-4	8.85E-4	-0.0144	-0.0151	-0.0168	-0.0131

Table 8. The CP and AW of δ ($\lambda = 0.05$).

Censored scheme	CP			AW		
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
$Q1, R1$	0.8838	0.9564	0.9993	0.3178	0.2518	0.1287
$Q1, R2$	0.8384	0.9075	0.9951	0.3723	0.2999	0.1288
$Q1, R3$	0.8855	0.9289	0.9976	0.3001	0.2714	0.1315
$Q2, R1$	0.9681	0.9456	0.9869	0.2493	0.2657	0.1341
$Q2, R2$	0.8468	0.9702	0.9840	0.3669	0.2500	0.1339
$Q2, R3$	0.9238	0.9396	0.9707	0.2783	0.2672	0.1374
$Q3, R1$	0.9059	0.9422	0.9766	0.2456	0.2599	0.1238
$Q3, R2$	0.8831	0.9747	0.9873	0.2831	0.2588	0.1287
$Q3, R3$	0.8298	0.9657	0.9634	0.2730	0.2588	0.1283

From Tables 2–8, the following conclusions may be drawn:

(1) When the effective sample sizes (m and t) increase, the MSEs of AMLE and BE decrease. Therefore, enlarging the effective sample size can appropriately enhance the accuracy of the estimation.

(2) The BEs under Prior 2 perform similarly to the AMLE in terms of MSEs. However, the BEs under Prior 1 perform worse than AMLE.

(3) Under the same prior, as the sample size increases, the available information increases. Therefore, the MSEs show a decreasing trend.

(4) Under Prior 1, the BE based on LINEX loss function with $d = -3$ has better behavior than the other BEs. Under Prior 2, the performance of all the BEs is comparable.

(5) With the increase of the effective sample sizes, the CPs gradually reach the confidence level of 95%.

6. Real data analysis

In this section, two real data sets are used to validate the feasibility of the proposed method. These data sets are reported by Nelson [34], indicating the time when the electrodes are broken down by the insulating fluids at different voltages. X represents the insulating fluid at a voltage of 34kV, and Y represents the insulating fluid at a voltage of 36kV. The data sets are:

X : 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89;

Y : 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50.

First, we need to check whether the IWD can fit these data sets. We know that if a random variable T follows Weibull distribution, $X = T^{-1}$ follows IWD. Set $X = T^{-1}$ and $Y = Z^{-1}$. The transformed data sets, Anderson-Darling (A-D) statistics and p-values are presented in Table 9.

Table 9. The transformed data sets and p-values.

	Data sets								A-D	p-values
T	0.0137	0.0272	0.0295	0.0308	0.0315	0.0829	0.1209	0.1248	0.6006	0.1132
	0.1361	0.1538	0.2062	0.2141	0.2410	0.3165	0.3597	0.7634		
	1.0417	1.2821	5.2632							
Z	0.0392	0.0726	0.1869	0.2506	0.2725	0.3448	0.3690	0.3876	0.3121	0.5858
	0.4831	0.5076	0.5917	1.0101	1.0417	1.6949	2.8571			

As we can see, the p-values are all greater than a 5% significance level, which means that the Weibull distribution can fit these data sets T and Z effectively. In other words, the IWD is suitable for fitting data sets X and Y. Figures 4 and 5 give the probability-probability (P-P) plot and quantile-quantile (Q-Q) plot to visually show the fitting.

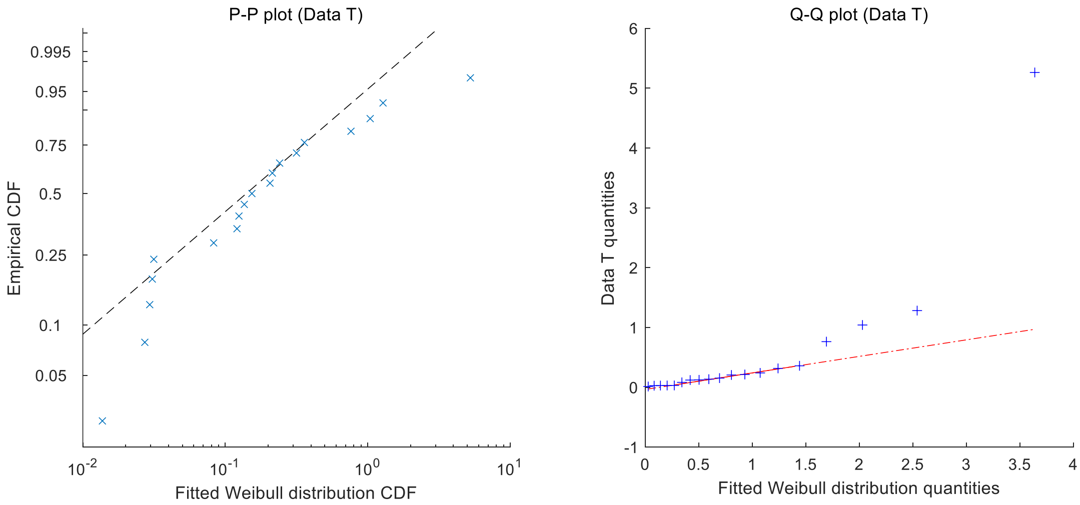

Figure 4. P-P and Q-Q plots for Data T.

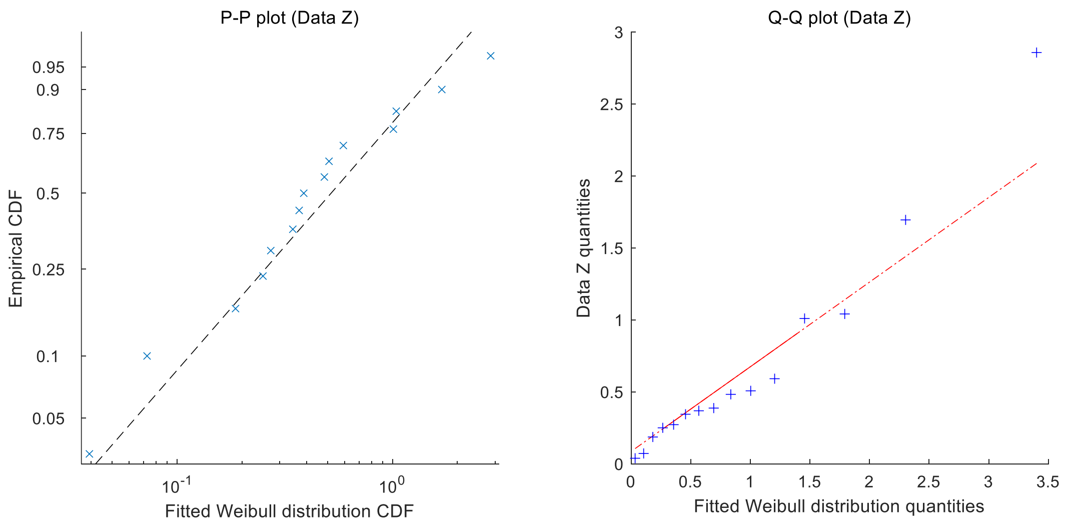

Figure 5. P-P and Q-Q plots for Data Z.

Next, Table 10 presents the different APT-II censored schemes. Since we cannot obtain any prior information, we take the hyperparameters of the prior distribution as $a_1 = b_1 = a_2 = b_2 = 0$. The approximate maximum likelihood estimates, the Bayesian estimates under symmetric entropy loss function and LINEX loss function with $d = 3$ and $d = -3$, and 95% ACIs are given in Table 11. We illustrate the existence and uniqueness of MLE through visual representations. Without the loss of generality, we choose censored scheme 3 in Table 10 to plot, as shown in Figure 6.

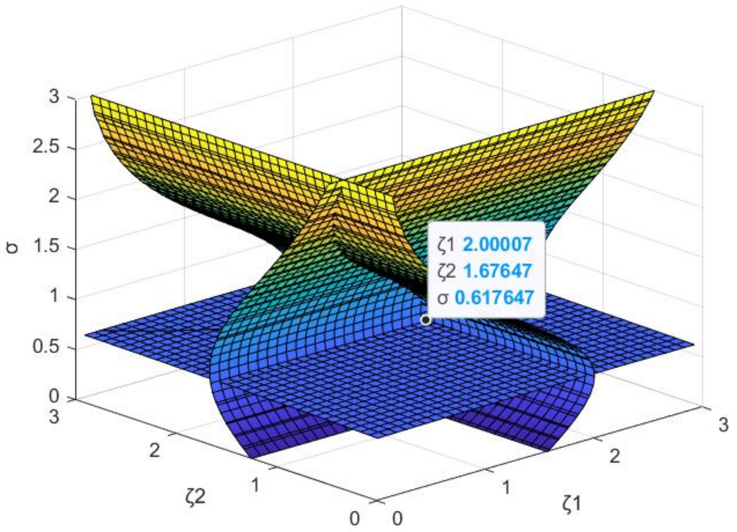

Figure 6. The graphs of partial derivatives of the log-likelihood function.

Table 10. Different censored schemes.

Censored scheme	Q	R
1	(2*5, 0*2, 1*1)	(1*5, 0*5)
2	(0*7, 11*1)	(0*9, 5*1)
3	(0*3, 5*1, 0*3, 6*1)	(0*4, 5*1, 0*3)

Table 11. The estimates and ACIs of δ.

Censored scheme	$\hat{\delta}_{AML}$	$\hat{\delta}_S$	$\hat{\delta}_E$		ACI
			$d = 3$	$d = -3$	
1	0.5179	0.5228	0.5080	0.5335	(0.3815, 0.6579)
2	0.5055	0.5131	0.4912	0.5244	(0.2808, 0.7303)
3	0.4425	0.4507	0.4315	0.4634	(0.2852, 0.5998)

7. Conclusions

The APT-II censored scheme allows more flexibility during the lifetime test, thereby providing more control on the test, leading to shorter test time and more failed observations. In this paper, we investigate the classical and Bayesian estimation of stress-strength reliability based on APT-II censored sample for IWD with the same shape but different scale parameters. The MLE can be obtained by the iteration algorithm. Note that the form of MLE is not explicit, and we propose AMLE and construct ACI. The BEs are also derived based on gamma prior under symmetric entropy loss function and LINEX loss function. Lindley's approximation is used to obtain the approximate Bayesian estimates. The simulation results show that MLE has the smaller MSE than BE under gamma prior. In addition, the censored scheme has a significant impact on the estimates. Yan et al. [35] proposed an improved adaptive progressive type-II censored scheme. Based on this censored scheme, we will consider the statistical inference of multi-component stress-strength reliability for other distributions, such as Weighted Exponential distribution and improved Lomax distribution.

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Conflict of interest

The authors declare no conflict of interest.

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